Aharonov–Bohm effect in curved space and cosmic strings

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A quantum theory is developed for the scattering of a nonrelativistic particle in the field of a cosmic string regarded as a combination of magnetic and gravitational strings. Allowance is made for the effects due to the finite transverse dimensions of the string under fairly general assumptions about the distribution of the magnetic field and the spatial curvature in the string. It is shown that in a definite range of angles the differential scattering cross section at all absolute values of the wave vector of the incident particle depends strongly on the magnetic flux of the string.

1. INTRODUCTION

It is well known that in classical mechanics a complete description of electromagnetic effects can be made by means of the electromagnetic field strength, which acts directly on charged particles. In 1959, Aharonov and Bohm,[1] using the Schrödinger equation, considered the scattering of an electron in an external static magnetic field produced by an infinitely long solenoid and found an effect that does not depend on the depth of penetration of the electrons into the region of magnetic force lines. This showed that in quantum mechanics the electromagnetic field acts on charged particles even in the case when the region in which the field is localized cannot be reached by the particles (see, for example, the reviews of Refs. 2 and 3).

In this paper, we investigate the effect of curvature of space on the Aharonov–Bohm effect. Namely, we consider a situation in which there is not only a tube of magnetic force lines but also an external static cylindrically symmetric gravitational field with symmetry axis that coincides with the axis of the magnetic tube. It is assumed that at large distances from the symmetry axis the space becomes locally flat and the region of spatial curvature (the gravitational tube) may either coincide with the magnetic tube or include it or, finally, be included in it. We pose and solve the problem of the scattering of a charged test particle in a space with gravitational and magnetic tubes.

It is appropriate to point out here that the simultaneous existence of magnetic and gravitational tubes is rather typical of models with spontaneous breaking of gauge symmetries. In such models, there arise vacuum structures in the form of strings (for example, Abrikosov–Nielsen–Olesen vortices[4,5]), which are characterized, on the one hand, by a certain flux that one can reasonably call magnetic since it corresponds to the spontaneous breaking of a gauge degree of freedom and, on the other, by the value of a condensate that spontaneously breaks a symmetry. To this condensate there corresponds a uniform distribution of mass along the string axis, and this is the source of the gravitational field of the type described above. We note that this latter field is rather weak and it is apparently for this reason that it has effectively escaped the attention of investigators. In particular, to the best of our knowledge the processes of scattering of particles by strings (vortices) usually take into account only the presence of the magnetic component, and the presence of the gravitational component is ignored (see, for example, Refs. 6 and 7). It is to the correction of this, in our view, shortcoming that the present paper is devoted. We shall show that it is possible to take into account gravitational components of a much more general form than those inherent in strings in models with spontaneous breaking of gauge symmetries.

In what follows, we shall, for definiteness, consider strings that are cosmological objects—so-called cosmic strings. According to modern ideas, the early stages in the evolution of the universe were characterized by high temperatures and a greater degree of symmetry than now, and as the universe cooled there was a series of phase transitions with spontaneous symmetry breaking.[8] The topological defects formed as a result (monopoles, strings, and domain walls) may be stable, and the mere fact of their existence leads to many important consequences in cosmology.[9] Particularly interesting cosmological objects are strings, which, in particular, can play the role of nucleation centers of galaxies and gravitational lenses.[10]

The energy–momentum tensor of a straight infinitesimally thin and infinitely long cosmic string is described by

$$T_n^{n'}=\begin{pmatrix}\varepsilon & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -p_z c\end{pmatrix},$$

$$\varepsilon=-p_z c=Mc^2\delta(x)\delta(y), \qquad (1)$$

where the parameter M is the linear mass density of the string. The spacetime metric generated by the source (1) can be represented in one of the following three equivalent forms:[11–13]

$$ds^2=-c^2dt^2+dz^2+\left(\frac{r}{r_0}\right)^{-8GMc^{-2}}(dr^2+r^2d\phi^2)$$
$$=-c^2dt^2+dz^2+(1-4GMc^{-2})^{-1}d\tilde{\rho}^2$$
$$+(1-4GMc^{-2})\tilde{\rho}^2d\phi^2$$

$$= -c^2dt^2 + dz^2 + d\rho^2 + \rho^2 d\tilde{\phi}^2, \qquad (2)$$

where

$$\rho = \frac{r_0}{1-4GMc^{-2}} \left(\frac{r}{r_0}\right)^{1-4GMc^{-2}},$$

$$\tilde{\rho} = \sqrt{1-4GMc^{-2}}\rho,$$

$$0 \leq \phi < 2\pi, \quad 0 \leq \tilde{\phi} < 2\pi(1-4GMc^{-2})$$

and r_0 is a parameter with the dimensions of length. In polar coordinates (r,ϕ), the metric of a surface orthogonal to the coordinate lines corresponding to the variable z has a conformal form; in the polar coordinates $\tilde{\rho}$, ϕ, the energy-momentum tensor takes the form (1) $(x=\tilde{\rho}\cos\phi$, $y=\tilde{\rho}\sin\phi)$; finally, in the polar coordinates ρ, $\tilde{\phi}$ with incomplete angle$(\rho,\tilde{\phi})$, the metric takes a manifestly flat form, and it becomes obvious that the surface z=const is isometric to the surface of a cone. It is also obvious that a fixed or slowly moving string does not interact with gravitating objects. At the same time, the conical nature of the space (2) leads to remarkable effects such as the doubling of the image of objects situated behind the string (which may be an explanation of the double structure of quasars),[10] fluctuations in the temperature of the microwave background,[14] and the formation of wakes behind moving strings (which may play an important role in the formation of the large-scale structure of the universe)[15] (see the review of Ref. 16).

It should be noted that spaces with the metric (2) have been known for quite a long time under the name of conical spaces.[17,18] Apparently, one of the first to draw attention to such spaces and their interesting properties was M. Fierz (see the footnote in Ref. 17). Comparatively recently, scattering of test scalar and spinor particles by the singularity corresponding to the cone apex ($r=0$) has been considered.[19–21] It was also shown long ago that the presence of a conical singularity in space leads to a gravitational analog of the Aharonov–Bohm effect.[22,23] As noted above, we believe it is of interest to consider the situation in which there is flux of magnetic field along a gravitational string and to study the scattering of a test charged particle by such an object, in other words, to consider the standard (magnetic) Aharonov–Bohm effect in a Fierz space.

This problem was actually posed for the first time in Ref. 24, in which some results were obtained; in particular, attention was drawn to the difference between the scattering for spinor and scalar particles. In this paper, we make a more systematic and complete study of the problem. We determine the S matrix for scattering in the case of combined singular magnetic and gravitational strings, establish the connection between the S matrix and the scattering amplitude in this and in the more general case of nonsingular strings, and also take into account effects due to the finite transverse dimensions of the strings.

More precisely, we consider the spacetime with metric

$$ds^2 = -c^2dt^2 + dz^2 + f(X,Y)(dX^2+dY^2), \qquad (3)$$

where $X=r\cos\phi$, $Y=r\sin\phi$, and f is the conformal factor of the metric of the surface z=const. This last surface has Gaussian curvature

$$K(X,Y) = -(2f)^{-1}(\partial_X^2 + \partial_Y^2)\ln f, \qquad (4)$$

which satisfies the condition

$$K(X,Y) = 0 \qquad (5)$$

for $r > r_K$. With regard to the closure of the region, $r \leq r_K$, we assume that here $K(X,Y)$ is a piecewise continuous function with singularities at isolated points or on isolated lines that are integrable with respect to the measure $f(X,Y)\,dXdY$. Then the total integrated curvature (in units of 2π) is given by

$$\Phi_K = \frac{1}{2\pi}\int_{r\leq r_K} dXdY f(X,Y)K(X,Y). \qquad (6)$$

As we showed in Ref. 25, for $r > r_K$ [when the condition (5) is satisfied] the conformal factor takes the form

$$f(X,Y) = (r/r_0)^{-2\Phi_K}. \qquad (7)$$

Thus, taking into account (2) and (3), we obtain a relationship between the total curvature of the surface and the linear mass density of the cosmic string:

$$\Phi_K = 4GMc^{-2}. \qquad (8)$$

In the case of a singular gravitational string, we have $r_K \to 0$ and

$$K(X,Y) = 2\pi\Phi_K \frac{\delta(X)\delta(Y)}{f(X,Y)}. \qquad (9)$$

Along the coordinate lines corresponding to the variable z in the spacetime (3) a static magnetic field $B(X,Y)$ that satisfies the condition

$$B(X,Y) = 0 \qquad (10)$$

for $r > r_B$ is directed. With regard to the closure of the region, $r \leq r_B$, we assume that here $B(X,Y)$ is a piecewise continuous function with singularities at isolated points or on isolated lines that are integrable with respect to the measure $f(X,Y)dXdY$. Then the total magnetic flux (in London units $2\pi\hbar c/e$) is given by

$$\Phi = \frac{e}{2\pi\hbar c}\int_{r\leq r_B} dXdY f(X,Y)B(X,Y), \qquad (11)$$

where e is the coupling constant of the matter to the gauge minus field ($-e$ is the charge of the test particle). In the case of a singular magnetic string, we have $r_B \to 0$ and

$$B(X,Y) = 2\pi\frac{\hbar c}{e}\Phi\frac{\delta(X)\delta(Y)}{f(X,Y)}. \qquad (12)$$

In accordance with what we have already said concerning the general case of strings in models with spontaneous breaking of a gauge symmetry, fairly small values of Φ_K are currently considered in the phenomenology of cosmic strings; for example, $\Phi_K \sim 10^{-5}$ for strings formed as a result of a phase transition at the scales of the symmetry breaking of the grand unification of the interactions, and $\phi_K \sim 10^{-31}$ for strings formed at the scales of the symmetry breaking of the electroweak interaction. We shall use the expression cosmic strings in a wider sense as possible configurations of space with cylindrical symmetry that may also be characterized by much larger (and also negative) values of Φ_K. With regard to the magnetic component and the actual gauge model of the string, we shall also make the most general assumptions, and the gauge symmetry group (Abelian or non-Abelian, spontaneously broken or exact) will not be in any way particularized. Putting it briefly, by a cosmic string we mean an object characterized by two parameters, Φ and Φ_K, which can take values in the ranges $-\infty<\Phi<\infty$ and $-\infty<\Phi_K<1$; values in the range $1\leq\Phi_K<\infty$ are not considered here (see Ref. 26).

In this paper, we study the quantum-mechanical scattering of nonrelativistic test particles by a cosmic string. Since the motion of the particles along the string axis z is free, we can make a restriction to considering the motion of particles on the surface $z=\text{const}$ (for more details, see, for example, Ref. 24). The Schrödinger equation for the wave function describing the stationary scattering state has the form

$$H\psi(X,Y)=\frac{\hbar^2k^2}{2m}\,\psi(X,Y), \tag{13}$$

where m and k are, respectively, the mass of the particle and the absolute value of its wave vector. Under the condition of axial symmetry of the magnetic field strength and the Gaussian curvature,

$$\partial_\phi B(X,Y)=0, \quad \partial_\phi K(X,Y)=0, \tag{14}$$

we obtain for the Hamiltonian the expression

$$H=-\frac{\hbar^2}{2m}\left\{\frac{1}{r^2f}\left[(r\partial_r)^2+\left(\partial_\phi-i\,\frac{e}{\hbar c}\,V+\frac{1}{2}\,i\sigma W\right)^2\right]+\sigma\,\frac{e}{\hbar c}\,B-\frac{1}{2}\,K\right\}, \tag{15}$$

where

$$V(r)=\int_0^r drrfB, \quad W(r)=\int_0^r drrfK, \tag{16}$$

$\sigma=\sigma_3=\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$ in the case of a spinor particle (ψ is a two-component column function) and $\sigma=0$ in the case of a scalar particle (ψ is a single-component function). As radial variable, it is convenient to use the geodesic length in the radial direction:

$$\rho=\int_0^r drf^{1/2}. \tag{17}$$

Then the metric (3) (under the condition of axial symmetry) takes the form

$$ds^2=-c^2dt^2+dz^2+d\rho^2+\gamma^2d\phi^2, \tag{18}$$

where

$$\gamma(\rho)=r(\rho)\sqrt{f\,[r(\rho)]}, \tag{19}$$

and the Hamiltonian (15) can be represented as follows:

$$H=-\frac{\hbar^2}{2m}\left\{\gamma^{-1}\partial_\rho\gamma\partial_\rho+\gamma^{-2}\left[\partial_\phi-i\,\frac{e}{\hbar c}\,V+\frac{1}{2}\,i\sigma(1-(\partial_\rho\gamma))\right]^2+\sigma\,\frac{e}{\hbar c}\,\gamma^{-i}(\partial_\rho V)+\frac{1}{2}\,\gamma^{-1}(\partial_\rho^2\gamma)\right\}. \tag{20}$$

The operator H acts in the space of functions with scalar product

$$(\psi,\psi')=\int_0^{2\pi}d\phi\int_0^\infty drrf\psi^*\psi'=\int_0^{2\pi}d\phi\int_0^\infty d\rho\gamma\psi^*\psi'. \tag{21}$$

In contrast, the Hamiltonian in the absence of a string (i.e., for $\Phi=\Phi_K=0$),

$$H_0=-\frac{\hbar^2}{2m}\,(\partial_\rho^2+\rho^{-1}\partial_\rho+\rho^{-2}\partial_\phi^2), \tag{22}$$

acts on the space of functions with scalar product

$$(\psi,\psi')=\int_0^{2\pi}d\phi\int_0^\infty d\rho\rho\psi^*\psi'. \tag{23}$$

We introduce a transformation of the operator and of the functions that leaves the form of Eq. (13) unchanged:

$$\tilde{H}=\Xi H\Xi^{-1}, \quad \tilde{\psi}=\Xi\psi, \quad \tilde{H}\tilde{\psi}=\frac{\hbar^2k^2}{2m}\,\tilde{\psi}. \tag{24}$$

Choosing

$$\Xi=\gamma^{1/2}\rho^{-1/2}, \tag{25}$$

we obtain for the transformed Hamiltonian the expression

$$\tilde{H}=-\frac{\hbar^2}{2m}\left\{\partial_\rho^2+\rho^{-1}\partial_\rho-\frac{1}{4}\,\rho^{-2}+\frac{1}{4}\,\gamma^{-2}(\partial_\rho\gamma)^2+\gamma^{-2}\left[\partial_\phi-i\,\frac{e}{\hbar c}\,V+\frac{1}{2}\,i\sigma(1-(\partial_\rho\gamma))\right]^2+\sigma\,\frac{e}{\hbar c}\,\gamma^{-1}(\partial_\rho V)\right\}; \tag{26}$$

then in the space of the transformed functions the scalar product is defined in accordance with (23), rather than (21).

We regard H_0 (22) as an unperturbed Hamiltonian and

$$\Delta H=\tilde{H}-H_0 \tag{27}$$

as an operator that describes a perturbing interaction, and we shall attempt to construct a scattering theory. Taking into account (22) and (26), we represent the operator (27) in the form

$$\Delta H=v(\mathbf{x})+v^j(\mathbf{x})\left(-i\,\frac{\partial}{\partial x^j}\right)+v^{jj'}(\mathbf{x})\left(-\frac{\partial^2}{\partial x^j\partial x^{j'}}\right), \tag{28}$$

where we have introduced the notation $\mathbf{x}=(x^1,x^2)$, $x^1=\rho\cos\phi$, $x^2=\rho\sin\phi$, and $j,j'=1,2$.

If the coefficient functions v, v^j, and $v^{jj'}$ decrease as $O(\rho^{-1-\epsilon})$ as $\rho\to\infty$ ($\epsilon>0$), then in accordance with Ref. 27 the perturbation ΔH (28) has a short range and scattering theory can be constructed in the usual manner (see, for example, Refs. 28 and 29). However, even for particle scattering by a purely magnetic string ($\Phi\neq0$ and $\Phi_K=0$) the perturbation ΔH has a long range since the coefficient function v^j decreases in the limit $\rho\to\infty$ as $O(\rho^{-1})$. Because of the interaction, it is impossible to choose a plane wave as the incident wave, as noted by Aharonov and Bohm.[1] Nevertheless, in this case it is possible to develop a scattering theory and obtain in its framework the Aharonov–Bohm scattering amplitude (see Ref. 30).

Hörmander[27] considered a class of perturbations of the form (28) containing both a short-range part and a long-range part characterized by real coefficient functions that decrease in the limit $\rho\to\infty$ as $O(\rho^{-\epsilon})$ ($0<\epsilon\leqslant 1$), and he formulated certain additional requirements under which scattering theory can be developed. As he notes on p. 417 of the translation of his monograph of Ref. 27, "the existence of modified wave operators is proved under the weakest sufficient conditions among all those known at the present time."

Hörmander's conditions are satisfied by the perturbation in the problem of scattering by a purely magnetic string,

$$v\sim O(\rho^{-2}) \text{ and } v^j\sim O(\rho^{-1}), \quad \rho\to\infty$$

(v and v^j are real functions, and $v^{jj'}=0$) and, for example, the perturbation in the problem of scattering by a Coulomb center:

$$v\sim O(\rho^{-1}), \quad \rho\to\infty$$

(v is a real function, and $v^j=v^{jj'}=0$). In contrast, the perturbation in the problem of scattering by a cosmic string ($\Phi\neq 0$ and $\Phi_K\neq 0$) does not satisfy Hörmander's conditions:

$$v\sim O(\rho^{-2}), \quad v^j\sim O(\rho^{-1}) \text{ and } v^{jj'}\sim O(1), \quad \rho\to\infty, \tag{29}$$

where v^j, in contrast to v and $v^{jj'}$, is a complex function (more precisely, the imaginary part of v^j of order ρ^{-1} is due to the nondecrease of the real quantity $v^{jj'}$ in the limit $\rho\to\infty$). Nevertheless, even in this last case it is possible to develop a scattering theory, and in the present paper we shall construct wave operators explicitly.

In Sec. 2, we determine the S matrix and the scattering amplitude in the case of a singular cosmic string ($r_B=0$ and $r_K=0$). In Sec. 3, we take into account the effects of the gravitational structure of the string (the finiteness of the transverse dimensions of the region of curvature of space, $r_K>0$) and in Sec. 4 the effects of the magnetic structure of the string (finiteness of the transverse dimensions of the region of magnetic flux, $r_B>0$). Section 5 is devoted to discussion of the results. Details of the derivation of the basic relations are given in Appendices A, B, and C.

2. THE *S* MATRIX AND SCATTERING AMPLITUDE IN THE CASE OF A SINGULAR COSMIC STRING

In the case $r_B=0$ and $r_K=0$, we have

$$\rho=\frac{r_0}{1-\Phi_K}\left(\frac{r}{r_0}\right)^{1-\Phi_K}, \quad \gamma=\rho(1-\Phi_K), \tag{30}$$

and the Hamiltonian H (20), which is identical to the transformed Hamiltonian $\tilde{H}$ (26), takes the form

$$H_1=-\frac{\hbar^2}{2m}\left[\partial_\rho^2+\rho^{-1}\partial_\rho+\rho^{-2}(1-\Phi_K)^{-2}\left(\partial_\phi-i\Phi+\frac{1}{2}i\sigma\Phi_K\right)^2\right]. \tag{31}$$

We consider the action of the operator H_1 (31) in the space of the transformed functions [with scalar product defined in accordance with (23)]; in what follows, we shall omit the tilde for brevity. In Appendix A we show how the resolvent of the operator H_1 (31) is found and then how, using it, one can find the evolution operator $\exp(-i\hbar^{-1}H_1t)$. We give here the result:

$$\exp(-i\hbar^{-1}H_1t)(\rho,\phi;\rho',\phi')=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\times\exp[in(\phi-\phi')]U_n(\rho,\rho',t), \tag{32}$$

where

$$U_n(\rho,\rho',t)=-\frac{im}{\hbar t}J_{\alpha_n}\left(\frac{m\rho\rho'}{\hbar|t|}\right)\exp\left\{i\left[\frac{m(\rho^2+\rho'^2)}{2\hbar t}-\frac{1}{2}\alpha_n\pi\,\mathrm{sgn}(t)\right]\right\}, \tag{33}$$

$$\alpha_n=\left|n-\Phi+\frac{1}{2}\sigma\Phi_K\right|(1-\Phi_K)^{-1}, \tag{34}$$

$J_\lambda(u)$ is a Bessel function,

$$\mathrm{sgn}(u)=\begin{cases}1, & u>0,\\ -1, & u<0,\end{cases}$$

and henceforth $\sigma=\pm 1$ for the upper or lower components of a spinor wave function and $\sigma=0$ for a scalar wave function.

By means of the evolution operator, we find

$$\exp(i\hbar^{-1}H_1t)\exp(-i\hbar^{-1}H_0t)(\rho,\phi;\rho',\phi')=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}e^{in(\phi-\phi')}\Omega_n(\rho,\rho',t), \tag{35}$$

where

$$\Omega_n(\rho,\rho',t)=\exp\left\{-i\left[\frac{m(\rho^2-\rho'^2)}{2\hbar t}-\frac{1}{2}(\alpha_n-|n|)\pi\times\mathrm{sgn}(t)\right]\right\}\int_0^\infty du\,uJ_{\alpha_n}(\rho u)J_{|n|}(\rho' u), \tag{36}$$

in which H_0 is the free Hamiltonian (22). We define the (Møller) wave operators

$$\Omega^{\pm}_{H_1,H_0}=\lim_{t\to\mp\infty}\exp(i\hbar^{-1}H_1t)\exp(-i\hbar^{-1}H_0t).$$

Going over to the corresponding limits in (35) and (36), we obtain

$$\Omega^{\pm}_{H_1,H_0}(\rho,\phi;\rho',\phi')=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\exp[in(\phi-\phi')]\times\Omega_n^{\pm}(\rho;\rho'), \tag{37}$$

where

$$\Omega_n^{\pm}(\rho;\rho')=\exp\left[\mp\frac{1}{2}i(\alpha_n-|n|)\pi\right]$$

$$\int_0^\infty du u J_{\alpha_n}(\rho u) J_{|n|}(\rho' u). \tag{38}$$

The S matrix is defined by

$$S=(\Omega^-_{H_1,H_0})^*\Omega^+_{H_1,H_0},$$

where the asterisk denotes the Hermitian conjugate.

Taking into account (37) and (38), we obtain

$$S(\rho,\phi;\rho',\phi')=\frac{\delta(\rho-\rho')}{\sqrt{\rho\rho'}}\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\exp\{i[n(\phi-\phi')-(\alpha_n-|n|)\pi]\}. \tag{39}$$

The same expression for the S matrix can be obtained by means of an analysis based on the Lippmann–Schwinger equation (see Appendix B).

Let ψ and ψ' be certain functions that belong to the space of functions with scalar product defined in accordance with (23). Taking into account the periodicity in ϕ of the functions

$$\psi(\rho,\phi)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\psi_n(\rho)e^{in\phi},$$

$$\psi'(\rho,\phi)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\psi'_n(\rho)e^{in\phi}$$

and using (39), we can readily obtain expressions for the function

$$(S\psi)(\rho,\phi)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\psi_n(\rho)\exp\{i[n\phi-(\alpha_n-|n|)\pi]\} \tag{40}$$

and the matrix element

$$(\psi',S\psi)\equiv\int_0^{2\pi}d\phi\int_0^\infty d\rho\rho\psi'^*(\rho,\phi)(S\psi)(\rho,\phi)$$
$$=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\exp[-i(\alpha_n-|n|)\pi]\times\int_0^\infty d\rho\rho\psi'^*_n(\rho)\psi_n(\rho). \tag{41}$$

The solution of Eq. (13) in which the operator H is replaced by H_0 (22) can be represented in the form

$$\psi^{(0)}(\mathbf{x},\mathbf{k})=\frac{1}{2\pi}\exp(i\mathbf{xk})=\frac{1}{2\pi}\exp[ik\rho\cos(\phi-\phi_k)]$$
$$=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}J_{|n|}(k\rho)\exp\left\{i\left[n(\phi-\phi_k)+\frac{1}{2}|n|\pi\right]\right\}, \tag{42}$$

where $\mathbf{x}=(\rho\cos\phi,\rho\sin\phi)$ and $\mathbf{k}=(k\cos\phi_k,\ k\sin\phi_k)$. Using (41) and taking into account (39), we determine the matrix element

$$(\psi^{(0)}(\mathbf{k}),S\psi^{(0)}(\mathbf{k}'))\equiv\int_0^{2\pi}d\phi\int_0^\infty d\rho\rho[\psi^{(0)}(\mathbf{x},\mathbf{k})]^*\times(S\psi^{(0)})(\mathbf{x},\mathbf{k}')$$
$$=S(k,\phi_k;k',\phi_{k'}). \tag{43}$$

To find the amplitude and scattering cross section, it is necessary to determine the transition matrix on the energy shell (see, for example, Refs. 28 and 29):

$$T=S-I, \tag{44}$$

where the identity operator I is characterized by the singular kernel

$$I(k,\phi;k',\phi')=\frac{\delta(k-k')}{\sqrt{kk'}}\Delta(\phi-\phi'), \tag{45}$$

$$\Delta(\phi)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}e^{in\phi}. \tag{46}$$

Note that the right-hand side of (45) is equal to the generalized function $\delta(\mathbf{k}-\mathbf{k}')$ rewritten in polar coordinates ($\mathbf{k}=(k\cos\phi,k\sin\phi)$ and $\mathbf{k}'=(k'\cos\phi',k'\sin\phi')$); the generalized function $\Delta(\phi)$ (46) is to be understood as a functional that acts on periodic (with period 2π) functions:

$$\int_0^{2\pi}d\phi'F(\phi')\Delta(\phi-\phi')=F(\phi),\qquad \text{if } F(\phi+2\pi)=F(\phi). \tag{47}$$

The scattering amplitude can be determined by means of the expression

$$f'(k,\phi-\phi')=-(2\pi)^{3/2}\frac{m}{\hbar^2}\left(\frac{i}{k}\right)^{1/2}t(k,\phi-\phi'), \tag{48}$$

where the t matrix is related to the T matrix by

$$T=-2\pi i\delta\left[\frac{\hbar^2}{2m}(k^2-k'^2)\right]t. \tag{49}$$

As a result, the relation between the S matrix and the scattering amplitude f' takes the form

$$(S-I)(k,\phi;k',\phi')=\delta(k-k')\sqrt{\frac{i}{2\pi k}}f'(k,\phi-\phi'). \tag{50}$$

We sum over n in the expression (39) for the S matrix. Using the elementary relation

$$\cot\frac{\phi}{2}[\sin(n+1)\phi-\sin n\phi]=\cos(n+1)\phi+\cos n\phi,$$

we readily obtain

$$\int_0^\pi d\phi\cot\frac{\phi}{2}\sin N\phi=\pi,\quad N=1,2,3,..., \tag{51}$$

whereupon

$$\frac{1}{2}\cot\frac{\phi}{2}=\sum_{n=1}^{\infty}\sin n\phi, \tag{52}$$

which in conjunction with the definition (46) gives

$$\sum_{n=N}^{\infty} e^{in\phi}=\pi\Delta(\phi)-e^{iN\phi}(e^{i\phi}-1)^{-1}, \quad N=1,2,3,\ldots . \tag{53}$$

In its turn, using the last relation, we obtain

$$\begin{aligned}S(k,\phi;k',\phi')=&\frac{1}{2}\frac{\delta(k-k')}{\sqrt{kk'}}\Bigg\{\Delta(\phi-\phi'+\beta\pi)\\&\times\exp[-i\mu(1+\beta)\pi]+\Delta(\phi-\phi'-\beta\pi)\\&\times\exp[i\mu(1+\beta)\pi]\\&+\frac{1}{\pi}\frac{\exp[i\nu(\phi-\phi'+\beta\pi)-i\mu(1+\beta)\pi]}{1-\exp[-i(\phi-\phi'+\beta\pi)]}\\&-\frac{1}{\pi}\frac{\exp[i\nu(\phi-\phi'-\beta\pi)+i\mu(1+\beta)\pi]}{1-\exp[-i(\phi-\phi'-\beta\pi)]}\Bigg\},\end{aligned} \tag{54}$$

where

$$\mu=\Phi-\frac{1}{2}\sigma\Phi_K,\quad \nu=\mathrm{integ}_-\left(\Phi-\frac{1}{2}\sigma\Phi_K\right),$$

$$\beta=\Phi_K(1-\Phi_K)^{-1}, \tag{55}$$

and $\mathrm{integ}_-(u)$ is the integer part of the number u (the integer that is closest to and smaller than u or is equal to u); the divergences in the two last terms in (54) in the limit $\phi-\phi'\to(\pm\beta\pi)(\mathrm{mod}\ 2\pi)$ [like the divergence in the final term in (53) in the limit $\phi\to 0$ (mod 2π)] is to be understood in the principal-value sense.

Taking into account (45), (50), and (54), we conclude that the scattering amplitude f' is a singular (generalized) function of the angle variable.

This is a manifestation of the long-range nature of the perturbing operator $\Delta H=H_1-H_0$. The fact that the scattering amplitude determined in the standard manner is a singular function was noted earlier in Ref. 30 for the case of a purely magnetic string and in Ref. 20 for the case of a purely gravitational string. As we have already noted in the Introduction, the fact, first established by Aharonov and Bohm,[1] that as incident wave in the case of scattering by a purely magnetic string it is necessary to choose the discontinuous function $\psi^{(0)}(\mathbf{x},\mathbf{k})\exp[i\Phi(\phi-\phi_k-\pi)]$ rather than the plane wave $\psi^{(0)}(\mathbf{x},\mathbf{k})$ (42) is also associated with the long-range interaction of the noted type. Since despite the long-range interaction an evolution operator and S matrix can be defined [see (32), (33), and (39)], it is necessary to modify the standard definition of the transition matrix and the scattering amplitude. The physical motivation is that, in contrast to the S matrix, the amplitude and scattering cross section must be interpreted in terms of ordinary functions rather than in terms of functionals (generalized functions). Thus, the problem reduces to a certain change in the form of the identity operator in order to include in it the entire singular (with respect to the angle variable) part of the S matrix.

We define the scattering amplitude by means of the relation

$$(S-I')(k,\phi;k',\phi')=\delta(k-k')\sqrt{\frac{i}{2\pi k}}f(k,\phi-\phi'), \tag{56}$$

where the modified identity operator I' in the case of an S matrix in the form (54) is characterized by the singular kernel

$$\begin{aligned}I'(k,\phi;k',\phi')=&\frac{1}{2}\frac{\delta(k-k')}{\sqrt{kk'}}\{\Delta(\phi-\phi'+\beta\pi)\\&\times\exp[-i\mu(1+\beta)\pi]+\Delta(\phi-\phi'-\beta\pi)\\&\times\exp[i\mu(1+\beta)\pi]\}.\end{aligned} \tag{57}$$

As a result, we obtain the scattering amplitude in the form

$$\begin{aligned}f(k,\phi)=&\frac{1}{\sqrt{2\pi ik}}\Bigg\{\frac{\exp[i\nu(\phi+\beta\pi)-i\mu(1+\beta)\pi]}{1-\exp[-i(\phi+\beta\pi)]}\\&-\frac{\exp[i\nu(\phi-\beta\pi)+i\mu(1+\beta)\pi]}{1-\exp[-i(\phi-\beta\pi)]}\Bigg\}.\end{aligned} \tag{58}$$

Note that an expression for the scattering amplitude, or rather for

$$f(k,\phi)-f(k,\phi)|_{\Phi=0}\exp[i\Phi(\phi-\pi)],$$

was obtained in Ref. 24. Upon the replacement of $\Phi-(\sigma/2)\Phi_K$ by $4EJc^{-2}$, (58) goes over into the expression obtained even earlier in Ref. 21 for the scattering amplitude of a relativistic test particle with energy E by a singular gravitational string with spin J. However, the very existence of gravitational strings with nonzero spin violates causality, as the authors of Ref. 21 themselves note (see also Ref. 31).

In Refs. 21 and 24, the scattering amplitude was determined by an analysis of the asymptotic behavior of the wave function at large distances from the string. Indeed, the solution of the Schrödinger equation (13) with the Hamiltonian H_1 (31) can be represented in the form

$$\begin{aligned}\psi(\mathbf{x},\mathbf{k}')=&(\Omega^{+}_{H_1,H_0}\psi^{(0)})(\mathbf{x},\mathbf{k}')=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}J_{\alpha_n}(k\rho)\\&\times\exp\left\{i\left[n(\phi-\phi')-\frac{1}{2}\alpha_n\pi+|n|\pi\right]\right\},\end{aligned} \tag{59}$$

where $\mathbf{x}=(\rho\cos\phi,\rho\sin\phi)$ and $\mathbf{k}'=(k\cos\phi',k\sin\phi')$. Going over to the $\rho\to\infty$ asymptotic behavior in (59), we obtain

$$\begin{aligned}\psi(\mathbf{x},\mathbf{k}')=&\frac{1-\Phi_K}{2\pi}\sum_{l}\exp\Bigg\{-ik\rho\cos[(1-\Phi_K)\\&\times(\phi-\phi'-\pi+2l\pi)]+i\left(\Phi-\frac{1}{2}\sigma\Phi_K\right)\\&\times(\phi-\phi'-\pi+2l\pi)\Bigg\}+f(k,\phi-\phi')\\&\times(2\pi\sqrt{\rho})^{-1}\exp(ik\rho)+O(\rho^{-1}),\end{aligned} \tag{60}$$

where the summation is over integer values of l that satisfy the condition

$$\frac{\phi'-\phi}{2\pi}-\frac{1}{2}\beta<l<\frac{\phi'-\phi}{2\pi}+1+\frac{1}{2}\beta, \tag{61}$$

and the function $f(k,\phi)$ is given by (58). In accordance with the generally accepted interpretation, the factor of the outgoing wave $(2\pi\sqrt{\rho})^{-1}\exp(ik\rho)$ is the scattering amplitude, while the term of order $O(1)$ is the incident wave. As can be seen from (60), the incident wave has discontinuities at $\phi-\phi'=(\pm\beta\pi)$ (mod 2π); the scattering amplitude (58) diverges at the same values of the angle variable. Thus, the asymptotic expansion (60) has meaning only when $\phi-\phi'\neq(\pm\beta\pi)$ (mod 2π). However, the original function (59) is finite and continuous for all values of the angle variable. In particular, for $(\phi-\phi'\pm\beta\pi)k\rho\ll 1$ and $k\rho\gg 1$ ($\beta\neq N-1$, where $N=1,2,3,...$) we have

$$\begin{aligned}\psi(\mathbf{x},\mathbf{k}')=&\frac{1}{2\pi}\exp[\pm i(1+\beta)\mu\pi+ik\rho]\\&\times\left\{\frac{1}{1+\beta}\sum_{l}\exp\left(2i\mu l\pi-2ik\rho\sin^2\frac{l\pi}{1+\beta}\right)\right.\\&+\frac{1}{2(1+\beta)}\mp\frac{1}{\sqrt{2\pi ik\rho}}\\&\times\left(\frac{\exp[\mp 2i(1+\beta)\mu\pi\pm 2i\beta(\nu+1)\pi]}{1-\exp(\pm 2i\beta\pi)}\right.\\&\left.+\mu-\nu-\frac{1}{2}\right)+O\left(\frac{1}{k\rho}\right)\\&+\frac{1}{(1+\beta)^2}\left[i\sum_{l}\exp\left(2i\mu l\pi\right.\right.\\&\left.-2ik\rho\sin^2\frac{l\pi}{1+\beta}\right)\sin\frac{2l\pi}{1+\beta}\pm\frac{1}{\sqrt{2\pi ik}}\\&\left.\left.+O\left(\frac{1}{k\rho}\right)\right](\phi-\phi'\mp\beta\pi)k\rho\right\}+O[(\phi-\phi'\\&\mp\beta\pi)^2(k\rho)^2],\end{aligned} \tag{62}$$

where the summation is over the integers l that satisfy the condition

$$-\frac{1}{2}(1+\beta)(1\pm 1)<l<\frac{1}{2}(1+\beta)(1\mp 1). \tag{63}$$

For $(\phi-\phi'+\pi-N\pi)k\rho\ll 1$ and $k\rho\gg 1$ ($\beta=N-1$, where $N=1,2,3,...$), we have

$$\begin{aligned}\psi(\mathbf{x},\mathbf{k}')=&\frac{\exp(ik\rho)}{2\pi}\left\{\frac{\exp(iN\mu\pi)}{N}\sum_{l}\exp\left(2i\mu l\pi\right.\right.\\&\left.-2ik\rho\sin^2\frac{l\pi}{N}\right)+\frac{1}{N}\cos(N\mu\pi)\\&-\sqrt{\frac{2i}{\pi k\rho}}\left(\mu-\nu-\frac{1}{2}\right)\sin(N\mu\pi)+O\left(\frac{1}{k\rho}\right)\\&+\frac{1}{N^2}\left[i\exp(iN\mu\pi)\sum_{l}\exp\left(2i\mu l\pi\right.\right.\\&\left.-2ik\rho\sin^2\frac{l\pi}{N}\right)\sin\frac{2l\pi}{N}+\sqrt{\frac{2i}{\pi k\rho}}\\&\left.\times\sin(N\mu\pi)+O\left(\frac{1}{k\rho}\right)\right](\phi-\phi'+\pi\\&\left.-N\pi)k\rho\right\}+O[(\phi-\phi'+\pi-N\pi)^2(k\rho)^2],\end{aligned} \tag{64}$$

where the summation is over the integers l that satisfy the condition

$$-N<l<0. \tag{65}$$

Thus, despite the extremely strong long-range interaction in the problem of the scattering of a nonrelativistic particle by a singular cosmic string (the perturbation $\Delta H=H_1-H_0$ does not even satisfy Hörmander's conditions for long-range perturbations[27]), the S matrix and scattering amplitude can be determined. A particular feature of this problem associated with the long-range interaction is that in the evolutionary description of the scattering process the definition of the transition matrix [in accordance with (56) and (57)] is modified, while in the steady-state description of the scattering process a discontinuous function is chosen as the incident wave [see (60)].

3. ALLOWANCE FOR THE GRAVITATIONAL STRUCTURE OF THE COSMIC STRING

Let us assume that the size of the region of curvature of space is appreciably greater than the size of the region of magnetic flux, $r_K\gg r_B$; if the effects of the magnetic structure are ignored, the magnetic field strength will be given by the expression (12). With regard to the Gaussian curvature, in this section we shall proceed from a more general form of it than was discussed in the Introduction, namely

$$K(X,Y)=K'(X,Y)+2\pi\Phi_K^{(0)}\frac{\delta(X)\delta(Y)}{f(X,Y)}, \tag{66}$$

where K' for $r>r_K$ satisfies the condition (5) and for $r\leq r_K$ is a piecewise continuous function with singularities at isolatedpoints or on isolated lines that are integrable with respect to the measure $f(X,Y)dXdY$; at the same time,

$$\frac{1}{2\pi}\int dXdYf(X,Y)K'(X,Y)=\Phi_K-\Phi_K^{(0)}. \tag{67}$$

We assume further that K' satisfies the condition (14) of axial symmetry and that $\Phi_K^{(0)}<1$; the latter means that the geodesic length in the radial direction in the region of curvature of space is finite:

$$\int_0^r dr\sqrt{f(r)}<\infty \tag{68}$$

for $r\leq r_K$. Then in the region outside the string, we have

$$\left.\begin{aligned}\rho&=\sqrt{f_K}\,\frac{r_K}{1-\Phi_K}\left(\frac{r}{r_K}\right)^{1-\Phi_K}-\xi,\\ \gamma&=(\rho+\xi)(1-\Phi_K),\end{aligned}\right\} r>r_K, \tag{69}$$

where

$$\xi=\sqrt{f_K}(1-\Phi_K)^{-1}r_K-\int_0^{r_K}dr\sqrt{f(r)},\quad f_K=\left(\frac{r_K}{r_0}\right)^{-2\Phi_K}. \tag{70}$$

In the present case, the Hamiltonian $\tilde{H}$ (26) takes the form

$$H_2=-\frac{\hbar^2}{2m}\left\{\partial_\rho^2+\rho^{-1}\partial_\rho-\frac{1}{4}\rho^{-2}+\frac{1}{4}\gamma^{-2}(\partial_\rho\gamma)^2+\gamma^{-2}\left[\partial_\phi-i\Phi+\frac{1}{2}i\sigma(1-(\partial_\rho\gamma))\right]^2\right\}, \tag{71}$$

and outside the string we have

$$H_2=-\frac{\hbar^2}{2m}\left[\partial_\rho^2+\rho^{-1}\partial_\rho-\frac{1}{4}\rho^{-2}+\frac{1}{4}(\rho+\xi)^{-2}+(\rho+\xi)^{-2}(1-\Phi_K)^{-2}\left(\partial_\phi-i\Phi+\frac{1}{2}i\sigma\Phi_K\right)^2\right],$$

$$r>r_K. \tag{72}$$

Taking into account the relation

$$\Omega^{\pm}_{H_2,H_0}=\Omega^{\pm}_{H_2,H_1}\Omega^{\pm}_{H_1,H_0},$$

we can represent the linearly independent solutions of the Schrödinger equation (13) with the Hamiltonian H in the form H_2 (71) as follows:

$$\psi^{(\pm)}=\Omega^{\pm}_{H_2,H_1}\chi^{(\pm)}, \tag{73}$$

where $\chi^{(+)}$ and $\chi^{(-)}$ are linearly independent solutions of the Schrödinger equation with Hamiltonian H in the form H_1 (31) determined in accordance with (38) and (42) by

$$\chi^{(\pm)}=\Omega^{\pm}_{H_1,H_0}\psi^{(0)}=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}J_{\alpha_n}(k\rho)\times\exp\left\{i\left[n(\phi-\phi_K)\mp\frac{1}{2}(\alpha_n-|n|)\pi+\frac{1}{2}|n|\pi\right]\right\} \tag{74}$$

[note that $\psi^{(\pm)}$ and $\chi^{(\pm)}$ belong the space of functions with the scalar product defined in accordance with (23)]. The S matrix can be represented in the form [cf. (43)]

$$S(k,\phi_k;k',\phi_{k'})=(\chi^{(-)}(\mathbf{k}),(\Omega^{-}_{H_2,H_1})^*\Omega^{+}_{H_2,H_1}\chi^{(+)}(\mathbf{k}'))=(\psi^{(-)}(\mathbf{k}),\psi^{(+)}(\mathbf{k}')). \tag{75}$$

Thus, the problem is reduced to finding the wave operators $\Omega^{+}_{H_2,H_1}$ and $\Omega^{-}_{H_2,H_1}$ or the functions $\psi^{(+)}$ and $\psi^{(-)}$. This problem could conveniently be solved with the help of the Lippmann–Schwinger equation

$$\psi^{(\pm)}(\mathbf{x},\mathbf{k})=\chi^{(\pm)}(\mathbf{x},\mathbf{k})-(G^{\pm}V_1\psi^{(\pm)})(\mathbf{x},\mathbf{k}), \tag{76}$$

where

$$V_1=H_2-H_1, \tag{77}$$

$$G^{\pm}=\lim_{\epsilon\to 0_+}\left(H_1-\frac{\hbar^2k^2}{2m}\mp i\epsilon\right)^{-1}. \tag{78}$$

Using the expression (A9) in Appendix A for the resolvent of the operator H_1, we can obtain for the Green's functions the expression

$$G^{\pm}(\rho,\phi;\rho',\phi')=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\exp[in(\phi-\phi')]\times G_n^{\pm}(\rho;\rho'), \tag{79}$$

where

$$G_n^{+}(\rho;\rho')=i\pi\frac{m}{\hbar^2}\begin{cases}J_{\alpha_n}(k\rho)H^{(1)}_{\alpha_n}(k\rho'), & \rho<\rho',\\ H^{(1)}_{\alpha_n}(k\rho)J_{\alpha_n}(k\rho'), & \rho>\rho',\end{cases} \tag{80}$$

$$G_n^{-}(\rho;\rho')=-i\pi\frac{m}{\hbar^2}\begin{cases}J_{\alpha_n}(k\rho)H^{(2)}_{\alpha_n}(k\rho'), & \rho<\rho',\\ H^{(2)}_{\alpha_n}(k\rho)J_{\alpha_n}(k\rho'), & \rho>\rho'\end{cases} \tag{81}$$

[$H^{(1)}_\lambda(u)$ and $H^{(2)}_\lambda(u)$ are Hankel functions].

The solution of the Schrödinger equation (13) with the Hamiltonian H in the form H_2 (71) can be represented as follows:

$$\kappa(\mathbf{x},\mathbf{k})=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\exp[in(\phi-\phi_k)]\kappa_n(\rho,k), \tag{82}$$

where the function $\kappa_n(\rho,k)$ satisfies the equation

$$\left\{\partial_\rho^2+\rho^{-1}\partial_\rho-\frac{1}{4}\rho^{-2}+\frac{1}{4}\gamma^{-2}(\partial_\rho\gamma)^2-\gamma^{-2}\left[n-\Phi+\frac{1}{2}\sigma(1-(\partial_\rho\gamma))\right]^2+k^2\right\}\kappa_n(\rho,k)=0. \tag{83}$$

By virtue of the conditions (7) for $r>r_K$ and (68) for $r\leqslant r_K$, Eq. (83) has a unique solution (up to a factor that depends only on k) that is regular at the point $\rho=0$; in what follows, we shall understand by $\kappa_n(\rho,k)$ precisely such a solution.

We define

$$\rho_K=\int_0^{r_K}dr\sqrt{f(r)} \tag{84}$$

and find the Wronskian of the functions $\mathscr{F}^{(1)}(\rho_K)$ and $\mathscr{F}^{(2)}(\rho_K)$:

$$W[\mathscr{F}^{(1)}(\rho_K),\mathscr{F}^{(2)}(\rho_K)]=\{\mathscr{F}^{(1)}(\rho)[\partial_\rho\mathscr{F}^{(2)}(\rho)]-[\partial_\rho\mathscr{F}^{(1)}(\rho)]\mathscr{F}^{(2)}\times(\rho)\}|_{\rho=\rho_K}. \tag{85}$$

The Lippmann–Schwinger equation (76) is solved in Appendix B. We give here the main results. The wave functions (73) in the region outside the string have the form

$$\psi^{(+)}(\mathbf{x},\mathbf{k})=\sqrt{1+\xi\rho^{-1}}\,\frac{e^{ik\xi}}{2\pi}\sum_{n=-\infty}^{\infty}\exp\left[in(\phi-\phi_K)-i\left(\frac{1}{2}\alpha_n-|n|\right)\pi\right]$$

$$\times\left\{J_{\alpha_n}(k(\rho+\xi))\right.$$

$$-\frac{W[\kappa_n(\rho_K,k),\sqrt{1+\xi\rho_K^{-1}}J_{\alpha_n}(k(\rho_K+\xi))]}{W[\kappa_n(\rho_K,k),\sqrt{1+\xi\rho_K^{-1}}H^{(1)}_{\alpha_n}(k(\rho_K+\xi))]}$$

$$\left.\times H^{(1)}_{\alpha_n}(k(\rho+\xi))\right\},\quad \rho>\rho_K,\quad \psi^{(-)}(\mathbf{x},\mathbf{k}) \tag{86}$$

$$=\sqrt{1+\xi\rho^{-1}}\,\frac{e^{-ik\xi}}{2\pi}$$

$$\times\sum_{n=-\infty}^{\infty}\exp\left[in(\phi-\phi_K)+\frac{i}{2}\alpha_n\pi\right]\left\{J_{\alpha_n}(k(\rho+\xi))\right.$$

$$-\frac{W[\kappa_n(\rho_K,k),\sqrt{1+\xi\rho_K^{-1}}J_{\alpha_n}(k(\rho_K+\xi))]}{W[\kappa_n(\rho_K,k),\sqrt{1+\xi\rho_K^{-1}}H^{(2)}_{\alpha_n}(k(\rho_K+\xi))]}$$

$$\left.\times H^{(2)}_{\alpha_n}(k(\rho+\xi))\right\},\quad \rho>\rho_K. \tag{87}$$

The S matrix has the form

$$S(k,\phi;k',\phi')$$

$$=\frac{\delta(k-k')}{\sqrt{kk'}}\frac{1}{2\pi}e^{2ik\xi}\sum_{n=-\infty}^{\infty}\exp[in(\phi-\phi')-i(\alpha_n-|n|)\pi]$$

$$\times\left\{1-2\frac{W[\kappa_n(\rho_K,k),\sqrt{1+\xi\rho_K^{-1}}J_{\alpha_n}(k(\rho_K+\xi))]}{W[\kappa_n(\rho_K,k),\sqrt{1+\xi\rho_K^{-1}}H^{(1)}_{\alpha_n}(k(\rho_K+\xi))]}\right\}. \tag{88}$$

Determining a modified identity operator characterized by a singular kernel of the form

$$I'(k,\phi;k',\phi')=\frac{1}{2}\frac{\delta(k-k')}{\sqrt{kk'}}e^{2ik\xi}\{\Delta(\phi-\phi'+\beta\pi)$$

$$\times\exp[-i\mu(1+\beta)\pi]+\Delta(\phi-\phi'-\beta\pi)\exp[i\mu(1+\beta)\pi]\}, \tag{89}$$

and using the relation (56), we obtain for the scattering amplitude the expression

$$f(k,\phi)=\frac{\exp(2ik\xi)}{\sqrt{2\pi ik}}\left\{\frac{\exp[i\nu(\phi+\beta\pi)-i\mu(1+\beta)\pi]}{1-\exp[-i(\phi+\beta\pi)]}\right.$$

$$-\frac{\exp[i\nu(\phi-\beta\pi)+i\mu(1+\beta)\pi]}{1-\exp[-i(\phi-\beta\pi)]}$$

$$-2\sum_{n=-\infty}^{\infty}\exp[in\phi-i(\alpha_n-|n|)\pi]$$

$$\left.\frac{W[\kappa_n(\rho_K,k),\sqrt{1+\xi\rho_K^{-1}}J_{\alpha_n}(k(\rho_K+\xi))]}{W[\kappa_n(\rho_K,k),\sqrt{1+\xi\rho_K^{-1}}H^{(1)}_{\alpha_n}(k(\rho_K+\xi))]}\right\}. \tag{90}$$

In contrast to the series in (88), the series in (90) is absolutely convergent and therefore does not contain divergences with respect to the angle variable; in addition, in the limit $k\to 0$ the series in (90) decreases at least as $O\{\ln^{-1}[k(\rho_K+\xi)]\}$, and therefore at small values of the absolute magnitude of the wave vector the effects of the gravitational structure of the string become unimportant.

To conclude this section, we give specific examples of gravitational structure of the string. We choose

$$\gamma(\rho)=(1-\Phi_K^{(0)})\rho,\quad 0<\rho<\rho_K \tag{91}$$

and

$$\xi=\rho_K\frac{\Phi_K-\Phi_K^{(0)}}{1-\Phi_K}. \tag{92}$$

Then the solution of Eq. (83) that is regular at the point $\rho=0$ takes the form

$$\kappa_n(\rho,k)=c_n(k)J_{\alpha_n^{(0)}}(k\rho), \tag{93}$$

where

$$\alpha_n^{(0)}=\left|n-\Phi+\frac{1}{2}\sigma\Phi_K^{(0)}\right|(1-\Phi_K^{(0)})^{-1}. \tag{94}$$

Note that the structure (91)–(92) in the case $\Phi_K^{(0)}=0$ is known as a structure of flower-pot type.[32]

Considerably more interesting is the example of a structure with continuous distribution of the curvature within the string. We choose

$$\gamma(\rho)=(\rho_K+\xi)\frac{(1-\Phi_K)(1-\Phi_K^{(0)})}{(\Phi_K-\Phi_K^{(0)})}\times\left[1-\left(\frac{1-\Phi_K}{1-\Phi_K^{(0)}}\right)^{\rho/\rho_K}\right],$$

$$0<\rho<\rho_K \tag{95}$$

and

$$\xi=\rho_K\left[\frac{\Phi_K-\Phi_K^{(0)}}{1-\Phi_K}\left(\ln\frac{1-\Phi_K^{(0)}}{1-\Phi_K}\right)^{-1}-1\right]. \tag{96}$$

Then Eq. (83) reduces to a hypergeometrical differential equation. Indeed, introducing the notation

$$\rho_{KO}=\rho_K\left(\left|\ln\frac{1-\Phi_K^{(0)}}{1-\Phi_K}\right|\right)^{-1},$$

$$\tilde{\alpha}_n^{(0)}=\left|n-\Phi+\frac{1}{2}\sigma\right|(1-\Phi_K^{(0)})^{-1} \tag{97}$$

and taking into account (94), we obtain

$$\left\{s(1-s)\partial_s^2+2[p_n^{\pm}-(p_n^{\pm}+q_n)s]\partial_s-\left[2p_n^{\pm}q_n+(\alpha_n^{(0)})^2\right.\right.$$

$$\left.\left.\pm(\tilde{\alpha}_n^{(0)})^2-\frac{1}{4}\right]\right\}u_n^{(\pm)}(s,k)=0, \tag{98}$$

where

$$u_n^{(\pm)}(s,k)=\rho_{KO}^{1/2}(-\ln s)^{1/2}s^{1/2-p_n^{\pm}}(1-s)^{-q_n}\kappa_n(\rho,k),\quad s=\exp\left(-\frac{\rho}{\rho_{KO}}\right),$$

$$p_n^{+}=\frac{1}{2}+\tilde{\alpha}_n^{(k)},\quad \tilde{\alpha}_n^{(k)}=\sqrt{(\tilde{\alpha}_n^{(0)})^2-(k\rho_{KO})^2},$$

$$p_n^{-}=\frac{1}{2}+ik\rho_{KO},\quad q_n=\frac{1}{2}-\alpha_n^{(0)},\tag{99}$$

where the plus and minus signs correspond to the cases $\Phi_K>\Phi_K^{(0)}$ or $\Phi_K<\Phi_K^{(0)}$. As a result, the solution of Eq. (83) that is regular at the point $\rho=0$ takes the form

$$\kappa_n(\rho,k)=c_n(k)\rho^{-1/2}\exp\left[-\tilde{\alpha}_n^{(k)}\frac{\rho}{\rho_{KO}}\right]\left[1-\exp\left(-\frac{\rho}{\rho_{KO}}\right)\right]^{1/2+\alpha_n^{(0)}}{}_2\mathscr{F}_1\left[\frac{1}{2}+\alpha_n^{(0)}+\tilde{\alpha}_n^{(k)}+ik\rho_{KO},\frac{1}{2}+\alpha_n^{(0)}+\tilde{\alpha}_n^{(k)}-ik\rho_{KO};1+2\alpha_n^{(0)};1-\exp\left(-\frac{\rho}{\rho_{KO}}\right)\right],\quad \Phi_K>\Phi_K^{(0)}\tag{100}$$

and

$$\kappa_n(\rho,k)=c_n(k)\rho^{-1/2}\exp(-ik\rho)\left[1-\exp\left(-\frac{\rho}{\rho_{KO}}\right)\right]^{1/2+\alpha_n^{(0)}}{}_2\mathscr{F}_1\left[\frac{1}{2}+\alpha_n^{(0)}+\tilde{\alpha}_n^{(k)}+ik\rho_{KO},\frac{1}{2}+\alpha_n^{(0)}-\tilde{\alpha}_n^{(k)}+ik\rho_{KO};1+2\alpha_n^{(0)};1-\exp\left(-\frac{\rho}{\rho_{KO}}\right)\right],\quad \Phi_K<\Phi_K^{(0)},\tag{101}$$

where ${}_2\mathscr{F}_1(a,b;c;s)$ is the hypergeometric function.[33]

4. ALLOWANCE FOR MAGNETIC STRUCTURE OF THE COSMIC STRING

In this section, in contrast to the previous one, we assume that the dimensions of the region of curvature of space do not exceed those of the region of magnetic flux: $r_K\leq r_B$. By means of the formalism presented earlier, it is possible to develop a scattering theory in this case too. Omitting the details, we give here the results.

Let $\kappa_n(\rho,k)$ be the solution of the following equation, which is regular at the point $\rho=0$:

$$\left\{\partial_\rho^2+\rho^{-1}\partial_\rho-\frac{1}{4}\rho^{-2}+\frac{1}{4}\gamma^{-2}(\partial_\rho\gamma)^2-\gamma^{-2}\left[n-\frac{e}{\hbar c}V+\frac{1}{2}\sigma(1-(\partial_\rho\gamma))\right]^2+\sigma\frac{e}{\hbar c}\gamma^{-1}(\partial_\rho V)+k^2\right\}\kappa_n(\rho,k)=0,\quad 0<\rho<\rho_K.\tag{102}$$

Let

$$\rho_B=\int_0^{r_B}dr\sqrt{f(r)},\tag{103}$$

and $\tau_n^{(1)}(\rho,k)$ and $\tau_n^{(2)}(\rho,k)$ be two linearly independent solutions of the equation

$$\left[\partial_\rho^2+\rho^{-1}\partial_\rho-\frac{1}{4}\rho^{-2}+\frac{1}{4}(\rho+\xi)^{-2}-(\rho+\xi)^{-2}\times(1-\Phi_K)^{-2}\left(n-\frac{e}{\hbar c}V+\frac{1}{2}\sigma\Phi_K\right)^2+\sigma\frac{e}{\hbar c}(\rho+\xi)^{-1}(1-\Phi_K)^{-1}(\partial_\rho V)+k^2\right]\times\tau_n(\rho,k)=0,\quad \rho_K<\rho<\rho_B.\tag{104}$$

We define

$$\tilde{\tau}_n(\rho,k)=\tau_n^{(1)}(\rho,k)-\frac{W[\kappa_n(\rho_K,k),\tau_n^{(1)}(\rho_K,k)]}{W[\kappa_n(\rho_K,k),\tau_n^{(2)}(\rho_K,k)]}\times\tau_n^{(2)}(\rho,k).\tag{105}$$

Then we obtain the following expressions for the S matrix and the scattering amplitude:

$$S(k,\phi;k',\phi')=\frac{\delta(k-k')}{\sqrt{kk'}}\frac{1}{2\pi}e^{2ik\xi}\times\sum_{n=-\infty}^{\infty}\exp[in(\phi-\phi')-i(\alpha_n-|n|)\pi]\times\left\{1-2\frac{W[\tilde{\tau}_n(\rho_B,k),\sqrt{1+\xi\rho_B^{-1}}J_{\alpha_n}(k(\rho_B+\xi))]}{W[\tilde{\tau}_n(\rho_B,k),\sqrt{1+\xi\rho_B^{-1}}H_{\alpha_n}^{(1)}(k(\rho_B+\xi))]}\right\},\tag{106}$$

$$f(k,\phi)=e^{2ik\xi}\left\{f_0(k,\phi)-\sqrt{\frac{2}{\pi ik}}\times\sum_{n=-\infty}^{\infty}\exp[in\phi-i(\alpha_n-|n|)\pi]\times\frac{W[\tilde{\tau}_n(\rho_B,k),\sqrt{1+\xi\rho_B^{-1}}J_{\alpha_n}(k(\rho_B+\xi))]}{W[\tilde{\tau}_n(\rho_B,k),\sqrt{1+\xi\rho_B^{-1}}H_{\alpha_n}^{(1)}(k(\rho_B+\xi))]}\right\}.\tag{107}$$

In (107), we have denoted by $f_0(k,\phi)$ the amplitude for scattering by the singular string (58).

In contrast to the series in (106), the series in (107) is absolutely convergent and therefore does not contain divergences with respect to the angle variable. In the limit $k\to 0$, the series in (107) decreases at least as $O\{\ln^{-1}[k(\rho_B+\xi)]\}$, and therefore at small values of the absolute magnitude of the wave vector the effects of the structure (both gravitational and magnetic) become unimportant.

If the region of curvature of space coincides with the region of magnetic flux, $r_K=r_B$, then we obtain

$$f(k,\phi)=e^{2ik\xi}\Bigg\{f_0(k,\phi)-\sqrt{\frac{2}{\pi ik}}$$

$$\times\sum_{n=-\infty}^{\infty}\exp[in\phi-i(\alpha_n-|n|)\pi]$$

$$\times\frac{W[\kappa_n(\rho_B,k),\sqrt{1+\xi\rho_B^{-1}}J_{\alpha_n}(k(\rho_B+\xi))]}{W[\kappa_n(\rho_B,k),\sqrt{1+\xi\rho_B^{-1}}H^{(1)}_{\alpha_n}(k(\rho_B+\xi))]}\Bigg\},\quad \rho_K=\rho_B. \tag{108}$$

If the size of the region of magnetic flux appreciably exceeds that of the region of spatial curvature, $r_B \gg r_K$, then it follows from (107) that the effects of the gravitational structure hidden deep within the magnetic structure are unimportant over a very extensive range of the wave vector: $0<k\ll(\rho_K+\xi)^{-1}$; at the same time, we obtain

$$f(k,\phi)=f_0(k,\phi)-\sqrt{\frac{2}{\pi ik}}\sum_{n=-\infty}^{\infty}\exp[in\phi-i(\alpha_n-|n|)\pi]\frac{W[\tau_n^{(1)}(\rho_B,k),J_{\alpha_n}(k\rho_B)]}{W[\tau_n^{(1)}(\rho_B,k),H^{(1)}_{\alpha_n}(k\rho_B)]},$$

$$k(\rho_K+\xi)\ll 1, \tag{109}$$

where as $\tau_n^{(1)}(\rho,k)$ we have chosen the solution of Eq. (104) that is regular in the limit $\rho_K\to 0$.

Note that the above expressions are also valid when the magnetic field strength and the Gaussian curvature are given in a more general form than was discussed in the Introduction. Namely, the Gaussian curvature is specified in the same form as in the preceding section [see (66)–(68)], and the magnetic field strength is given in the form

$$B(X,Y)=B'(X,Y)+2\pi\frac{\hbar c}{e}\Phi^{(0)}\frac{\delta(X)\delta(Y)}{f(X,Y)}, \tag{110}$$

where B' for $r>r_B$ satisfies the condition (10) and for $r\leq r_B$ is a piecewise continuous function with singularities at isolatedpoints or on isolated lines that are integrable with respect to the measure $f(X,Y)dXdY$; at the same time

$$\frac{e}{2\pi\hbar c}\int dXdYf(X,Y)B'(X,Y)=\Phi-\Phi^{(0)}; \tag{111}$$

both B' and K' satisfy the condition of axial symmetry (14).

We consider specific examples of string structure. Let the gravitational structure be specified in accordance with (95) and (96) and the magnetic structure be given as follows:

$$B'=\begin{cases}0, & 0<\rho<\rho_K,\\ 2\dfrac{\hbar c}{e}\dfrac{\Phi-\Phi^{(0)}}{1-\Phi_K}\dfrac{1}{\rho_{BK}^2}, & \rho_K<\rho<\rho_B,\end{cases} \tag{112}$$

or

$$V=\begin{cases}\dfrac{\hbar c}{e}\Phi^{(0)}, & 0<\rho<\rho_K,\\ \dfrac{\hbar c}{e}\left[(\Phi-\Phi^{(0)})\dfrac{(\rho+\xi)^2-(\rho_K+\xi)^2}{\rho_{BK}^2}+\Phi^{(0)}\right], & \\ \rho_K<\rho<\rho_B & \end{cases} \tag{113}$$

where

$$\rho_{BK}=[(\rho_B+\xi)^2-(\rho_K+\xi)^2]^{1/2}. \tag{114}$$

Then the regular solution of Eq. (102) can be represented in the form of (100) and (101) (in which it is necessary to replace Φ by $\Phi^{(0)}$), and the two linearly independent solutions of Eq. (104) can be represented in the form

$$\tau_n^{(1)}(\rho,k)=\tilde{c}_n(k)\rho^{-1/2}(\rho+\xi)^{1/2+\tilde{\alpha}_n}\times\exp\left[-\frac{|\Phi-\Phi^{(0)}|}{2(1-\Phi_K)}\left(\frac{\rho+\xi}{\rho_{BK}}\right)^2\right]{}_1\mathscr{F}_1\left(\frac{1}{2}+\frac{1}{2}\tilde{\alpha}_n-\frac{1}{2}\theta_n;1+\tilde{\alpha}_n;\frac{|\Phi-\Phi^{(0)}|}{1-\Phi_K}\left(\frac{\rho+\xi}{\rho_{BK}}\right)^2\right) \tag{115}$$

and

$$\tau_n^{(2)}(\rho,k)=\tilde{c}_n(k)\rho^{-1/2}(\rho+\xi)^{1/2+\tilde{\alpha}_n}\times\exp\left[\frac{|\Phi-\Phi^{(0)}|}{2(1-\Phi_K)}\left(\frac{\rho+\xi}{\rho_{BK}}\right)^2\right]\times\Psi\left(\frac{1}{2}+\frac{1}{2}\tilde{\alpha}_n+\frac{1}{2}\theta_n;1+\tilde{\alpha}_n;-\frac{|\Phi-\Phi^{(0)}|}{1-\Phi_K}\left(\frac{\rho+\xi}{\rho_{BK}}\right)^2\right), \tag{116}$$

where

$$\tilde{\alpha}_n=\left|n-\Phi^{(0)}+(\Phi-\Phi^{(0)})(\rho_K+\xi)^2\rho_{BK}^{-2}+\frac{1}{2}\sigma\Phi_K\right|\times(1-\Phi_K)^{-1},\quad \theta_n=\mathrm{sgn}(\Phi-\Phi^{(0)})\times\left[n-\Phi^{(0)}+\sigma\left(1-\frac{1}{2}\Phi_K\right)\right]\times(1-\Phi_K)^{-1}+|\Phi-\Phi^{(0)}|(1-\Phi_K)^{-1}(\rho_K+\xi)^2\rho_{BK}^{-2}+(1-\Phi_K)(2|\Phi-\Phi^{(0)}|)^{-1}k^2\rho_{BK}^2. \tag{117}$$

Here ${}_1\mathscr{F}_1(a;b;s)$ is the Kummer confluent hypergeometric function, and $\Psi(a;b;s)$ is the Tricomi confluent hypergeometric function;[33] at the same time, by definition,

$$\Psi(a;b;-s)=\lim_{\epsilon\to 0_+}\Psi(a;b;-s+i\epsilon).$$

Note that in the limit $\rho_K\to 0$ the solution (115) goes over into a function that is regular at the point $\rho=0$.

As a further example, we consider the situation in which the string is surrounded by a potential barrier, i.e., the Hamiltonian $\tilde{H}$ (26) is replaced by $\tilde{H}+\tilde{V}$, where

$$\tilde{V}=\begin{cases}\frac{\hbar^2}{2m}k_c^2, & 0<r<r_B,\\ 0, & r_B<r<\infty.\end{cases} \tag{118}$$

In the limit $k_c\to\infty$, the structure of the string is completely screened, and the wave function of the test particle satisfies the Dirichlet condition on the boundary of the string:

$$\psi|_{r=r_B}=0. \tag{119}$$

In this case, the scattering amplitude takes the form

$$f(k,\phi)=e^{2ik\xi}\left\{f_0(k,\phi)-\sqrt{\frac{2}{\pi ik}}\sum_{n=-\infty}^{\infty}\exp[in\phi-i(\alpha_n-|n|)\pi]\frac{J_{\alpha_n}(k(\rho_B+\xi))}{H^{(1)}_{\alpha_n}(k(\rho_B+\xi))}\right\}. \tag{120}$$

The dependence of the modulus of the scattering amplitude (120) on the magnetic flux of the string describes a purely quantum (not possessing classical analogs) effect, since by virtue of the condition (119) a direct effect of the magnetic field on test particles is completely excluded. In accordance with what we have already noted in the more general case, the expression (120) goes over for $k(\rho_B+\xi)\ll 1$ into the expression for the amplitude of scattering by the singular string (58). For $k(\rho_B+\xi)\gg 1$, we obtain (see Appendix C)

$$\begin{aligned}f(k,\phi)=&-e^{2ik\xi}\sqrt{\frac{\rho_B+\xi}{2}}(1-\Phi_K)\\&\times\sum_l\sqrt{\cos\left[\frac{1-\Phi_K}{2}(\phi-\pi+2l\pi)\right]}\\&\times\exp\left\{i\left(\Phi-\frac{1}{2}\sigma\Phi_K\right)(\phi-\pi+2l\pi)\right.\\&\left.-2ik(\rho_B+\xi)\cos\left[\frac{1-\Phi_K}{2}(\phi-\pi+2l\pi)\right]\right\}\\&+\sqrt{\rho_B+\xi}\,O\{[k(\rho_B+\xi)]^{-1/6}\},\quad \phi\neq(\pm\beta\pi)(\mathrm{mod}\,2\pi),\end{aligned} \tag{121}$$

where the summation is over the integers l that satisfy the condition [cf. (61)]

$$-\frac{\phi}{2\pi}-\frac{1}{2}\beta<l<-\frac{\phi}{2\pi}+1+\frac{1}{2}\beta, \tag{122}$$

$$\begin{aligned}f(k,\phi)=&-\sqrt{\frac{k}{2\pi i}}(\rho_B+\xi)(1-\Phi_K)\\&\times\exp\left[2ik\xi\pm i\left(\Phi-\frac{1}{2}\sigma\Phi_K\right)\frac{\pi}{1-\Phi_K}\right]\\&\times\{1+O[(\phi\mp\beta\pi)k(\rho_B+\xi)]\}\\&+\sqrt{\rho_B+\xi}\,O(1),\\&(\phi\mp\beta\pi)k(\rho_B+\xi)\ll 1,\end{aligned} \tag{123}$$

where $\beta\neq N-1$ ($N=1,2,3,\ldots$),

$$\begin{aligned}f(k,\phi)=&-\sqrt{\frac{2k}{\pi i}}\frac{\rho_B+\xi}{N}\exp(2ik\xi)\\&\times\cos\left\{\left[\Phi N-\frac{1}{2}\sigma(N-1)\right]\pi\right\}\{1+O\\&\times[(\phi+\pi-N\pi)k(\rho_B+\xi)]\}+\sqrt{\rho_B+\xi}\,O(1),\\&\times(\phi+\pi-N\pi)k(\rho_B+\xi)\ll 1,\end{aligned} \tag{124}$$

where $\beta=N-1$ ($N=1,2,3,\ldots$).

5. DISCUSSION OF THE RESULTS

In this paper, we have developed the quantum theory of the scattering of nonrelativistic particles with spin 0 and 1/2 in the field of a cosmic string ($-\infty<\Phi<\infty$ and $-\infty<\Phi_K<1$). The operator ΔH (27), which describes a perturbing interaction in the given case, is a differential operator of the same order as the unperturbed Hamiltonian H_0 (22); moreover, only the part of ΔH having the form of an interaction potential can be regarded as having a short range; the remaining part of ΔH containing differentiation describes a fairly strong long-range interaction (so strong that the Hörmander conditions[29] are not satisfied). Despite the actual nondecrease of the perturbing interaction at large distances from the string, an S matrix can be defined both in the case of the singular string (54) and when allowance is made for gravitational (88) or magnetic and gravitational (106) structures of the string. In the framework of the theory that we have developed, we have obtained the scattering amplitude [see (58), (90), and (107), respectively]. The effects of the structure of the cosmic string are taken into account in the most general form; we have also given specific examples of structure, including an example of a structure that cannot be reached by test particles.

If in any of the expressions (58), (90), and (107) we make the substitution $\Phi\to\Phi+N$ (where N is an integer), then we obtain the relation

$$f|_{\Phi+N}=e^{iN(\phi-\pi)}f|_{\Phi}, \tag{125}$$

from which it follows that the differential scattering cross section $d\sigma/d\phi=|f|^2$ is a periodic function (with period 1) of the magnetic flux Φ. Therefore, as in the case of scattering by a purely magnetic string ($\Phi\neq 0$ and $\Phi_K=0$), only the fractional part of the magnetic flux, and not its entire value, has physical meaning.

In contrast to scattering by a purely magnetic string, the scattering by the cosmic string depends on the spin of the scattered nonrelativistic particle, as follows from the explicit form of the expressions obtained for the scattering amplitude (where $\sigma=\pm 1$ for the upper or lower component of a spinor particle and $\sigma=0$ for a scalar particle). Another difference is that the scattering amplitude, as a function of the scattering angle, does not diverge in the forward direction but in two directions symmetric with respect to the forward direction with deflection determined by the integrated curvature Φ_K; a divergence is contained only in the term that describes the scattering by a singular string. The angular divergence of the scattering amplitude is due to the long-range interaction; the long-range interaction, which is probably stronger than in the case of scattering by a purely magnetic string, leads to divergence at one rather than two values of the angle.

At small values of the absolute magnitude of the wave vector of the test particle ($k \to 0$), the effects of the string structure become unimportant, and the differential cross section is given by the expressions that describe the scattering by a singular string. With allowance for (58), these can be represented in the form

$$\frac{d\sigma}{d\phi}=\frac{1}{4\pi k}\left\{\frac{1}{2\sin^2[(\phi+\beta\pi)/2]}+\frac{1}{2\sin^2[(\phi-\beta\pi)/2]}-\frac{\cos[2\mu(1+\beta)\pi-(2\nu+1)\beta\pi]}{\sin[(\phi+\beta\pi)/2]\sin[(\phi-\beta\pi)/2]}\right\} \tag{126}$$

for scalar ($\sigma=0$) and polarized spinor ($\sigma=\pm 1$) particles and

$$\frac{d\sigma}{d\phi}=\frac{1}{4\pi k}\left\{\frac{1}{2\sin^2[(\phi+\beta\pi)/2]}+\frac{1}{2\sin^2[(\phi-\beta\pi)/2]}-\frac{\cos[2\Phi(1+\beta)\pi-(\nu_+ +\nu_- +1)\beta\pi]\cos[(\nu_+ -\nu_- +1)\beta\pi]}{\sin[(\phi+\beta\pi)/2]\sin[(\phi-\beta\pi)/2]}\right\} \tag{127}$$

for unpolarized spinor particles; here μ, ν, and β are determined by the relations (55), and

$$\nu_\pm=\mathrm{integ}_-(\Phi\mp\Phi_K/2). \tag{128}$$

Note that because of the divergence of the expressions (126) and (127) as $\phi\to(\pm\beta\pi) \pmod{2\pi}$ the total cross section for scattering by a singular string

$$\sigma_{\mathrm{tot}}=\int_0^{2\pi} d\phi\,\frac{d\sigma}{d\phi}$$

is infinite.

At large absolute values of the wave vector of the test particle ($k\to\infty$), the structure effects become predominant, and the term that describes the scattering by the singular string can be ignored. In this region, the difference between scattering by a purely magnetic string ($\Phi_K=0$) and scattering by a cosmic string with $\Phi_K\neq 0$ is, in our view, most remarkable.

In the case $\Phi_K=0$, the contribution of the string structure to the scattering amplitude in the limit $k\to\infty$ is, like the incident wave, proportional to a phase factor that depends on Φ; therefore, the cross section in this region does not depend on Φ. In the case $\Phi_K\neq 0$, the contribution of the structure to the scattering amplitude as $k\to\infty$ is, like the incident wave [see (60)], equal to a sum of terms that is each proportional to a phase factor that depends on Φ; therefore, the cross section in this region depends on Φ as a consequence of the interference of the different phase factors. This last result appears somewhat surprising, since the region $k\to\infty$ corresponds to the classical limit $\hbar\to 0$, and one might expect that in this limit the quantum effects, which include the dependence of the cross section on Φ, would vanish (as happens in the case $\Phi_K=0$).

In this connection, the most interesting situation appears to be the one in which the cosmic string is contained in an impenetrable shell (119), i.e., direct effect of the magnetic field and the Gaussian curvature on test particles is completely ruled out. Taking into account (121), we obtain for the differential scattering cross section in the limit $k\to\infty$ the expression

$$\frac{d\sigma}{d\phi}=\frac{1}{2}(\rho_B+\xi)(1-\Phi_K)^2 \times\left|\sum_l\sqrt{\cos\left[\frac{1-\Phi_K}{2}(\phi-\pi+2l\pi)\right]} \times\exp\left\{i(2\Phi-\sigma\Phi_K)l\pi-2ik(\rho_B+\xi) \times\cos\left[\frac{1-\Phi_K}{2}(\phi-\pi+2l\pi)\right]\right\}\right|^2, \tag{129}$$

where the summation is over the integers l that satisfy the condition (122), and ξ and ρ_B are determined by the relations (70) and (103) ($r_B\geqslant r_K$). In the case $\Phi_K=0$, we obtain

$$\frac{d\sigma}{d\phi}=\frac{1}{2}r_B\sin\frac{\phi}{2}\quad(0<\phi<2\pi) \tag{130}$$

the differential cross section for scattering of a classical pointlike particle by an impenetrable cylinder of radius r_B. Before we turn to analysis of the relation (129) in the case $\Phi_K\neq 0$, we make a small digression concerning the scattering of test particles by a cosmic string in classical mechanics.

If we ignore structure effects or enclose the string in an impenetrable shell, the magnetic component of the string does not affect the classical trajectories of the motion of test particles. With regard to the gravitational component, its effect on the classical trajectories of the motion is purely kinematic in the case of a singular string, since the mere presence of the string changes the global properties of the space [see (2)]. Bearing in mind that in the polar coordinates $\rho,\tilde{\phi}$ there is no scattering, and going over to the polar coordinates ρ,ϕ [with allowance for the relation (8)], we obtain the of classical trajectories shown in Figs. 1 and 2, in which the string is directed perpendicular to the plane of the figure and the position at which it intersects the plane is indicated by the dot. The scattering angle does not depend on the impact parameter and is equal to ω_β or $-\omega_\beta$ ($0\leqslant\omega_\beta\leqslant\pi$) depending on the side from which the particle approaches the string. Depending on the value of Φ_K, the trajectories either do not intersect (Fig. 1) or do intersect (Fig. 2); the value of the

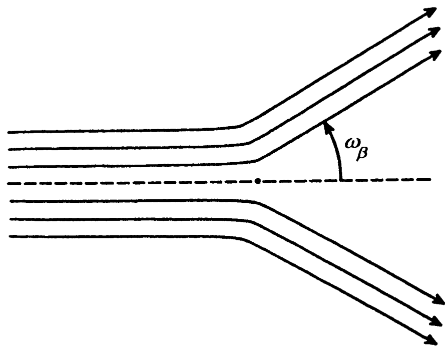

FIG. 1. Scattering of test particles by a singular cosmic string in classical mechanics for $-\infty<\Phi_K<0$: $\omega_\beta=-\beta\pi$, $(2n-1)/2n<\Phi_K<2n/(2n+1)$ $(n\geq 1)$: $\omega_\beta=(2n-\beta)\pi$.

angle ω_β itself depends on Φ_K (or β). It is natural to call the region of angles $-\omega_\beta<\phi<\omega_\beta$ the region of the classical shadow of the string.

Returning to the quantum-mechanical scattering, we note first that it is $\Delta(\phi-\phi'-\omega_\beta)$ and $\Delta(\phi-\phi'+\omega_\beta)$ that occur in the definition of the modified identity operator in the case of scattering by both a singular string and by a string possessing structure [see (57) and (89)]. Note also that the amplitude and scattering cross section in the case of a singular string diverge as $\phi\to\pm\omega_\beta$ [see (58), (126), and (127)], while the incident wave has discontinuities at $\phi-\phi'=\pm\omega_\beta$ [see (60)]. Finally, taking into account (122), we find that the number of terms in the expression for the amplitude of scattering by the structured string (121) that do no decrease as $k\to\infty$—in what follows we denote this number by n_l—is even in the region of the classical shadow of the string and odd outside this region; moreover, in the case of Φ_K values corresponding to Fig. 1 the value of n_l outside the shadow is one greater than the value of n_l in the shadow, while in the case of Φ_K values corresponding to Fig. 2 the value of n_l in the shadow exceeds by unity the value of n_l outside the shadow. More precisely, in the case $-\infty<\Phi_K<0$ we have $n_l=1$ outside the shadow and $n_l=0$ in the shadow [at the same time, the main contribution to the scattering cross section in this region is the small quantity $(\rho_B+\xi)O\{[k(\rho_B+\xi)]^{-1/3}\}$]; in the case $0<\Phi_K<1/2$, we have $n_l=1$ outside the shadow and $n_l=2$ in the shadow; in the case $1/2<\Phi_K<2/3$, we have $n_l=3$ outside the shadow and $n_l=2$ in the shadow; in the case $2/3<\Phi_K<3/4$, we have $n_l=3$ outside the shadow and $n_l=4$ inside the shadow and so forth with increasing value of n_l. In particular, in the case $0<\Phi_K<1/2$, which is the most interesting from the phenomenological point of view, we obtain the following expressions for the differential cross section for scattering of scalar $(\sigma=0)$ and polarized spinor $(\sigma=\pm 1)$ particles in the limit $k\to\infty$:

$$\frac{d\sigma}{d\phi}=\frac{1}{2}(\rho_B+\xi)(1-\Phi_K)^2\cos\left[(1-\Phi_K)\frac{\phi-\pi}{2}\right],$$

$$\omega_\beta<\phi<2\pi-\omega_\beta \tag{131}$$

and

$$\frac{d\sigma}{d\phi}=(\rho_B+\xi)(1-\Phi_K)^2\left(\cos\left[(1-\Phi_K)\frac{\phi}{2}\right]\sin\left(\Phi_K\frac{\pi}{2}\right)+\sqrt{\sin^2\left(\Phi_K\frac{\pi}{2}\right)-\sin^2\left[(1-\Phi_K)\frac{\phi}{2}\right]}\times\cos\left\{(2\Phi-\sigma\Phi_K)\pi+4k(\rho_B+\xi)\times\sin\left[(1-\Phi_K)\frac{\phi}{2}\right]\cos\left(\Phi_K\frac{\pi}{2}\right)\right\}\right),$$

$$-\omega_\beta<\phi<\omega_\beta. \tag{132}$$

In the strictly forward direction, we have

$$\frac{d\sigma}{d\phi}=2(\rho_B+\xi)(1-\Phi_K)^2\sin\left(\Phi_K\frac{\pi}{2}\right)\times\cos^2\left[\left(\Phi-\frac{1}{2}\sigma\Phi_K\right)\pi\right],\quad \phi=0. \tag{133}$$

In conclusion, we note also that in the limit $k\to\infty$ the total scattering cross section is finite:

$$\sigma_{\rm tot}=2(\rho_B+\xi)(1-\Phi_K). \tag{134}$$

This last relation is valid not only in the case $0<\Phi_K<1/2$ but also in the case of all possible values $-\infty<\Phi_K<1$ considered in the present study; as in (129)–(133), we have omitted in (134) terms $(\rho_B+\xi)O\{[k(\rho_B+\xi)]^{-1/3}\}$ that decrease as $k\to\infty$.

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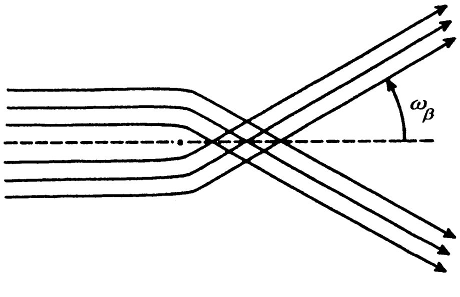

FIG. 2. The same as in Fig. 1 for $0<\Phi_K<1/2$: $\omega_\beta=\beta\pi$; $2n/(2n+1)<\Phi_K<(2n+1)/(2n+2)$ $(n\geq 1)$: $\omega_\beta=(\beta-2n)\pi$.

APPENDIX A

Resolvent of the operator H_1

We define the resolvent of the operator H_1 (31) as follows:

$$R^\omega=\left(H_1-\frac{\hbar^2}{2m}\omega\right)^{-1}. \tag{A1}$$

The kernel of the resolvent

$$R^\omega(\rho,\phi;\rho',\phi')=\langle \mathbf{x}|R^\omega|\mathbf{x}'\rangle$$

satisfies the equation

$$\left(H_1-\frac{\hbar^2}{2m}\,\omega\right)R^{\omega}(\rho,\phi;\rho',\phi')=\frac{\delta(\rho-\rho')}{\sqrt{\rho\rho'}}\,\Delta(\phi-\phi'), \tag{A2}$$

where $\Delta(\phi-\phi')$ is determined by the expression (46). Taking into account the explicit form of the operator H_1 (31) and representing the kernel of the resolvent in the form

$$R^{\omega}(\rho,\phi;\rho',\phi')=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\times\exp[in(\phi-\phi')]R_n^{\omega}(\rho;\rho'), \tag{A3}$$

we find that the function R_n^{ω} satisfies the equation

$$(-\partial_\rho^2-\rho^{-1}\partial_\rho+\rho^{-2}\alpha_n^2-\omega)R_n^{\omega}(\rho;\rho')=\frac{2m\,\delta(\rho-\rho')}{\hbar^2\sqrt{\rho\rho'}}, \tag{A4}$$

where α_n is determined by the expression (34). We determine

$$\sqrt{\omega}=\sqrt{|\omega|}\exp\left(\frac{i}{2}\arg\omega\right),$$

$$0<\arg\omega<2\pi,\quad \mathrm{Im}\sqrt{\omega}>0. \tag{A5}$$

The general solution of Eq. (A4) has the form

$$R_n^{\omega}(\rho;\rho')=\frac{i\pi}{2}\frac{m}{\hbar^2}\{\theta(\rho-\rho')[H_{\alpha_n}^{(1)}(\rho\sqrt{\omega})H_{\alpha_n}^{(2)}(\rho'\sqrt{\omega})-H_{\alpha_n}^{(2)}(\rho\sqrt{\omega})H_{\alpha_n}^{(1)}(\rho'\sqrt{\omega})]+H_{\alpha_n}^{(1)}\times(\rho\sqrt{\omega})C_n^{(1)}(\rho')+H_{\alpha_n}^{(2)}(\rho\sqrt{\omega})C_n^{(2)}(\rho')\}, \tag{A6}$$

where

$$\theta(u)=\begin{cases}1, & u>0,\\ 0, & u<0.\end{cases}$$

From the condition of regularity of the solution as $\rho\to 0$, we obtain

$$C_n^{(1)}(\rho')=C_n^{(2)}(\rho'), \tag{A7}$$

and from the condition of regularity of the solution as $\rho\to\infty$ [namely, taking into account (A5), from the condition that the solution in the limit $\rho\to\infty$ has asymptotic behavior in the form $\sim(2\pi\sqrt{\rho})^{-1}\exp(i\rho\sqrt{\omega})$], we obtain

$$C_n^{(2)}(\rho')=H_{\alpha_n}^{(1)}(\rho'\sqrt{\omega}). \tag{A8}$$

Taking (A7) and (A8) into account, we find

$$R_n^{\omega}(\rho;\rho')=i\pi\frac{m}{\hbar^2}[\theta(\rho-\rho')H_{\alpha_n}^{(1)}(\rho\sqrt{\omega})J_{\alpha_n}(\rho'\sqrt{\omega})+\theta(\rho'-\rho)J_{\alpha_n}(\rho\sqrt{\omega})H_{\alpha_n}^{(1)}(\rho'\sqrt{\omega})]. \tag{A9}$$

We determine the inverse Hilbert transform for the function R_n^{ω}:

$$\sigma_n^s(\rho;\rho')=\frac{1}{2\pi i}\lim_{\epsilon\to 0_+}[R_n^{s+i\epsilon}(\rho;\rho')-R_n^{s-i\epsilon}(\rho;\rho')]=\frac{m}{\hbar^2}\,\theta(s)J_{\alpha_n}(\rho\sqrt{s})J_{\alpha_n}(\rho'\sqrt{s}). \tag{A10}$$

We also define the function

$$\sigma^s(\rho,\phi;\rho',\phi')=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\times\exp[in(\phi-\phi')]\sigma_n^s(\rho;\rho'). \tag{A11}$$

For the decomposition of the identity of the operator H_1 we have

$$\left(\frac{d}{ds}\,E^s\right)=\frac{\hbar^2}{2m}\,\sigma^s, \tag{A12}$$

from which we obtain for a function of the operator H_1 the relation

$$\mathscr{F}(H_1)(\rho,\phi;\rho',\phi')=\frac{\hbar^2}{2m}\int_{-\infty}^{\infty}ds\,\mathscr{F}\left(\frac{\hbar^2 s}{2m}\right)\sigma^s(\rho,\phi;\rho',\phi'). \tag{A13}$$

Using the last relation for the evolution operator (32), we find

$$U_n(\rho,\rho',t)=\frac{\hbar^2}{2m}\int_{-\infty}^{\infty}ds\,\exp\left(-\frac{i\hbar st}{2m}\right)\sigma_n^s(\rho;\rho'). \tag{A14}$$

Taking (A10) into account and integrating over s in (A14), we obtain (33).

APPENDIX B

Lippmann–Schwinger equation in the case of short- and long-range perturbations

Let

$$\tilde{H}\psi(\mathbf{x},\mathbf{k})=\frac{\hbar^2k^2}{2m}\,\psi(\mathbf{x},\mathbf{k}),\quad \tilde{H}_0\chi(\mathbf{x},\mathbf{k})=\frac{\hbar^2k^2}{2m}\,\chi(\mathbf{x},\mathbf{k}), \tag{B1}$$

where ψ and χ belong to the space of functions with scalar product defined in accordance with (23). The two linearly independent solutions of the first of Eqs. (B1) can be expressed in terms of two linearly independent solutions of the second of Eqs. (B1):

$$\psi^{(\pm)}(\mathbf{x},\mathbf{k})=\Omega_{\tilde{H},\tilde{H}_0}^{(\pm)}\chi^{(\pm)}(\mathbf{x},\mathbf{k}). \tag{B2}$$

The last relation can be represented in the form of the Lippmann–Schwinger equation

$$\psi^{(\pm)}(\mathbf{x},\mathbf{k})=\chi^{(\pm)}(\mathbf{x},\mathbf{k})-(G^{\pm}\tilde{V}\psi^{(\pm)})(\mathbf{x},\mathbf{k}), \tag{B3}$$

where

$$\tilde{V}=\tilde{H}-\tilde{H}_0, \tag{B4}$$

$$G^{\pm}=\lim_{\epsilon\to 0_+}\left(\tilde{H}_0-\frac{\hbar^2k^2}{2m}\mp i\epsilon\right)^{-1}. \tag{B5}$$

Defining the S matrix as

$$S=(\Omega^{(-)}_{\tilde{H},\tilde{H}_0})^*\Omega^{(+)}_{\tilde{H},\tilde{H}_0},$$

we can obtain

$$(\chi^{(-)}(\mathbf{k}),(S-I)\chi^{(+)}(\mathbf{k}'))=-2\pi i\delta\left[\frac{\hbar^2}{2m}(k^2-k'^2)\right]\times(\chi^{(-)}(\mathbf{k}),\tilde{V}\psi^{(+)}(\mathbf{k}')), \quad (B6)$$

from which, taking into account the definition (49) of the t matrix, we find

$$t(k,\phi-\phi')=(\chi^{(-)}(\mathbf{k}),\tilde{V}\psi^{(+)}(\mathbf{k}')). \quad (B7)$$

We choose as operator $\tilde{H}_0$ the operator H_1 (31). Then the Green's functions G^+ and G^- are described by the expressions (79)–(81).

Let the operator $\tilde{V}$ (B4) be the operator of multiplication by a square integrable function:

$$(\tilde{V}\psi)(\mathbf{x})=v(\mathbf{x})\psi(\mathbf{x}), \quad (B8)$$

where

$$\int d^2\mathbf{x}|v(\mathbf{x})|^2<\infty. \quad (B9)$$

Then, taking into account the asymptotic behavior of the Green's function as $\rho\to\infty$ and $\rho'<\infty$,

$$G^+(\rho,\phi;\rho',\phi')=\frac{m}{\hbar^2}\sqrt{\frac{i}{2\pi k\rho}}e^{ik\rho}\sum_{n=-\infty}^{\infty}J_{\alpha_n}(k\rho')\times\exp\left[in(\phi-\phi')-\frac{1}{2}i\alpha_n\pi\right]+O(\rho^{-1})$$
$$=\frac{m}{\hbar^2}\sqrt{\frac{2\pi i}{k\rho}}e^{ik\rho}[\chi^{(-)}(\mathbf{x}',\mathbf{k})]^*+O(\rho^{-1}), \quad (B10)$$

where we have used the relation (74), we obtain

$$\psi^{(+)}(\mathbf{x},\mathbf{k}')=\chi^{(+)}(\mathbf{x},\mathbf{k}')-\frac{m}{\hbar^2}\sqrt{\frac{2\pi i}{k\rho}}\times e^{ik\rho}(\chi^{(-)}(\mathbf{k}),\tilde{V}\psi^{(+)}(\mathbf{k}')), \quad (B11)$$

where $\mathbf{x}=(\rho\cos\phi,\rho\sin\phi)$, $\mathbf{k}=(k\cos\phi,k\sin\phi)$, and $\mathbf{k}'=(k\cos\phi',k\sin\phi')$. If in place of H_1 (31) we choose H_0 (22) as $\tilde{H}_0$, then in the above expressions we must replace $\chi^{(\pm)}$ by $\psi^{(0)}$ (42).

Defining the scattering amplitude f' by means of the relation

$$\psi^{(+)}(\mathbf{x},\mathbf{k}')=\chi^{(+)}(\mathbf{x},\mathbf{k}')+f'(k,\phi-\phi')(2\pi\sqrt{\rho})^{-1}e^{ik\rho}, \quad (B12)$$

and taking (B7) and (B11) into account, we arrive at (48) and (50). We emphasize that these last relations are also valid in the case when instead of H_0 (22) we choose H_1 (31). Of decisive importance here is the short-range interaction of the perturbing operator $\tilde{V}$ determined, for example, by the conditions (B8) and (B9).

Neither the operator V_1 (77) nor the operator

$$V_0=H_1-H_0 \quad (B13)$$

satisfies the conditions (B8) and (B9); moreover, if the operator V_1 satisfies the Hörmander conditions for long-range perturbations,[27] the operator V_0 does not satisfy the last conditions. Nevertheless, even in the case of $\tilde{V}$ in the form (77) or (B13) an S matrix can be defined, as will be shown below by means of the formalism of the Lippmann–Schwinger equation.

First of all, we represent the functions in the relation (73) in the form

$$\psi^{(\pm)}(\mathbf{x},\mathbf{k})=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\exp[in(\phi-\phi_k)]a_n^{(\pm)}\times(k)\kappa_n(\rho,k), \quad (B14)$$

$$\chi^{(\pm)}(\mathbf{x},\mathbf{k})=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\exp[in(\phi-\phi_k)]b_n^{(\pm)}\times(k)\lambda_n(\rho,k), \quad (B15)$$

where the function κ_n satisfies Eq. (83), and the function λ_n the equation

$$(\partial_\rho^2+\rho^{-1}\partial_\rho-\rho^{-2}\alpha_n^2+k^2)\lambda_n(\rho,k)=0. \quad (B16)$$

The Lippmann–Schwinger equations for the radial components of the functions $\psi^{(\pm)}$ and $\chi^{(\pm)}$ have the form

$$a_n^{(\pm)}(k)\kappa_n(\rho,k)=b_n^{(\pm)}(k)\lambda_n(\rho,k)-\frac{\hbar^2}{2m}\int_0^\infty d\rho'\rho'G_n^{\pm}(\rho,\rho')V_n^{(1)}\times(\rho')a_n^{(\pm)}(k)\kappa_n(\rho',k), \quad (B17)$$

$$b_n^{(\pm)}(k)\lambda_n(\rho,k)=\frac{i^{|n|}}{2\pi}J_{|n|}(k\rho)-\frac{\hbar^2}{2m}\int_0^\infty d\rho'\rho'G_n^{(0)\pm}(\rho,\rho')V_n^{(0)}\times(\rho')b_n^{(\pm)}(k)\lambda_n(\rho',k), \quad (B18)$$

where G_n^+ and G_n^- are determined by the expressions (80) and (81),

$$V_n^{(1)}(\rho)=\gamma^{-2}\left\{\left[n-\Phi+\frac{1}{2}\sigma(1-(\partial_\rho\gamma))\right]^2-\frac{1}{4}(\partial_\rho\gamma)^2\right\}-\rho^{-2}\left(\alpha_n^2-\frac{1}{4}\right), \quad (B19)$$

$$G_n^{(0)+}(\rho;\rho')=i\pi\frac{m}{\hbar^2}\begin{cases}J_{|n|}(k\rho)H_{|n|}^{(1)}(k\rho'), & \rho<\rho',\\ H_{|n|}^{(1)}(k\rho)J_{|n|}(k\rho'), & \rho>\rho',\end{cases} \quad (B20)$$

$$G_n^{(0)-}(\rho;\rho')=-i\pi\frac{m}{\hbar^2}\begin{cases}J_{|n|}(k\rho)H_{|n|}^{(2)}(k\rho'), & \rho<\rho',\\ H_{|n|}^{(2)}(k\rho)J_{|n|}(k\rho'), & \rho>\rho',\end{cases} \quad (B21)$$

$$V_n^{(0)}(\rho)=\rho^{-2}(\alpha_n^2-n^2). \quad (B22)$$

Equation (B18) has been written down with allowance for the explicit form of the expansion (42) of the plane wave $\psi^{(0)}$; note also the relation

$$V_n^{(1)}(\rho)=[(\rho+\xi)^{-2}-\rho^{-2}]\left(\alpha_n^2-\frac{1}{4}\right),\quad \rho>\rho_K. \quad \text{(B23)}$$

Substituting the regular distribution of Eq. (B16) in explicit form [the function $J_{\alpha_n}(k\rho)$] in (B18) and integrating, we determine the coefficients $b_n^{(\pm)}(k)$ and obtain for the function $\chi^{(\pm)}(\mathbf{x},\mathbf{k})$ the expression (74). We now turn to the determination of the coefficients $a_n^{(\pm)}(k)$.

Taking into account the definition of the Wronskian (85) and the relation

$$\int_{\rho_1}^{\rho_2} d\rho\rho V_n^{(1)}(\rho)\kappa_n(\rho,k)\lambda_n(\rho,k)$$
$$=\rho_1 W[\kappa_n(\rho_1,k),\lambda_n(\rho_1,k)]$$
$$-\rho_2 W[\kappa_n(\rho_2,k),\lambda_n(\rho_2,k)], \quad \text{(B24)}$$

we obtain from Eq. (B17)

$$a_n^+(k)\lim_{\rho\to\infty}\rho W[H_{\alpha_n}^{(1)}(k\rho),\kappa_n(\rho,k)]$$
$$=-\frac{2i}{\pi}\exp\left(-\frac{i}{2}\alpha_n\pi\right),$$
$$a_n^-(k)\lim_{\rho\to\infty}\rho W[H_{\alpha_n}^{(2)}(k\rho),\kappa_n(\rho,k)]=\frac{2i}{\pi}\exp\left(\frac{i}{2}\alpha_n\pi\right). \quad \text{(B25)}$$

Taking into account (72), we obtain in the region outside the string

$$\kappa_n(\rho,k)=\sqrt{1+\xi\rho^{-1}}[c_n^{(1)}(k)H_{\alpha_n}^{(1)}(k(\rho+\xi))+c_n^{(2)}$$
$$\times(k)H_{\alpha_n}^{(2)}(k(\rho+\xi))]. \quad \text{(B26)}$$

Substituting (B26) in (B25), we obtain

$$a_n^+(k)c_n^{(2)}(k)=\frac{1}{2}\exp\left[-i\left(\frac{1}{2}\alpha_n-|n|\right)\pi+ik\xi\right],$$
$$a_n^-(k)c_n^{(1)}(k)=\frac{1}{2}\exp\left[\frac{1}{2}i\alpha_n\pi-ik\xi\right]. \quad \text{(B27)}$$

From the condition of smoothness of the solution $\kappa_n(\rho,k)$ on the boundary of the string

$$\sqrt{1+\xi\rho_K^{-1}}[c_n^{(1)}(k)H_{\alpha_n}^{(1)}(k(\rho_K+\xi))$$
$$+c_n^{(2)}(k)H_{\alpha_n}^{(2)}(k(\rho_K+\xi))]$$
$$=\kappa_n(\rho_K,k),$$
$$\{\partial_\rho\sqrt{1+\xi\rho^{-1}}[c_n^{(1)}(k)H_{\alpha_n}^{(1)}(k(\rho+\xi))+c_n^{(2)}(k)H_{\alpha_n}^{(2)}$$
$$\times(k(\rho+\xi))]\}|_{\rho=\rho_K}$$
$$=[\partial_\rho\kappa_n(\rho,k)]|_{\rho=\rho_K}, \quad \text{(B28)}$$

we find

$$c_n^{(1)}(k)=i\frac{\pi}{4}\rho_K W[\kappa_n(\rho_K,k),\sqrt{1+\xi\rho_K^{-1}}H_{\alpha_n}^{(2)}$$
$$\times(k(\rho_K+\xi))],$$
$$c_n^{(2)}(k)=-i\frac{\pi}{4}\rho_K W[\kappa_n(\rho_K,k),\sqrt{1+\xi\rho_K^{-1}}H_{\alpha_n}^{(1)}$$
$$\times(k(\rho_K+\xi))], \quad \text{(B29)}$$

from which, taking into account (B27), we obtain

$$a_n^+(k)c_n^{(1)}(k)=-\frac{1}{2}\exp\left[-i\left(\frac{1}{2}\alpha_n-|n|\right)\pi+ik\xi\right]$$
$$\times\frac{W[\kappa_n(\rho_K,k),\sqrt{1+\xi\rho_K^{-1}}H_{\alpha_n}^{(2)}(k(\rho_K+\xi))]}{W[\kappa_n(\rho_K,k),\sqrt{1+\xi\rho_K^{-1}}H_{\alpha_n}^{(1)}(k(\rho_K+\xi))]},$$
$$a_n^-(k)c_n^{(2)}(k)=-\frac{1}{2}\exp\left[\frac{1}{2}i\alpha_n\pi-ik\xi\right]$$
$$\times\frac{W[\kappa_n(\rho_K,k),\sqrt{1+\xi\rho_K^{-1}}H_{\alpha_n}^{(1)}(k(\rho_K+\xi))]}{W[\kappa_n(\rho_K,k),\sqrt{1+\xi\rho_K^{-1}}H_{\alpha_n}^{(2)}(k(\rho_K+\xi))]}. \quad \text{(B30)}$$

Taking into account (B27) and (B30), we find that in the region outside the string the functions $\psi^{(+)}(\mathbf{x},\mathbf{k})$ and $\psi^{(-)}(\mathbf{x},\mathbf{k})$ are described by the expressions (86) and (87).

We determine the S matrix:

$$(\psi^{(-)}(\mathbf{k}),\psi^{(+)}(\mathbf{k}'))=(2\pi)^{-1}\sum_{n=-\infty}^{\infty}\exp[in(\phi-\phi')]$$
$$\times[a_n^{(-)}(k)]^*a_n^{(+)}(k')$$
$$\times\int_0^\infty d\rho\rho\kappa_n(\rho,k)\kappa_n(\rho,k'). \quad \text{(B31)}$$

By means of Eq. (83), we can obtain

$$\{\partial_\rho\rho W[\kappa_n(\rho,k),\kappa_n(\rho,k')]\}=(k^2-k'^2)$$
$$\times\rho\kappa_n(\rho,k)\kappa_n(\rho,k'). \quad \text{(B32)}$$

Integrating (B32) with allowance for the regularity of κ_n at the point $\rho=0$ and the asymptotic behavior of κ_n as $\rho\to\infty$, we find

$$\int_0^\infty d\rho\rho\kappa_n(\rho,k)\kappa_n(\rho,k')$$
$$=\frac{2\delta(k-k')}{\sqrt{kk'}}[|c_n^{(1)}(k)|^2+|c_n^{(2)}(k)|^2], \quad \text{(B33)}$$

from which, taking into account (B27) and (B30), we obtain for the S matrix (B31) the expression (88).

APPENDIX C

Asymptotic behavior of the scattering amplitude as $k\to\infty$

We denote the infinite sum that occurs in the definition of the scattering amplitude (120) by

$$\Sigma(s,\phi)=\sum_{n=-\infty}^{\infty}\exp[in\phi-i(\alpha_n-|n|)\pi]\frac{J_{\alpha_n}(s)}{H^{(1)}_{\alpha_n}(s)},\tag{C1}$$

where $s=k(\rho_B+\xi)$. We consider first the sum

$$\Sigma_1(s,\phi)=\sum_{\alpha_n>s}\exp[in\phi-i(\alpha_n-|n|)\pi]\frac{J_{\alpha_n}(s)}{H^{(1)}_{\alpha_n}(s)}$$
$$=\sum_{n=s^+}^{\infty}\frac{J_{(n-\mu)(1+\beta)}(s)}{H^{(1)}_{(n-\mu)(1+\beta)}(s)}\exp[in(\phi-\pi)$$
$$-i(n-\mu)(1+\beta)\pi]+\sum_{n=s^-}^{\infty}\frac{J_{(n+\mu)(1+\beta)}(s)}{H^{(1)}_{(n+\mu)(1+\beta)}(s)}$$
$$\times\exp[-in(\phi-\pi)-i(n+\mu)(1+\beta)\pi],\tag{C2}$$

where

$$s^{\pm}=\text{integ}_-\left(\pm\mu+\frac{s}{1+\beta}\right)+1.\tag{C3}$$

Taking into account the asymptotic behavior of the cylinder functions

$$J_\omega(s)=(2\pi)^{-1/2}(\omega^2-s^2)^{-1/4}\exp\left[(\omega^2-s^2)^{1/2}\right.$$
$$\left.-\omega\tanh^{-1}\left(1-\frac{s^2}{\omega^2}\right)^{1/2}\right][1+O(\omega-1)],$$

$$H^{(1)}_\omega(s)=-2i(2\pi)^{-1/2}(\omega^2-s^2)^{-1/4}\exp\left[-(\omega^2-s^2)^{1/2}\right.$$
$$\left.+\omega\tanh^{-1}\left(1-\frac{s^2}{\omega^2}\right)^{1/2}\right][1+O(\omega^{-1})]\tag{C4}$$

for $0<s\leqslant s_{\max}<\omega$, we obtain

$$\Sigma_1(s,\phi)=\frac{i}{2}\sum_{n=s^+}^{\infty}\exp\left\{2[(n-\mu)^2(1+\beta)^2-s^2]^{1/2}\right.$$
$$-2(n-\mu)(1+\beta)\tanh^{-1}$$
$$\left.\times\left[1-\frac{s^2}{(n-\mu)^2(1+\beta)^2}\right]^{1/2}\right\}$$
$$\times\left\{1+O\left(\frac{1}{(n-\mu)(1+\beta)}\right)\right\}$$
$$\times\exp[in(\phi-\pi)-i(n-\mu)(1+\beta)\pi]$$
$$+\frac{i}{2}\sum_{n=s^-}^{\infty}\exp\left\{2[(n+\mu)^2(1+\beta)^2-s^2]^{1/2}\right.$$
$$-2(n+\mu)(1+\beta)\tanh^{-1}$$
$$\left.\times\left[1-\frac{s^2}{(n+\mu)^2(1+\beta)^2}\right]^{1/2}\right\}$$
$$\times\left\{1+O\left(\frac{1}{(n+\mu)(1+\beta)}\right)\right\}$$
$$\times\exp[-in(\phi-\pi)-i(n+\mu)(1+\beta)\pi].\tag{C5}$$

Taking into account the relation

$$\frac{\partial}{\partial\omega}\left[(\omega^2-s^2)^{1/2}-\omega\tanh^{-1}\left(1-\frac{s^2}{\omega^2}\right)^{1/2}\right]$$
$$=-\ln\frac{\omega+(\omega^2-s^2)^{1/2}}{s}<0\tag{C6}$$

for $\omega>s$, we obtain the relation

$$\sum_{n=s^{\pm}}^{\infty}\exp\left\{2[(n\mp\mu)^2(1+\beta)^2-s^2]^{1/2}-2(n\mp\mu)\right.$$
$$\left.\times(1+\beta)\tanh^{-1}\left[1-\frac{s^2}{(n\mp\mu)^2(1+\beta)^2}\right]^{1/2}\right\}$$
$$=\int_{s^{\pm}}^{\infty}dn\,\exp\left\{2[(n\mp\mu)^2(1+\beta)^2-s^2]^{1/2}-2(n\mp\mu)\right.$$
$$\left.\times(1+\beta)\tanh^{-1}\left[1-\frac{s^2}{(n\mp\mu)^2(1+\beta)^2}\right]^{1/2}\right\}+O(1)$$
$$=\frac{s}{1+\beta}\int_0^{\infty}d\tau\,\sinh\tau\,\exp[2s(\sinh\tau-\tau\cosh\tau)]$$
$$+O(1)=\frac{1}{1+\beta}\Gamma\left(\frac{2}{3}\right)\left(\frac{s}{12}\right)^{1/3}+O(1)\tag{C7}$$

as $s\to\infty$, where $\Gamma(z)$ is the gamma function. By means of the last relation, we obtain the estimate

$$\Sigma_1(s,\phi)\leqslant c_1 s^{1/3},\tag{C8}$$

where c_1 depends neither on s nor on ϕ.

We now consider the finite sum

$$\Sigma_2(s,\phi)=\sum_{\alpha_n\leqslant s}\exp[in\phi-i(\alpha_n-|n|)\pi]\frac{J_{\alpha_n}(s)}{H^{(1)}_{\alpha_n}(s)}$$
$$=\frac{1}{2}\sum_{n=\nu+1}^{s_+}\left[1+\frac{H^{(2)}_{(n-\mu)(1+\beta)}(s)}{H^{(1)}_{(n-\mu)(1+\beta)}(s)}\right]\exp[in(\phi$$
$$-\pi)-i(n-\mu)(1+\beta)\pi]+\frac{1}{2}\sum_{n=-\nu}^{s_-}$$
$$\times\left[1+\frac{H^{(2)}_{(n+\mu)(1+\beta)}(s)}{H^{(1)}_{(n+\mu)(1+\beta)}(s)}\right]\exp[-in(\phi-\pi)$$
$$-i(n+\mu)(1+\beta)\pi],\tag{C9}$$

where

$$s_{\pm}=\text{integ}_+\left(\pm\mu+\frac{s}{1+\beta}\right)-1,\tag{C10}$$

and $\mathrm{integ}_+(u)$ is the next integer number greater than u, is equal to u if u is an integer. Taking into account the asymptotic behavior

$$H_\omega^{(1)}(s)=\left(\frac{2}{i\pi}\right)^{1/2}(s^2-\omega^2)^{-1/4}\exp\left[i(s^2-\omega^2)^{1/2}-i\omega\cos^{-1}\frac{\omega}{s}\right][1+O(s^{-1})],$$

$$H_\omega^{(2)}(s)=\left(\frac{2i}{\pi}\right)^{1/2}(s^2-\omega^2)^{-1/4}\exp\left[-i(s^2-\omega^2)^{1/2}+i\omega\cos^{-1}\frac{\omega}{s}\right][1+O(s^{-1})], \tag{C11}$$

for $\omega<s_{\min}\leqslant s<\infty$, we obtain

$$\begin{aligned}\Sigma_2(s,\phi)&=\frac{1}{2}\frac{\exp[i\nu(\phi+\beta\pi)]-\exp[-i(s_-+1)(\phi+\beta\pi)]}{1-\exp[-i(\phi+\beta\pi)]}\\&\times\exp[-i\mu(1+\beta)\pi]\\&-\frac{1}{2}\frac{\exp[i\nu(\phi-\beta\pi)]-\exp[is_+(\phi-\beta\pi)]}{1-\exp[-i(\phi-\beta\pi)]}\\&\times\exp[i\mu(1+\beta)\pi]+\frac{i}{2}\sum_{n=\nu+1}^{s_+}\exp[2\pi if_+(n,s)]\\&+\frac{i}{2}\sum_{n=-\nu}^{s_-}\exp[2\pi if_-(n,s)]+O(s^{-1}),\end{aligned}\tag{C12}$$

where

$$\begin{aligned}f_\pm(n,s)=\pi^{-1}\Big\{&-[s^2-(n\mp\mu)^2(1+\beta)^2]^{1/2}\\&+(n\mp\mu)(1+\beta)\cos^{-1}\frac{(n\mp\mu)(1+\beta)}{s}\\&+\frac{1}{2}n(\pm\phi-\beta\pi)\pm\frac{1}{2}\mu(1+\beta)\pi\Big\}.\end{aligned}\tag{C13}$$

Taking the Fourier expansion, we can obtain the relation

$$\begin{aligned}\sum_{n=n_1}^{n_2}\exp[2\pi if(n)]=\sum_{l=-\infty}^{\infty}\int_{n_1}^{n_2}du\,\exp\{2\pi i[f(u)&-ul]\}+\frac{1}{2}\{\exp[2\pi if(n_1)]\\&+\exp[2\pi if(n_2)]\}.\end{aligned}\tag{C14}$$

Let $n_2-n_1\gg 1$ and $f(u)$ be a function that is convex upward on the interval $n_1<u<n_2$:

$$\frac{d^2f}{du^2}<0. \tag{C15}$$

We can now show that only a finite number of terms of the series on the right-hand side of (C14) makes the main contribution, and we can obtain the relations

$$\sum_{n=n_1}^{n_2}\exp[2\pi if(n)]=\sum_{l=l_1}^{l_2}\int_{n_1}^{n_2}du\,\exp\{2\pi i[f(u)-ul]\}+O(1),\quad l_1<0<l_2, \tag{C16}$$

$$\begin{aligned}\sum_{n=n_1}^{n_2}\exp[2\pi if(n)]&=\sum_{l=l_1}^{l_2}\int_{n_1}^{n_2}du\,\exp\{2\pi i[f(u)-ul]\}\\&+\int_{n_1}^{n_2}du\,\exp[2\pi if(u)]+O(1),\end{aligned}$$

$$0<l_1<l_2\ (\text{or } l_1<l_2<0), \tag{C17}$$

where

$$l_1\approx(df/du)|_{u=n_2}\quad\text{and}\quad l_2\approx(df/du)|_{u=n_1}.$$

Let $s\gg 1$ and $g(\tau)$ be a function that is convex upward on the interval $\tau_1<\tau<\tau_2$,

$$d^2g/d\tau^2<0, \tag{C18}$$

$$(dg/d\tau)|_{\tau=\tau_0}=0. \tag{C19}$$

Then by means of the method of stationary phase (see, for example, Ref. 34), we can obtain the relations

$$\int_{\tau_1}^{\tau_2}d\tau\,\exp[isg(\tau)]=\exp[isg(\tau_0)]\left(\frac{2\pi i}{s(d^2g/d\tau^2)|_{\tau=\tau_0}}\right)^{1/2}+O(s^{-1}),\quad\tau_1<\tau_0<\tau_2, \tag{C20}$$

$$\int_{\tau_1}^{\tau_2}d\tau\,\exp[isg(\tau)]=\exp[isg(\tau_0)]\left(\frac{\pi i}{2s(d^2g/d\tau^2)|_{\tau=\tau_0}}\right)^{1/2}+O(s^{-1}),\quad\tau_0=\tau_1\ \text{or}\ \tau_0=\tau_2. \tag{C21}$$

Using (C14)–(C21), we obtain

$$\begin{aligned}\sum_{n=\nu+1}^{s_+}\exp[2\pi if_+(n,s)]&=\frac{s^{1/2}}{1+\beta}\left\{\sum_l\exp\left[i\mu(\phi+\pi-2l\pi)-2is\cos\frac{\phi+\pi-2l\pi}{2(1+\beta)}\right]\right.\\&\left.\times\left[-i\pi\cos\frac{\phi+\pi-2l\pi}{2(1+\beta)}\right]^{1/2}+a\right\}+O(1).\end{aligned}\tag{C22}$$

where the summation is over the integers l that satisfy the condition

$$\frac{\phi}{2\pi}-\frac{1}{2}\beta<l<\frac{\phi}{2\pi}+\frac{1}{2}, \tag{C23}$$

$$\sum_{n=-\nu}^{s_-}\exp[2\pi if_-(n,s)]$$

$$=\frac{s^{1/2}}{1+\beta}\left\{\sum_l\exp\left[i\mu(\phi-\pi+2l\pi)-2is\right.\right.$$

$$\cos\frac{\phi-\pi+2l\pi}{2(1+\beta)}\Bigg]\Bigg[-i\pi\cos\frac{\phi-\pi+2l\pi}{2(1+\beta)}\Bigg]^{1/2}+a\Bigg\}$$

$$+O(1), \tag{C24}$$

where the summation is over the integers l that satisfy the condition

$$-\frac{\phi}{2\pi}-\frac{1}{2}\beta<l<-\frac{\phi}{2\pi}+\frac{1}{2}. \tag{C25}$$

At the same time

$$a=\begin{cases}0, & \phi\neq\pi(\mathrm{mod}\ 2\pi),\\ (-i\pi)^{1/2}/2, & \phi=\pi(\mathrm{mod}\ 2\pi).\end{cases} \tag{C26}$$

Combining (C22) and (C24) and taking into account (C12), we obtain

$$\Sigma_2(s,\phi)=\frac{(i\pi s)^{1/2}}{2(1+\beta)}\sum_l \exp\Bigg[i\mu(\phi-\pi+2l\pi)$$

$$-2is\cos\frac{\phi-\pi+2l\pi}{2(1+\beta)}\Bigg]\Bigg[\cos\frac{\phi-\pi+2l\pi}{2(1+\beta)}\Bigg]^{1/2}$$

$$+O(1),\quad \phi\neq(\pm\beta\pi)(\mathrm{mod}\ 2\pi), \tag{C27}$$

where the summation is over the integers l that satisfy the condition (122). Similarly, for $(\phi\mp\beta\pi)s\ll 1$ we obtain

$$\Sigma_2(s,\phi)=\frac{s}{N}\cos(\mu N\pi)\{1+O[(\phi+\pi-N\pi)s]\}$$

$$+\frac{(i\pi s)^{1/2}}{2N}\sum_l \exp\Bigg[i\mu(N-2l)\pi$$

$$-2is\sin\frac{l\pi}{N}\Bigg]\Bigg(\sin\frac{l\pi}{N}\Bigg)^{1/2}+O(1),$$

$$\beta=N-1\quad (N=1,2,3,\ldots), \tag{C28}$$

$$\Sigma_2(s,\phi)=\frac{s}{2(1+\beta)}\exp[\pm i\mu(1+\beta)\pi]\{1+O[(\phi$$

$$\mp\beta\pi)s]\}+\frac{(i\pi s)^{1/2}}{2(1+\beta)}\sum_l \exp\Bigg[\pm i\mu(1+\beta$$

$$-2l)\pi-2is\sin\frac{l\pi}{1+\beta}\Bigg]$$

$$\times\Bigg(\sin\frac{l\pi}{1+\beta}\Bigg)^{1/2}+O(1), \tag{C29}$$

$$\beta\neq N-1\quad (N=1,2,3,\ldots),$$

where the summation is over the integers l that satisfy the condition

$$0<l<1+\beta. \tag{C30}$$

Taking into account (C8) and (C27)–(C29), we obtain the relations (121), (123), and (124).

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