Soliton interaction with an impurity in the $\lambda\phi_2^4$ theory

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The interaction of the kink of the $\lambda\phi_2^4$ theory with an attractive δ-function impurity is discussed. It is shown that, depending on the initial kink velocity, the kink either passes over the impurity, is captured by it, or is reflected backward. This last effect has a resonant nature and can be explained using the effective Lagrangian obtained in the paper that takes into account the interaction of the translational mode of the kink excitation and the discrete mode of its excitation. The stochastization of the problem when the kink is captured by the impurity is discussed. It is shown that the spectrum of excitations of a kink at an inhomogeneity does not contain the so-called discrete impurity mode that is realized in the spectrum of the theory as a discrete mode if excitations over the vacuum are considered.

1. INTRODUCTION

Problems of the interaction of solitons (kinks) with impurities have been fairly widely discussed in recent years. For these purposes, one most frequently considers the sine–Gordon equation and the Ginzburg–Landau–Higgs (GLH) equation. The sine–Gordon equation describes many phenomena in different branches of physics, for example, the propagation of magnetic flux in Josephson junctions, domain wall dynamics in magnetic crystals, the propagation of dislocations through a crystal, the propagation of ultrashort optical pulses through a two-level medium, etc. It is clear from this that the problem of solving the sine–Gordon equation in the presence of nonlocal inhomogeneities arises very naturally. Depending on the geometry of the problem, either one-dimensional or multidimensional equations may be appropriate for the physical situation. The behavior of the solitons of the sine–Gordon equation in the presence of a variety of inhomogeneities has been repeatedly discussed in the literature (see, for example, the reviews of Refs. 1–4).

With regard to the GLH $\lambda\phi^4$ theory, it also has numerous applications in solid-state physics. For example, this model describes phase transitions in ferromagnetic materials, conductivity in quasi-one-dimensional systems, nonlinear excitations in polymer chains, etc. (see, for example, Ref. 5 and the references in it). In field theory, the GLH equation is used as the simplest model of a theory with spontaneously broken symmetry to describe the evolution of domain walls and vacuum bubbles. The behavior of the scalar GLH field in the presence of massive point fermion, which can be regarded as an inhomogeneity, was discussed in Ref. 6. A discrete (lattice) approximation of the $\lambda\phi^4$ theory with local mass inhomogeneities was considered in Ref. 7. Therefore, study of the behavior of the solutions of the sine–Gordon and GLH equations with inhomogeneities is of undoubted interest.

In this paper, we consider the interaction of a kink in the $\lambda\phi_2^4$ theory with a δ-function impurity. We consider a Lagrangian of the form

$$\mathscr{L}=1/2\int_{-\infty}^{+\infty}dx\left(\phi_t^2-\phi_x^2-\frac{\lambda}{2}(\phi^2-m^2\lambda^{-1})^2\times[1-\mu\delta(x-X_0)]\right). \tag{1}$$

For the field theory determined by (1), at the classical level we can set $\lambda=1$, $m=1$, after which the equation for the field $\phi(x,t)$ takes the form

$$\phi_{tt}-\phi_{xx}+(\phi^3-\phi)[1-\mu\delta(\delta-X_0)]=0. \tag{2}$$

Here μ is a constant that characterizes the strength of the interaction of the field $\phi(x,t)$ with the impurity. In the case $\mu=0$, Eq. (2) has the solution (kink)

$$\phi(x,t)=\tanh[\gamma(x-x_1-Vt)], \tag{3}$$

where $\gamma=[2(1-V^2)]^{-1/2}$, V is a continuous parameter ($0\leq V<1$) that determines the kink velocity, and x_1 is the initial position of the kink.

In this paper, we consider what happens to the kink in the case $\mu\neq0$. It is well known that a δ-function impurity acts on the kink in the approximation of an undeformable kink as a potential. For the case of the sine–Gordon equation, this question was first discussed in Ref. 8, where it was shown that for the appropriate sign of μ a microinhomogeneity acts on the kink of the theory as an attractive potential, so that the soliton can be localized. In Refs. 9–11, questions relating to the deformability of a soliton solution localized at a microinhomogeneity were studied for the case of the sine–Gordon equation. It was shown that as the constant μ is increased, in addition to the vibrational motion of the kink in the potential produced by the inhomogeneity, kink deformation effects of a resonant nature arise. For example, it was found in Ref. 11 that there is a strong change in the shape of the soliton in the presence of vibrations, the change occurring at a definite value of the constant μ. The problem of scattering of the kink by the impurity for the problem (1) was first studied in Ref. 12, and some resonance effects were found which are discussed below.

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In the question of kink–impurity interaction, a recent development has been discussion of the role of the specific degree of freedom known as the discrete excitation mode of the impurity. Interest in this mode increased after the publication of Ref. 7, although this excitation mode was apparently first mentioned in Ref. 6. We elucidate the position of the discrete impurity mode in the excitation spectrum of the problem (1). For any μ, the solutions $\phi_{\pm}=\pm 1$, which are vacuum solutions in the case $\mu=0$, persist for Eq. (2). We now seek the spectrum of small excitations around the solution ϕ_+:

$$\phi(x,t)=\phi_+ + \delta\phi(x,t), \quad |\delta\phi(x,t)|\ll 1. \tag{4}$$

Substituting (4) in Eq. (2) and linearizing with respect to $\delta\phi$, we obtain for $\delta\phi$ the equation

$$\delta\phi_{tt}-\delta\phi_{xx}+2\delta\phi=2\mu\delta\phi\delta(x). \tag{5}$$

Here the impurity is assumed to be situated at the point $X_0=0$. Making the substitution $\delta\phi=\exp(-i\tilde{\omega}t)\chi(x)$, we obtain for the function $\chi(x)$ the equation

$$-\chi_{xx}-2\mu\delta(x)\chi=(\tilde{\omega}^2-2)\chi. \tag{6}$$

Equation (6) has the unique normalizable solution

$$\chi(x)=A\exp(-\mu|x|), \quad \tilde{\omega}^2=2-\mu^2. \tag{7}$$

It is natural to call (7) a discrete impurity mode. We shall consider the influence of this mode on the behavior of solitons near the inhomogeneity. In Ref. 13, this was done for the sine–Gordon equation. In Ref. 13, scattering of the soliton by the impurity was described using an effective Lagrangian that takes into account the potential interaction of the soliton with the impurity and the possibility of excitation of the discrete impurity mode. Comparison of such a model with calculations of the exact field problem indicated that the behavior of the soliton could be described successfully with the effective Lagrangian.

The aim of this paper is to investigate the problem of interaction of the kink with the impurity for the theory (1). We discuss: 1) the influence of the discrete impurity mode on the motion of the kink; 2) an effective Lagrangian that describes approximately the kink–impurity interaction. We shall also compare the exact solutions of Eq. (2) with the solutions that follow from the effective Lagrangian; in particular, we shall obtain a description of the "escape windows" and the regions where the kink is captured by the impurity. Preliminary results were partly discussed in our Refs. 12 and 14.

2. BOUND STATE OF THE KINK AND IMPURITY

It is convenient to begin the investigation of the kink–impurity interaction by discussing the behavior of the kink near the impurity. In Fogel's approximation,[8] it can be shown[12] that the impurity in the theory (1) acts on the kink like an attractive potential ($\mu>0$):

$$V(x)=-\frac{\mu}{4\cosh^4(\alpha x)}, \tag{8}$$

where $\alpha=1/\sqrt{2}$. Thus, in this approximation there is a finite motion of the kink for energy $E<0$ and an infinite motion for $E>0$. However, the picture of the soliton motion in the exact field problem (1) is more complicated. The soliton may be not only captured by the attractive impurity but also reflected by it. This effect has a resonant nature and was first discussed in Ref. 12. Later, the same effect was also discovered for the sine–Gordon equation in Ref. 13. Resonant interaction of the soliton with the impurity cannot be obtained in the potential approximation (8) and requires a more detailed study.

We note first of all that a kink situated at the impurity position,

$$\phi_0(x,X_0)=\tanh[\alpha(x-X_0)], \tag{9}$$

is an exact solution of the problem (2). We attempt to find solutions of Eq. (2) in the form

$$\phi(x,t;X_0)=\tanh[\alpha(x-X_0)]+\delta\phi(x,t;X_0), \tag{10}$$

where $|\delta\phi(x,t;X_0)|\ll 1$. Choosing $X_0=0$, we obtain by analogy with (5)

$$\delta\phi_{tt}-\delta\phi_{xx}+(3\tanh^2\alpha x-1)\delta\phi=-\mu\delta\phi(x,t)\delta(x). \tag{11}$$

From this, making the substitution $\delta\phi=\exp(-i\omega t)\chi(x)$, we obtain for the function $\chi(x)$ the equation

$$-\chi_{xx}+[-3ch^{-2}(\alpha x)+\mu\delta(x)]\chi=(\omega^2-2)\chi(x). \tag{12}$$

The potential of the problem consists of the attractive potential $-3\cosh^{-2}(\alpha x)$ and a repulsive δ-functional potential. The ground state of the problem is determined by the equation

$$\mu=-\frac{2\sqrt{2}\Gamma(-1/2+\varepsilon/2)\Gamma(2+\varepsilon/2)}{\Gamma(-1+\varepsilon/2)\Gamma(3/2+\varepsilon/2)}, \tag{13}$$

where $\Gamma(y)$ is the gamma function, and $\varepsilon^2=-2(\omega_0^2-2)$. In the limit $\mu\to 0$, we obtain from (13)

$$\omega_0^2=\frac{3}{4\sqrt{2}}\mu, \tag{14}$$

and this reproduces the frequency of the kink vibrations in the harmonic approximation for the potential $V(x)$ in (8). For $\mu=0$, we have $\omega_0=0$, i.e., the frequency ω_0 arises from the zero (shear) mode of kink excitations in the absence of interaction with the impurity. In the limit $\mu\to\infty$, we have $\varepsilon\to 1$ and $\omega_0^2\to 3/2$.

The first excited state of the problem (12) has a node at $x=0$, is identical to the solution of the problem for $\mu=0$, and corresponds to the eigenvalue $\omega_1^2=3/2$. Thus, in the presence of the impurity the discrete excitation mode of the kink exists and has the same eigenfunction and eigenvalue as in the case $\mu=0$. There are also solutions of the problem (12) corresponding to a continuum with $\omega^2\geqslant 2$ (Ref. 14).

Thus, we have listed the excitation spectrum of the kink situated at the point of the impurity. It consists of vibrations of the kink around the point X_0 with frequency ω_0, variations in the shape of the kink with frequency ω_1, and a continuum. There is no excitation of the discrete impurity mode of the type (7) in the case when the kink is at the point of the impurity. We now consider the excitation spectrum of the system in the case when the kink is far from the impurity. In this case, the discrete impurity mode (7) localized near the impurity is added to the zero mode $\omega_0=0$ and the first discrete mode $\omega_1=\sqrt{3/2}$ of excitations of the kink localized near it. The discrete impurity mode splits off from the continuum at distances between the impurity and kink of order unity and does not exist at shorter distances. As is shown by the experience gained from study of the kink–antikink ($K\bar{K}$) interaction in the $\lambda\phi_2^4$ theory,[15–18] the continuum excitations are strongly suppressed compared with the excitations of the discrete kink mode with $\omega_1=\sqrt{3/2}$ at not too high collision energies. Therefore, it appears logical to construct initially the effective Lagrangian of the kink–impurity interaction with allowance for only the modes ω_0 and ω_1. The procedure for constructing such an effective Lagrangian is completely equivalent to the problem of constructing the effective Lagrangian for the $K\bar{K}$ interaction.[16–18]

3. EFFECTIVE LAGRANGIAN OF KINK–IMPURITY INTERACTION

We shall seek solutions of the theory described by the Lagrangian (1) at all separations X between the impurity and the center of the kink in the form

$$\phi(x,t)=\tanh[\alpha(x-X)]+A\chi_1[\alpha(x-X)]. \tag{15}$$

Here $\chi_1(z)$ is the solution of Eq. (12) corresponding to the eigenvalue $\omega_1=\sqrt{3/2}$, $z\equiv\alpha(x-X)$:

$$\chi_1(z)=\frac{\sqrt{9/8}\tanh z}{\cosh z}. \tag{16}$$

We substitute the function $\phi(x,t)$ in the form (15) in the Lagrangian (1) and regard $X(t)$ and $\mathbf{A}(t)$ as dynamical variables in what follows. Integrating the expression (1) with respect to x, we obtain the effective Lagrangian

$$\mathscr{L}(X,\dot{X},A,\dot{A})=\mathscr{L}_0+\mathscr{L}_1+\mathscr{L}_{\text{int}} \tag{17}$$

up to terms quadratic in A, where

$$\mathscr{L}_0=M_K(\dot{X}^2/2-1)-V(X);\quad V(X)=-\frac{\mu}{4\cosh^4\alpha X},$$

$$\mathscr{L}_1=\dot{A}^2/2-\omega_1^2A^2/2,$$

$$\mathscr{L}_{\text{int}}=-\frac{\mu A\sqrt{3/2}}{\sqrt{2}\cosh^3\alpha X}\tanh^2\alpha X+\frac{\mu A^2 3/4}{\sqrt{2}\cosh^2\alpha X}\tanh^2\alpha X(3\tanh^2\alpha X-1),$$

where $M_K=2\sqrt{2}m^3/3\lambda=2\sqrt{2/3}$ is the kink mass. The standard variational procedure for the Lagrangian (17) leads to dynamical equations for $X(t)$ and $A(t)$ by analogy with the procedure used earlier for the $K\bar{K}$ interaction.[18]

If we take into account only the term with $\mathscr{L}_0$ in (17), then we obtain a potential model for the kink–impurity interaction—the analog of Fogel's approximation[8] for the sine–Gordon equation. In this approximation, the kink either passes above the impurity or executes a finite motion in the region of the impurity. The expression $\mathscr{L}_1$ is the Lagrangian that describes the free vibration of the first mode with $\omega_1=\sqrt{3/2}$. The term $\mathscr{L}_{\text{int}}$ describes the interaction of the zeroth and first modes due to the presence of the impurity. Making the change of variables

$$C=A/\sqrt{2M_K},\quad y=X\sqrt{2},\quad t'=t/\sqrt{2}$$

in (17), we obtain for the effective Lagrangian (17) the system of dynamical equations

$$\begin{cases}\ddot{y}+\mu(dv_0/dy+Cdv_1/dy+C^2dv_2/dy)=0\\ \ddot{C}+3C+\mu(v_1+2Cv_2)=0.\end{cases} \tag{18}$$

Here

$$v_0(y)=-\frac{(3/8)/\sqrt{2}}{\cosh^4 y};\quad v_1=\frac{3/2\tanh^2 y}{\cosh^3 y};$$

$$v_2(y)=-\frac{(3/2)/\sqrt{2}\tanh^2 y}{\cosh^3 y}(3\tanh^2 y-1).$$

Note that if the first excited mode of the kink is frozen, i.e., we set $C\equiv 0$, we would obtain instead of (18) the potential problem

$$\ddot{y}+\mu dv_0/dy=0, \tag{19}$$

which, with allowance for the energy conservation law, leads to solutions in quadratures. In contrast to (19), Eqs. (18) describe the motion of a system with two degrees of freedom (y,C), which are related by one conservation law. Thus, in the 4-dimensional phase space of the dynamical system (18) a 3-dimensional manifold of trajectories is distinguished. It is well known that in such dynamical systems stochastization can occur. The motion may be either finite or infinite. We have studied the solutions of the system (18) corresponding to the evolution of the unexcited kink scattered by the impurity. We studied the evolution in time of the functions $X(t)$ and $A(t)$ corresponding to the initial conditions

$$X(0)=1.5,\quad \dot{X}(0)=V,\quad A(0)=\dot{A}(0)=0.$$

Here $\dot{X}(0)$ is the initial velocity V of the kink (a free parameter of the problem).

We studied the system (18) for the different values μ=0.3, 0.5, 1.0, 2.0 of the coupling parameter and different initial velocities V. For example, for μ=0.3 the range of velocities $0.025\leqslant V\leqslant 0.075$ was calculated with velocity step $\Delta V=2.10^{-4}$. In this case, the relative error in the energy was $\sim 10^{-6}$. Without allowance for the coupling of the channels, i.e., for the problem (19), the infinite motion corresponds to passage of the kink from $-\infty$ to $+\infty$ over the attractive well $v_0(y)$.

If the coupling of the channels is accounted for, the kink may be captured by the well, and it can also be scattered backward. All three different forms of kink motion are shown in Figs. 1–3. Figure 1 shows projections of the phase trajectories onto the planes $(X,\dot{X})$ and $(A,\dot{A})$ for the case when the kink passes over the well, Fig. 2 is for the case of capture, and Fig. 3 for backward scattering. Backward scattering by the impurity is a new element inherent in systems of the type (18). Indeed, the capture effect could be ascribed to the presence of friction in the system. This was precisely how the discovered capture effect in kink–antikink collisions was originally explained.[15] However, the observation of "es-

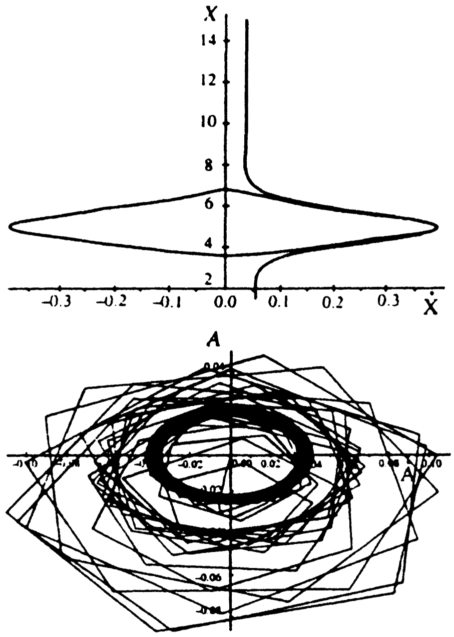

FIG. 1. Projections of phase trajectories onto the $(X,\dot{X})$ plane (at the top) and the $(A,\dot{A})$ plane (at the bottom) for $\mu=0.3$, $V=0.0544$. A case in which the kink passes near the inhomogeneity after it has made one complete motion around it (here and in what follows, the impurity is situated at the point $X=5.0$). The lower figure shows internal excitation of the kink, and one can clearly see the elliptical polarization of the outgoing kink.

cape windows" (see Ref. 17 and the references cited in it) indicated an existing mechanism for returning the energy to the zero mode. In the case of kink–impurity interaction, the same effect exists in the reflection of the kink by the attract-

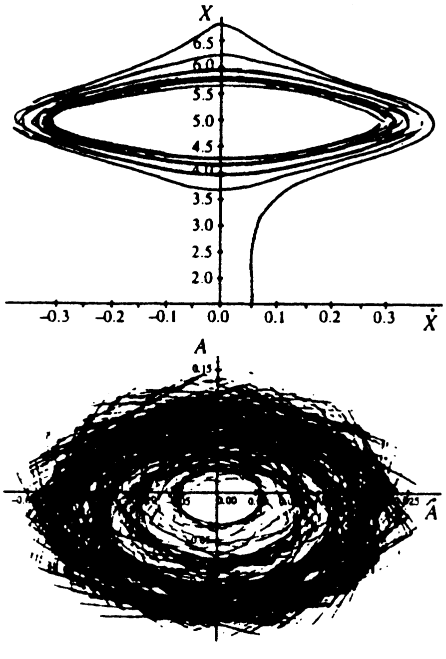

FIG. 2. Projections of the phase trajectories onto the $(X,\dot{X})$ plane (at the top) and the $(A,\dot{A})$ plane (at the bottom) for $\mu=0.3$, $V=0.0551$. A case of capture of the kink by the inhomogeneity situated at the point $X_0=5.0$. The internal excitation of the kink in the lower part of the figure suggests the existence of quasilevels in the kink–impurity system.

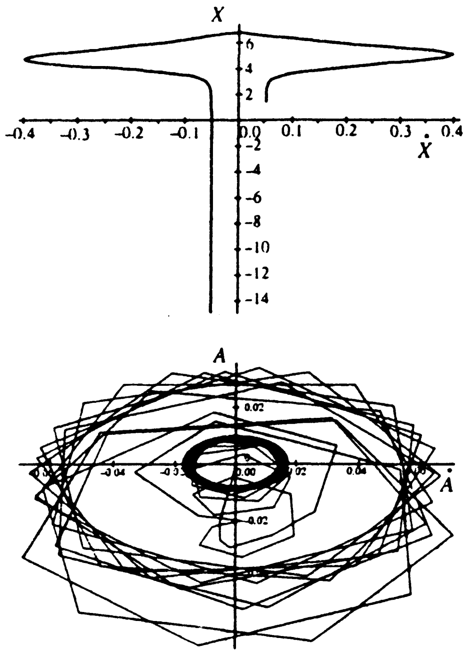

FIG. 3. Projections of the phase trajectories onto the $(X,\dot{X})$ plane (at the top) and the $(A,\dot{A})$ plane (at the bottom) for $\mu=0.3$, $V=0.0537$. A case of reflection of the kink by the inhomogeneity. The reflected kink is elliptically polarized.

ing impurity. The further study of the model leads to the discovery of several windows of transmission and capture (see Fig. 4). We mention that the system of equations (8) was calculated to $t'=10^3$. It is entirely possible that at times $t>t'$ some of the trajectories from the capture windows may make a contribution to the region of escape forward or backward. The difference between the trajectories is manifested particularly clearly when one studies the Poincaré plots in the $(X,\dot{A})$ plane made for the case of reflection of the kink by the inhomogeneity (Fig. 5) and for capture (Fig. 6). It can be seen that in the case of capture the plane of the plots is filled rather uniformly, in contrast to the backward scattering. Thus, in the capture windows the motion is nearly stochastic,

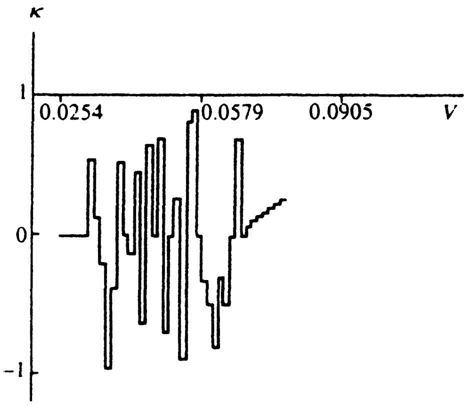

FIG. 4. Windows of transmission and capture for $\mu=0.3$ shown as the coefficient of inelasticity $\kappa=V_f^2/V_{in}^2$ as a function of the initial velocity V: A plus sign of $\kappa(V)$ corresponds to transmission of the kink, a minus sign to reflection, and $\kappa=0$ to capture of the kink by the impurity.

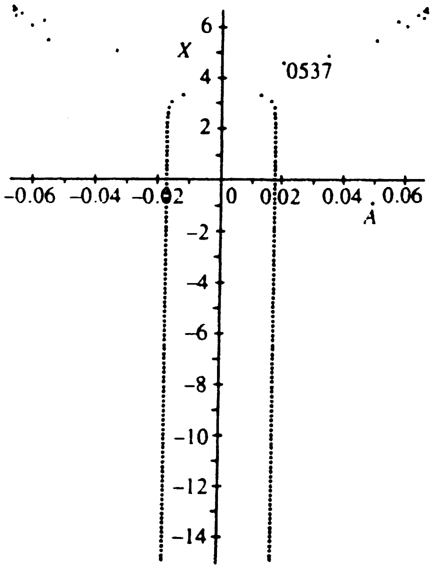

FIG. 5. Poincaré plot $X(\dot{A})|_{A=0}$ for the case of reflection of the kink by the inhomogeneity, $\mu=0.3$, $V=0.0537$. Regular behavior.

while in the transmission and reflection windows it is nearly integrable.

It was also found that there is a critical value V_{cr} of the velocity above which the only kink passes around the inhomogeneity occurs, while below it all the above effects occur. Thus, for $\mu=0.3$, $V_{cr}\cong 0.0686$; for $\mu=0.5$, $V_{cr}\cong 0.151$; for $\mu=1$, $V_{cr}\cong 0.322$; and for $\mu=2$, $V_{cr}\cong 0.149$. Note that with the effective Lagrangian (17) the critical velocity V_{cr} increases with increasing μ in the region of small μ. However, in the region $\mu>1$ the regime is different, and for large μ V_{cr} is observed to decrease.

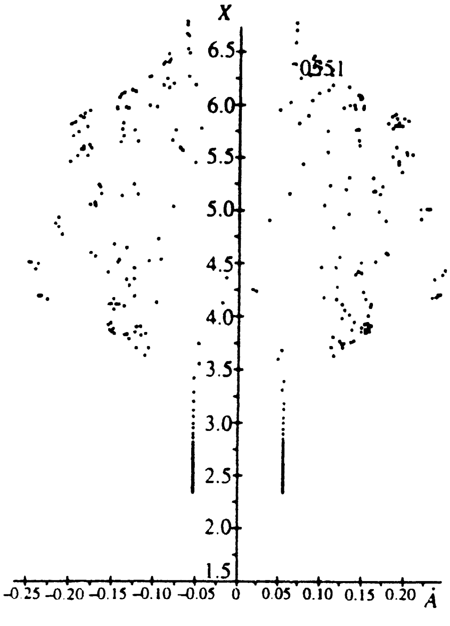

FIG. 6. Poincaré plot of $X(\dot{A})|_{A=0}$ for the case of capture of the kink by the inhomogeneity, $\mu=0.3$, $V=0.0551$. Stochastization of the process.

Thus, analysis of the field problem (1) using the effective Lagrangian (17) leads to a very varied picture of solutions not found in the potential approximation: The kink can be captured by the impurity and reflected backward from the attractive impurity. There are windows in which the kink passes over the impurity. There exists a critical kink velocity, above which the kink is not captured by the impurity but passes above it, merely losing some of its energy. All these predictions are confirmed by direct calculations of Eq. (2) of the exact field problem.

4. EXACT FIELD PROBLEM

The solution of the Cauchy problem for Eq. (2) requires much more computing time than the integration of the system of ordinary differential equations (18). As initial conditions, we chose a kink moving with constant velocity V far from the impurity:

$$\phi(x,0)=\tanh\,\gamma(x-x_a),\quad \phi_t=-V\gamma\,\cosh^{-2}\gamma(x-x_a).$$

In the numerical experiments, we used a Gaussian approximation for the δ function in Eq. (2), defining it so that $\delta(x,\alpha)\xrightarrow[\alpha\to\infty]{}\delta(x)$:

$$\delta(x,\alpha)=(\alpha/\sqrt{\pi})\,\exp[-\alpha^2(x-X)^2]. \tag{20}$$

In connection with this substitution, the results of the calculations begin to depend, in general, on the parameter α. In our computational scheme, the limit $\alpha\to\infty$, with which one should compare the results of the calculation in accordance with the effective Lagrangian (17), cannot be realized literally. Therefore, although Eq. (2) was calculated for different fairly large values of the parameter α, such that the diameter of the impurity was much less than the diameter of the kink, we cannot require complete numerical agreement of the results of the exact theory and the model (17). However, in the exact problem we discuss the observables that depend weakly on the particular value of the parameter α. For example, the difference between the critical velocities when α was changed from $\alpha=5$ to $\alpha=10$ was found to be less than 5%.

The results of solving the exact equation (2) with allowance for the substitution (20) confirm the main results obtained with the model effective Lagrangian (17). When the kink passes over the impurity, the exact theory determines the critical velocity V_{cr} for capture of the kink by the impurity, the passage of the kink over the impurity for $V>V_{cr}$, and the reflection of the kink by the attractive impurity, and also windows of transmission below V_{cr}. The value of the critical velocity is reproduced rather accurately. For example, for $\mu=0.3$ the critical velocity for the effective potential is $V_{cr}^{\text{eff}}\cong 0.0686$, while for the exact equation we have $V_{cr}^{\text{exact}}\cong 0.069$ for $\alpha=5$ and $V_{cr}^{\text{exact}}\cong 0.0730$ for $\alpha=10$.

Nevertheless, the exact calculation leads to some important differences for several predictions. Thus, for $\mu=0.3$ and $\mu=0.5$ we found in each case only one window of kink reflection by the inhomogeneity. It appears that study of the kink in the field problem leads to smearing of the windows. A similar effect was also observed in the case of $K\bar{K}$ interactions for velocities near V_{cr}; see Ref. 18. In addition, as

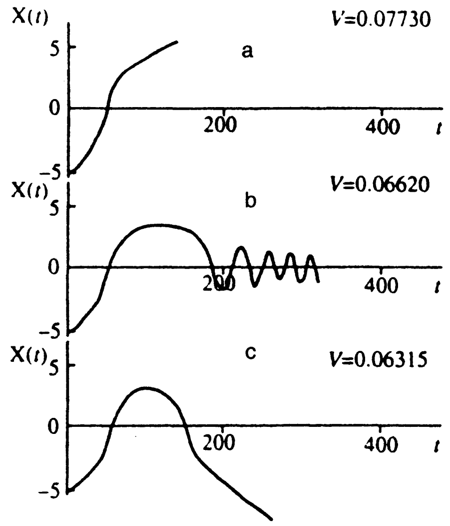

FIG. 7. Behavior of the kink in the exact field theory (2) for $\mu=0.3$, $\alpha=5$ for different initial velocities: a—transmission, b—capture, c—reflection.

can be seen from Figs. 1–3, in this problem there is a strong mismatch of the periods of the motions with respect to the "slow" $X(t)$ and "fast" $A(t)$ coordinates. In this connection, the escape windows, which impose the condition that one period of the motion with respect to the two coordinates is multiple of the other, are very narrow. Therefore, on the one hand, energy loss to the continuum of excitations may lead to disappearance of some of the windows. On the other hand, some of the remaining windows in the exact problem may be too narrow and may be missed in the numerical calculation.

It is here worth noting that in the exact field problem the step in the initial velocity, $\Delta V=2\cdot 10^{-3}$, was four times greater than the step for the model problem (17) ($\Delta V=5\cdot 10^{-4}$). This was dictated by the limitations on the computing time. Because of this, some of the narrow "windows" may have been lost. In addition, a detailed study was made only for the case $\mu=0.3$. For $\mu\geqslant 0.5$, no study of the complete spectrum was made. The main aim of the numerical solution of the field problem (2) made here was to confirm the qualitative conclusions concerning the diversity of solutions that follow from the effective Lagrangian (17). It can be seen from analysis of our solutions that the detailed information, i.e., the position of the windows of reflection and transmission, is fairly sensitive to the choice of the approximation of the δ function and requires more detailed investigation. It appears that the same is true of the results of the numerical calculation of Eq. (2) published recently in Ref. 5, where the δ function was replaced by a rectangular well without a discussion of the consequences of such a replacement.

Examples of the behavior of the kink of the exact problem (2) for three different initial velocities V are given in Fig. 7 ($\mu=0.3$).

5. CONCLUSIONS

As can be seen from the above calculations, further improvement in the description of the behavior of the kink near the inhomogeneity must take into account the effects of energy loss to the continuum excitation modes and to the discrete impurity mode. In the case of the sine–Gordon equation, an attempt to take into account the discrete impurity mode was made in Ref. 13. A formalism similar to the one discussed above was used, i.e., a solution to the problem with the effective Lagrangian was sought for an ansatz in the form a sum of a kink and the discrete impurity mode for all separations between the kink and the impurity.

As follows from Sec. 2 of this paper, at short distances between the kink and the impurity the impurity mode is absent altogether, and therefore in the ansatz of Ref. 13 the derivative of the field is discontinuous in x in the form of the discrete mode, which should not happen in this region between the kink and the impurity. Therefore, the success of the effective description of the motion of the kink near the inhomogeneity for the sine–Gordon equation of Ref. 13 requires further investigation. With regard to the $\lambda\phi^4$ theory, the role of the discrete mode of kink excitation is decisive. The same conclusion was drawn by the authors of Ref. 5 by analyzing the distribution of the energy between the discrete mode of the kink and the impurity mode when the kink interacts with the impurity. Unfortunately, we were not able to make a detailed comparison of the results of our calculations with those of Ref. 5, since the case $\mu=0.3$ that we studied in detail in Ref. 12 was not discussed in Ref. 5.

The results obtained in the present paper continue a series of studies on classical nonintegrable field theories in which there are solitary waves possessing at least two degrees of freedom, for example, the frequency of vibrations of the solitary wave in a potential and a discrete mode of excitation of it. If one frequency is a multiple of the other, the solution of the problem exhibits the specific resonance effects that were first found in the $\lambda\phi_2^4$ theory[17] and then for the modified sine–Gordon equation[19] and for the double sine–Gordon equation.[20] In Ref. 12, we found for the first time the same effect for the kink of the $\lambda\phi_2^4$ theory interacting with an impurity, and in Ref. 13 the same effect was found later for interaction with an impurity of the kink of the sine–Gordon equation.

At the same time, the investigation begun in Ref. 18 into the properties of a model effective Lagrangian with a finite number of degrees of freedom provides a qualitative understanding of the fact that outside the escape windows the motion is nearly stochastic. Here we should also mention the studies of Refs. 21–23, which discussed problems of stochastization when solitonlike solutions interact in systems that are not integrable by the inverse scattering method.

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