Fluctuations of passive scalar concentration with nonzero mean gradient in random velocity fields

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(Submitted 19 April 1995)
Zh. Éksp. Teor. Fiz. **108**, 1403–1410 (October 1995)

Fluctuations of a passive scalar concentration in a random velocity field are considered. Statistical quantities related to the spatial derivatives of the concentration field are calculated in the presence of a nonzero mean concentration gradient in the delta-correlation approximation. This permits an estimation of the stationary variance of the concentration, which has a logarithmic dependence on the molecular diffusion coefficient. Conditions for the applicability of the delta-correlated approximation in this problem are obtained. © *1995 American Institute of Physics.*

1. GOVERNING EQUATIONS

The governing equations for the problem of diffusing passive tracers in random velocity fields are of the following two types:

$$\left(\frac{\partial}{\partial t}+\mathbf{u}(\mathbf{r},t)\cdot\frac{\partial}{\partial \mathbf{r}}\right)q(\mathbf{r},t)=\kappa\frac{\partial^2}{\partial \mathbf{r}^2}q(\mathbf{r},t),$$

$$q(\mathbf{r},0)=q_0(\mathbf{r}), \tag{1}$$

$$\left(\frac{\partial}{\partial t}+\mathbf{u}(\mathbf{r},t)\cdot\frac{\partial}{\partial \mathbf{r}}\right)p_i(\mathbf{r},t)=-\frac{\partial u_k(\mathbf{r},t)}{\partial r_i}p_k(\mathbf{r},t)+\kappa\frac{\partial^2}{\partial \mathbf{r}^2}p_i(\mathbf{r},t),\quad i=1,2,3, \tag{2}$$

$$\mathbf{p}(\mathbf{r},0)=\mathbf{p}_0(\mathbf{r})=\frac{\partial}{\partial \mathbf{r}}q_0(\mathbf{r}).$$

Equation (1) describes the evolution of the concentration field $q(\mathbf{r},t)$ of a passive scalar, and (2) describes its spatial gradient $\mathbf{p}(\mathbf{r},t)=\partial q(\mathbf{r},t)/\partial\mathbf{r}$. The constant κ is the molecular diffusion coefficient. In this paper, the velocity field $\mathbf{u}(\mathbf{r},t)$ is assumed to be a zero-mean random Gaussian incompressible field (div $\mathbf{u}=0$), homogeneous and isotropic in space and stationary in time with the covariance tensor

$$B_{ij}(\mathbf{r}-\mathbf{r}',t-t')=\langle u_i(\mathbf{r},t)u_j(\mathbf{r}',t')\rangle, \tag{3}$$

and the spectral density $E(k,t)$ of the energy of the fluid flow. The covariance tensor is determined by the formula

$$B_{ij}(\mathbf{r},t)=\int d\mathbf{k}E(k,t)\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)e^{i\mathbf{kr}}. \tag{4}$$

For a general initial distribution $q_0(\mathbf{r})$ of the passive scalar, statistical analysis of the solutions of Eq. (1) is quite involved. The linearity of (1) makes it relatively easy to study such simple statistical characteristics as the mean concentration $\langle q(\mathbf{r},t)\rangle$ and the covariance function $\langle q(\mathbf{r}_1,t)q(\mathbf{r}_2,t)\rangle$ which, for the simplest models of the velocity field $\mathbf{u}$, are also described by linear equations (see, e.g., Ref. 1). Here $\langle ...\rangle$ denotes statistical averaging over the ensemble of realizations of the field $\mathbf{u}$. However, because the diffusion term contains second-order derivatives, it is much more difficult to derive an equation describing the evolution of, for example, the one-point probability densities of solutions of (1).

A description of the probability density of the concentration requires consideration of Eqs. (2) for its spatial gradient, and generally, equations for derivatives of higher order. Also, one can write equations in variational derivatives (Hopf equations) for the characteristic functional (see, e.g., Ref. 2). However, at this time there are no satisfactory methods for solving and analyzing them. For these reasons, it is necessary to study approximate methods of analyzing statistical solutions of equation (1) and (2). Some of these methods have been recently developed in Ref. 3.

In the present paper we discuss the evolution of the statistical characteristics of a passive scalar concentration with a nonzero mean concentration gradient, i.e., we study Eqs. (1), (2) with the initial conditions

$$q_0(\mathbf{r})=\mathbf{G}\cdot\mathbf{r},\quad \mathbf{p}_0(\mathbf{r})=\mathbf{G}.$$

Recently, this problem has attracted considerable attention, both theoretically and experimentally in Refs. 4, 5. These papers used numerical modeling and construction of phenomenological models to analyze the behavior of the stationary (as $t\to\infty$) probability density of the passive scalar gradient, and the occurrence of distributions with "slowly decaying tails" of the exponential type. Note that in Ref. 5 it was demonstrated that the stationary probability density of the scalar concentration also has a slowly decaying tail.

2. SPLITTING THE COVARIANCE

Representing concentration $q(\mathbf{r},t)$ in the form

$$q(\mathbf{r},t)=\mathbf{G}\cdot\mathbf{r}+\tilde{q}(\mathbf{r},t), \tag{5}$$

we obtain equation

$$\left[\frac{\partial}{\partial r}+\mathbf{u}(\mathbf{r},t)\cdot\frac{\partial}{\partial \mathbf{r}}\right]\tilde{q}(\mathbf{r},t)=-\mathbf{G}\cdot\mathbf{u}(\mathbf{r},t)+\kappa\frac{\partial^2}{\partial \mathbf{r}^2}\tilde{q}(\mathbf{r},t),\quad \tilde{q}(\mathbf{r},0)=0 \tag{6}$$

for the fluctuating component $\tilde{q}$ of the passive scalar concentration. Because of the statistical spatial homogeneity of field $\tilde{q}(\mathbf{r},t)$, the analysis of Eq. (6) is substantially simpler than the analysis of problem (1). In particular, the statistical moment $\langle\tilde{q}^n(\mathbf{r},t)\rangle$, which is independent of $\mathbf{r}$, satisfies the equation

$$\frac{d}{dt}\langle\tilde{q}^n(\mathbf{r},t)\rangle=-n\mathbf{G}\langle\mathbf{u}(\mathbf{r},t)\tilde{q}^{n-1}(\mathbf{r},t)\rangle-\kappa n(n-1)\times\langle\tilde{q}^{n-2}(\mathbf{r},t)\tilde{\mathbf{p}}^2(\mathbf{r},t)\rangle, \tag{7}$$

where

$$\tilde{\mathbf{p}}(\mathbf{r},t)=\frac{\partial}{\partial \mathbf{r}}\tilde{q}(\mathbf{r},t)=\mathbf{p}(\mathbf{r},t)-\mathbf{G}.$$

To split the covariance $\langle\mathbf{u}\tilde{q}^{n-1}\rangle$ on the right-hand side of (7), we will use the Furutsu–Novikov formula,[5,6]

$$\langle u_i(\mathbf{r},t)F[\mathbf{u}]\rangle=\int d\mathbf{r}'\int dt' B_{ij}(\mathbf{r}-\mathbf{r}',t-t')\times\left\langle\frac{\delta F[\mathbf{u}]}{\delta u_j(\mathbf{r}',t')}\right\rangle, \tag{8}$$

which is valid for any zero mean Gaussian field $\mathbf{u}$ and an arbitrary functional $F[\mathbf{u}]$ of this field. Now, Eq. (7) can be rewritten in the form

$$\frac{d}{dt}\langle\tilde{q}^n(\mathbf{r},t)\rangle=-n(n-1)G_i\int d\mathbf{r}'\int_0^t dt' B_{ij}(\mathbf{r}-\mathbf{r}',t-t')\left\langle\tilde{q}^{n-2}(\mathbf{r},t)\frac{\delta\tilde{q}(\mathbf{r},t)}{\delta u_j(\mathbf{r}',t')}\right\rangle-\kappa n(n-1)\langle\tilde{q}^{n-2}(\mathbf{r},t)\tilde{\mathbf{p}}^2(\mathbf{r},t)\rangle. \tag{9}$$

In the delta-correlated approximation, the covariance function $B_{ij}(\mathbf{r},t)$ in (9) can be replaced by

$$B_{ij}(\mathbf{r},t)=2B_{ij}^{\text{eff}}(\mathbf{r})\delta(t), \tag{10}$$

where

$$B_{ij}^{\text{eff}}(\mathbf{r})=\frac{1}{2}\int_{-\infty}^{\infty}dt B_{ij}(\mathbf{r},t)=\int d\mathbf{k}\tilde{E}(k)\times\left(\delta_{ij}-\frac{k_ik_j}{\mathbf{k}^2}\right)e^{i\mathbf{kr}},$$

$$\tilde{E}(k)=\frac{1}{2}\int_{-\infty}^{\infty}dt E(k,t).$$

A justification of this approximation can be found, for example, in Ref. 1; later on we shall give the conditions for its applicability to the problem (6). The variational derivative on the right-hand side of (9) is expressed in terms of the quantity $\delta\tilde{q}(\mathbf{r},t)/\delta u_j(\mathbf{r}',t-0)$, which can be determined directly from the original Eq. (6):

$$\frac{\delta\tilde{q}(\mathbf{r},t)}{\delta u_j(\mathbf{r}',t-0)}=-\delta(\mathbf{r}-\mathbf{r}')\frac{\partial}{\partial r_j}\tilde{q}(\mathbf{r},t)-G_j\delta(\mathbf{r}-\mathbf{r}'). \tag{11}$$

Substituting (10) and (11) into (9), we obtain the equation

$$\frac{d}{dt}\langle\tilde{q}^n(\mathbf{r},t)\rangle=n(n-1)D_1\mathbf{G}^2\langle\tilde{q}^{n-2}(\mathbf{r},t)\rangle-\kappa n(n-1)\times\langle\tilde{q}^{n-2}(\mathbf{r},t)\tilde{\mathbf{p}}^2(\mathbf{r},t)\rangle, \tag{12}$$

where

$$B_{ij}^{\text{eff}}(0)=D_1\delta_{ij},\quad D_1=\frac{N-1}{N}\int d\mathbf{k}\tilde{E}(k), \tag{12'}$$

and N is the spatial dimension (i.e., $N=3$ or 2).

3. STATIONARY REGIME

In the stationary regime ($t\to\infty$), we obtain from (12) the following equation for the time-independent statistical characteristics:

$$\langle\tilde{q}^{n-2}(\mathbf{r},t)\tilde{\mathbf{p}}^2(\mathbf{r},t)\rangle=\frac{D_1\mathbf{G}^2}{\kappa}\langle\tilde{q}^{n-2}(\mathbf{r},t)\rangle. \tag{13}$$

In particular, for $n=2$, it follows from (13) that the stationary value of the second moment of the gradient of the fluctuation is

$$\langle\tilde{\mathbf{p}}^2(\mathbf{r},t)\rangle=\langle[\nabla\tilde{q}(\mathbf{r},t)]^2\rangle=D_1\mathbf{G}^2/\kappa \tag{13'}$$

and consequently, Eq. (13) can be rewritten in the form

$$\langle\tilde{q}^{n-2}(\mathbf{r},t)\tilde{\mathbf{p}}^2(\mathbf{r},t)\rangle=\langle\tilde{\mathbf{p}}^2(\mathbf{r},t)\rangle\langle\tilde{q}^{n-2}(\mathbf{r},t)\rangle. \tag{13''}$$

i.e., in the stationary regime, the quantities $\tilde{q}(\mathbf{r},t)$ and $\tilde{\mathbf{p}}^2(\mathbf{r},t)$ are statistically uncorrelated. Equation (12) can be rewritten in the form

$$\frac{d}{dt}\langle\tilde{q}^n(\mathbf{r},t)\rangle=n(n-1)D_1\mathbf{G}^2\langle f(\mathbf{r},t)\tilde{q}^{n-2}(\mathbf{r},t)\rangle, \tag{14}$$

where

$$f(\mathbf{r},t)=1-\frac{\kappa}{D_1\mathbf{G}^2}\tilde{\mathbf{p}}^2(\mathbf{r},t).$$

It follows from (14) that the variance of concentration field $\tilde{q}(\mathbf{r},t)$ ($\langle\tilde{q}(\mathbf{r},t)\rangle=0$) is

$$\langle\tilde{q}^2(\mathbf{r},t)\rangle=2D_1\mathbf{G}^2\int_0^t d\tau\langle f(\mathbf{r},\tau)\rangle. \tag{15}$$

In the absence of molecular diffusion we have $f(\tau)=1$, and

$$\langle\tilde{q}^2(\mathbf{r},t)\rangle=2D_1\mathbf{G}^2t. \tag{16}$$

In this case, the one-point distribution of the field $\tilde{q}(\mathbf{r},t)$ is Gaussian $\tilde{q}(\mathbf{r},t)$ is uncorrelated with its spatial derivatives. In the general case, the solution (16) is valid for sufficiently small times.

Notice that, in the above approximation, the covariance function

$$\Gamma(\mathbf{r},t)=\langle\tilde{q}(\mathbf{r}_1,t)\tilde{q}(\mathbf{r}_2,t)\rangle,\quad \mathbf{r}=\mathbf{r}_1-\mathbf{r}_2,$$

satisfies the equation

$$\frac{\partial}{\partial t}\Gamma(\mathbf{r},t)=2[B_{ij}^{\text{eff}}(0)-B_{ij}^{\text{eff}}(\mathbf{r})+\kappa\delta_{ij}]\frac{\partial^2}{\partial r_i\partial r_j}\Gamma(\mathbf{r},t)+2G_iG_jB_{ij}^{\text{eff}}(\mathbf{r}),$$

and, consequently, the stationary values $\Gamma(\mathbf{r})=\lim_{t\to\infty}\Gamma(\mathbf{r},t)$ are described by

$$G_iG_j\,B_{ij}^{\text{eff}}(\mathbf{r})=-[B_{ij}^{\text{eff}}(0)-B_{ij}^{\text{eff}}(\mathbf{r})+\kappa\delta_{ij}]\frac{\partial^2}{\partial r_i\partial r_j}\Gamma(\mathbf{r}).$$

Setting $\mathbf{r}=0$ in this equation, we obtain Eq. (13′). Differentiating the equation twice with respect to $\mathbf{r}$ and putting $\mathbf{r}=0$, we get

$$\kappa^2\left\langle\left(\frac{\partial^2}{\partial\mathbf{r}^2}\tilde{q}(\mathbf{r},t)\right)^2\right\rangle=(N+2)(N-1)D_2(D_1+\kappa)\mathbf{G}^2,$$

where the coefficient D_2 is determined by the equation

$$-\left.\frac{\partial^2B_{ij}^{\text{eff}}(\mathbf{r})}{\partial r_k\partial r_l}\right|_{\mathbf{r}=0}=D_2((N+1)\delta_{ij}\delta_{kl}-\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}),$$

that is

$$D_2=\frac{1}{N(N+2)}\int d\mathbf{k}k^2\tilde{E}(k).$$

To evaluate $\langle\tilde{q}^2(\mathbf{r},t)\rangle|_{t\to\infty}$ in (15) we need information about the time evolution of the second moment $\langle\tilde{\mathbf{p}}^2(\mathbf{r},t)\rangle$. That information can be obtained by considering Eq. (2), in which the molecular diffusion term is neglected. Although this approximation for Eq. (2) is good only for a finite time interval,[7] knowledge of the exact value (13′) will give us an opportunity to estimate the stationary value of the variance of the passive scalar concentration in the presence of molecular diffusion.

We introduce the function

$$\Phi_{t,\mathbf{r}}(\mathbf{p})=\delta(\mathbf{p}(\mathbf{r},t)-\mathbf{p}),$$

satisfying the Liouville equation

$$\left[\frac{\partial}{\partial t}+\mathbf{u}(\mathbf{r},t)\cdot\frac{\partial}{\partial\mathbf{r}}\right]\Phi_{t,\mathbf{r}}(\mathbf{p})=\frac{\partial}{\partial p_i}\frac{\partial u_k(\mathbf{r},t)}{\partial r_i}p_k\Phi_{t,\mathbf{r}}(\mathbf{p}),\tag{17}$$

with the initial condition

$$\Phi_{0,\mathbf{r}}(\mathbf{p})=\delta(\mathbf{G}-\mathbf{p}).\tag{18}$$

Averaging Eq. (17) over the ensemble of realizations of the random field $\mathbf{u}$, and using the Furutsu–Novikov formula (8), the delta-correlated approximation (10) for the random field $\mathbf{u}$, a formula for the variational derivative

$$\frac{\delta\Phi_{t,\mathbf{r}}(\mathbf{p})}{\delta u_j(\mathbf{r}',t-0)}=\left\{-\delta(\mathbf{r}-\mathbf{r}')\frac{\partial}{\partial r_j}+\frac{\partial}{\partial p_i}\frac{\partial\delta(\mathbf{r}-\mathbf{r}')}{\partial r_i}p_j\right\}\Phi_{t,\mathbf{r}}(\mathbf{p})$$

derived from (17) and, finally, the statistical homogeneity of the field $\mathbf{p}(\mathbf{r},t)$, we obtain the Fokker–Planck equation

$$\frac{\partial P_t(\mathbf{p})}{\partial t}=D_2\left[(N+1)\frac{\partial^2}{\partial\mathbf{p}^2}\mathbf{p}^2-2\frac{\partial}{\partial\mathbf{p}}\mathbf{p}-2\left(\frac{\partial}{\partial\mathbf{p}}\mathbf{p}\right)^2\right]P_t(\mathbf{p}),$$
$$P_0(\mathbf{p})=\delta(\mathbf{G}-\mathbf{p}),\tag{19}$$

of the probability density $P_t(\mathbf{p})=\langle\Phi_{t,\mathbf{r}}(\mathbf{p})\rangle$ of the gradient of the passive scalar concentration.

4. CLOSED SET OF MOMENT EQUATIONS

The specific structure of Eq. (19) permits us to derive a closed set of equations

$$\begin{aligned}&\frac{\partial}{\partial t}\langle|\mathbf{p}(\mathbf{r},t)|^n\rangle=D_2n(N+n)(N-1)\langle|\mathbf{p}(\mathbf{r},t)|^n\rangle,\\&\frac{\partial}{\partial t}\langle\mathbf{p}(\mathbf{r},t)|\mathbf{p}(\mathbf{r},t)|^n\rangle=D_2n(N+n+2)(N-1)\times\langle\mathbf{p}(\mathbf{r},t)|\mathbf{p}(\mathbf{r},t)|^n\rangle\end{aligned}\tag{20}$$

for the moment functions of the gradient field $\mathbf{p}(\mathbf{r},t)$ and, consequently,

$$\begin{aligned}&\langle|\mathbf{p}|^n\rangle=|\mathbf{G}|^n\ \exp[D_2n(N+n)(N-1)t],\\&\langle\mathbf{p}|\mathbf{p}|^n\rangle=\mathbf{G}|\mathbf{G}|^n\ \exp[D_2n(N+n+2)(N-1)t]\end{aligned}\tag{21}$$

and, in particular,

$$\langle\mathbf{p}(\mathbf{r},t)\rangle=\mathbf{G},\quad\langle|\tilde{\mathbf{p}}(\mathbf{r},t)|^2\rangle=\mathbf{G}^2\{\exp[2D_2(N+2)\times(N-1)t]-1\},\tag{22}$$

where $\tilde{\mathbf{p}}(\mathbf{r},t)=\partial\tilde{q}(\mathbf{r},t)/\partial\mathbf{r}$ is the gradient of the fluctuation of the passive scalar concentration. Thus the moments of the magnitude of the concentration gradient grow exponentially in time, and the normalized quantity $|\mathbf{G}|^{-1}|\mathbf{p}(\mathbf{r},t)|$ has a log-normal probability distribution, i.e., the quantity $\chi(\mathbf{r},t)=\ln\{|\mathbf{G}|^{-1}|\mathbf{p}(\mathbf{r},t)|\}$ has a Gaussian distribution with parameters

$$\langle\chi(\mathbf{r},t)\rangle=D_2N(N-1)t,\quad\sigma_\chi^2(t)=2D_2(N-1)t.\tag{23}$$

The log-normal probability distribution of the concentration gradient was, apparently, first discovered in Ref. 7 (although equations like (19) do not appear there), and its properties can be found, for example, in Ref. 8.

As we indicated earlier, the log-normal approximation of the probability density (in the absence of molecular diffusion) and, in particular, the exponential growth of (22) in time are valid only for a finite time interval. However, the exact knowledge of (13′) allows us to use the exponential law (22) to approximate the time behavior of the quantity $\langle\tilde{\mathbf{p}}^2(\mathbf{r},t)\rangle$ in the presence of molecular diffusion up to the time

$$t_0\sim\frac{1}{2D_2(N+2)(N-1)}\ln\frac{D_1+\kappa}{\kappa}\tag{24}$$

when it attains its stationary value $\langle \tilde{\mathbf{p}}^2 \rangle$ described by Eq. (13′). In other words, we use Eq. (2) without the molecular diffusion term for $t<t_0$ and Eq. (13′) for $t>t_0$. Hence, we have $\int_0^\infty dt f(t) \sim t_0$ and, by (15), we get the stationary variance of the field $\tilde{q}(\mathbf{r},t)$, i.e.,

$$\lim_{t\to\infty} \langle \tilde{q}^2(\mathbf{r},t)\rangle \sim \frac{1}{(N+2)(N-1)} \frac{D_1}{D_2} \mathbf{G}^2 \ln \frac{D_1+\kappa}{\kappa}. \tag{25}$$

Using $D_1 \sim \sigma_u^2 \tau_0$ and $D_1/D_2 \sim r_0^2$, where σ_u^2 is the variance of velocity fluctuations, and τ_0 and r_0 are, respectively, its time and space correlation radii, we find from (24), (25) that the time t_0, in view of its logarithmic dependence on κ, cannot be too large. In addition,

$$\langle \tilde{q}^2 \rangle \sim \mathbf{G}^2 r_0^2 \ln \frac{\sigma_u^2 \tau_0}{\kappa}, \quad \kappa \ll \sigma_u^2 \tau_0.$$

Earlier, we observed that the passive scalar concentration is uncorrelated with the square of its gradient, both in the stationary regime and in the initial time interval (in which molecular diffusion effects are not important). If this were true over the whole time interval, we would have found from Eq. (14) that the one-point distribution of the field $\tilde{q}$ is Gaussian. However, as follows from Ref. 5 (see also Ref. 9), this is not the case, and the existence of such a correlation is the most important factor in the formation of the stationary regime for fluctuations of $\tilde{q}$.

5. APPLICABILITY CONDITIONS

In conclusion, we present conditions for the applicability of the delta-correlated approximation of a random filed $\mathbf{u}$.

We begin with a more general diffusion approximation (see, e.g., Ref. 1), which neglects the influence of the field $\mathbf{u}$ at scales of order τ_0. In this case Eq. (9) for the quantity $\langle q^n(\mathbf{r},t)\rangle$ is exact. Since the principal contribution in the time integral on the right-hand side of (9) comes from $t-t' \sim \tau_0$, we can neglect the fluctuation terms in the corresponding equation for the variational derivative. As a result, we find that

$$\frac{\delta \tilde{q}(\mathbf{r},t)}{\delta u_j(\mathbf{r}',t')} = - \exp[\kappa(t-t')\Delta] \times \left[\delta(\mathbf{r}-\mathbf{r}') \frac{\partial}{\partial r_j} \tilde{q}(\mathbf{r},t') - G_j \delta(\mathbf{r}-\mathbf{r}') \right], \tag{26}$$

where Δ is the Laplacian. In the same time interval

$$\tilde{q}(\mathbf{r},t') = \exp[-\kappa(t-t')\Delta] \tilde{q}(\mathbf{r},t).$$

In view of the spatial homogeneity, the first term on the right-hand side of (26) makes no contribution, and we obtain the equation

$$\frac{d}{dt} \langle \tilde{q}^n(\mathbf{r},t)\rangle = n(n-1)\mathbf{G}^2 D_1(t,\kappa) \langle \tilde{q}^{n-2}(\mathbf{r},t)\rangle - \kappa n(n-1) \langle \tilde{q}^{n-2}(\mathbf{r},t) \tilde{\mathbf{p}}^2(\mathbf{r},t)\rangle, \tag{27}$$

where

$$D_1(t,\kappa) = \frac{N-1}{N} \int_0^t d\tau \int d\mathbf{k} E(k,\tau) \exp(-\kappa\tau \mathbf{k}^2). \tag{28}$$

Hence the condition for the applicability of the diffusion approximation is

$$D_1(t,\kappa)\mathbf{G}^2 \tau_0 \ll 1. \tag{29}$$

For the delta-correlated approximation, in addition to (29), one assumes

$$\kappa\tau_0/r_0^2 \ll 1, \quad \text{and } t \gg \tau_0. \tag{30}$$

Then, from (12′) the coefficient $D_1(t,\kappa) \sim D_1$, and condition (29) can be rewritten in the form

$$\sigma_u^2 \mathbf{G}^2 \tau_0^2 \ll 1. \tag{31}$$

Also, recall that in the derivation of the stationary value of $\langle \tilde{q}^2 \rangle$ [see (15)], we needed information about the evolution of the random field $\mathbf{p}$. This was provided by Eq. (22), whose validity depended on the condition $D_2 \tau_0 \ll 1$, or equivalently

$$\sigma_u^2 \tau_0^2 \ll r_0^2. \tag{32}$$

Thus, we finally conclude that Eqs. (30)–(32) are the conditions for the applicability of the delta-correlated approximation used throughout this paper. They require that both the fluctuations of the velocity field and the molecular diffusion coefficient be small. These requirements are not severe limitations in the analysis of, for instance, geophysical flows.

This work was supported in part by grants from the U.S. Office of Naval Research, Grants No MBP00, MBP300 from the International Science Foundation and Russian Government, and Projects No 94-05-16151 and 95-05-14247 of the Russian Fund for Fundamental Research.

[1] V. I. Klyatskin, Usp. Fiz. Nauk, **164**, 531 (1994).
[2] E. Hopf, J. Ration Mech. Anal. **1**, 87 (1952); A. Monin and A. Yaglom, *Statistical Fluid Mechanics*, MIT Press, Cambridge, MA (1980).
[3] Ya. G. Sinai and V. Yakhot, Phys. Rev. Lett. **63**, 1962 (1989); H. Chen, S. Chen and R. H. Kraichnan, Phys. Rev. Lett. **63**, 2657 (1989); Y. Kimura and R. H. Kraichnan, Phys. Fluids A **5**, 2264 (1993).
[4] A. Pumir, B. Shraiman, and E. Siggia, Phys. Rev. Lett. **66**, 2984 (1991); J. Gollub, J. Clarke, M. Gharib *et al.*, Phys. Rev. Lett. **67**, 3507 (1991); M. Holzer and A. Pumir, Phys. Rev. E **47**, 202 (1993); A. Pumir, Phys. Fluids **6**, 2118 (1994); M. Holzer and E. Siggia, Phys. Fluids **6**, 1820 (1994).
[5] B. I. Shraiman and E. D. Siggia, Phys. Rev. E **49**, 2912 (1994).
[6] K. Furutsu, J. Res. NBS. D **67**, 303 (1963); E. A. Novikov, Zh. Éksp. Teor. Fiz. **47**, 1919 (1964) [Sov. Phys. JETP **20**, 1290 (1964)].
[7] R. Kraichnan, J. Fluid Mech. **64**, 737 (1974).
[8] V. I. Klyatskin and A. I. Saichev, Usp. Fiz. Nauk. **162**, 161 (1992) [Sov. Phys. Usp. **34**, 185 (1992)].
[9] R. Kraichnan, Phys. Rev. Lett. **72**, 1016 (1994).

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