Rutherford scattering in the presence of a monochromatic light wave

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A closed expression for the amplitude of two-photon free–free electron transitions in a Coulomb field is obtained in the form of a single integral of the hypergeometric functions $_2F_1$. The singular part of the amplitude for the case of elastic scattering is identified. Also, the intensity-linear correction to the Rutherford scattering in the presence of a light wave is calculated. Finally, the specific "asymmetry" of the cross sections of elastic scattering and processes of stimulated bremsstrahlung and inverse bremsstrahlung caused by the dependence of the cross sections on the degree of circular polarization of the light field is studied.

1. INTRODUCTION

The Rutherford formula for the scattering cross section of a nonrelativistic electron with momentum **p** scattered through an angle ϑ in the Coulomb field of a fixed charge Ze,

$$d\sigma_{\rm R}=\left(\frac{mZe^2}{2p^2}\right)^2\frac{d\Omega}{\sin^4(\vartheta/2)},\qquad(1)$$

is a fundamental physical formula, and calculations of corrections to this formula caused by various factors are of obvious interest. For instance, allowing for relativistic effects for an unpolarized electron leads to multiplication of the right-hand side of Eq. (1) by the Mott factor

$$1-\left(\frac{v}{c}\sin\frac{\vartheta}{2}\right)^2,$$

which significantly suppresses backscattering in the ultrarelativistic limit (see Ref. 1, §80). Radiative corrections to (1) caused by the virtual interaction between the electron and the vacuum, which are of order α, the fine-structure constant, have also been calculated (Ref. 1, §122). When Coulomb scattering takes place in the presence of a monochromatic (laser) wave of intensity I and frequency ω, induced radiative corrections to $d\sigma_{\rm R}$ emerge. These corrections are caused by stimulated processes of reradiation of real photons by an electron without any change in the absolute value of momentum **p**. For intensities I small compared to the characteristic "atomic" intensity

$$I_0=\frac{ce^2}{8\pi a_0^4}=3.51\times10^{16}\text{W/cm}^{-2},$$

where a_0 is the Bohr radius, the induced corrections are linear in I, so that the total scattering cross section is

$$d\sigma=d\sigma_{\rm R}+\frac{I}{I_0}d\sigma_{\rm ind}.\qquad(2)$$

Potential scattering in the presence of a light wave has been studied by many researchers (the most complete review of the theoretical and experimental results is given in Ref. 2) in connection with problems of multiphoton stimulated bremsstrahlung and inverse bremsstrahlung (SBIB) in a laser field, i.e., free–free transitions of an electron with momentum **p** into a state with momentum $\mathbf{p}_n'$ defined by the energy conservation law

$$\frac{\mathbf{p}_n'^2}{2m}=\frac{\mathbf{p}^2}{2m}\pm n\hbar\omega.\qquad(3)$$

The most general results were obtained by Bunkin and Fedorov[3] and Kroll and Watson.[4] In Ref. 3 it was shown that if the scattering potential is taken into account in perturbation theory (the Born approximation), the cross section $d\sigma^n$ of n-photon SBIB has the form

$$d\sigma^n=\frac{p_n'}{p}J_n^2\left(\frac{|e\mathbf{F}(\mathbf{p}-\mathbf{p}_n')|}{m\hbar\omega^2}\right)d\sigma_{\rm B},\qquad(4)$$

where **F** is the amplitude of the light wave, J_n is the Bessel function, and $d\sigma_{\rm B}$ is the Born cross section in the absence of a light wave. Kroll and Watson[4] established that in the low-frequency limit ($\omega\to0$) the cross section $d\sigma^n$ also has the form of (4), with $d\sigma_{\rm B}$ replaced by the exact scattering cross section in the absence of a light field. Since for a Coulomb potential $d\sigma_{\rm B}=d\sigma\equiv d\sigma_{\rm R}$, expanding the Bessel function J_n in Eq. (4) in a power series for small values of F, we arrive in both cases considered at the following simple result for the cross section (2) corresponding to $n=0$:

$$d\sigma=d\sigma_{\rm R}\left(1-\frac{e^2}{2m^2\hbar^2\omega^4}|\mathbf{F}(\mathbf{p}-\mathbf{p}')|^2\right).\qquad(5)$$

Berson[5] found an expression for $d\sigma^n$ using an approximation in which electron motion was described classically and the emission or absorption of photons quantum mechanically. Consistent allowance for the Coulomb potential for a finite ω is exceptionally difficult, and has been done only for the case of double bremsstrahlung and inverse bremsstrahlung in a weak field[6,7] (see also Ref. 8). Numerical calculations of the cross section of double bremsstrahlung for an electron scattered by atoms were done by Korol.[9] A number of researchers did numerical calculations of the elastic scattering cross section in a strong field via numerical integration of the Schrödinger equation. Van de Ree, Kaminski, and Gavrila[10] did their calculation for an ultrastrong high-frequency field. Dimou and Faisal[11] used the method of strong channel cou-

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pling to calculate the scattering cross section in a circularly polarized field with a frequency $\omega = 0.472\text{Ry}$. The R-matrix method of calculation was developed by Dörr *et al.*[12]

In this paper we obtain an analytic expression for the amplitude of two-photon free–free transitions in a Coulomb field and do a consistent calculation of the correction $d\sigma_{\text{ind}}$ to the Rutherford cross section with exact allowance for the Coulomb potential. We also analyze the dependence of $d\sigma_{\text{ind}}$ on the frequency and polarization of the light wave.[1)] In contrast to processes of the SBIB type, there are additional difficulties in calculating the elastic scattering cross section caused by the singularity of the amplitude of a free–free transition between states with the same energy. These problems are analyzed in Sec. 3 both for the Coulomb potential and for a simpler case of a short-range potential (a delta-like well). Section 4 discusses the specific polarization "asymmetry" of the cross sections of elastic scattering and SBIB-type processes, an asymmetry related to the dependence of $d\sigma$ on the degree of circular polarization of the light wave. This effect is absent in the Born and low-frequency approximations, and carries new information about the effect of laser radiation on collision processes. Section 5 examines a number of limits and gives the results of numerical calculations of corrections to the Rutherford formula.

2. A GENERAL EXPRESSION FOR THE CORRECTIONS TO THE ELASTIC SCATTERING CROSS SECTION

The scattering of an electron on a static potential $U(r)$ in an electromagnetic field with an electric vector

$$\mathbf{F} = F\ \text{Re}\{\mathbf{e}\exp(-i\omega t)\}, \quad \mathbf{e}\cdot\mathbf{e}^* = 1,$$

is described by the Schrödinger equation (in what follows we use natural units: $e = \hbar = m = 1$)

$$i\frac{\partial \Psi}{\partial t} = (H_0 + V)\Psi, \tag{6}$$

where

$$H_0 = \hat{\mathbf{p}}^2/2 - U(r).$$

It is convenient to express the operator V describing the electron–wave interaction in the dipole approximation in terms of the momentum operator:

$$V = V^{(+)}\exp(-i\omega t) + V^{(-)}\exp(i\omega t),$$

$$V^{(+)} = -i\frac{F}{2\omega}\mathbf{e}\hat{\mathbf{p}}, \quad V^{(-)} = i\frac{F}{2\omega}\mathbf{e}^*\hat{\mathbf{p}}.$$

The scattering function Ψ has a quasienergy structure:

$$\Psi(\mathbf{r},t) = e^{-iEt}[\psi_0(\mathbf{r}) + \psi_{\pm 1}(\mathbf{r})e^{\mp i\omega t} + \psi_{\pm 2}(\mathbf{r})e^{\mp 2i\omega t} + \cdots]. \tag{7}$$

The terms with $\psi_{\pm n}$ describe the scattering of an electron accompanied by absorption $(+n)$ or emission $(-n)$ of n photons, while $\psi_0(\mathbf{r})$ corresponds to elastic scattering. In a weak field,

$$\psi_{\pm n} \sim F^n$$

and

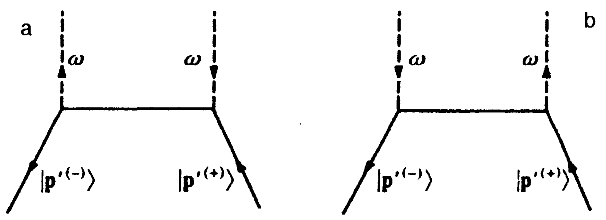

FIG. 1.

$$\psi_0 = |\mathbf{p}^{(+)}\rangle + \psi_0^{(2)} + \cdots,$$

where $|\mathbf{p}^{(+)}\rangle$ is the scattering function for the Hamiltonian H_0, and $\psi_0^{(2)} \sim F^2$. Calculating $\psi_0^{(2)}$ perturbatively, we represent the elastic scattering amplitude f, which determines, as usual, the asymptotic behavior of $\psi_0(\mathbf{r})$ as $r \to \infty$, as the sum of the elastic scattering amplitude f_0 in the absence of the field of the wave and a correction term $f_2 \sim F^2$:

$$f = f_0 + f_2, \tag{8}$$

$$f_2 = \frac{1}{2\pi}\langle \mathbf{p}'^{(-)}|V^{(-)}G_{E+\omega}V^{(+)} + V^{(+)}G_{E-\omega}V^{(-)}|\mathbf{p}^{(+)}\rangle. \tag{9}$$

Here $|\mathbf{p}^{(-)}\rangle$ and $|\mathbf{p}^{(+)}\rangle$ are the wave functions (normalized to unit density) of the continuous spectrum of the Hamiltonian H_0 that asymptotically behave like incoming and outgoing waves, and $G_{\mathscr{E}}$ is the Green's function of H_0 that asymptotically behaves like outgoing waves for $\mathscr{E} > 0$. The two terms in Eq. (9) correspond to the process of absorption of a photon that is subsequently emitted and to the reverse process. Graphically they can be depicted by a "direct" (Fig. 1(a)) and "reverse" (Fig. 1(b)) Feynman diagram.

For the scattering cross section that allows for terms $\sim F^2$, Eq. (8) yields

$$d\sigma = d\sigma_0 + d\sigma_{\text{corr}}, \tag{10}$$

where

$$d\sigma_0 = |f_0|^2 d\Omega, \quad d\sigma_{\text{corr}} = 2\ \text{Re}(f_0^* f_2)d\Omega.$$

Thus, the corrections (induced by the light field) to the elastic scattering cross section have an interference structure (similar to vacuum radiative corrections), in contrast to the structure of the SBIB cross section, which is given by the square of the absolute values of the amplitudes $f_{\pm n}$, which are determined by the asymptotic behavior of the $\psi_{\pm n}$ in (7).

For the Coulomb potential $U(r) = -Z/r$ we have

$$d\sigma_0 = d\sigma_{\text{R}},$$

$$f_0 = \frac{a}{2p}\frac{\Gamma(1-ia)}{\Gamma(1+ia)}\left(\sin\frac{\vartheta}{2}\right)^{-2+2ia}, \quad a = \frac{Z}{p}, \tag{11}$$

and $G_{\mathscr{E}}$ and $|p^{(\pm)}\rangle$ in (9) are the Coulomb Green's function and the corresponding Coulomb wave functions:

$$|\mathbf{p}^{(+)}\rangle \equiv \psi_{\mathbf{p}}^{(+)} = A_{\mathbf{p}}^{(+)}\exp(i\mathbf{pr})\Phi(ia,1;i(pr-\mathbf{pr}))$$
$$= A_{\mathbf{p}}^{(+)}\exp(ipr)\Phi(1-ia,1;i(pr-\mathbf{pr})),$$

$$|\mathbf{p}'^{(-)}\rangle \equiv \psi_{\mathbf{p}'}^{(-)} = A_{\mathbf{p}'}^{(-)}\exp(i\mathbf{p}'\mathbf{r})\Phi(-ia',1;-i(p'r+\mathbf{p}'\mathbf{r})) = A_{\mathbf{p}'}^{(-)}\exp\{-ip'r\}\Phi(1+ia',1;i(p'r+\mathbf{p}'\mathbf{r})), \tag{12}$$

with $A_{\mathbf{p}}^{(\pm)} = \exp\{\pi a/2\}\Gamma(1\mp ia)$ the normalization factors, and $\Phi(\alpha,\beta;x)$ the confluent hypergeometric function.

3. CALCULATING THE AMPLITUDE OF A TWO-PHOTON FREE-FREE TRANSITION AND REMOVING SINGULARITIES IN THE ELASTIC SCATTERING AMPLITUDE

The divergences in the matrix elements (9) at $p'=p$ present an additional difficulty in studying elastic scattering, in comparison to SBIB processes related to absorption or emission of energy. The simplest way to establish the origin of the divergences is to examine the matrix element of a two-photon transition between the states ψ_{Elm} of the continuous spectrum with certain values of energy and orbital angular momentum l. The asymptotic behavior, as $|\mathbf{r}|\to\infty$, of the integral of the product of the Green's function $G_{\mathscr{E}}(\mathbf{r},\mathbf{r}')$ and ψ_{Elm} has the following form:

$$\int d\mathbf{r}' G_{\mathscr{E}}(\mathbf{r},\mathbf{r}')(\mathbf{e}\hat{\mathbf{p}}')\psi_{Elm}(\mathbf{r}') \sim c_1 r^{Z/\sqrt{-2\mathscr{E}}-1} \times \exp(-\sqrt{-2\mathscr{E}}r) + \frac{c_2}{r}\cos\left[pr - \frac{\pi}{2}l + \delta_l(E)\right],$$

where c_1 and c_2 are constants, and δ_l is the scattering phase for the potential $U(r)$. The presence of the second term oscillating with the same frequency as the final-state wave function leads to a divergence in the matrix element

$$\langle \psi_{E'lm}|(\mathbf{e}',\hat{\mathbf{p}})G_{\mathscr{E}}(\mathbf{e}\hat{\mathbf{p}})|\psi_{Elm}\rangle$$

at $p'=p$. [2)] The two terms in Eq. (9) diverge for the same reason. In view of this, when calculating $d\sigma_{\rm corr}$, we assume that in Eq. (9) $p'\neq p$ and then pass to the limit $p'=p$ only in the final analytic result.

3.1. Amplitude of a two-photon free-free transition in a Coulomb field

Consider the two-photon matrix element of the general form

$$M \equiv M(\mathbf{e}',\mathbf{e},\mathscr{E}) = \langle \psi_{\mathbf{p}'}^{(-)}|(\mathbf{e}'\hat{\mathbf{p}})G_{\mathscr{E}}(\mathbf{e}\hat{\mathbf{p}})|\psi_{\mathbf{p}}^{(+)}\rangle, \tag{13}$$

with the Coulomb Green's function $G_{\mathscr{E}}$. The result of $\hat{\mathbf{p}}$ operating on the Coulomb functions (12) can be written as follows:

$$(\mathbf{e}\hat{\mathbf{p}})\psi_{\mathbf{p}}^{(+)} = (\mathbf{e}\mathbf{p})\psi_{\mathbf{p}}^{(+)} + \left(i\mathbf{e}\frac{\partial}{\partial\mathbf{q}}\right)\varphi^{(+)}\Big|_{\mathbf{q}=\mathbf{p}}, \tag{14}$$

where

$$\varphi^{(+)} = A_{\mathbf{p}}^{(+)}\frac{p}{r}\exp(ipq)\Phi(-ia,1;-i(qr-\mathbf{q}\mathbf{r})).$$

Here the variables $\mathbf{q}$ and $\mathbf{p}$ are assumed to be independently differentiable, and then we must put $\mathbf{q}=\mathbf{p}$. Note that the transformation (14) is essentially similar to that used earlier in bremsstrahlung theory,[1,15] and proves to be convenient for analytic calculations of matrix elements not only of the first order but also of the second.

Substituting (14) and a similar expression with $\psi^{(-)}$ into (13) and applying the operator $G_{\mathscr{E}}=(H_0-\mathscr{E})^{-1}$ to $\psi^{(\pm)}$, which are the eigenvectors of H_0, we arrive at the following expression for M:

$$M = M_1 + M_2 + M_3,$$

$$M_1 = \frac{1}{p'^2/2-\mathscr{E}}(\mathbf{e}'\mathbf{p}')\left(i\mathbf{e}\frac{\partial}{\partial\mathbf{q}}\right)\langle\psi^{(-)}|\varphi^{(+)}\rangle,$$

$$M_2 = \frac{1}{p^2/2-\mathscr{E}}(\mathbf{e}\mathbf{p})\left(i\mathbf{e}'\frac{\partial}{\partial\mathbf{q}'}\right)\langle\varphi^{(-)}|\psi^{(+)}\rangle, \tag{15}$$

$$M_3 = -\left(\mathbf{e}'\frac{\partial}{\partial\mathbf{q}'}\right)\left(\mathbf{e}\frac{\partial}{\partial\mathbf{q}}\right)\langle\varphi^{(-)}|G|\varphi^{(+)}\rangle.$$

It is convenient to calculate M_1, M_2, and M_3 using parabolic coordinates $\{\xi,\eta,\varphi\}$, in which the vectors $\mathbf{p}$ and $\mathbf{q}$ are directed along the z axis and the vectors $\mathbf{p}'$ and $\mathbf{q}'$ lie in the xz plane. We write the Green's function as a series expansion in the eigenfunctions of the operator $\hat{L}_z$:

$$G_{\mathscr{E}}(\xi,\eta,\varphi;\xi',\eta',\varphi') = \frac{1}{2\pi}\sum_m G_{\mathscr{E}}^m(\xi,\eta;\xi',\eta')\exp[im(\varphi-\varphi')]. \tag{16}$$

In the chosen system of coordinates $\varphi^{(+)}$ is independent of φ, with the result that terms with $m\neq 0$ in (16) vanish after integration with respect to φ. For $G_{\mathscr{E}}^{m=0}$ we use an integral representation obtained on the basis of the results of Ref. 16:

$$G_{\mathscr{E}}^{m=0} = \frac{1}{2\pi\nu}\int_0^1 dx\frac{2x^{-Z\nu}}{(1-x)^2} \times\exp\left(-\frac{\xi+\xi'+\eta+\eta'}{2\nu}\frac{(1+x)}{(1-x)}\right) \times I_0\left(\frac{2}{\nu(1-x)}\sqrt{x\xi\xi'}\right)I_0\left(\frac{2}{\nu(1-x)}\sqrt{x\eta\eta'}\right), \tag{17}$$

where $\nu=1/\sqrt{-2\mathscr{E}}$, with $\nu=i|\nu|$ for $\mathscr{E}>0$, and I_0 is the modified Bessel function.

The calculation of M in Appendix A yields the following results:

$$M_1 = ZA_{\mathbf{p}'}^{(-)*}A_{\mathbf{p}}^{(+)}\frac{16\pi|\nu|^2}{1-p'^2|\nu|^2}(\mathbf{e}'\mathbf{n}')p' \times\frac{(p-p'+i0)^{ia'-1}(p'-p+i0)^{ia-1}}{(p+p')^{ia+ia'+1}} \times\left[(\mathbf{e}\mathbf{n})p\frac{1-ia}{p-p'}\,{}_2F_1(2-ia,1-ia';2;\lambda_0) + (\mathbf{e}\mathbf{n}')p'\frac{1-ia'}{p'-p}\,{}_2F_1(1-ia,2-ia';2;\lambda_0)\right], \tag{18}$$

where

$$\mathbf{n}=\frac{\mathbf{p}}{p},\quad \mathbf{n}'=\frac{\mathbf{p}'}{p'},\quad \lambda_0=-\frac{2pp'}{(p-p')^2}(1-\cos\vartheta),$$

and ${}_2F_1(a,b;c;x)$ is the hypergeometric function.

The term M_2 is obtained from (18) by interchanging primed and unprimed quantities (except for the normalizing factors $A_{\mathbf{p}'}^{(-)*}$ and $A_{\mathbf{p}}^{(+)}$):

$$M_2=M_1(\mathbf{p}\leftrightarrows\mathbf{p}',\mathbf{e}\leftrightarrows\mathbf{e}').$$

The term M_3 can be expressed in terms of integrals of ${}_2F_1$:

$$M_3=-i64\pi Z^2A_{\mathbf{p}'}^{(-)*}A_{\mathbf{p}}^{(+)}|\nu|^5$$
$$\times\int_0^1 dx x^{-Z\nu}A^{ia-1}B^{ia'-1}C^{-ia-ia'-2}$$
$$\times\{(\mathbf{e}'\mathbf{n}')(\mathbf{e}\mathbf{n})\alpha\beta_2F_1(1-ia,1-ia';1;\lambda)$$
$$+(\mathbf{e}',\mathbf{n}'-\mathbf{n})(\mathbf{e}\mathbf{n})\frac{4p|\nu|\beta x}{A}(1-ia)$$
$$\times{}_2F_1(2-ia,1-ia';2;\lambda)$$
$$+(\mathbf{e}'\mathbf{n}')(\mathbf{e},\mathbf{n}-\mathbf{n}')\frac{4p'|\nu|\alpha x}{B}(1-ia')$$
$$\times{}_2F_1(1-ia,2-ia';2;\lambda)$$
$$+[(\mathbf{e}',\mathbf{n}')(\mathbf{e}\mathbf{n})-(\mathbf{e}'\mathbf{e})]2x_2F_1(1-ia,1-ia';2;\lambda)$$
$$+(\mathbf{e}',\mathbf{n}'-\mathbf{n})(\mathbf{e},\mathbf{n}-\mathbf{n}')\frac{8x^2pp'|\nu|^2}{AB}(1-ia)(1-ia')$$
$$\times{}_2F_1(2-ia,2-ia';3;\lambda)\}. \tag{19}$$

Above we have employed the following notation:

$$\lambda=\frac{8xpp'|\nu|^2}{AB}(1-\cos\vartheta),$$
$$A=(1-p|\nu|+i0)(1+p'|\nu|+i0)$$
$$-x(1+p|\nu|-i0)(1-p'|\nu|-i0),$$
$$B=(1+p|\nu|+i0)(1-p'|\nu|+i0)$$
$$-x(1-p|\nu|-i0)(1+p'|\nu|-i0), \tag{20}$$
$$C=(1+p|\nu|)(1+p'|\nu|)$$
$$-x(1-p|\nu|)(1-p'|\nu|),$$
$$\alpha=1+p|\nu|+x(1-p|\nu|),\quad \beta=1+p'|\nu|+x(1-p'|\nu|).$$

Equations (18)–(20) are written on the assumption that the intermediate-state energy is positive: $\mathscr{E}>0$. For $\mathscr{E}<0$ we have $\nu=1/\sqrt{-2\mathscr{E}}$, which is real, so that in Eqs. (18)–(20) we must substitute $-i\nu$ for $|\nu|$. The infinitesimal imaginary terms, which indent the integration contour about singular points, result from the regularization of integrals with oscillating functions in (9), a process achieved by the substitution

$$G_{\mathscr{E}}(\mathbf{r},\mathbf{r}')\rightarrow\exp[-\varepsilon(r+r')]G_{\mathscr{E}}(\mathbf{r},\mathbf{r}'),\quad \varepsilon>0,$$

The above formulas are highly symmetric in relation to the momenta $\mathbf{p}$ and $\mathbf{p}'$ and fully determine the matrix element of a two-photon transition in the continuous spectrum with $p'\neq p$. They can be used, in particular, to calculate the cross sections of double bremsstrahlung and inverse bremsstrahlung. Note that to within a multiplicative factor, the terms M_1 and M_2 coincide with the amplitudes of ordinary bremsstrahlung involving photons with polarizations $\mathbf{e}$ and $\mathbf{e}'$, respectively, written in symmetric form in relation to the variables $\mathbf{p}$ and $\mathbf{p}'$ (a similar expression for one-photon amplitudes can be found in Gavrila's review[2]. The matrix elements M_1 and M_2 contain the Born limit of M, since the integral term M_3, as Eq. (19) shows, is proportional to Z^2. Interestingly, the two-photon amplitude contains integrals of hypergeometric functions with the same parameters as in the bremsstrahlung amplitude. The two-photon transition amplitude between the continuous-spectrum states with fixed orbital angular momenta l and $l'=l,\ l\pm2$ has a similar form.[17] Gavrila *et al.*[6] and Florescu and Djamo[7] calculated the matrix element (13) using the double bremsstrahlung and inverse bremsstrahlung data with a different technique (involving the momentum representation of the Green's function). Their results are also expressed in terms of integrals like (19), but have a more complicated structure, e.g., do not contain integral-free terms like M_1 and M_2.

3.2. Removing singularities from the elastic scattering amplitude

Sending p' to p in Eqs. (18) and (19), we can easily see that M_1, M_2, and M_3 diverge, whereas the final expression in (10) for $d\sigma_{\mathrm{corr}}$ must remain finite. To resolve the indeterminacy, one must isolate the divergent terms in M_1, M_2, and M_3 explicitly. For the integral-free terms M_1 and M_2 this is easily done by expanding the hypergeometric function ${}_2F_1$ in a series in the inverse powers of the argument. As a result we arrive at the following expression for the singular part:

$$(M_1+M_2)_{\mathrm{sing}}$$
$$=C_1\frac{|\nu|^2}{1-p^2|\nu|^2}[(\mathbf{e}'\mathbf{n}')(\mathbf{e}\mathbf{n}')-(\mathbf{e}'\mathbf{n})(\mathbf{e}\mathbf{n})]$$
$$\times\frac{1}{p-p'}+C_2\frac{|\nu|^2}{1-p^2|\nu|^2}[2(\mathbf{e}'\mathbf{n}')(\mathbf{e}\mathbf{n})$$
$$+(\mathbf{e}'\mathbf{n}')(\mathbf{e}\mathbf{n}')+(\mathbf{e}'\mathbf{n})(\mathbf{e}\mathbf{n})]\ln(-\lambda_0), \tag{21}$$

where

$$C_{1,2}=C_{1,2}(Z,p,\vartheta).$$

Isolating the singularities in M_3, which is done in Appendix B, requires more subtle transformations and leads to the following expression:

$$(M_3)_{\mathrm{sing}}=C_2\frac{|\nu|^2}{1-p^2|\nu|^2}[-2(\mathbf{e}'\mathbf{n}')(\mathbf{e}\mathbf{n})$$
$$+(\mathbf{e}'\mathbf{n}')(\mathbf{e}\mathbf{n}')+(\mathbf{e}'\mathbf{n})(\mathbf{e}\mathbf{n})]\ln(-\lambda_0). \tag{22}$$

Thus, the logarithmic divergences in M_1, M_2, and M_3 are partially compensated for, and the remaining singularities cancel out when the two matrix elements are added in the amplitude (9):

$$M(\mathbf{e}^*,\mathbf{e},p^2/2+\omega)+M(\mathbf{e},\mathbf{e}^*,p^2/2-\omega'), \quad \omega'\to\omega.$$

The final expression for the correction term in (10) has the form

$$d\sigma_{\rm corr}=-d\sigma_{\rm R}\frac{F^2p^2}{2\omega^4}|\mathbf{e},(\mathbf{n}-\mathbf{n}')|^2 + 2\,\mathrm{Re}[f_0^*(h_2^{(+)}+h_2^{(-)})]d\Omega, \tag{23}$$

where

$$h_2^{(-)}=h_2^{(+)}(\omega\to-\omega,\mathbf{e}\to\mathbf{e}^*),$$

$$h_2^{(+)}=(\mathbf{e}^*\mathbf{n}')(\mathbf{en})N_1+[(\mathbf{e}^*,\mathbf{n}'-\mathbf{n})](\mathbf{en}) + (\mathbf{e}^*\mathbf{n}')(\mathbf{e},\mathbf{n}-\mathbf{n}')]N_2 + [1-(\mathbf{e}^*\mathbf{n}')(\mathbf{en})]N_3+|\mathbf{e},\mathbf{n}-\mathbf{n}'|^2N_4,$$

$$N_1=K\int_0^1 dx x^{-Z\nu}A^{2ia-2}C^{-2ia-2}\alpha^2{}_2F_1 \times(1-ia,1-ia;1;\lambda),$$

$$N_2=-\frac{2p|\nu|}{1-p^2|\nu|^2}K\int_0^1 dx x^{-Z\nu}A^{2ia-2}C^{-2ia-2} \times\Bigg[\alpha(1-ia)_2F_1(1-ia,2-ia;2;\lambda) +(1-p|\nu|)\left[x+ia\alpha+\frac{2x\alpha(1+ia)}{C}(1-p|\nu|)\right] \times{}_2F_1(1-ia,1-ia;2;\lambda)\Bigg],$$

$$N_3=-2K\int_0^1 dx x^{1-Z\nu}A^{2ia-2}C^{-2ia-2} \times{}_2F_1(1-ia,1-ia;2;\lambda),$$

$$N_4=-8(1-ia)^2p^2|\nu|^2K\int_0^1 dx x^{2-Z\nu}A^{2ia-4}C^{-2ia-2} \times{}_2F_1(2-ia,2-ia;3;\lambda).$$

Here

$$K=-8iF^2\omega^{-2}Z^2|\nu|^5\exp(\pi a)[\Gamma(1-ia)]^2,$$

and A, C, α, and λ are defined in (20), where we must set $p'=p$. In this case

$$B=A=(1-p^2|\nu|^2)(1-x)+i0, \qquad \lambda=\frac{8xp^2|\nu|^2}{A^2}(1-\cos\vartheta). \tag{25}$$

The functions ${}_2F_1$ in the integrals in (24) have a branch point at $\lambda=1$, which lies on the integration path. The choice of the necessary analytic branch is determined by the imaginary indentation term in (25).

Note that the first (integral-free) term in (23) is exactly the correction to $d\sigma_{\rm R}$ in (5) and hence determines the Born and low-frequency limits of $d\sigma_{\rm corr}$.

3.3. Short-range potential

To examine the emergence and removal of divergences in the elastic process, we take a simple example and study the correction to the cross section of elastic scattering by a delta-like potential. As is known, the amplitude f_0 in (8) in this case is given by the following relationship:

$$f_0=-\frac{1}{\kappa+ip},$$

where $\kappa=\sqrt{-2E_0}$, and E_0 is the energy of the single bound state in the delta-like well. The matrix element $M=M(\mathbf{e}',\mathbf{e},\mathcal{E})$ in (13) can be expressed in terms of elementary functions as follows:[18]

$$\frac{M}{8\pi}=-\frac{(\mathbf{e}'\mathbf{p}')(\mathbf{ep})}{\kappa+i\sqrt{2\mathcal{E}}}\frac{1}{2\mathcal{E}-p^2}\frac{1}{2\mathcal{E}-p'^2} -\frac{(\mathbf{e}'\mathbf{p}')(\mathbf{ep}')}{\kappa+ip}\frac{1}{2\mathcal{E}-p'^2}\frac{1}{p^2-p'^2} +\frac{(\mathbf{e}'\mathbf{p})(\mathbf{ep})}{\kappa+ip'}\frac{1}{2\mathcal{E}-p^2}\frac{1}{p^2-p'^2} +\frac{i}{3}\frac{(\mathbf{e}'\mathbf{e})}{(\kappa+ip)(\kappa+ip')}\left(-\frac{p^3}{(2\mathcal{E}-p^2)(p^2-p'^2)} +\frac{p'^3}{(2\mathcal{E}-p'^2)(p^2-p'^2)}+\frac{(2\mathcal{E})^{3/2}}{(2\mathcal{E}-p^2)(2\mathcal{E}-p'^2)}\right),$$

so one can easily establish that divergences $\sim 1/(p'-p)$ appear as $p'\to p$ (in contrast to the Coulomb case, no logarithmic singularities appear here), and that these singularities disappear when the two diagrams of Figs. 1(a) and (b) are added. The final expression for the cross section (10) is

$$d\sigma_0=\frac{1}{\kappa^2+p^2}d\Omega,$$

$$d\sigma_{\rm corr}=-d\sigma_0\frac{F^2p^2}{2\omega^4}\mathrm{Re}\Bigg[(\mathbf{e}^*\mathbf{n})(\mathbf{en})+(\mathbf{e}^*\mathbf{n}')(\mathbf{en}') -(\mathbf{e}^*\mathbf{n}')(\mathbf{en})\frac{\kappa+ip}{\kappa+i(p^2+2\omega)^{1/2}} -(\mathbf{e}^*\mathbf{n})(\mathbf{en}')\frac{\kappa+ip}{\kappa+i(p^2-2\omega)^{1/2}} +\frac{i}{3p^2(\kappa+ip)}((p^2+2\omega)^{3/2}+(p^2-2\omega)^{3/2} -2p^3)\Bigg]. \tag{26}$$

4. POLARIZATION DEPENDENCE OF THE CROSS SECTION

We start with the polarization dependence of the matrix element of a two-photon transition of the general type (13). Since $M(\mathbf{e}',\mathbf{e},\mathcal{E})$ is a linear function of the polarization vectors $\mathbf{e}$ and $\mathbf{e}'$, the amplitude can obviously be written as a sum of products of linearly independent combinations of vectors $\mathbf{e}$, $\mathbf{e}'$ and $\mathbf{n}$, $\mathbf{n}'$ with invariant amplitudes that depend solely on p, p', ϑ, and ω. For the independent combinations of $\mathbf{e}$, $\mathbf{e}'$ and $\mathbf{n}$, $\mathbf{n}'$ we can take, for example, the following five combinations:

$$\Phi_1=(\mathbf{ee})', \quad \Phi_2=(\mathbf{en})(\mathbf{e}'\mathbf{n}'), \quad \Phi_3=(\mathbf{en}')(\mathbf{e}'\mathbf{n}),$$

$$\Phi_4=(\mathbf{en})(\mathbf{e}'\mathbf{n}), \quad \Phi_5=(\mathbf{en}')(\mathbf{e}'\mathbf{n}'),$$

which enter into the matrix element (18), (19) for the case of a Coulomb field. In the general case of distinct photons $\mathbf{e}$ and $\mathbf{e}'$, all the above combinations are complex-valued, so that the inelastic scattering cross section exhibits complicated polarization and angular structure.

For elastic scattering, where $\mathbf{e}'=\mathbf{e}^*$, we have $\Phi_1=1$ and $\Phi_2=\Phi_3^*$, while the coefficients of the real-valued combinations Φ_4 and Φ_5 must be equal because of the invariance of the amplitude under time reversal, i.e., under the following substitutions:

$$\mathbf{n}\to-\mathbf{n}', \quad \mathbf{n}'\to-\mathbf{n}, \quad \mathbf{e}\to\mathbf{e}^*.$$

As a result, the polarization dependence of the amplitude f_2 and the cross section $d\sigma_{\text{corr}}$ of (10) are determined by four terms, and $d\sigma_{\text{corr}}/d\Omega$ for scattering by an arbitrary potential $U(r)$ can be written as

$$d\sigma_{\text{corr}}/d\Omega=Q_1+Q_2|\mathbf{e},\mathbf{n}-\mathbf{n}'|^2+Q_3\,\text{Re}[(\mathbf{en})(\mathbf{e}^*\mathbf{n}')] + Q_4\,\text{Im}[(\mathbf{en})(\mathbf{e}^*\mathbf{n}')], \tag{27}$$

where the Q_i for the case of Coulomb scattering can be expressed in an obvious way in terms of f_0 and the coefficients N_i in (23) and (24).

The easily verifiable relationship

$$\text{Im}\,[(\mathbf{en})(\mathbf{e}^*\mathbf{n}')]=-\frac{1}{2}\xi\left[\frac{\mathbf{k}}{|\mathbf{k}|}(\mathbf{nn}')\right], \tag{28}$$

where $\mathbf{k}$ is the photon's wave vector, shows that the last term on the right-hand side of (27) leads to a dependence of the cross section on the sign of ξ, the degree of the wave's circular polarization:

$$\xi=i\frac{\mathbf{k}}{|\mathbf{k}|}[\mathbf{ee}^*], \quad -1\leqslant\xi\leqslant1.$$

The effect is at its maximum for scattering through an angle of $\frac{1}{2}\pi$ and propagation of light at right angles to the scattering plane. The difference in the scattering cross sections for light with left- and right-circular polarizations,

$$\Delta\left(\frac{d\sigma}{d\Omega}\right)\equiv\frac{d\sigma(\xi=+1)}{d\Omega}-\frac{d\sigma(\xi=-1)}{d\Omega} = -Q_4\left(\frac{\mathbf{k}}{|\mathbf{k}|}\,[\mathbf{nn}']\right), \tag{29}$$

is determined by a distinctive interference of the amplitudes f_0 and f_2:

$$Q_4=2\,\text{Re}(f_0^*)\,\text{Re}(P_4)-2\,\text{Im}(f_0^*)\,\text{Im}(P_4),$$

where P_4 is the coefficient of $\text{Im}\{(\mathbf{e}\cdot\mathbf{n})(\mathbf{e}^*\cdot\mathbf{n}')\}$ in the expansion of f_2 similar to (27). For Coulomb scattering, P_4 can be expressed in terms of the coefficients N_i in (24):

$$P_4=P_4^{(+)}-P_4^{(-)},$$

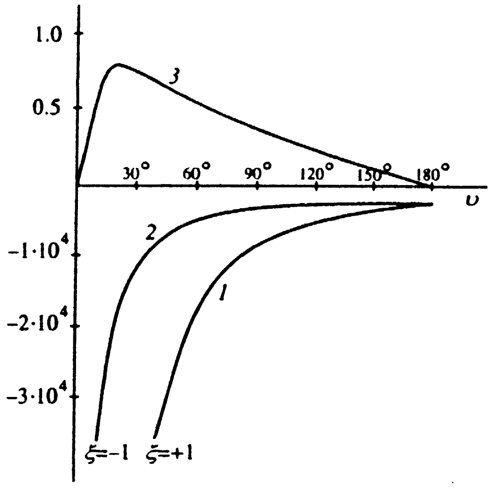

FIG. 2. Angular dependence of $F^{-2}Z^8(d\sigma/d\Omega)$ for right- ($\xi=1$) and left- ($\xi=-1$) circularly polarized light waves (curves *1* and *2*), and of the asymmetry ratio $(d\sigma(\xi=+1)-d\sigma(\xi=-1))/(d\sigma(\xi=+1)+d\sigma(\xi=-1))$ (curve *3*) at $\omega/p^2=0.1$ and $Z/p=1$. The light propagates at right angles to the electron scattering plane.

$$P_4^{(+)}=N_1+2N_2-N_3, \quad P_4^{(-)}=P_4^{(+)}(\omega\to-\omega).$$

An illustration of the numerical value of $\Delta(d\sigma/d\Omega)$ in given in Fig. 2, which depicts the dependence of $d\sigma_{\text{corr}}/d\Omega$ on the angle ϑ for left- and right-circular polarization of the light wave, as well as the angular dependence of the ratio

$$\frac{d\sigma(\xi=+1)-d\sigma(\xi=-1)}{d\sigma(\xi=+1)+d\sigma(\xi=-1)}.$$

For the selected values of the parameters, $\omega/p^2=0.1$ and $Z/p=1$, the "asymmetry" of the correction term $d\sigma_{\text{corr}}/d\Omega$ is perfectly clear.

Asymmetry of the cross section similar to (29) also emerges in other collisions in the presence of a light wave and, we believe, is of interest for experimental observation. Note that this effect was not discussed in the work on SBIB cited above, since it is absent in the first Born and low-frequency approximations. The reason is that the combination of vectors (28) involves the vectors $\mathbf{p}_1$ and $\mathbf{p}_2$, while in these approximations the amplitude depends only on the momentum transfer $\mathbf{q}=\mathbf{p}_2-\mathbf{p}_1$. Formally this follows from the invariance of the cross section calculated in the first Born approximation under reversal of the signs of electron and photon momenta (see Ref. 1, §93).

The expression for $\Delta(d\sigma/d\Omega)$ has an especially simple form in the case of scattering by a short-range potential, which follows from Eqs. (26) and (27):

$$\Delta\left(\frac{d\sigma}{d\Omega}\right) = d\sigma_0\frac{F^2p^2}{4\omega^4}\text{Im}\left(\frac{\kappa+ip}{\kappa+i(p^2+2\omega)^{1/2}}-\frac{\kappa+ip}{\kappa+i(p^2-2\omega)^{1/2}}\right) \times\xi\left(\frac{\mathbf{k}}{|\mathbf{k}|}\,[\mathbf{nn}']\right).$$

Note the complicated frequency dependence of $\Delta(d\sigma/d\Omega)$. Specifically, $\Delta(\omega)$ has a break at $\omega = \frac{1}{2}p^2$ because the energy of the Green's function in the second term in (9) becomes negative and the "virtual recombination" channel opens. In a certain sense this is similar to the situation in which the amplitude of light scattering on a bound state at above-threshold frequencies, when the one-photon ionization channel opens, acquires an imaginary part. Also, the cross section (26) increases drastically as the frequency ω approaches $\omega_{\rm rec}=|E_0|+\frac{1}{2}12p^2$, which corresponds to electron recombination into a bound state with energy E_0. At $\omega=\omega_{\rm rec}$ the cross section (26) has a pole, whose removal requires introducing a ground-state width, which determines the probability of photoemission.

The vector combination (28) also enters into the cross section of ordinary bremsstrahlung and describes the emission of a circularly polarized photon by an unpolarized electron. It is not forbidden by general symmetry considerations, and in this connection is discussed in Ref. 1, §93, but we do not know of any quantitative calculations of the effect, although they could be done on the basis of the known results of the theory of bremsstrahlung in a Coulomb field (see Ref. 1, §92). In particular, if under conditions in which the Born approximation is valid we keep only the first two Born terms, the differential cross section for emission in the frequency interval $d\omega$ has the form

$$\frac{d\sigma}{d\Omega_{\mathbf{p}'}d\Omega_{\mathbf{k}}d\omega} = \frac{\alpha(\alpha Z)^2}{\pi^2\omega}\frac{p'}{p}\frac{1}{(\mathbf{p}'-\mathbf{p})^4} \times \exp\{\pi(a'-a)\}\left\{|\mathbf{e},\mathbf{p}-\mathbf{p}'|^2+Z\omega\xi\left(\frac{\mathbf{k}}{|\mathbf{k}|}[\mathbf{pp}']\right) \times\frac{(\mathbf{p}-\mathbf{p}')^2}{(pp')^2(p+p')(1-\cos\vartheta)}\ln\frac{(\mathbf{p}-\mathbf{p}')^2}{(p-p')^2}\right\}, \tag{30}$$

where

$$p'^2/2 = p^2/2-\omega. \tag{31}$$

The term with ξ comes from the interference of the first and second Born amplitudes. Equation (30) also demonstrates the smallness (of order ω) of the "asymmetric" term in the low-frequency limit $\omega\to 0$. SBIB in the scattering of electrons with an energy of approximately 11 eV by argon atoms was first observed by Andrick and Langhaus,[19] who used linearly polarized radiation from a CO_2 laser. Under such conditions the Born approximation is invalid, and as estimates have shown, the difference between the cross sections with left- and right-circular polarized light can be observed experimentally.

5. ANALYSIS OF LIMITING CASES AND NUMERICAL RESULTS

In analyzing the dependence of the Q_i in (27) on Z, p, ω, and ϑ, it is convenient to use the dimensionless quantities Z, $a=Z/p$, ω^2/p^2, and $\varepsilon=\sin^2(\vartheta/2)$. Then from (23) it follows that

$$Q_i=Z^{-8}\tilde{Q}_i(a,\omega/p^2,\varepsilon), \tag{32}$$

with no further factorization of variables possible. It is interesting to compare the dependence on the variables in (32) with the results of approximate calculations.[3–5] In the Born region, $d\sigma_{\rm corr}$ can be expressed as a power-law function of the foregoing parameters,

$$\tilde{Q}_i\sim a^{10}\varepsilon^{-2}(\omega/p^2)^{-4}$$

[see Eq. (33)], while the semiclassical approximation[5] gives for $\tilde{Q}_i$ an expression of the self-similar type,

$$\tilde{Q}_i=a^{14}f_i(a\omega/p^2,\varepsilon).$$

The complicated structure of (23) does not, in particular, enable one to approximate $d\sigma_{\rm corr}$ by a simple expression over a broad range of the variable values.

For the same reason, finding the limits for $d\sigma_{\rm corr}$ is a cumbersome problem, requiring fairly intricate asymptotic estimates in the different parameter ranges. Below we examine some of the limiting domains of parameter values in which $d\sigma_{\rm corr}$ becomes considerably simpler.

5.1. Low frequencies ($\omega/p^2\ \sin^2(\vartheta/2)\ll 1$)

The leading term in the series expansion of $d\sigma_{\rm corr}$ in powers of ω is specified by the Kroll–Watson low-frequency asymptotic behavior:

$$\frac{d\sigma_{\rm corr}}{d\Omega}=-\frac{F^2a^2}{8\omega^4\varepsilon^2}|\mathbf{e},\mathbf{n}'-\mathbf{n}|^2. \tag{33}$$

Since the accuracy of the Kroll–Watson approximation has long been under discussion (see, e.g., Ref. 2), we give expressions for Q_i that determine, according to (27), not only the leading term (33) but also two corrections to this term:

$$Q_2=-B\left[1+\pi\frac{a\omega}{p^2}+\left(\frac{\pi^2}{3}-\frac{1-L}{2\varepsilon}\right)\frac{a^2\omega^2}{p^4}\right],$$

$$Q_3=B\left(\ln^2\frac{\omega^2}{p^4\varepsilon}-2L\ \ln\frac{\omega^2}{p^4\varepsilon}-4(\gamma-2\ \ln 2)^2 -4\ \mathrm{Re}\psi(ia)[\ \mathrm{Re}\psi(ia)+2\gamma-2\ \ln 2]\right)\frac{a^2\omega^2}{p^4},$$

$$Q_4=-2BL\frac{a\omega}{p^2}\left(1+\pi\frac{a\omega}{p^2}\right), \tag{34}$$

where

$$B=\frac{F^2a^2}{8\omega^4\sin^4(\vartheta/2)}$$

is the Born factor,

$$L=\left[\ln\frac{\omega^2}{p^4\varepsilon}-2\ \ln 2+2\gamma+2\ \mathrm{Re}\psi(ia)\right],$$

$\psi(x)=d(\ln\Gamma(x))/dx$ is the digamma function, and γ=0.57721... is Euler's constant.

The term with Q_1 in (27) has a higher order of smallness,

$$Q_1\sim B\frac{a^3\omega^3}{p^6}\left(\ln\frac{\omega^2}{p^4\varepsilon}+\mathrm{const}\right),$$

and contributes nothing to the cross section in this approximation.

The above results show that the expansion of the Q_i is in the parameters $a\omega/p^2$, $a\omega/p^2\varepsilon$, and $(a\omega/p^2)\ln(a\omega/p^2\varepsilon)$, whose values determine the accuracy of the Kroll–Watson approximation for a Coulomb potential. Note that no logarithmic terms appear in the corrections if one deals with a short-range potential. For instance, in the second order,

$$d\sigma_{\rm corr}=-d\sigma_0\frac{F^2p^2}{2\omega^4}\left(|\mathbf{e},\mathbf{n}'-\mathbf{n}|^2-\frac{2\omega}{\kappa^2+p^2}\,\frac{\kappa}{p}\,\mathrm{Im}\{(\mathbf{en})(\mathbf{e}^*\mathbf{n}')\}\right).$$

5.2. High momenta ($a \ll 1$)

The leading term in the asymptotic behavior is again (33). That the leading terms $d\sigma_{\rm corr}$ in the low-frequency and Born limits coincide is characteristic only of the Coulomb field and can be explained by the fact that the Born approximation yields an expression for the Rutherford scattering cross section coinciding with the exact expression. Using (26), one can easily verify that in the case of a delta-like potential the expressions for $d\sigma_{\rm corr}$ in these regions differ. The terms with integrals in (24) are of order Z^3 and determine the correction to the first Born approximation. If in the integrands we put $Z=0$, the integrals can easily be expressed in terms of elementary functions, but the results prove to be very cumbersome. Hence, to simplify our discussion we examine several special cases where the corrections to the first Born approximation are simple.

5.2.1. Low frequencies $[\omega/p^2\sin^2(\vartheta/2)\ll 1]$

$$Q_2=-B\left(1+\frac{\pi a\omega}{p^2}\right),\quad Q_4=-2B\frac{a\omega}{p^2}\ln\left(\frac{\omega^2}{4p^4\varepsilon}\right).$$

This coincides with (34) if in the latter we leave only the first-order correction in ω/p^2 and pass to the limit $a\ll 1$.

5.2.2. Low frequencies ($\omega/p^2\ll 1$) *and small angles* ($\vartheta/(\omega/p^2)\ll 1$)

$$Q_2=-F^2\frac{2a^2}{\omega^4\vartheta^4}\left(1+\frac{a\vartheta}{2}\right),\quad Q_4=F^2\frac{2a^3p^2}{\omega^5\vartheta^2}.$$

To determine the contribution of these terms to the cross section (27) we must bear in mind that for small angles, $|\mathbf{e}(\mathbf{n}-\mathbf{n}')|^2\sim\vartheta^2$ and $\mathrm{Im}\{(\mathbf{en})(\mathbf{e}^*\mathbf{n}')\}\sim\vartheta$.

5.2.3. High frequencies ($\omega/p^2\gg 1$)*, small angles* ($\vartheta\ll 1$)*, and* $a\ln\vartheta\ll 1$

$$Q_2=-F^2\frac{2a^2}{\omega^4\vartheta^4},\quad Q_1=-F^2\frac{3\pi a^3}{2\omega^4\vartheta^2},$$

$$Q_3=F^2\frac{9\pi a^3}{2\omega^4\vartheta^2}.$$

In the above limits the correction $d\sigma_{\rm corr}$, as $\vartheta\to 0$, has a smaller singularity than the Rutherford cross section $d\sigma_{\rm R}$. The situation is the same for arbitrary values of the parameters a and ω/p^2.

5.3. High frequencies ($\omega/p^2 2\gg 1$)

The expression for the Q_i has the form $Q_i=F^2R_i/\omega^4$, but the functions $R_i(Z,p,\vartheta)$ are still complicated. Note that the dependence on ω is the same as in the low-frequency and Born limits, but the terms Q_1 and Q_3, together with Q_2, contribute to the cross section (27). In this limit the asymmetric term with Q_4 is absent.

5.4. Resonant frequencies

If the photon energy is higher than the electron energy and

$$\frac{p^2}{2}-\omega=-\frac{Z^2}{2n^2},\quad n=1,\,2,\ldots,$$

$d\sigma_{\rm corr}$ contains resonances caused by virtual recombination and ionization processes. The resonant structure of the cross section (capture–escape resonances) was studied by Dimou and Faisal[11] by numerical methods.

In the resonant frequency range $1\leq Z\nu<\infty$, the integrals in (24) diverge at $x=0$ and must be transformed into contour integrals by the substitution

$$\int_0^1 x^{-Z\nu}f(x)dx=\frac{1}{\exp(-2i\pi Z\nu)-1}\int_1^{(0+)}x^{-Z\nu}f(x)dx. \tag{35}$$

The resonance terms can be found from (24) and (35) by the residue theorem. For instance, at $n=1$ we have

$$d\sigma_{\rm corr}/d\Omega=F^2\frac{2^6\pi a^{13}}{Z^6(1+a^2)^4}\,\frac{\exp(-4a\,\mathrm{arccot}\,a)}{1-\exp(-2\pi a)}\times\mathrm{Re}\left(\frac{(\sin^2(\vartheta/2))^{-1-ia}}{(1-ia)^2}\,\frac{(\mathbf{en}')(\mathbf{e}^*\mathbf{n})}{\Delta}\right),$$

where $\Delta=\omega-Z^2/2-p^2/2$. The same result follows from the general expression (10) for the cross section if in the amplitude (9) we leave only the resonance term

$$\frac{d\sigma_{\rm corr}}{d\Omega}=2\mathrm{Re}\left\{\frac{1}{\Delta}f_0^*\frac{1}{2\pi}\langle\mathbf{p}'^{(-)}|V^{(+)}|1s\rangle\times\langle 1s|V^{(-)}|\mathbf{p}^{(+)}\rangle\right\}.$$

When $\Delta\to 0$, we must substitute $\Delta-i\Gamma/2$ for Δ, where Γ is the width of ground-state level (the ionization width associated with the possibility of photoemission).

For arbitrary values of p, ϑ, and ω, the integrals N_i with hypergeometric functions in (24) can quite easily be evaluated numerically. Calculations show a fairly strong dependence of $d\sigma_{\rm corr}/d\Omega$ on the variables Z/p and ω/p^2. For instance, in Figs. 3(a) and 3(b) the values of Z/p are the same, while the values of ω/p^2 differ by a factor of ten. As the results show, when the frequency varies by a factor of ten, the characteristic values of Q_i vary by a factor of 10^3 to 10^4 (the Rutherford cross section does not change in the process). Considered as functions of the scattering angle ϑ, the Q_i must oscillate near the point $\vartheta=0$ owing to the phase factor $[\sin(\vartheta/2)]^{-2ia}$ that appears in the product $f_0^*f_2$ in the small-angle limit. As ϑ varies from 5° to 180°, the Q_i

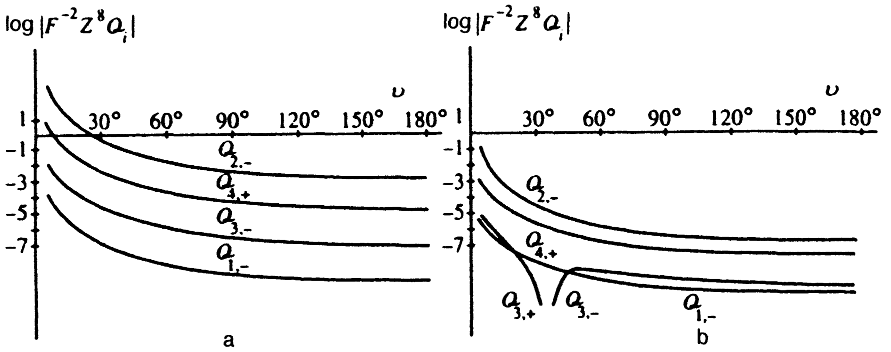

FIG. 3. Angular dependence of $\log|F^{-2}Z^8Q_i|$ for the following values of the parameters: a) $\omega/p^2=0.01$ and $Z/p=0.1$; b) $\omega/p^2=0.1$ and $Z/p=0.1$. The sign of $Q_i(\vartheta)$ is displayed near each curve.

smoothly decrease by several orders of magnitude. Here, since the Q_i are not of fixed sign, they can reverse sign at finite values of ϑ, say, as Q_3 does in Fig. 3(b).

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APPENDIX A

We calculate M_3 of (15), assuming that $\omega<p^2/2$ and $\nu=i|\nu|$. Without altering the notation of Sec. 3, we introduce the change of variables

$|\nu|\mathbf{p}\rightarrow\mathbf{p}$ (and similarly for $\mathbf{p}'$, $\mathbf{q}$, and $\mathbf{q}'$),

$r/|\nu|\rightarrow r$ (and similarly for r', ξ, ξ', η,η')

and drop the normalizing factors. Then in parabolic coordinates $\varphi^{(+)}$ assumes the form

$$\varphi^{(+)}=|\nu|^2\frac{2p}{\xi+\eta}\exp\left(ip\frac{\xi+\eta}{2}\right)\Phi(-ia,1,-iq\eta).$$

For the function $\varphi^{(-)}$ we use the following integral representation:

$$\varphi^{(-)*}=|\nu|^2\frac{2p'}{\xi'+\eta'}\exp\left(ip'\frac{\xi'+\eta'}{2}\right)\frac{1}{\Gamma(1+ia')\Gamma(-ia')}\times\int_0^1 dt\,t^{-1-ia'}(1-t)^{ia'}\exp\left(-i\left(q'\frac{\xi'+\eta'}{2}+q'_\parallel\frac{\xi'-\eta'}{2}+q'_\perp\sqrt{\xi'\eta'}\cos\varphi'\right)t\right). \quad \text{(A1)}$$

Here we have allowed for the fact that in the chosen system of coordinates (see Sec. 3),

$$\mathbf{q}'\mathbf{r}'=q'_\parallel z'+q'_\perp\sqrt{r'^2-z'^2}\cos\varphi'=q'_\parallel\frac{\xi'-\eta'}{2}+q'_\perp\sqrt{\xi'\eta'}\cos\varphi'.$$

In terms of the new variables,

$$M_3=-|\nu|^8\left(\mathbf{e}'\frac{\partial}{\partial\mathbf{q}'}\right)\left(\mathbf{e}\frac{\partial}{\partial\mathbf{q}}\right)\tilde{M}_3,\qquad \tilde{M}_3=\langle\varphi^{(-)}|G|\varphi^{(+)}\rangle. \quad \text{(A2)}$$

By integrating with respect to φ' (in $\tilde{M}_3$ only $\varphi^{(-)}$ depends on the polar angles) via the well-known relationship

$$\int_0^{2\pi}\exp(i\alpha\cos\varphi)d\varphi=2\pi J_0(\alpha),$$

we write $\tilde{M}_3$ as a multiple integral:

$$\tilde{M}_3=A\int_0^1 dx\,x^{-Z\nu}(1-x)^{-2}\int_0^1 dt\,t^{-1-ia'}(1-t)^{ia'}\times\int_0^\infty d\eta\exp\left\{\frac{ik\eta}{2}\right\}\Phi(-ia,1,-iq\eta)\times\int_0^\infty d\eta'\exp\left[i\eta'\frac{k'-(q'-q'_\parallel)t}{2}\right]J_0\left(\frac{2}{1-x}\sqrt{x\eta\eta'}\right)\times\int_0^\infty d\xi'\exp\left[i\xi'\frac{k'-(q'+q'_\parallel)t}{2}\right]J_0(q'_\perp\sqrt{\xi'\eta't})\times\int_0^\infty d\xi\exp\left(\frac{ik\xi}{2}\right)J_0\left(\frac{2}{1-x}\sqrt{x\xi\xi'}\right).$$

Here we have introduced the notation

$$k=p+\frac{1+x}{1-x}+i0,\quad k'=p'+\frac{1+x}{1-x}+i0,$$

$$A=-i\frac{pp'}{|\nu|^5}\frac{\pi}{\Gamma(ia')\Gamma(1-ia')}.$$

The integrals with respect to the parabolic coordinates ξ, ξ', η, and η are evaluated via the following well-known formulas[20]:

$$\int_0^\infty dx\exp(\gamma x)J_0(\alpha\sqrt{x})=-\frac{1}{\gamma}\exp\left\{\frac{\alpha^2}{4\gamma}\right\},\qquad \int_0^\infty dx\exp(\gamma x)\Phi(b,1,-\alpha x)=-\frac{1}{\gamma}\left(1-\frac{\alpha}{\gamma}\right)^{-b}. \quad \text{(A3)}$$

As a result, $\tilde{M}_3$ assumes the form

$$\tilde{M}_3 = 16A\int_0^1 dx\, x^{Z\nu}(1-x)^{-2}s^{-1-ia}\int_0^1 dt\, t^{-1-ia}(1-t)^{ia'}$$
$$\times(s-2q'kt)^{-ia-1}(s(s-2qk')-2t(ksq'-qq'(kk'+s)-qq'_{\parallel}(kk'-s)))^{ia},$$
$$s=kk'-\frac{4x}{(1-x)^2}.$$

The integral with respect to t can be found by using the integral representation of the Appel function $F_1(-ia',-ia,1+ia,1;x,y)$, which can be reduced[21] to ${}_2F_1$:

$$\tilde{M}_3=-16i\pi pp'|\nu|^{-5}\int_0^1 dx\, x^{-Z\nu}(1-x)^{-2}s^{-2-ia-ia'}$$
$$\times(s-2q'k)^{ia'}(s-2qk')^{ia}\,{}_2F_1(-ia,-ia;1;\lambda),$$
$$\lambda=2\frac{(kk'-s)(qq'-\mathbf{qq}')}{(s-2qk')(s-2q'k)}.$$

If in (A2) we differentiate with respect to $\mathbf{q}$ and $\mathbf{q}'$, revert to the original variables, and restore the normalizing factors, we arrive at the expression (19) for M_3. The same approach can be used to calculate M_1. Representing $\psi^{(-)}$ in a way similar to (A1) and integrating via (A2), we arrive at (18). Note that a matrix element like M_1 appears in bremsstrahlung theory and can be calculated by applying methods of contour integration.[15,22]

APPENDIX B

We isolate the singular part in the integral terms in the elastic scattering amplitude. As Eq. (19) shows, the second and third terms in M_3 diverge as $p'\to p$; the sum of these two terms we denote by M'. Since the divergence is logarithmic, we can set $p'=p$ in the integrand everywhere except at A and B:

$$M'=-i\cdot 256\pi p^2|\nu|^6 a^2(1-ia)\int_0^1 dx f(x)$$
$$\times\left(\frac{(\mathbf{e}',\mathbf{n}'-\mathbf{n})(\mathbf{en})}{A}+\frac{(\mathbf{e}'\mathbf{n}')(\mathbf{e},\mathbf{n}-\mathbf{n}')}{B}\right),$$
$$f(x)=x^{1-Z\nu}A^{ia-1}B^{ia-1}C^{-2ia-2}\alpha_2F_1(1-ia,2-ia;2;\lambda).$$

Using the identity

$$\frac{x}{A}+\frac{y}{B}=\frac{x-y}{2}\left(\frac{1}{A}-\frac{1}{B}\right)+\frac{x+y}{2}\left(\frac{1}{A}+\frac{1}{B}\right),$$

we write M' in the form

$$M'=C_-I_-+C_+I_+,$$

where

$$I_\pm=\int_0^1 dx f(x)\left(\frac{1}{A}\pm\frac{1}{B}\right).$$

It can be verified that I_- leads at $p'=p$ to a finite expression, which drops out when the two matrix elements (13) corresponding to the diagrams in Figs. 1(a) and (b) are added. We transform the integrand in I_+ via the following relationship:

$$(1-ia)A^{ia-1}B^{ia-1}F(1-ia,2-ia,2,\lambda)\left(\frac{1}{A}\frac{dA}{dx}+\frac{1}{B}\frac{dB}{dx}\right)$$
$$=-\frac{d}{dx}[A^{ia-1}B^{ia-1}F(1-ia,1-ia,2,\lambda)]$$
$$+\frac{\lambda}{x}A^{ia-1}B^{ia-1}F'(1-ia,1-ia,2,\lambda)$$

($F(a,b,c,x)\equiv{}_2F_1(a,b;c;x)$, and F' is the derivative of F with respect to its argument), which can be verified by turning to the formula[21]

$$\lambda F'(a,b,c,\lambda)=aF(a+1,b,c,\lambda)-aF(a,b,c,\lambda).$$

Noting that

$$\frac{dA}{dx}=\frac{dB}{dx}=-(1-p^2|\nu|^2),$$

at $p'=p$ and integrating in I_+ by parts, we arrive at formula (22) for the divergent part $(M_3)_{\text{sing}}$ of the expression (19) for M_3.

[1)]A brief description of the present work is given in Ref. 13.

[2)]Note that Korol[14] obtained all the singular terms of the matrix element of a one-photon transition in the continuous spectrum, $\langle\psi_{E'l'm'}|\mathbf{e}\cdot\hat{\mathbf{p}}|\psi_{Elm}\rangle$, which emerge at $p'=p$ for the same reason as noted earlier in the text.

[1] V. B. Berestetskiĭ, E. M. Lifshitz, and L. P. Pitaevskiĭ, *Quantum Electrodynamics*, 3rd ed., Pergamon Press, Oxford (1990).
[2] M. Gavrila, in *Collision Theory for Atoms and Molecules*, F. A. Gianturco (ed.), Plenum Press, New York (1989), p. 139.
[3] F. V. Bunkin and M. V. Fedorov, Zh. Éksp. Teor. Fiz. **49**, 1215 (1965) [Sov. Phys. JETP **22**, 844 (1966)].
[4] N. M. Kroll and K. M. Watson, Phys. Rev. A **8**, 804 (1973).
[5] I. Berson, Zh. Éksp. Teor. Fiz. **80**, 1727 (1981) [Sov. Phys. JETP **53**, 891 (1981)].
[6] M. Gavrila, A. Maquet, and V. Veniard, Phys. Rev. A **32**, 2537 (1985); **42**, 236 (1990).
[7] V. Florescu and V. Djamo, Phys. Lett. A **119**, 73 (1986).
[8] N. L. Manakov, S. I. Marmo, and A. G. Faĭnshteĭn, in *Abstracts of Papers of the 9th All-Union Conf. on Phys. of Electron and Atomic Collisions*, Riga (1984), Part 2, p. 118.
[9] A. V. Korol, J. Phys. B **27**, 155 (1994).
[10] J. van de Ree, J. Z. Kaminski, and M. Gavrila, Phys. Rev. A **37**, 4536 (1988).
[11] L. Dimou and F. H. M. Faisal, Phys. Rev. Lett. **59**, 872 (1987).
[12] M. Dörr, C. J. Joachain, R. M. Potvliege, and S. Vučič, Phys. Rev. A **49**, 4852 (1994).
[13] A. G. Faĭnshteĭn (Fainshtein), N. L. Manakov, and S. I. Marmo, Phys. Lett. A **195**, 358 (1994).
[14] A. V. Korol, J. Phys. B **27**, L103 (1994).
[15] A. Sommerfeld, *Atomic Structure and Spectral Lines*, Methuen, London (1923).
[16] N. L. Manakov and L. P. Rapoport, Opt. Spektrosk. **33**, 988 (1972).
[17] N. L. Manakov, S. I. Marmo, and A. V. Shaposhnikov, in *Atoms and Molecules in Strong Fields of Laser Radiation*, F. V. Bunkin and I. I. Tugov (eds.), Wiley-Nauka, Moscow (1992).

[18] S. I. Marmo and A. G. Faĭnshteĭn, Izv. Vyssh. Uchebn. Zaved., Fiz. No. 4, 52 (1984).
[19] D. Andrick and L. Langhaus, J. Phys. B **11**, 2355 (1978).
[20] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products*, Academic Press, New York (1980).
[21] A. Erdélyi, *Higher Transcendental Functions*, Vol. 1 (Bateman Project), McGraw-Hill, New York (1953).
[22] A. Nordsieck, Phys. Rev. **93**, 785 (1954).

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