Some applications of the large river effect to fluctuation theory

A. É. Filippov

Physicotechnical Institute, National Academy of Sciences of the Ukraine, 340114 Donetsk, Ukraine

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We consider the evolution of the renormalized parameters of the Ginzburg–Landau–Wilson (GLW) free energy functional in the critical region. The renormalization process is described by means of the exact renormalization group (RG) equation, using the recently observed fast relaxation of parameters in the vicinity of the universal trajectory, the so-called "large river," in phase space. We show that all earlier results obtained in the scaling limit of the fixed GLW functional are automatically reproduced in the renormalization process of arbitrary bare parameters; this removes the problem of the choice of physical solutions.

1. INTRODUCTION

The conceptual side of the renormalization group (RG) method is rather simple. In the vicinity of critical points, where the correlation radius of the fluctuations is larger than other characteristic scales of the system, one integrates the partition function by parts over the small-scale variables so as to obtain a new effective form of the free energy functional. As expected, when the integration is repeated ad infinitum, one loses essentially all information about the details of the system, except perhaps some general properties such as the symmetry group or the number of interacting components of the order parameter.[1]

In fact, the realization of this program encounters considerable mathematical difficulties. The traditional methods for an analytic solution of the problem, which are based on one form or another of perturbation theory, create, moreover, the problem of the convergence of series, which is surmounted by various artificial means.[2–4] Moreover, the procedure for cutting off the perturbation series is itself based on an additional assumption about the existence of an expansion in an appropriate parameter (which in reality is not small) such as $\varepsilon=4-d$, where d is the dimensionality of the space, the magnitude of the coupling constant, or the reciprocal of the number of components in the order parameter, $1/n$.[5,6] This introduces an uncontrollable restriction into the theory and closes the path to its further improvement.

The approach based on using the exact RG equation removes these difficulties in principle, but the mathematical complexity of an equation in functional derivatives again raises problems concerning the use of approximations and the choice of physical branches of the solution.[7,8] The most promising approach is to use a local version of the exact RG equation as the initial approximation, based on the smallness of the Fisher index η, reflecting the insignificant generation of nonlocal corrections to the interaction of fluctuations in the critical region. The RG equation then takes the form of an "ordinary" differential equation, and one can solve this, at least numerically.

In the limit of a fixed GLW functional, the solution of the local RG equation gives an infinite set of branches, reproducing in this form the old problem of "redundant operators" and the choice of the physical branch of the solution. One can show that only one of them (hereafter called the "physical" one) satisfies the requirements of the theory of critical phenomena.[7] The good calculated values of the critical indices based on this branch, as well as the verified smallness of the expansion parameter for nonlocal corrections,[8] confirms its appropriateness as the zeroth approximation. Moreover, the main requirement used as a decision criterion, namely the absence of a logarithmic divergence for finite values of the order parameter field φ, and accordingly the "good" asymptotic behavior $\approx\varphi^2/2$ as $\varphi\rightarrow\infty$, is unphysical, leaving open the main problem of how a nonuniversal trial GLW function can be turned into a fixed universal distribution that "independently chooses the correct solution" from among all others.

For an answer to this problem one must turn to a version of the RG equation which depends on the renormalized time l and use it to study the relaxation of the parameters of arbitrary GLW functionals.[9–11] Taking into account the inherent saddle-point nature of the required fixed solution one must then look for the critical surface (along which the system moves to the fixed solution) by the shooting method, using sufficiently arbitrary functions (!)$f(\varphi)$ as starting points, and the adjustable temperature $\tau=(T-T_c)/T_c$ as a parameter, i.e., the coefficient of the term $\tau\varphi^2$ which is quadratic in φ.

In implementing this program, we found that on the critical surface, the renormalized parameters rapidly reach the vicinity of some universal trajectory (the "large river") along which they afterwards relax slowly to the unique fixed solution.[11] The present author has recently[12] shown that this fixed solution is the same as the earlier found physical branch of the solution of the RG equation for the fixed point. This result removes the problem of the choice of the solution since it was shown that on the critical surface, any GLW trial free energy density is automatically transformed into a universal fixed distribution.

Besides solving the problem in principle, the large river effect can be constructively applied to study a number of specific problems. The main idea of applying it is based on the use of the attractor nature of the large river. Basically, this means that all characteristic properties of the system in the critical region must mathematically be reflected in the attraction of the appropriate functions to the attractor. This considerably simplifies the conceptual side of all calculations, in fact, reducing it directly to the original motivation for the RG method. In the present paper, we demonstrate the efficiency of the method for a number of actual problems in the theory of critical phenomena.

2. LOCAL VERSION OF THE EXACT RG EQUATION AND THE LARGE RIVER EFFECT

The exact RG equation is in general highly nonlocal, and can be written out in functional derivatives. In order to bypass the associated mathematical difficulties, we can use as the initial approximation an appreciably simpler local version of this equation, and the nonlocal corrections can then be taken into account via perturbation theory. How one takes such corrections into account will be discussed in the next section. Here, however, the exact RG equation[1] will be used in its local version, which is often sufficient for a discussion of the qualitative predictions of the theory[13–18]:

$$\dot{f}=\hat{R}f=df+\sum_i\left[f_{\varphi_i\varphi_i}-\frac{d-2}{2}f_{\varphi_i}\varphi_i-f^{\,2}_{\varphi_i}\right]. \tag{2.1}$$

Here $f\{\varphi_i\}$ is the density of the local free energy functional,

$$H[\varphi]=\int d^dr\left[\sum_i(\nabla\varphi_i)^2+f\{\varphi_i\}\right], \tag{2.2}$$

and the summation is over the n components of the vector $\boldsymbol{\varphi}=\{\varphi_i\}$. The multicomponent nature of the vector $\boldsymbol{\varphi}$ and the presence of various invariants in the expansion of $f\{\varphi_i\}$ is very important from the point of view of the numerous applications of the theory to real physical (as a rule, anisotropic or multicomponent) systems. Equation (2.1) was studied in just this context in a recent paper.[12] However, for an analysis of the general aspects of the theory, one can restrict oneself to its most transparent scalar and even version $f=f(\varphi^2)$ (as will be done everywhere in the present paper that the opposite is not stated explicitly).

We stated in the Introduction that the critical surface for

$$f(\varphi;l)=\sum_k^{\infty}g_{2k}(l)\left(\frac{\varphi_i}{2}\right)^{2k}$$

can be obtained by the shooting method.[9–12] The vertices $g_{2k}(l)$ of the GLW functional are then determined numerically as the coefficients in a Taylor series. We show in Fig. 1 the projections of several RG trajectories in the g_4g_6 plane. For all starting points, g_{2k} initially rapidly approach some universal curve, along which they later evolve slowly to a fixed point $f_l^*(\varphi_i)$. A similar effect was recently observed on the basis of a somewhat different version of the local RG equation;[9–11] it was called the "large river effect". This phenomenon is typical of the relaxing nonlinear equations, of which Eq. (2.1) is one. In particular, the latter are associated with reaching a dissipation minimum on stationary (attractor) trajectories in problems of physical kinetics.[19,20] This phenomenon reflects in the general case the fast "flow" of the phase trajectories into some (quasi-) potential valley along which the variables afterwards slowly "creep" to a fixed point. For saddle points, this "creeping" is anomalously slow (power-law rather than exponential). However, the RG fixed points are always saddle points, since for small deviations of the temperature from the critical surface, the RG trajectories passing through a flow minimum always go away from the fixed point.

In Ref. 12 we calculated the evolution with time of the combinations $A=g_2-g_2^*$ and $B=g_4-g_2(2-g_2)/3$, and also the quantities $\ln A$ and $\ln B$ as functions of $\ln l$ and l, respectively. We observed a fast exponential drop in the difference B and a slow attraction of the system to the fixed saddle point, characterized by a power-law dependence of the quantity $A(l)$.

A "large river" trajectory can be estimated analytically. The equation for $g_{2k}(l)$ has the form

$$\frac{\partial g_{2k}}{\partial l}=[d-(d-2)k]g_{2k}-\sum_{m=2}^{k+1}m(k+1-m)g_{2m}g_{2(k+1-m)}+(2k+1)(k+1)\frac{g_{2(k+1)}}{2}. \tag{2.3}$$

The flow minimum is reached for $\partial g_{2k}/\partial l\approx 0$, where we have

$$g_{2(k+1)}\approx\frac{2}{(2k+1)(k+1)g_{2(k+1)}}\times\left[[(d-2)k-d]g_{2k}+\sum_{m=1}^{k+1}m(k+1-m)g_{2m}g_{2(k+1-m)}\right]. \tag{2.4}$$

The recursion relations (2.4) determine $g_{2k}=g_{2k}(g_2)$ as a function of the single parameter g_2 on the curve passing through the fixed point which is the same as the physical branch of the solution of the equation $\hat{R}f=0$ found earlier:[7]

$$g_0\approx 0.076,\quad g_2\approx -0.456,\quad g_4\approx 0.373,$$
$$g_6\approx -0.141,\quad g_8\approx 0.067,\ \ldots\ .$$

The projection of this curve in the g_4g_6 plane is shown by small circles in Fig. 1, and it is clearly very close to the numerically determined "large river" trajectory.

The fact that the fixed solution is the same as the physical branch found earlier, and is unique for different trial functions $f(\varphi)$, must be considered a decisive demonstration that this branch of the solution of $\hat{R}f=0$ is the correct choice. Physically, this means that the bare structure of the GLW functional is determined by the microscopic interactions of the system.[21] If these interactions are such that, in general, a second order phase transition is possible in the system, it will take place when the characteristic interaction energy is the same as the temperature. In that case, fluctuations develop in the system, the correlation radius increases, and all effective parameters are renormalized. In turn, this means that a single

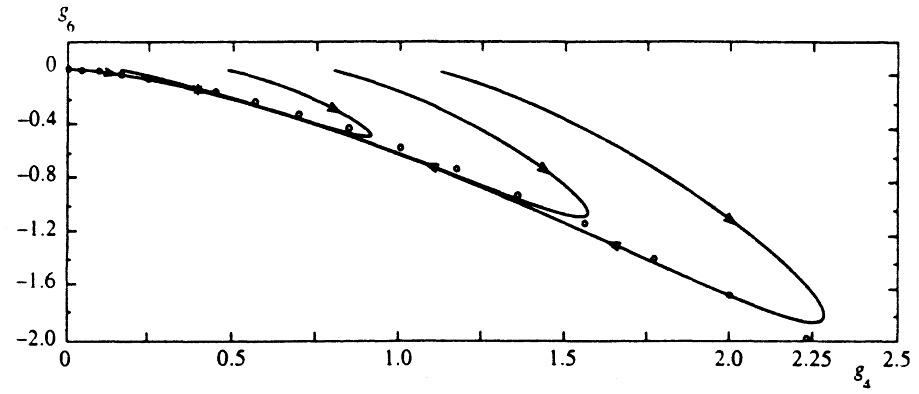

FIG. 1. Projection of phase trajectories in the g_4, g_6 plane. The analytic estimate for the "large river" trajectory is shown by small circles.

parameter (the temperature) ought to be sufficient to "place" the system on the critical surface, where it spontaneously arrives at the universal structure $f(\varphi)$. The shooting method, based on the use of a given trial form $f(\varphi; l=0)$ and the fitting of a single parameter τ, in fact mimics this physical process in a nonperturbative approach and with no additional hypotheses.

The attractor nature of the "large river" guarantees the universalization of the critical behavior long before the system reaches the true fixed point. This renders the main vertices in the qualitative structure of the phase portrait unimportant, since they rapidly "readjust" during variations in the important vertex $\tau = g_2$ (and some of the vertices of $\hat{g}_4$, which are also important in anisotropic models). Moreover, the slow change in the parameters along the large river gives a phase transition which is essentially indistinguishable from a continuous one, even in those cases when the RG trajectories never reach the desired fixed point.

One can estimate the rate of change of the parameters by considering the flow strength. To this end, Eq. (2.1) was rewritten in Ref. 22 in the form

$$\dot{\mu} = \hat{R}\mu = d\mu \ln \mu + \sum_i \left[\mu_{\varphi_i \varphi_i} - \frac{d-2}{2} \mu_{\varphi_i} \varphi_i \right], \qquad (2.5)$$

where $\mu = \exp(-f)$, with the quasilinear differential operator

$$\sum_i \left[\partial_{\varphi_i} \partial_{\varphi_i} - \frac{d-2}{2} \partial_{\varphi_i} \varphi_i \right].$$

The flow rate can be characterized by the norm $\|\dot{\mu}\| = \langle \dot{\mu} | \dot{\mu} \rangle^{1/2}$, in which the scalar product

$$\langle \psi | \xi \rangle = \int d^n \varphi \rho(\varphi) \psi(\varphi) \xi(\varphi)$$

contains the weight function $\rho(\varphi) = \exp[-(d-2)\varphi^2/4]$, which is determined by the structure of the quasilinear operator. If there exists a flow minimum $\mu = \mu^*$ (for which $\partial \|\dot{\mu}\|^2 / \partial \mu(\varphi')|_{\mu=\mu^*} = 0$), decomposing $\dot{\mu}$ in the ψ_k basis, i.e., writing it in the form

$$\dot{\mu} = \sum_k \beta_k \psi_k,$$

and afterwards taking its component along the vector ψ_j, it can easily be shown that

$$\int d^n \varphi' \frac{\partial \|\dot{\mu}\|^2}{\partial \mu(\varphi')} \psi_j(\varphi') = 2\beta_j \lambda_j = 0,$$

where the λ_j are the eigenvalues of the operator $\hat{L}$. Since there are at least some $\beta_j \neq 0$, there are some $\lambda_j = 0$. The evolution of the RG parameters is anomalously slow in the corresponding directions. Linearizing μ near μ^* and carrying out the standard operations of fluctuation theory, one easily reaches a behavior of the physical quantities similar to their behavior in the case of an ordinary second-order phase transition.

The existence of a flow minimum was confirmed in Ref. 12 by direct numerical integration of Eq. (2.3) for a tetragonal system with a trial function $\mu(\varphi_1, \varphi_2) = \exp[-f(\varphi_1, \varphi_2)]$, which depends on the invariant $u(\varphi_1^4 + \varphi_2^4)$ with a small admixture of the interaction $v(\varphi_1^2 + \varphi_2^2)^2$ (where $u \sim 10^{-2} v$).

3. GENERATION OF NONLOCALITY AND NATURAL DEFINITION OF ANOMALOUS DIMENSIONALITY

In the general case, the exact RG equation is nonlocal. Following Refs. 8 and 23, we write this equation out for an arbitrary Ginzburg–Landau (GL) functional,

$$H = H_{\text{total}} - H_0 = \sum_{k=1}^{\infty} 2^{1-2k} \int_{\{q_i, q_{i'}\}} \times (2\pi)^d \delta\left(\sum_{i=1}^{k} (\mathbf{q}_i + \mathbf{q}_{i'}) \right) g_k\{\mathbf{q}_i, \mathbf{q}_{i'}\} \prod_{i=1}^{k} \varphi_{q_i} \varphi_{q_i} \qquad (3.1)$$

in the form

$$\dot{H}[\varphi] = \frac{1}{2} \int_q \eta(q) [V - G_0^{-1}(q) |\varphi(q)|^2] + dV \frac{\partial H}{\partial V} - \int d^d r \left[\frac{(d-2)\varphi(\mathbf{r})}{2} + \mathbf{r} \nabla_r \varphi(\mathbf{r}) \right] \frac{\delta H}{\delta \varphi(\mathbf{r})} + \int_{rr'} \left\{ h(\mathbf{r} - \mathbf{r}') \left[\frac{\delta^2 H}{\delta \varphi(\mathbf{r}) \delta \varphi(\mathbf{r}')} - \frac{\delta H}{\delta \varphi(\mathbf{r})} \frac{\delta H}{\delta \varphi(\mathbf{r}')} \right] - \frac{1}{2} \eta(\mathbf{r} - \mathbf{r}') \varphi(\mathbf{r}) \frac{\delta H}{\delta \varphi(\mathbf{r}')} \right\}. \qquad (3.2)$$

We used here as H_0 the Gaussian functional

$$H_0=\int_q G_0^{-1}(q)|\varphi(q)|^2, \quad \int_q \equiv \int d^dq/(2\pi)^d \qquad (3.3)$$

and introduced the following notation: $h(q)=\exp(-q^2/2\Lambda^2)$ is the smooth cutoff factor, $\Delta_\varphi(\varphi)$ is the scale dimensionality of the field φ, and $\eta(q)$ is the anomalous dimensionality function, $\eta(q)=\Delta_\varphi(q)-(d+2)/2$. One can show that the function $\eta(q)$ determines the structure of the physically measurable correlation function $G_q=\langle\varphi_q,\varphi_{-q}\rangle$ at the critical point, and its limit $\eta(q\to 0)=\eta(0)$ is the same as the Fisher index η in the asymptotic expression $G_q\sim q^{-2+\eta(0)}$ for $\tau=0$ and $q\to 0$ (see Ref. 8).

The main difficulty in applying this equation is that even for a purely local trial functional, nonlocality will arise as a result of the RG transformation. If one formally lets the cutoff momentum Λ tend to infinity, we have $h(q)\to$const $=h(0)=1$ and, hence, the function

$$h(\mathbf{r}-\mathbf{r}')=\left(\frac{\Lambda}{\sqrt{2\pi}}\right)^d \exp\left[-\frac{(\mathbf{r}-\mathbf{r}')^2\Lambda^2}{2}\right]$$

tends to a δ function: $h(\mathbf{r}-\mathbf{r}')\to\delta(\mathbf{r}-\mathbf{r}')$. In that case no nonlocal terms will be generated in the free energy functional if they are not present in its bare form. This makes it possible formally to use the local version of the exact RG equation.

Indeed, the original RG equation is written for the normalization $\Lambda=1$, and the nonlocality is generated due to the term

$$-\int_{rr'}\bar{h}(\mathbf{r}-\mathbf{r}')\frac{\delta H}{\delta\varphi(\mathbf{r})}\frac{\delta H}{\delta\varphi(\mathbf{r}')}, \qquad (3.4)$$

where $\bar{h}(\mathbf{r}-\mathbf{r}')=h(\mathbf{r}-\mathbf{r}')-\delta(\mathbf{r}-\mathbf{r}')$.

We encounter here the problem of determining $\eta(q)$. Above it was purely formally defined as the difference between the dimensionality of the field $\varphi(q)$ and the naive dimensionality $(d+2)/2$. If we use the fact that $\Delta_\varphi(q)=(d+2)/2=$const, the Gaussian functional H_0 of (3.3) turns out to be unchanged, and the form of the RG equation is greatly simplified, since there are no terms proportional to $\eta(q)=\Delta_\varphi(q)-(d+2)/2$. This suggests that if we formally take all nonlocal corrections exactly into account, this will give the required nonlocal form of $H[\varphi]$, the correct function $G(q)$, and so on. However, such a solution is to date impossible.

Moreover, the requirement that the functional

$$H_0=\int_q G_0^{-1}(q)|\varphi(q)|^2$$

be constant under RG transformations is as artificial as splitting it off from the total form H_{total}, since when $\Delta_\varphi(q)=(d+2)/2=$const, terms proportional to $|\varphi(q)|^2$ are inevitably generated in $H=H_{\text{total}}-H_0$. More natural is a choice of $\Delta_\varphi(q)$ that preserves the coefficient of $|\varphi(q)|^2$ in the total GLW functional, and transfers the source of structural change in the (exact) correlation function $G(q)$ from the generation of corrections to this coefficient to the generation of the anomalous dimensionality $\eta(q;l)=\Delta_\varphi(q;l)-(d+2)/2$ of the field φ with time l. In fact, it is just this choice that occurs in conventional RG theory, and that was also used in previous papers in the limit $l\to\infty$ (i.e., it was assumed, in fact, that $\eta(q)\equiv\eta[(q;l\to\infty)]$. This made it possible to develop a perturbation theory in a verifiably small η. However, it was made clear that this choice does not fix η uniquely and, in turn, requires a selection of this quantity starting from the requirement for a physical branch of the solution for the nonlocal corrections.[8] Taking now into account knowledge of the large river effect and the uniqueness of the fixed point for the local approximation, we may expect that the required quantity η is obtained automatically when we place the system on the critical surface of the local equation and undertake a parallel solution of the time (l-) dependent equation(s) for the nonlocal correction.

One can obtain the equation for the nonlocal correction by writing H as the sum $H=\Phi_0+\Phi_1$ of a local part $\Phi_0=\int_r f[\varphi(\mathbf{r})]$ and an additional term Φ_1. If, as expected, $\Phi_1\ll\Phi_0$ (to the degree that η is small), we have, after linearizing in Φ_1,

$$\dot{\Phi}_1=-\frac{1}{2}\int\eta(\nabla\varphi(\mathbf{r}))^2-\int d^dr\left[\frac{(d-2)\varphi(\mathbf{r})}{2}+\mathbf{r}\nabla_r\varphi(\mathbf{r})\right]$$
$$\times\frac{\delta\Phi_1}{\delta\varphi(\mathbf{r})}+\int_{rr'}\left\{h(\mathbf{r}-\mathbf{r}')\left[\frac{\delta^2\Phi_1}{\delta\varphi(\mathbf{r})\delta\varphi(\mathbf{r}')}\right.\right.$$
$$\left.-2\frac{\delta\Phi_1}{\delta\varphi(\mathbf{r})}\frac{\delta\Phi_0}{\delta\varphi(\mathbf{r}')}\right]-\int_{rr'}\bar{h}(\mathbf{r}-\mathbf{r}')\frac{\delta H}{\delta\varphi(\mathbf{r})}\frac{\delta H}{\delta\varphi(\mathbf{r}')}$$
$$\left.-\frac{1}{2}\eta(\mathbf{r}-\mathbf{r}')\varphi(\mathbf{r})\frac{\delta\Phi_1}{\delta\varphi(\mathbf{r}')}\right\}. \qquad (3.5)$$

Describing the variation of the correlation function in terms of the generation of η instead of in terms of corrections to the coefficient of $|\varphi(q)|^2$ makes it possible to look for Φ_1 in the form of a gradient expansion. To do so, we expand the factor h in powers of gradients, and retain to first order the lowest nonvanishing contributions from $\nabla\varphi$:

$$\Phi_1=\int_r\chi(\varphi(\mathbf{r}))(\nabla\varphi)^2. \qquad (3.6)$$

It can easily be shown that in this approximation it is sufficient to retain the anomalous dimensionality function $\eta(\mathbf{r}-\mathbf{r}')$ in the form $\eta(\mathbf{r}-\mathbf{r}')\simeq$const$=\eta$. The RG equations were worked out in this approximation in Ref. 8, and take the form

$$\dot{f}=df-\frac{d-2+\eta}{2}\varphi f_\varphi+f_{\varphi\varphi}-f_\varphi^2+B\chi, \qquad (3.7a)$$

$$\dot{\chi}=-[\eta+4f_{\varphi\varphi}]\chi-\left[\frac{d-2+\eta}{2}\varphi+2f_\varphi\right]\chi_\varphi$$
$$+\chi_{\varphi\varphi}+\frac{1}{2}[f_{\varphi\varphi}^2-\eta], \qquad (3.7b)$$

where the coefficient B is $\int_p p^2h(p)=d/(2\pi)^{d/2}\ll 1$. The set of Eqs. (3.7) was numerically integrated in the same paper at the fixed point $\dot{f}=\dot{\chi}=0$.

The main difficulty here is that the physical branch of the solution for $f(\varphi(\mathbf{r}))$ itself is known only numerically, and

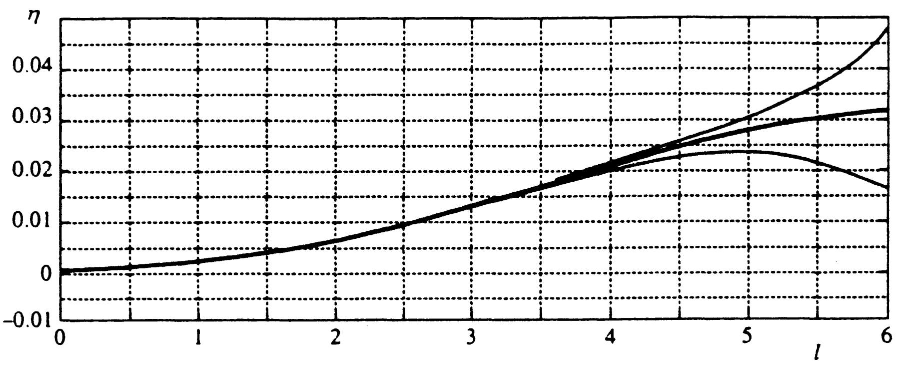

FIG. 2. The quantity η as function of l for the physically most interesting $d=3$ case. Further on (for $l\geqslant 6$), the quantity η remains essentially constant, coinciding in $l\gg 1$ limit with the fixed value found earlier by means of the l-independent equations for the fixed point. We also show two close phase trajectories obtained for deviations of the system on the two sides of the critical surface.

varies as a function of the parameter η in (3.7a). To bypass this difficulty, we put $\varphi = \varphi'/\sqrt{2\Delta_\varphi}$, removing the η dependence in (3.7a) to the effective dimensionality $d_{\text{eff}}=2d/(2-\eta)$, and neglecting the small quantity $B\chi$. Moreover, we have already stated that when one works only with the equations for the fixed point, the physical constraints do not completely determine the numerical value of the index η (since in the given approximation, they simply reduce to the boundary conditions $\chi(0)=\chi_\varphi(0)=0$). With these constraints and with a fixed branch of the solution of $\dot{f}(\varphi)=0$, the equation for the fixed point $\dot{\chi}=0$ has a one-parameter family of solutions $\chi(\varphi;\eta)$. As in the purely local case with $f(\varphi(\mathbf{r}))$, this again led to problems in choosing a physical branch *of the* solution for $\chi(\varphi;\eta)$. This choice, strictly, also fixed the magnitude of $\eta\neq 0$.

The new approach radically changes the philosophy behind the search for η. If at the initial time the function $\chi(\varphi,l=0)$ satisfies the requirement $\chi(0,0)=\chi_\varphi(0,0)=0$, these boundary conditions will also be satisfied for arbitrary l if the function $f(\varphi,l)$ is even and $\dot{\chi}(0,l)=0$. Hence,

$$\eta(l)=\chi_{\varphi\varphi}(0,l)+\frac{1}{2}f_{\varphi\varphi}^2(0,l), \tag{3.8}$$

which determines the function $\eta(l)$ at any time l. As a result, we have the following prescription for calculating the functions f, χ, and η. For a given structure of the seed $f(\varphi,l=0)$ we choose $\tau=g_2$ such that we fall on the critical surface (deformed by the presence of η and $B\chi$ in the equations for f, which in turn are calculated in the case of a parallel solution of Eqs. (3.7) taking condition (3.8) into account). Nevertheless, since η is small, it is convenient to begin the search for the desired τ, starting with τ as found in the local approximation.

This program was implemented when we prepared the present paper. We show in Fig. 2 as an example, the behavior of $\eta(l)$ calculated in this manner for the physically most interesting case $d=3$. As l increases, so does η, and in the limit as $l\to\infty$ it reaches a fixed value which is the same as the one found earlier using the (l-independent) equations for the fixed point.

4. EVOLUTION OF THE CORRELATION FUNCTIONS IN THE CRITICAL REGION

Although the natural determination of the index η that does not rest upon additional hypotheses is also of interest from the point of view of the foundations of the general theory, by itself this result does not enhance traditional approaches. The starting idea of the derivation of the exact RG equation, resting upon the use of the scaling equations for the physical correlation functions, assumed broader possibilities thanks to the connection between RG equation and the scaling equations.[23] In the present section, we shall demonstrate that such possibilities are, indeed, opened up by the new approach to solving the RG equation.

The idea of the derivation of the scaling equations for the correlation functions is simple, but writing them down is very cumbersome (one can find a detailed exposition in Ref. 23). Without going into details, we note that the scaling equations can be written in the form of an infinite hierarchy of partial differential equations, the characteristic set for which is the same as the exact RG equation. This means, in particular, that the singular points at which the correlation functions diverge (which must be at the physical critical point) are the same as the fixed points of the RG equation.

Defining the mth order correlation function by

$$\left\langle \prod_{j=1} \varphi(q_j)\right\rangle=(2\pi)^d\delta\left(\sum_{j=1}^{m} q_j\right)A_m\{q_j\}, \tag{4.1}$$

we can write the scaling equations in the form

$$\begin{aligned}&\left\{\frac{m(d+2)}{d}-d-d\frac{V\partial}{\partial V}+\sum_{j=1}^{m}\left[q_j\frac{\partial}{\partial q_j}+2G_0^{-1}h(q_j)\right.\right.\\&\left.\left.+\hat{\Phi}[g_k]\right]\right\}\delta\left(\sum_{j=1}^{m} q_j\right)A_m(q_j,\Lambda;g_k\{q_j\})\\&=\sum_{l=1}^{m}\sum_{l\neq t}^{m}\delta(q_j+q_l)h(q_j)\left\langle\prod_{l\neq j\neq t}\varphi(q_t)\right\rangle.\end{aligned} \tag{4.2}$$

The operator $\hat{\Phi}[g_k]$ is here given by

$$\hat{\Phi}[g_k]=\sum_{k=1}^{\infty}\int_{\{q_i\}}\hat{U}_k[g_j\{q_i\}]\frac{\delta}{\delta g_k}\equiv\sum_{k=1}^{\infty}\int_{\{q_i\}}\frac{\partial g_k\{q_i\}}{\partial l}\frac{\delta}{\delta g_k}, \tag{4.3}$$

$$\hat{U}_k[g_j\{q_i\}]=\left(\varepsilon_k+\sum_{j=1}^{k}q_j\frac{\partial}{\partial q_j}\right)+\int_{qq'}(2\pi)^d\delta(\mathbf{q}+\mathbf{q}')h(q)\times\left\{\frac{1}{2}(k+2)(k+1)g_{(k+2)}(q,q',\{q_j\})+\frac{1}{2}\sum_{m=1}^{k+1}m(k-m+2)g_m(q,q_1,q_2,\ldots,q_{2m-1})\times g_{k-m+1}(q',q_{2m'},\ldots,q_{2k})\right\}. \tag{4.4}$$

In view of the fact that the operator $\hat{\Phi}[g_k]$ describes the derivative of the correlation functions with respect to l, i.e.,

$$\hat{\Phi}[g_k]A=\sum_{k=1}^{\infty}\int_{\{q_i\}}\frac{\partial g_k\{q_i\}}{\partial l}\frac{\delta A}{\delta g_k}=\frac{\partial A}{\partial l}, \tag{4.5}$$

and restricting ourselves for definiteness to the two-point function

$$A_2(\mathbf{q},-\mathbf{q})=\langle\varphi(\mathbf{q})\varphi(-\mathbf{q})\rangle/V\equiv G(q), \tag{4.6}$$

we obtain for it a compact version of Eq. (4.2):

$$\left[2-\eta(q)+q\frac{\partial}{\partial q}+4G_0^{-1}h(q)+\frac{\partial}{\partial l}\right]G(q;\{g_k(l)\})=2. \tag{4.7}$$

By virtue of the symmetry of $G(q;\{g_k(l)\})=G(-q;\{g_k(l)\})$, this correlation function depends only on $q=|q|$, and in particular, in the limit as $G(q;\{g_k(l)\})\to\infty$, we have simply

$$G(q;\{g_k(l)\})=G(0;\{g_k(l=0)\})/(q^{2-\eta}+e^{2l}). \tag{4.8}$$

Equations (4.8) and (4.9) look like a surprisingly simple solution of the problem. However, it is so simple only in terms of the formal variable l. In actual fact, we are interested in how G depends on the physical quantities, i.e., in the present case on q and τ:

$$G(q;\tau)=G(0;1)/\{q^{2-\eta}+\exp[2l(\tau)]\}.$$

In turn, in order to find the function $l(\tau)$, we must first solve the whole system of equations for the infinite number of vertices $g_k(l)$ (i.e., the equation for $\partial f/\partial l$), and after that express them all, including l, in terms of the single parameter τ. Nonetheless, this procedure is realizable in the approach described here.

We combine the equations for f, χ, and G into a single system as follows:

$$\frac{\partial f}{\partial l}=df-\frac{d-2+\eta}{2}\varphi f_\varphi+f_{\varphi\varphi}-f_\varphi^2+B\chi, \tag{4.9a}$$

$$\frac{\partial\chi}{\partial l}=-[\eta+4f_{\varphi\varphi}]\chi-\left[\frac{d-2+\eta}{2}\varphi+2f_\varphi\right]\chi_\varphi+\chi_{\varphi\varphi}+\frac{1}{2}[f_{\varphi\varphi}^2-\eta], \tag{4.9b}$$

$$\frac{\partial G}{\partial l}=2-\left[2-\eta+q\frac{\partial}{\partial q}+4G_0^{-1}h(q)\right]G(q;\{g_k(l)\}). \tag{4.9c}$$

We shall consider in parallel the evolution on the large river of the functions f and χ in the space of the variable φ, of the quantity $\eta(l)$ given by Eq. (4.9b), together with the condition (3.8) in the form

$$\eta(l)=\chi_{\varphi\varphi}(0,l)+\frac{1}{2}f_{\varphi\varphi}^2(0,l),$$

and of the function $G(q;l)$ defined in the space of the wave vector q. Essentially, we then simply copy the smoothing and universalization process of the correlation function, which proceeds in parallel with the renormalization of the effective parameters of the system. Since g_k and G are evaluated at the same times l, there is a one-to-one correspondence between the numerically determined $\tau=g_2$ and the other quantities, and that correspondence can be explicitly calculated and visualized.

In order to study the system (4.9) analytically, it may be useful to consider it in the context of some form of perturbation theory. In particular, such a study is carried out in Appendix A in the spirit of the ε expansion. It reproduces the well-known results of the ε expansion, and enables us to see directly how the required solution for the correlation function $G=G(\tau)$ can be obtained in the context of Eqs. (4.9).

The program of numerical calculations described here was also implemented, and the corresponding result is shown for $d=3$ in Fig. 3, where we show the evolution with time l of the function $G(q;l)$, which is put into correspondence with the renormalized quantity τ. One easily performs also the inverse Fourier transformation and obtains the form of the correlation function $G(\mathbf{r};l)$ in real space at an arbitrary stage of the renormalization. The result of such a transformation is shown in Fig. 3b for the critical point (i.e., for $l\gg 1$, when this function has essentially ceased changing). Plotting the function $G(\mathbf{r};l)$ on a log–log scale, we can verify that (with time l) it acquires a scaling shape arriving at the appropriate distribution $G(\mathbf{r},\infty)$ as at an attractor.

The whole philosophy behind the method is such that it does not make any fundamental distinction between the search for a solution for the simplest (scalar and even) GLW functional, and the analogous procedure in more complex cases such as, for instance, the study of tricritical behavior or the behavior of systems with a symmetry that allows for the existence of odd terms in the GLW expansion, i.e., $g_{2k+1}\neq 0$.

In the interests of brevity and given that the physics of this problem is of interest in its own right, we do not present here the corresponding material, and it makes sense to consider only that which is necessary for completing the discussion of the system of scaling equations containing arbitrary vertices, including also $g_{2k+1}\neq 0$. Part of the necessary in-

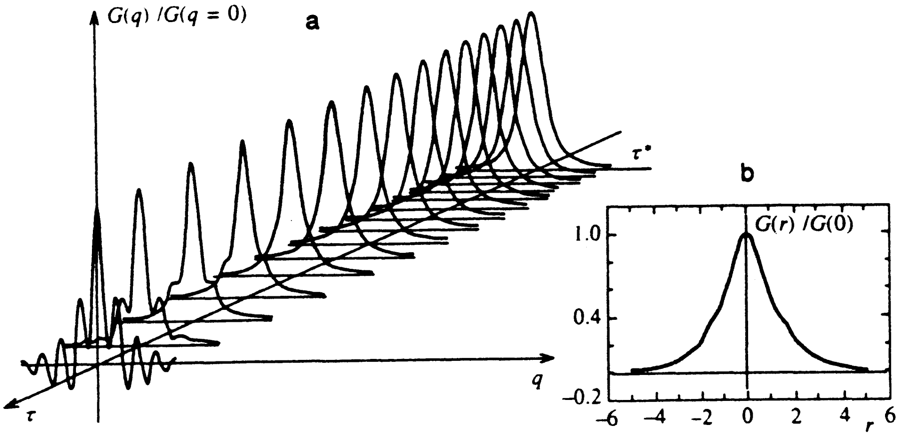

FIG. 3. a) Evolution of the Fourier transform of the correlation function $G(q,l(\tau))$ placed in correspondence with variations in the temperature parameter τ. b) The shape of the correlation function $G(\mathbf{r};l)$ in real space, given for the critical point (i.e., for $l \gg 1$ when this function essentially ceases to change). This shape $G(\mathbf{r},\infty)$ is an attractor to which any trial functions $G(\mathbf{r};0)$ will be attracted.

formation about the RG equation with $g_{2k+1} \neq 0$ is relegated to Appendix B. Using the general scaling equation for arbitrary correlation functions,

$$\left\langle \prod_{j=1}^{m} \varphi(q_j) \right\rangle = (2\pi)^d \delta\left(\sum_{j=1}^{m} q_j \right) A_m\{q_j\},$$

we have for the lowest odd numbers $m=1,3$ the following equations (written here for the sake of compactness in the local approximation):

$$\left[\frac{2-d}{2} + \sum_{k=0} \frac{\partial g_k}{\partial l} \frac{\partial}{\partial g_k} \right] \langle \varphi_{q=0} \rangle = 0,$$

$$\left\{ \frac{6+d}{2} + \sum_{j=1}^{3} \left[q_j \frac{\partial}{\partial q_j} + 2G_0^{-1}(q_j) h(q_j) \right] + \sum_{k=0} \frac{\partial g_k}{\partial l} \frac{\partial}{\partial g_k} \right\} \langle \varphi_{q_1} \varphi_{q_2} \varphi_{q_3} \rangle \delta\left(\sum_{i=1}^{3} q_i \right) = 3 \langle \varphi_{q=0} \rangle. \tag{4.10}$$

The first of these equations can be solved independently in the form

$$\left(\frac{2-d}{2} + \frac{d}{dl} \right) \langle \varphi_{q=0} \rangle = 0 \tag{4.11}$$

immediately giving $\langle \varphi_{q=0} \rangle = \exp[(d-2)l/2]$.

In particular, in the ε expansion (see Appendix A), we have

$$2l(\tilde{\tau}) = \ln|\tilde{\tau}|/(1-3g_4^*)$$

and hence,

$$\langle \varphi_{q=0} \rangle = |\tilde{\tau}|^{(d-2)/4(1-3g_4^*)}.$$

Since the equation for the fixed point has, as before, only an (Ising) nontrivial solution with $g_{2k+1}=0$ (see Appendix B), we obtain for β a magnitude which is the same as the classical result for the index β:

$$\beta = \frac{d-2}{4(1-3g_4^*)} \approx \frac{1}{2} - \frac{\varepsilon}{6}, \tag{4.12}$$

or, in more general form, $\beta = 1/2 - 3\varepsilon/2(n+8)$. Using $\langle \varphi_{q=0} \rangle$, one can solve the next equation, which determines $\langle \varphi_{q_1} \varphi_{q_2} \varphi_{q_3} \rangle$, and so on.

In conclusion, note that we are, in fact, dealing with a coherent hierarchy of scaling equations, including the RG equation, which can be interpreted as an additional scaling equation for the partition function, considered in this context as a special case of the zeroth-order correlation function $A_{m=0}$, so that the most refined method for studying critical phenomena turns out to be at the same time both the most convenient and the most natural one. If we ignore the frightening complexity of Eqs. (4.9), the prescription for solving them looks like a naive copy of the experimental prescription. One need merely let the temperature of the system approach the (renormalized) critical temperature, and the system itself will produce a scale-invariant structure with a fluctuating field with the corresponding correlation function and other parameters that are automatically acquired during its approach to the critical point.

I am grateful to J. P. Badiali, V. Russier, and D. Di Caprio for hospitality and collaboration at the Université Pierre et Marie Curie (Paris VI), during which the "large river" effect was discussed many times and used to analyze various problems in fluctuation theory. I am also grateful to C. Bagnuls and C. Bervillier for useful discussions and kind information on the latest results of their study of the exact RG equation.

APPENDIX A

Perturbation Theory Solution of the Equations for the Renormalized Functional and the Correlation Function

References to the numerical solution of equations, especially unfamiliar ones like (4.9), usually leave some gaps in the exposition since the numerical solution cannot be reproduced in all details in a journal. Taking this into account, we consider a simplified form of (4.9) in the present Appendix, reduced in the spirit of the first ε expansion approximation, and solve it analytically.

We restrict ourselves to the local approximation, and we truncate the system of equations for the vertices g_{2k} as follows:

$$\partial g_0/\partial l = dg_0 + 2g_2,$$

$$\partial g_2/\partial l = 2g_2 + 12g_4 - 4g_2^2,$$

$$\partial g_4/\partial l=(4-d)g_4+30g_6-16g_4g_2,$$

$$\partial g_6/\partial l\approx g_6+16g_4^2+\dots \quad (A1)$$

The quantity $\varepsilon=4-d$ is assumed here to be small. In the vicinity of the fixed point, where the trajectories follow the large river and hence $\hat{U}_k[g_j]\approx 0$, we have

$$\partial\tilde{\tau}/\partial l=2\tilde{\tau}(1-3g_4^*)+\dots,$$

$$\partial g_4/\partial l=\varepsilon g_4-18g_4^2+\dots,$$

$$\partial g_6/\partial l\approx -g_6+\dots, \quad (A2)$$

in which we have introduced the renormalized temperature $\tilde{\tau}=\tau+3g_2/2$. For a given value of $\varepsilon\neq 0$, the vertex g_4 tends to $g_4^*=\varepsilon/18\neq 0$, and only the equation $\partial\tilde{\tau}/\partial l=2\tilde{\tau}(1-3g_4^*)$ remains important. Integrating it with the boundary condition $\tilde{\tau}(0)=1$,

$$2l(\tilde{\tau})=\ln|\tilde{\tau}|/(1-3g_4^*) \quad (A3)$$

and substituting this result into the equation for the correlation function (4.8), we have finally

$$C(q;\tilde{\tau})=G(0;1)/(q^2+|\tilde{\tau}|^\gamma), \quad (A4)$$

where the index γ is the same as the quantity well known from the ε expansion

$$\gamma=1/(1-3g_4^*)\approx 1+\varepsilon/6, \quad (A5)$$

(in the more general $n\neq 1$ case, one easily obtains in a similar way $\gamma=1+(n+2)/2(n+8)$).[21]

It is also of interest to calculate the temperature dependence of the G function as $d\to 4$. In that limit, $g_4^*=\varepsilon/18$ tends to zero as $\tilde{\tau}\to 0$ (or, accordingly, as $l\to\infty$), and the solution is not so trivial, but more intuitive, as the program of reducing all quantities to their dependence on a single parameter $\tilde{\tau}$ is realized. Instead of Eqs. (A2), we now have

$$\partial\tilde{\tau}/\partial l=2\tilde{\tau}[1-3g_4(l)],$$

$$\partial g_4(l)/\partial l=-18g_4^2(l). \quad (A.6)$$

We have already stated that it is convenient to use $\tilde{\tau}$ as the independent parameter, and to consider first the equation

$$\partial g_4/\partial\tilde{\tau}\approx -9g_4^2/\tilde{\tau}, \quad (A7)$$

with solution in the form

$$g_4(\tilde{\tau})=-[9\ln(\tilde{\tau})]^{-1}, \quad (A8)$$

which one now must substitute into the first of Eqs. (A6) to obtain the desired integral:

$$l=\ln[\tilde{\tau}(\ln(\tilde{\tau}))^{-1/3}]. \quad (A9)$$

This relation gives the well known logarithmic correction to the mean-field critical asymptotic form:

$$G(q=0;\tilde{\tau})=\tilde{\tau}(\ln(\tilde{\tau}))^{1/3}. \quad (A10)$$

The results obtained for $d\neq 4$ and for $d=4$ are somewhat trivial in and of themselves, but the process of obtaining them demonstrates how the formal l-dependence of the vertices of the free energy functional and of the function G is transformed into the real temperature dependence of the correlation function. For an analytic solution, we have in this case used essentially the same prescription for finding the required dependence as was described in the main text for the numerical procedure, with the only difference being that here we succeeded explicitly in transforming the parametric data $G(l)$ and $g_{2k}(l)$ to the function $G(\tilde{\tau})$.

APPENDIX B

Lack of a Nontrivial Solution for a Model with φ^{2k+1} Terms

Here we consider some properties (of the local version) of the exact RG equation

$$\dot{f}\equiv\hat{R}f=df-\frac{d-2}{2}\varphi f_\varphi+f_{\varphi\varphi}-f_\varphi^2 \quad (B.1)$$

when there are odd terms in the GLW functional. It is convenient to split the local GLW functional density f into two parts, $f(\varphi)=a(\varphi)+s(\varphi)$, where $a(\varphi)=[f(\varphi)-f(-\varphi)]/2$ and $s(\varphi)=[f(\varphi)+f(-\varphi)]/2$, and rewrite Eq. (B1) in the form of a set of two equations:

$$\frac{\partial s}{\partial l}\equiv\hat{R}s=ds-\frac{d-2}{2}\varphi s_\varphi+s_{\varphi\varphi}-s_\varphi^2-a_\varphi^2,$$

$$\frac{\partial a}{\partial l}\equiv\hat{R}a=da-\left(\frac{d-2}{2}\varphi+2s_\varphi\right)a_\varphi+a_{\varphi\varphi}. \quad (B2)$$

This pair of equations can be studied both numerically and analytically. In the present Appendix, we concentrate on an analysis of static solutions $\hat{R}s=\hat{R}a=0$, which are of interest from the stand point of solving the equations for the correlation functions. Many problems connected with applying the large river method, of course, turn out to be omitted.

First and foremost, we analyze the mutually consistent asymptotic behavior of the functions $s(\varphi)$ and $a(\varphi)$ as $\varphi\to\infty$. If the presence of the term a_φ^2 in the equation $\hat{R}s=0$ is unimportant, the asymptotic behavior of the function $s(\varphi)$ remains the same as for $a=0$, and hence $s_\varphi\approx\varphi$ as $\varphi\to\infty$. Substituting this asymptotic form into the equation $\hat{R}a=0$ we have

$$\hat{R}a=da-\left(\frac{d-2}{2}\varphi+2s_\varphi\right)a_\varphi+a_{\varphi\varphi}$$

$$\approx da-\frac{d+2}{2}\varphi a_\varphi+a_{\varphi\varphi}. \quad (B3)$$

The simplest possibility of satisfying this equation in the limit as $\varphi\to\infty$ consists in assuming that

$$\left(\frac{d-2}{2}\varphi+2s_\varphi\right)a_\varphi\gg a_{\varphi\varphi}.$$

This yields

$$\hat{R}a\approx da-\frac{d+2}{2}\varphi a_\varphi \quad (B4)$$

and, hence, $a\sim\varphi^{2d/(d+2)}$. For $2\leqslant d\leqslant 4$, $1\leqslant 2d/(d-2)\leqslant 2$, and the corresponding term $a_\varphi^2\sim\varphi^{2(d-2)/(d+2)}\ll\varphi^2$ in the equation $\hat{R}s=0$ is indeed unimportant. However, this is a special solution found, in fact, for a (rather special) choice of the boundary conditions. To control the asymptotic behavior in the more general case, we need to solve Eq. (B4). Using the substitution $a(\varphi)=\varphi\kappa(\varphi^2)$, we have

$$\frac{d-2}{2}\kappa-\frac{d+2}{2}\varphi\kappa_\varphi+\left(\kappa_{\varphi\varphi}+\frac{2\kappa_\varphi}{\varphi}\right)=0. \tag{B5}$$

It is then easy to reduce this equation to the standard form of the equation for the hypergeometric functions

$$z\kappa_{zz}+(\beta-z)\kappa_z-\alpha\kappa=0, \tag{B6}$$

where $z=(d+2)\varphi^2/4$, $\beta=3/2$, and $\alpha=(2-d)/[2(d+2)]$, and to use its well known asymptotic form. Turning then to the function $a(\varphi)$, we obtain the general asymptotic form for the family of solutions of the initial equation

$$a\sim\varphi^{2d/(d+2)}\exp[(d+2)\varphi^2/4].$$

The general solution which we have obtained diverges much faster than $s\propto\varphi^2$ as $\varphi\to\infty$. Moreover, the special solution obtained earlier cannot be used simultaneously $\varphi>0$ and $\varphi<0$, since it requires a special choice of the constant $a_\varphi(0)$ which cannot be satisfied simultaneously for $\varphi>0$ and $\varphi<0$ by virtue of the continuity of the function $a(\varphi)$.

This fact is reflected in the perturbation theory as follows. Introducing a cutoff as was done above, we obtain a set of equations for the parameters g_k resembling the ε expansion in the presence of odd-order quantities g_{2k+1}:

$$\partial g_0/\partial l=dg_0+2g_2-g_1^2,$$

$$\partial g_1/\partial l=(d+2)g_1/2+6g_3-g_1g_2,$$

$$\partial g_2/\partial l=2g_2+12g_4-4g_2^2-6g_1g_3,$$

$$\partial g_3/\partial l=(6-d)g_3/2+20g_5-12g_2g_3-8g_4g_1,$$

$$\partial g_4/\partial l=(4-d)g_4+30g_6-16g_4g_2-10g_5g_1-8g_3^2,$$

$$\partial g_5/\partial l\approx(10-3d)g_5/2+24g_3g_4-12g_6g_1,$$

$$\partial g_6/\partial l\approx(6-2d)g_6+16g_4^2-24g_2g_6-30g_3g_5. \tag{B.7}$$

Using this cutoff of the system, one in fact assumes that $g_4\sim4-d=\varepsilon$, $g_6\sim\varepsilon^2$, and $g_{k\geqslant7}\to0$ or, in other words, that $f(\varphi)\sim\varepsilon$, and that it tends to zero as $\varepsilon\to0$. But that, in turn, means that $g_{2k+1}\to0$, and hence $g_3g_4\ll\varepsilon$ and $g_6g_1\ll\varepsilon^2$, so that $g_5\ll\varepsilon\Rightarrow g_3\ll\varepsilon\Rightarrow g_1\ll\varepsilon$. In other words, $g_{2k+1}\ll\varepsilon\sim g_{2k}$, so that only the even part of the function $f(\varphi)=s(\varphi)$, which is small as ε, is important in the framework of the ε expansion.

This result is the same as the well known result of the standard field theoretical approach to the theory of critical phenomena (see Refs. 24 and 25). Moreover, it is of great import in the framework of the method described here for solving simultaneously the exact RG equations and the equations for the correlation functions, since it reduces the problem of finding the critical surface in the presence of g_{2k+1} vertices to the simpler $g_{2k+1}=0$ case. All nonvanishing corrections to the quantities g_{2k+1} play the role of deviations from the critical surface, and can be taken into account in the appropriate manner, i.e., in the language of the appropriate eigenfunctions, instabilities, and so on. Such a study has been carried out, but it lies beyond the scope of the present paper.

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