Symmetry theory of Anderson's transition

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The Vollhardt–Wölfle hypothesis that the irreducible vertex $U_{\mathbf{kk}'}(\mathbf{q})$ appearing in the Bethe–Salpeter equation contains a diffusion pole in the limit $\mathbf{k}+\mathbf{k}'\to 0$, which includes the observed diffusion coefficient $D(\omega,q)$, is proved. In the quantum kinetic equation the quantity $U_{\mathbf{kk}'}(\mathbf{q})$ plays the role of a transition probability $W_{\mathbf{kk}'}$ and its anomalous growth as $D(\omega,q)$ decreases is the physical reason for localization. As $\omega\to 0$, in the localized phase $D(\omega,q)=(-i\omega)d(q)$ holds, where $d(q)$ is a regular function of q^2, associated with the properties of a typical wave function. The presence of a diffusion pole in $U_{\mathbf{kk}'}(\mathbf{q})$ makes it possible to represent the quantum "collision operator" $\hat{L}$ as a sum of a singular operator $\hat{L}_{\text{sing}}$, which has an infinite number of zero modes, and a regular operator $\hat{L}_{\text{reg}}$ of a general form. Investigation of the response of the system to a change in $\hat{L}_{\text{reg}}$ leads to a self-consistency equation, which replaces the rough Vollhardt–Wölfle equation. Its solution shows that $D(0,q)$ vanishes at the transition point simultaneously for all q; the spatial dispersion of $D(\omega,q)$ at $\omega\to 0$ is found to be ~ 1 in relative units, it is determined by the atomic scale, and it has no manifestations on the scale $q\sim\xi^{-1}$ associated with the correlation length ξ. The values obtained for the critical exponent (s) of the conductivity and the critical exponent (ν) of the localization length in a d-dimensional space, $s=1$ $(d>2)$ and $\nu=(d-2)^{-1}$ $(2<d<4)$, $\nu=1/2$ $(d>4)$, agree with most existing results. With respect to the character of the change in the symmetry, the Anderson transition is found to be similar to the Curie point of an isotropic ferromagnet with an infinite number of components. For such a magnet the critical exponents are known exactly and they agree with the exponents indicated above: This suggests that the symmetry of the critical point has been completely determined and that the exponents have been determined exactly.

1. INTRODUCTION

It is now widely acknowledged (see, for example, Ref. 1, p. 76) that the theory of phase transitions should, in principle, be constructed as a symmetry theory. Specifically, the effective Hamiltonian of the system is represented in the form

$$H=H_c+\tau H_{\text{int}} \tag{1}$$

where H_c is the critical-point Hamiltonian, possessing a high symmetry; H_{int} is a general operator which is compatible with the symmetry of the total Hamiltonian H; and, τ is a parameter that measures the distance to the transition. The most general motivation for the separation (1) is that the set of Hamiltonians H_c (for example, Hamiltonians of different ferromagnets at the Curie point) should be separated from the set of all Hamiltonians H by imposing some kind of additional conditions which can be interpreted as generalized symmetry requirements.

In this approach the problem consists of determining the complete symmetry of the Hamiltonian H_c, which thus far it has been impossible to do for most phase transitions. For example, Landau's well-known theory[2] starts from the obvious symmetry of the Hamiltonian and does not take into account scale invariance and other symmetry elements arising as a result of the fluctuations near the critical point (Ref. 3, Chap. 9, §2). Landau's theory is exact, giving an example of a complete symmetry theory, only in high-dimension spaces where the additional symmetry associated with fluctuations does not arise. Another example is the conformal theory of phase transitions for the two-dimensional case,[4] which, proceeding from the conformal invariance of the system at the critical point and the finiteness of the number of strongly fluctuating quantities, fixes a discrete series of sets of critical exponents.

In the present paper we adopt the symmetry approach to the investigation of Anderson's transition,[5–10] making a separation of the type (1) not for the Hamiltonian H but for an operator $\hat{L}$ which is the quantum analog of the Boltzmann collision operator. The theory is based on the following initial assumptions.

1. The Schrödinger equation in a space of dimension d

$$[\epsilon(\hat{\mathbf{p}})+V(\mathbf{r})]\psi(\mathbf{r})=E\psi(\mathbf{r}) \tag{2}$$

describing the motion of noninteracting electrons with an arbitrary spectrum $\epsilon(\mathbf{p})$ in a random potential $V(\mathbf{r})$ is studied. With regard to the random potential, it is assumed only that the averages with respect to its realizations can be calculated by the diagrammatic technique. The existence of a diagrammatic technique for the standard models of a random potential is proved directly in Refs. 1, 11, and 12. In the general case, the question of the limits of applicability of the technique—some problems obviously arise for quasirandom systems[13,14]—has been little studied.

The exact Green's function of Eq. (2) is expressed in

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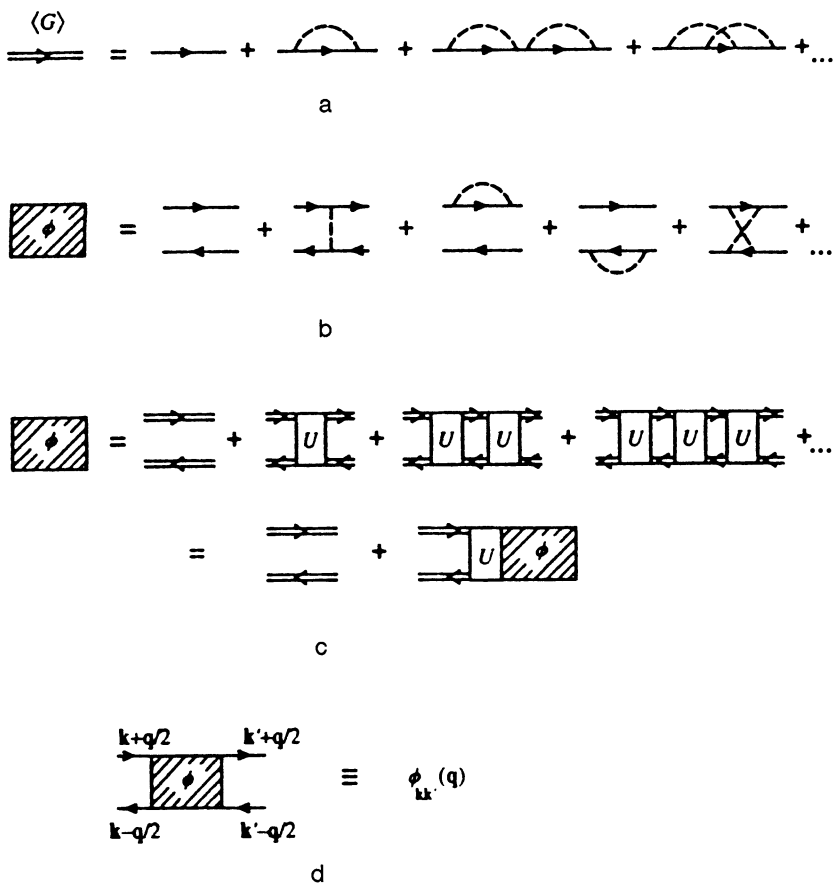

FIG. 1. a, b—Diagrams for the average Green's function (a) and the quantity ϕ (b), which correspond to a Gaussian random potential[1] or a Born approximation for randomly distributed impurities[11] (in what follows, their specific form is not used). c—Graphical representation of the Bethe–Salpeter equation; d—explanation of the three-momentum notation.

terms of the eigenfunctions $\psi_s(\mathbf{r})$ and eigenvalues ϵ_s $(s=1,2,\ldots,N)$ of the equation:

$$G_E^{R,A}(\mathbf{r},\mathbf{r}')=\sum_s \frac{\psi_s(\mathbf{r})\psi_s^*(\mathbf{r}')}{E-\epsilon_s\pm i\delta}. \tag{3}$$

The averaged Green's function $\langle G(\mathbf{r},\mathbf{r}')\rangle$ is determined by a diagrammatic series (Fig. 1a), and in accordance with current ideas[8,15] it is assumed to be analytic at the point of the Anderson transition: For $d\geqslant 4$ this was recently proved by the present author.[16,17] The quantity

$$\phi(\mathbf{r}_1\mathbf{r}_2,\mathbf{r}_3\mathbf{r}_4)=\langle G_{E+\omega}^R(\mathbf{r}_1\mathbf{r}_2)G_E^A(\mathbf{r}_3\mathbf{r}_4)\rangle \tag{4}$$

which contains information about the kinetic properties, has a singularity at the transition point. This quantity is determined by a collection of diagrams with four legs, constructed on G^R and G^A lines (Fig. 1b), and its properties are similar to those of the two-particle Green's function in the theory of interacting particles.[11] It satisfies the Bethe–Salpeter equation, containing an irreducible vertex U (Fig. 1c).

2. The following symmetry elements are assumed:

(a) *Spatial uniformity in the mean.* This leads to a conservation law for the external momenta in the diagrams. This makes it possible to express $\langle G\rangle$ in terms of the self-energy Σ

$$\langle G_E^{R,A}(\mathbf{k})\rangle\equiv G_{\mathbf{k}}^{R,A}=\frac{1}{E-\epsilon_{\mathbf{k}}-\Sigma_{\mathbf{k}}^{R,A}}, \tag{5}$$

and to introduce for the function ϕ the three-momentum notation $\phi_{\mathbf{k}\mathbf{k}'}(\mathbf{q})$ (Fig. 1d) and to write the Bethe–Salpeter equation (Fig. 1c) in the form

$$\phi_{\mathbf{k}\mathbf{k}'}(\mathbf{q})=G_{\mathbf{k}+\mathbf{q}/2}^R G_{\mathbf{k}-\mathbf{q}/2}^A\left\{N\delta_{\mathbf{k}-\mathbf{k}'}+\frac{1}{N}\sum_{\mathbf{k}_1}U_{\mathbf{k}\mathbf{k}_1}(\mathbf{q})\phi_{\mathbf{k}_1\mathbf{k}'}(\mathbf{q})\right\}. \tag{6}$$

Here and below the energy variable is equal to $E+\omega$ for the functions G^R and E for the functions G^A.

(b) *Isotropy in the mean.* When this isotropy is taken into account, the function $\phi_{\mathbf{k}\mathbf{k}'}(\mathbf{q})$ depends only on scalar products constructed from $\mathbf{k}$, $\mathbf{k}'$ and $\mathbf{q}$, whence, specifically,

$$\phi_{\mathbf{k}\mathbf{k}'}(\mathbf{q})=\phi_{-\mathbf{k},-\mathbf{k}'}(-\mathbf{q}). \tag{7}$$

Similarly, $G_{\mathbf{k}}^R$ and $G_{\mathbf{k}}^A$ depend on $\mathbf{k}^2$ and are even functions of $\mathbf{k}$.

(c) *Time-reversal invariance.* This property makes it possible to choose eigenfunctions $\psi_s(\mathbf{r})$ which are real and to drop the conjugation sign in Eq. (3). Then $G(\mathbf{r},\mathbf{r}')=G(\mathbf{r}',\mathbf{r})$ and interchanging $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$, $\mathbf{r}_4$ in Eq. (4) on switching at the same time to the momentum representation gives

$$\phi_{\mathbf{k}\mathbf{k}'}(\mathbf{q})=\phi_{-\mathbf{k}',-\mathbf{k}}(-\mathbf{q}) \tag{8}$$

$$\phi_{\mathbf{k}\mathbf{k}'}(\mathbf{q})=\phi_{(\mathbf{k}-\mathbf{k}'+\mathbf{q})/2,(\mathbf{k}'-\mathbf{k}+\mathbf{q})/2}(\mathbf{k}+\mathbf{k}') \tag{9}$$

Comparing Eqs. (7) and (8), we obtain

$$\phi_{\mathbf{kk}'}(\mathbf{q})=\phi_{\mathbf{k}'\mathbf{k}}(\mathbf{q}). \tag{10}$$

Solving Eq. (6) formally for the function $U_{\mathbf{kk}'}(\mathbf{q})$ and using Eqs. (7) and (10), it is easy prove similar properties for this function:

$$U_{\mathbf{kk}'}(\mathbf{q})=U_{-\mathbf{k},-\mathbf{k}'}(-\mathbf{q}), \quad U_{\mathbf{kk}'}(\mathbf{q})=U_{\mathbf{k}'\mathbf{k}}(\mathbf{q}). \tag{11}$$

3. It is conventionally assumed that Anderson's transition proceeds from a phase with exponential localization of the wave functions into a phase with a finite diffusion coefficient. The existence of exponential localization in the limit $E\to-\infty$ and finite diffusion for large positive values of E (for $d>2$ and an unbounded spectrum $\epsilon(\mathbf{p})$, $0\leqslant\epsilon(\mathbf{p})\leqslant\infty$) has been firmly established, as a result of many investigations, for Eq. (2). The proof of the existence of a mobility threshold is based mainly on Mott's argument:[7] The existence of states with different degree of localization and the same energy is impossible because of instability with respect to an infinitesimal general perturbation. Mott's argument does not forbid, however, the existence of intermediate states—with exponential localization, hybrid states, and so on—and correspondingly different types of "Anderson transitions" (for example, in the quasirandom systems[13,14] the transition occurs from exponential localization to a ballistic regime). In the present paper the first instability, arising with a motion from deep in an exponentially localized phase, is investigated and it is shown that it does indeed correspond to a transition into a phase with finite diffusion.

4. The general ideas of the modern theory of critical phenomena[1]—parametric space, critical surface, important and unimportant parameters—are used.

5. The theory is based on the physical idea that the localization phenomenon is associated with a diffusion pole in the irreducible four-leg diagram

$$U_{\mathbf{kk}'}(\mathbf{q})=U^{\mathrm{reg}}_{\mathbf{kk}'}(\mathbf{q})+U^{\mathrm{sing}}_{\mathbf{kk}'}(\mathbf{q})=U^{\mathrm{reg}}_{\mathbf{kk}'}(\mathbf{q})+\frac{F(\mathbf{k},\mathbf{k}',\mathbf{q})}{-i\omega+D(\omega,\mathbf{k}+\mathbf{k}')(\mathbf{k}+\mathbf{k}')^2} \tag{12}$$

proposed by Vollhardt and Wölfle in the so-called "self-consistent theory of localization" (see Ref. 18, and also Refs. 10 and 19). This idea agrees with the theory of weak localization,[20–22] according to which the diffusion pole in $U_{\mathbf{kk}'}(\mathbf{q})$ determines the main quantum corrections to the conductivity which in turn determine the scaling behavior in a space with dimension $d=2+\epsilon$. The diffusion pole in $U_{\mathbf{kk}'}(\mathbf{q})$ with the classical diffusion coefficient D_{cl} arises as a result of summation of fan-shaped diagrams;[20] Vollhardt and Wölfle conjectured that when these diagrams are taken into account, D_{cl} is replaced by the exact diffusion coefficient $D(\omega,\mathbf{q})$. Then they approximated $U_{\mathbf{kk}'}(\mathbf{q})$ with $U^{\mathrm{reg}}_{\mathbf{kk}'}(\mathbf{q})=$ const, $F(\mathbf{k},\mathbf{k}',\mathbf{q})=$ const and solved approximately the Bethe–Salpeter equation (6), which, using the Ward identities[18]

$$\Delta\Sigma_{\mathbf{k}}(\mathbf{q})=\frac{1}{N}\sum_{\mathbf{k}_1}U_{\mathbf{kk}_1}(\mathbf{q})\Delta G_{\mathbf{k}_1}(\mathbf{q}), \tag{13}$$

$$\Delta G_{\mathbf{k}}(\mathbf{q})\equiv G^R_{\mathbf{k}+\mathbf{q}/2}-G^A_{\mathbf{k}-\mathbf{q}/2},$$
$$\Delta\Sigma_{\mathbf{k}}(\mathbf{q})\equiv\Sigma^R_{\mathbf{k}+\mathbf{q}/2}-\Sigma^A_{\mathbf{k}-\mathbf{q}/2} \tag{14}$$

was rewritten in the form

$$[-\omega+(\epsilon_{\mathbf{k}+\mathbf{q}/2}-\epsilon_{\mathbf{k}-\mathbf{q}/2})]\phi_{\mathbf{kk}'}(\mathbf{q})+\frac{1}{N}\sum_{\mathbf{k}_1}U_{\mathbf{kk}'}(\mathbf{q})\times[\Delta G_{\mathbf{k}_1}(\mathbf{q})\phi_{\mathbf{kk}'}(\mathbf{q})-\Delta G_{\mathbf{k}}(\mathbf{q})\phi_{\mathbf{k}_1\mathbf{k}'}(\mathbf{q})]=\Delta G_{\mathbf{k}}(\mathbf{q})N\delta_{\mathbf{k}-\mathbf{k}'}. \tag{15}$$

There exists a simple estimate which gives the same results. We note that the second term in Eq. (15) is reminiscent of a Boltzmann collision integral and indeed transforms into this integral in the limit of weak disorder (Sec. 3). It is significant that in the quantum region the quantity $U_{\mathbf{kk}'}(\mathbf{q})$ plays the role of a "transition probability." Using a τ-type approximation, $D\propto l\propto\langle U\rangle^{-1}$ (l is the mean free path, $\langle\ldots\rangle$ denotes averaging over the momenta), and taking into account Eq. (12), we obtain the self-consistency equation of the Vollhardt–Wôlfle theory

$$D\sim\mathrm{const}\left(U_0+F_0\int\frac{d^dq}{-i\omega+D(\omega,q)q^2}\right)^{-1}. \tag{16}$$

This estimate is no less accurate and demonstrates more clearly the crux of the matter than the approximate solution given in Ref. 18 for Eq. (15): As the degree of disorder increases, the "transition probability" increases anomalously as a result of a decrease of the diffusion coefficient, making it possible for the coefficient to vanish. Neglecting the spatial disperion $D(\omega,\mathbf{q})$, Eq. (16) makes it possible to determine the critical exponents for the conductivity σ and the localization length ξ

$$\sigma\sim\tau^s, \quad \xi\sim\tau^{-\nu} \tag{17}$$

(τ is the distance to the transition); setting $D=$ const $(\omega)\sim\sigma$ in the metal phase and $D\sim(-i\omega)\xi^2$ in the localized phase, we obtain

$$s=1,\ d>2; \quad \nu=\begin{cases}\dfrac{1}{d-2}, & 2<d<4\\[2ex] \dfrac{1}{2}, & d>4\end{cases}. \tag{18}$$

The drawbacks of the self-consistent localization theory given in Ref. 18 can already be seen from the exposition given above:

a) The method used to solve the Bethe–Salpeter equation is rough;

b) the spatial dispersion $D(\omega,\mathbf{q})$, which can change substantially the estimate of the integral in Eq. (16), is ignored; and,

c) an approximation is used for $U_{\mathbf{kk}'}(\mathbf{q})$ that leads to a singularity $\sim1/\omega$ on the right-hand side of Ward's identity (13) in the localized phase, and this is incompatible with regularity of Σ at the transition point.

One of the most interesting questions in the theory of localization is connected with the drawback b. It follows from the Berezinskiĭ–Gor'kov criterion[24] that in the local-

ized phase $D(0,\mathbf{q})\equiv 0$ (Sec. 4). There arises the question of the character of the change in the spatial dispersion of D near a transition. Vollhardt and Wölfle proceeded from the fact that $D(0,\mathbf{q})$ vanishes at the transition point immediately for all values of $\mathbf{q}$. Efetov stated a similar hypothesis.[21] The vanishing of the entire function cannot occur accidently and it must be supported by a deep symmetry: Does this symmetry exist? What is its nature? Another fundamental question of the theory is touched upon in the drawback c: If the diffusion pole in $U_{\mathbf{kk}'}(\mathbf{q})$ exists, then why is there no $\sim 1/\omega$ singularity on the right-hand side of Ward's identity (13)? The condition for this pole to be cancelled imposes stringent requirements on the approximation employed, while the satisfaction of Ward's identity has actually never been checked in any of the existing theories.[10]

A theory free of the drawbacks a–c and answering the questions raised is expounded below. Part 1 of this paper follows the scheme of the Vollhardt–Wölfle theory and contains a proof of the relations obtained in Ref. 18 by a chain of hypotheses or doubtful approximations. In Secs. 2 and 3 the diffusion poles of $\phi_{\mathbf{kk}'}(\mathbf{q})$ are separated and the result (12) is proved. In Sec. 4 the general properties of the diffusion coefficient and its connection with the localization of the wave functions are determined. The content of Secs. 5–7 replaces the rough solution of the Bethe–Salpeter equation:[18] In Sec. 5 a hierarchical structure of the spectrum is obtained for the quantum collision operator $\hat{L}$; a separation of the type (1), convenient for a symmetry analysis, is established; and, a condition on the transition point is found and a self-consistency equation, replacing Eq. (16), is derived. The diffusion coefficient $D(\omega,\mathbf{q})$ is sought under the assumption that the spatial dispersion is arbitrary (Secs. 6 and 7), but it is found that only the solution with a weak $\mathbf{q}$ dependence, not affecting the estimate of the integral in Eq. (16) and leading to the result (18) for the critical exponents, is found to be internally consistent. In summary, all basic results of Ref. 18 are found to be correct, which is surprising for such a rough theory.

The theory expounded starts from the obvious symmetry of the system and the additional symmetry of the critical point is determined in the course of the analysis. The inevitable question is: Are the hidden symmetry elements completely determined? A serious argument indicating that the determination is complete is (Sec. 8) that with respect to the character of the change in symmetry, Anderson's transition is found to be similar to the Curie point for an isotropic n-component ferromagnet in the limit $n\to\infty$. This model of a ferromagnet is the basis of the $1/n$ expansion,[1] its critical exponents are known exactly, and they are in exact agreement with Eq. (18). The isotropy of the equivalent ferromagnet is the symmetry that makes $D(0,\mathbf{q})$ vanish simultaneously for all $\mathbf{q}$; The approximate (to an accuracy $\sim\omega$) orthoganality of the singular part of $U_{\mathbf{kk}'}(\mathbf{q})$ and $\Delta G_{\mathbf{k}}(\mathbf{q})$ means that there is no singularity in the right-hand side of the Ward identity [Sec. 5.3, Eq. (13)].

Another method for checking the completeness of the symmetry found is to compare with the results of model investigations. The hypothesis that the exponents (18) are exact was actually stated in Ref. 25 on the basis of an analysis of all known results:

(a) For $d=2+\epsilon$ Wegner's relation $s=(d-2)\nu$, following from the existence of one-parameter scaling,[21] is valid and the ϵ expansion for the exponent ν has the form[26]

$$\nu=\frac{1}{\epsilon}+0\cdot\epsilon^0+0\cdot\epsilon^1+O(\epsilon^2), \tag{19}$$

which agrees with Eq. (18), if the coefficients of the higher order powers of ϵ are also zero.

(b) The result (18) separates the dimensions of the space $d_{c1}=2$ and $d_{c2}=4$, which on the basis of independent considerations are considered to be the lower[21] and the upper (see the discussion and references in Ref. 16) critical dimensions.

(c) The entire experience of the theory of phase transitions shows that for $d>d_{c2}$ the critical exponents do not depend on d, which is the case in Eq. (18).

(d) The exponents (18) agree with the results for $d=\infty$: $\nu=1/2$ (Refs. 27 and 28) and $s=1$ (Ref. 29); the disagreement with the result $s=\infty$, obtained in Ref. 28, is discussed in Sec. 9.

The value $\nu=1$ of the exponent for $d=3$ agrees satisfactorily with the results of numerical calculations ($\nu=1.2\pm 0.3$ (Ref. 30) and $\nu=1.5\pm 0.2$ (Ref. 31) and the qualitatively behavior of ν as a function of d agrees with the estimates from hierarchical models.[14] In Wegner's work[32] a finite contribution $\sim\epsilon^2$ is obtained in Eq. (19). This makes the agreement with the numerical calculations of Refs. 30 and 31 much worse. However, this result was derived for the zero-component σ-model, whose agreement with the initial disordered system is controversial Wegner[32] himself does not reject this), and it is apparently correct only in the lowest orders in ϵ (Sec. 9).

A qualitative result of this work, which can be checked experimentally, is the assertion that there is no spatial dispersion of $D(\omega,\mathbf{q})$ on the scale ξ^{-1} (compare with Refs. 33, 34, and 35). The interaction between the electrons can change this result and the experifmental investigation of $D(\omega,\mathbf{q})$ with $q\gtrsim\xi^{-1}$ in the limit $\omega\to 0$ could resolve the question of the applicability of the single-particle picture for describing localization. The absence of significant spatial dispersion does not contradict the strong dependence of the diffusion coefficient D_L of a finite system on the size of the system L.[21] This dependence is related with the temporal dispersion and is determined according to the known function $D(\omega,q)$ from the relation $D_L\sim D(D_L/L^2,0)$.[18]

2. RELATIONS BETWEEN THE QUANTITIES IN THE PRESENCE OF SPATIAL AND TEMPORAL DISPERSION

In this section the existence of a diffusion pole in the quantity

$$\phi(\mathbf{q})=\frac{1}{N^2}\sum_{\mathbf{kk}'}\phi_{\mathbf{kk}'}(\mathbf{q}), \tag{20}$$

which is the Fourier transform of the quantity (4) for coinciding arguments $\mathbf{r}_1=\mathbf{r}_4$, $\mathbf{r}_2=\mathbf{r}_3$, will be proved. In contrast to Ref. 18 and other works, it will not be assumed that q is

small. In view of the great confusion in the literature, we shall give a complete summary of the formulas which are relevant here.

We shall consider the response of a system to an electric field $\mathbf{E}(\mathbf{r},t)\sim e^{i\mathbf{qr}-i\omega t}$. The frequency ω is assumed to be finite only in order to reveal the uncertainties which appear; the limit $\omega\to 0$ is taken in the final results. Neglecting magnetic effects, the field $\mathbf{E}$ is a purely potential field. This makes it possible to confine attention to the longitudinal components of the susceptibilities (Ref. 36, §103). In the presence of spatial dispersion two definitions of the conductivity are possible:

$$\begin{aligned}&\mathbf{j}(\omega,\mathbf{q})=\tilde{\sigma}(\omega,\mathbf{q})\mathbf{E}(\omega,\mathbf{q})\\&\mathbf{j}_e(\omega,\mathbf{q})=\sigma(\omega,\mathbf{q})\mathbf{E}(\omega,\mathbf{q})\end{aligned}\qquad \mathbf{j}=\mathbf{j}_e+\mathbf{j}_{\text{diff}}, \tag{21}$$

which relate $\mathbf{E}$ with the total current $\mathbf{j}$ or its electric component $\mathbf{j}_e$; the diffusion component of the current $\mathbf{j}_{\text{diff}}(\omega,\mathbf{q})=-i\mathbf{q}D(\omega,\mathbf{q})\rho(\omega,\mathbf{q})$ is due to the deviation of the electron density ρ from the equilibrium density. This deviation is determined by the polarizability α (φ is a scalar potential):

$$\rho(\omega,\mathbf{q})=\alpha(\omega,\mathbf{q})\varphi(\omega,\mathbf{q}). \tag{22}$$

The conductivity $\tilde{\sigma}$ appears in Kubo's formulas (see below), which determine the total response of the system to the field $\mathbf{E}$. The conductivity σ is related with the diffusion coefficient D by the Einstein relation

$$\sigma(\omega,\mathbf{q})=e^2N(\epsilon_F)D(\omega,\mathbf{q}) \tag{23}$$

since the change in the scalar potential φ and the chemical potential μ with $\mu(\mathbf{r},t)+e\varphi(\mathbf{r},t)=$ const does not destroy the thermodynamic equilibrium ($N(\epsilon_F)$ is the density of states at the Fermi level). A relation between $\sigma,\tilde{\sigma}$, and α follows from the continuity equation:

$$\begin{aligned}&-i\omega\tilde{\sigma}(\omega,\mathbf{q})=[-i\omega+D(\omega,\mathbf{q})q^2]\sigma(\omega,\mathbf{q}),\\&\omega\alpha(\omega,\mathbf{q})=-iq^2\tilde{\sigma}(\omega,\mathbf{q}),\end{aligned} \tag{24}$$

so that the difference between σ and $\tilde{\sigma}$ is important only for $\mathbf{q}\neq 0$. Using the relation (23), we obtain for the polarizability α and the permittivity ϵ

$$\begin{aligned}&\alpha(\omega,\mathbf{q})=-\frac{e^2N(\epsilon_F)D(\omega,\mathbf{q})q^2}{-i\omega+D(\omega,\mathbf{q})q^2},\\&\epsilon(\omega,\mathbf{q})=1-\frac{4\pi}{q^2}\alpha(\omega,\mathbf{q})=1-\frac{4\pi\tilde{\sigma}(\omega,\mathbf{q})}{i\omega}.\end{aligned} \tag{25}$$

It is clear from Eqs. (23–25) that if the diffusion coefficient $D(\omega,\mathbf{q})$ is given, then all quantities introduced above can be determined.

The quantities $\tilde{\sigma}$ and α are given by Kubo's formulas (Ref. 2, §126, and Ref. 37, §75)

$$\begin{aligned}\tilde{\sigma}(\omega,\mathbf{q})=\frac{1}{\omega}\int_0^\infty dte^{i\omega t}\int d\mathbf{r}e^{-i\mathbf{qr}}\langle\hat{j}(\mathbf{r},t)\hat{j}(0,0)\\-\hat{j}(0,0)\hat{j}(\mathbf{r},t)\rangle,\end{aligned} \tag{26}$$

$$\begin{aligned}\tilde{\sigma}(\omega,\mathbf{q})=\frac{1}{q}\int_0^\infty dte^{i\omega t}\int d\mathbf{r}e^{-i\mathbf{qr}}\langle\hat{j}(\mathbf{r},t)\hat{\rho}(0,0)\\-\hat{\rho}(0,0)\hat{j}(\mathbf{r},t)\rangle,\end{aligned} \tag{27}$$

$$\begin{aligned}\alpha(\omega,\mathbf{q})=-i\int_0^\infty dte^{i\omega t}\int d\mathbf{r}e^{-i\mathbf{qr}}\langle\hat{\rho}(\mathbf{r},t)\hat{\rho}(0,0)\\-\hat{\rho}(0,0)\hat{\rho}(\mathbf{r},t)\rangle\end{aligned} \tag{28}$$

which determine, respectively, the response of the current to a vector potential, the response of a current to the scalar potential, and the response of the density to a scalar potential. The equivalence of Eqs. (26) and (27) and the relation (24) between $\tilde{\sigma}$ and α follow from the continuity equation for the density operator $\hat{\rho}$ and the longitudinal component of the current operator $\hat{j}$ and the asymptotic expressions for $\tilde{\sigma}$ and α in the limit $\omega\to\infty$ (Ref. 36, §78).

We note that according to the precise meaning of Kubo's formula (see the detail discussion in Ref. 38), the response of the system to the field $\mathbf{D}$ produced by external charges must be calculated. In this approach the Coulomb interaction between the electrons must be necessarily included in the Hamiltonian to avoid contradictions in the Maxwell's equations; Kubo's formulas have a form that is somewhat different from Refs. 26–28 (Ref. 37, p. 413), and the correlation functions appearing in them must be calculated taking into account the Coulomb interaction. A different approach[38] is more convenient: The interaction between the electrons is divided into a short-range and slowly-varying long-range parts; the first part is included explicitly in the Hamiltonian and the second part is taken into account as a self-consistent field, leading to screening of the field $\mathbf{D}$; for this reason, the response to a real physical field $\mathbf{E}$ is studied and the correlation functions appearing in Eqs. (26–28) are calculated only taking into account the short-range part of the interaction. The latter part can be taken into account in the spirit of the Fermi-liquid theory. We shall neglect it completely, since in its classical formulation Anderson's problem is a problem of noninteracting electrons. We underscore the fact that the word "noninteracting" must be understood precisely in the sense indicated above, since otherwise the concept of conductivity cannot be introduced in a consistent fashion.

The correlation function in Eq. (28) for noninteracting electrons in a random potential is calculated similarly to the correlation function for a Fermi gas (Ref. 2, §117) using, instead of the plane-wave representation, a representation in terms of the eigenfunctions $\psi_s(r)$ of Eq. (2):[1)]

$$\begin{aligned}\alpha(\omega,\mathbf{q})=e^2\int_{-\infty}^{\infty}d\epsilon\int_{-\infty}^{\infty}d\omega'\frac{f_0(\epsilon)-f_0(\epsilon+\omega')}{\omega-\omega'+i\delta}N(\epsilon)\\\times\langle\rho_\epsilon\rho_{\epsilon+\omega'}\rangle_{\mathbf{q}}.\end{aligned} \tag{29}$$

Here $f_0(\epsilon)$ is the Fermi function, and $\langle\rho_\epsilon\rho_{\epsilon+\omega}\rangle_{\mathbf{q}}$ is the Fourier transform of the Berezinskiĭ–Gor'kov spectral density[24]

$$\begin{aligned}\langle\rho_E(\mathbf{r})\rho_{E+\omega}(\mathbf{r}')\rangle=\frac{1}{N(E)}\left\langle\sum_{ss'}\psi_s^*(\mathbf{r})\psi_{s'}(\mathbf{r})\psi_{s'}^*(\mathbf{r}')\right.\\\left.\times\psi_s(\mathbf{r}')\delta(E-\epsilon_s)\delta(E-\epsilon_{s'}+\omega)\right\rangle.\end{aligned} \tag{30}$$

For small ω and zero temperature, taking the imaginary part of Eq. (29), we obtain the inverse function

$$\langle \rho_{\epsilon_F} \rho_{\epsilon_F+\omega'} \rangle_{\mathbf{q}} = -\frac{\operatorname{Im} \alpha_{\epsilon_F}(\omega,\mathbf{q})}{\pi e^2 \omega N(\epsilon_F)} \tag{31}$$

(in the absence of interaction $\psi_s(\mathbf{r})$ and ϵ_s do not depend on ϵ_F and ϵ_F can be replaced by E). The standard diffusion form for $\langle \rho_E \rho_{E+\omega} \rangle_{\mathbf{q}}$ (Refs. 10 and 24) is obtained by substituting the expression (25) into Eq. (31), assuming $D(\omega,\mathbf{q})$ is real, which in general is not the case.

The following expression can be easily obtained for the function $\phi(\mathbf{q})$ (E and ω are parameters appearing in Eq. (4)):

$$\phi(\mathbf{q}) = \int_{-\infty}^{\infty} d\epsilon \int_{-\infty}^{\infty} d\omega' \, \frac{N(\epsilon)\langle \rho_\epsilon \rho_{\epsilon+\omega'} \rangle_{\mathbf{q}}}{(E+\omega-\epsilon+i\delta)(E-\omega'-\epsilon-i\delta)}. \tag{32}$$

The polarizability $\alpha(\omega,\mathbf{q})$ is a generalized susceptibility (Ref. 2, §123; Ref. 36, §103) and the oddness of Im $\alpha(\omega,\mathbf{q})$ as a function of the frequency makes it possible to write

$$\operatorname{Im} \alpha(\omega,\mathbf{q}) = \frac{\tilde{\alpha}(\omega,\mathbf{q}) - \tilde{\alpha}(-\omega,\mathbf{q})}{2i},$$

$$\tilde{\alpha}(\omega,\mathbf{q}) = \alpha(\omega,\mathbf{q}) - \alpha(0,\mathbf{q}). \tag{33}$$

Substituting the expressions (31) and (33) into Eq. (32), we obtain integrals with $\tilde{\alpha}(\omega',\mathbf{q})$ and $\tilde{\alpha}(-\omega',\mathbf{q})$ that converge separately. Making the substitution $\omega' \to -\omega'$ in the second of the integrals and shifting upwards the contour integration over ω', and taking into account the fact that $\alpha(\omega,\mathbf{q})$ is analytic in the upper half-plane, we obtain

$$\phi(\mathbf{q}) = \frac{1}{e^2} \int_{-\infty}^{\infty} d\omega'' \, \frac{\tilde{\alpha}_{E+\omega''}(\omega'',\mathbf{q})}{(\omega''-\omega-i\delta)(\omega''+i\delta)} \simeq \frac{2\pi i}{e^2 \omega} \tilde{\alpha}_E(\omega,\mathbf{q}), \tag{34}$$

where $\omega'' = \epsilon - E$. The second equality follows by neglecting ω'' in the argument $E+\omega''$, as a function of which appreciable changes in $\tilde{\alpha}$ occur on an atomic scale and are not important in the region $\omega'' \sim \omega$, which makes the main contribution to the integral. Substituting $\alpha(\omega,\mathbf{q})$ in the form (25), we obtain

$$\phi(\mathbf{q}) = \frac{2\pi N(E)}{-i\omega + D(\omega,\mathbf{q})q^2} + \phi_{\text{reg}}(\mathbf{q}), \tag{35}$$

where the contribution $\phi_{\text{reg}}(\mathbf{q})$ originates from the region of large values of ω'' in Eq. (34) and is regular in the limit $\omega,\mathbf{q} \to 0$. In the localized phase, when $D(\omega,\mathbf{q}) \sim (-i\omega)$ (Sec. 4), to obtain the expression (35), in separating $\alpha(0,\mathbf{q})$ from $\alpha(\omega,\mathbf{q})$ the existence of a small real frequency-independent additive term in the denominator, necessary in order for all expressions to be meaningful, must be taken into account. The quantity $\phi(\mathbf{q})$ has a diffusion pole which contains the observed diffusion coefficient.

3. SEPARATION OF DIFFUSION POLES FROM THE BETHE–SALPETER EQUATION

We introduce the operator $\hat{L}$, which is the symmetrized version of the operator on the left-hand side of Eq. (15), which arises as a result of the replacement $\phi_{\mathbf{k}\mathbf{k}'}(\mathbf{q}) \to \phi_{\mathbf{k}\mathbf{k}'}(\mathbf{q})\sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})}$ and division of Eq. (15) by $\sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})}$:

$$\begin{aligned} &\hat{L}(\mathbf{q}) = \hat{L}_0(\mathbf{q}) + \hat{M}(\mathbf{q}) \\ &\hat{L}_0 \psi_{\mathbf{k}} \equiv \frac{1}{N} \sum_{\mathbf{k}_1} U_{\mathbf{k}\mathbf{k}_1}(\mathbf{q}) [\Delta G_{\mathbf{k}_1}(\mathbf{q}) \psi_{\mathbf{k}} \\ &\qquad - \sqrt{\Delta G_{\mathbf{k}}(\mathbf{q}) \Delta G_{\mathbf{k}_1}(\mathbf{q})} \psi_{\mathbf{k}_1}], \\ &\hat{M} \psi_{\mathbf{k}} \equiv (\epsilon_{\mathbf{k}+\mathbf{q}/2} - \epsilon_{\mathbf{k}-\mathbf{q}/2}) \psi_{\mathbf{k}}. \end{aligned} \tag{36}$$

The operator $\hat{L}$ acts in the complex space and, by virtue of Eq. (11), it is self-conjugate with respect to the scalar product

$$(\phi,\psi) = \frac{1}{N} \sum_{\mathbf{k}} \phi_{\mathbf{k}} \psi_{\mathbf{k}}. \tag{37}$$

Its eigenfunctions $e_{\mathbf{k}}^{(s)}(\mathbf{q})$ form a complete orthonormal basis, and the eigenvalues $\lambda_s(\mathbf{q})$ are, generally speaking, complex. In terms of λ_s and $e^{(s)}$ the formal solution of the Bethe–Salpeter equation (15) has the form

$$\begin{aligned} &\phi_{\mathbf{k}\mathbf{k}'}(\mathbf{q}) = \sum_s \frac{f_{\mathbf{k}}^{(s)}(\mathbf{q}) f_{\mathbf{k}'}^{(s)}(\mathbf{q})}{-\omega + \lambda_s(\mathbf{q})}, \\ &f_{\mathbf{k}}^{(s)}(\mathbf{q}) = \sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})} e_{\mathbf{k}}^{(s)}(\mathbf{q}). \end{aligned} \tag{38}$$

One eigenvalue—for definiteness $\lambda_0(\mathbf{q})$—behaves as $\lambda_0(\mathbf{q}) \sim q^2$ for small $\mathbf{q}$. Indeed, the operator $\hat{L}_0$ has a zero mode $\psi_{\mathbf{k}}(\mathbf{q}) = \sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})}$ and, treating the operator $\hat{M} \sim q$ as a perturbation, we can construct the iterative series

$$\begin{aligned} &e_{\mathbf{k}}^{(0)}(\mathbf{q}) = \text{const}[\psi_{\mathbf{k}}^{(0)}(\mathbf{q}) + \psi_{\mathbf{k}}^{(1)}(\mathbf{q}) + \ldots], \\ &\psi_{\mathbf{k}}^{(0)}(\mathbf{q}) = \sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})}, \\ &\lambda_0(\mathbf{q}) = \lambda_0^{(1)}(\mathbf{q}) + \lambda_0^{(2)}(\mathbf{q}) + \ldots, \quad \lambda_0^{(n)}, \psi^{(n)} \sim q^n, \\ &(\psi^{(0)}, \psi^{(n)}) = 0, \quad n \neq 0 \end{aligned} \tag{39}$$

in the Brillouin–Wigner form.[39] The eigenvalues $\lambda_s(\mathbf{q})$ are even with respect to $\mathbf{q}$ (see Appendix) and the correction $\lambda_0^{(1)}$ is equal to zero, which can be easily verified directly. To second order in q we have

$$\lambda_0(\mathbf{q}) = \frac{(\psi^{(0)}, \hat{M}\psi^{(0)}) + (\psi^{(0)}, \hat{M}\psi^{(1)})}{(\psi^{(0)}, \psi^{(0)})} \tag{40}$$

where $\psi^{(1)}$ satisfies the equation

$$-\hat{P}_\perp \hat{M} \psi^{(0)} = \hat{L}_0 \psi^{(1)} \tag{41}$$

[$\hat{P}_\perp$ is a projection operator onto the space orthogonal to $\psi^{(0)}$ (Ref. 39)]. Making the substitution

$$\psi_{\mathbf{k}}^{(1)}(\mathbf{q}) = -i\sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})} \mathbf{q} \mathbf{l}_{\mathbf{k}} \tag{42}$$

and noting that

$$\frac{1}{N}\sum_{\mathbf{k}}\Delta G_{\mathbf{k}}(\mathbf{q})=\frac{1}{N}\sum_{\mathbf{k}}2i\ \mathrm{Im}\ G^R_{\mathbf{k}}=-2\pi iN(E) \qquad (43)$$

we rewrite (40) in the form ($\mathbf{v_k}$ is the velocity of electrons with momentum $\mathbf{k}$)

$$\lambda_0(\mathbf{q})=\frac{i}{2\pi N(E)}\left[\frac{1}{N}\sum_{\mathbf{k}}(\mathbf{qv_k})(\mathbf{ql_k})(-i)\Delta G_{\mathbf{k}}(\mathbf{q})\right.$$
$$\left.+\frac{1}{N}\sum_{\mathbf{k}}(\mathbf{qv_k})\Delta G_{\mathbf{k}}(\mathbf{q})\right]. \qquad (44)$$

For an isotropic spectrum $\epsilon(\mathbf{k})=k^2/2m$ the expressions (44) and (41) assume, to lowest order in $\mathbf{q}$, the form

$$\lambda_0(\mathbf{q})=-iD(0,0)q^2,\quad D(0,0)=\sigma(0,0)e^{-2}N^{-1}(E), \qquad (45)$$

$$\sigma(0,0)=\frac{e^2}{2\pi d}\frac{1}{N}\sum_{\mathbf{k}}(\mathbf{v_kl_k})i\Delta G_{\mathbf{k}}(0)$$
$$+\frac{e^2}{2\pi m}\frac{1}{N}\sum_{\mathbf{k}}\mathrm{Re}\ G^R_{\mathbf{k}}, \qquad (46)$$

$$\mathbf{v_k}=\frac{1}{N}\sum_{\mathbf{k}'}iU_{\mathbf{kk}'}(0)\Delta G_{\mathbf{k}'}(0)(\mathbf{l_k}-\mathbf{l_{k'}}). \qquad (47)$$

In the limit of weak disorder, when

$$\Delta G_{\mathbf{k}}(0)=G^R_{\mathbf{k}}-G^A_{\mathbf{k}}=2i\ \mathrm{Im}\frac{1}{E-\epsilon_{\mathbf{k}}+i\gamma}$$
$$\approx-2\pi i\delta(E-\epsilon_{\mathbf{k}}) \qquad (48)$$

we obtain from Eqs. (46) and (47)

$$\sigma(0,0)=\frac{e^2}{d}\frac{1}{N}\sum_{\mathbf{k}}\mathbf{v_kl_k}\delta(E-\epsilon_{\mathbf{k}}), \qquad (49)$$

$$\mathbf{v_k}=\frac{1}{N}\sum_{\mathbf{k}'}2\pi U_{\mathbf{kk}'}(0)(\mathbf{l_k}-\mathbf{l_{k'}})\delta(E-\epsilon_{\mathbf{k}'}), \qquad (50)$$

i.e., $\sigma(0,0)$ is the classical conductivity, $D(0,0)$ is the classical diffusion coefficient, and $\mathbf{l_k}$ is the vector mean free path length, determined by the standard classical equation (50) for scattering by impurities.[40] The results (46) and (47) extend the concept of a kinetic equation and a mean free path into the quantum region. The differences from the classical equations reduce to the following:

(a) The δ-function expressing the law of conservation of energy is smeared;

(b) the transition probabilities are replaced with $2\pi U_{\mathbf{kk}'}(0)$; and,

(c) σ acquires a quantum correction [last term in Eq. (46)] of the order of the Mott minimum conductivity.[7]

It is obvious from Eq. (45) that the diffusion pole is related with the zeroth term in the sum in Eq. (38). To compare with Eq. (35), we sum the expression (38) over $\mathbf{k}$ and $\mathbf{k}'$:

$$\phi(\mathbf{q})=\frac{A_0^2(\mathbf{q})}{-\omega+\lambda_0(\mathbf{q})}+\sum_{s\neq0}\frac{A_s^2(\mathbf{q})}{-\omega+\lambda_s(\mathbf{q})},$$

$$A_s(\mathbf{q})=\frac{1}{N}\sum_{\mathbf{k}}\sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})}e^{(s)}_{\mathbf{k}}(\mathbf{q}). \qquad (51)$$

Neglecting in Eq. (36) the operator $\hat{M}$, we have $e^{(0)}_{\mathbf{k}}(\mathbf{q})\sim\sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})}$, whence $A_0^2(\mathbf{q})=-2\pi iN(E)$, $A_s(\mathbf{q})=0$, $s\neq0$. Taking into account $\hat{M}$ by perturbation theory, we obtain

$$A_0^2(\mathbf{q})=\frac{-2\pi iN(E)}{1+B(\mathbf{q})},\quad B(\mathbf{q})\sim q^2;$$

$$A_s^2(\mathbf{q})\sim q^2,\quad s\neq0 \qquad (52)$$

and comparing the expressions (51) with Eq. (35) gives

$$D(\omega,\mathbf{q})q^2=i\lambda_0(\mathbf{q})[1+B(\mathbf{q})]-i\omega B(\mathbf{q}),$$

$$\phi_{\mathrm{reg}}(\mathbf{q})\sim q^2. \qquad (53)$$

The decomposition into regular and irregular parts is not unique and admits a "gauge transformation"

$$\tilde{\phi}_{\mathrm{reg}}(\mathbf{q})=\phi_{\mathrm{reg}}(\mathbf{q})-2\pi N(E)C(\mathbf{q}),$$

$$\tilde{D}(\omega,\mathbf{q})q^2=\frac{D(\omega,\mathbf{q})q^2+i\omega C(\mathbf{q})[-i\omega+D(\omega,\mathbf{q})q^2]}{1+C(\mathbf{q})[-i\omega+D(\omega,\mathbf{q})q^2]},$$

$$C(\mathbf{q})\sim q^2 \qquad (54)$$

up to which the identity (53) is valid. For this reason, it is convenient to set by definition

$$\lambda_0(\mathbf{q})=-iD(\omega,\mathbf{q})q^2 \qquad (55)$$

making the assumption that the diffusion coefficient $D(\omega,\mathbf{q})$ determined in this manner is related to the observed diffusion coefficient $D_{\mathrm{obs}}(\omega,\mathbf{q})$ by relations of the type (53) and (54). For any $B(\mathbf{q})$ and $C(\mathbf{q})$, we have $D(0,0)=D_{\mathrm{obs}}(0,0)$, and $D(0,\mathbf{q})$ and $D_{\mathrm{obs}}(0,\mathbf{q})$ vanish simultaneously. In practice, the difference between $D(\omega,\mathbf{q})$ and $D_{\mathrm{obs}}(\omega,\mathbf{q})$ is not important. The point is that the spatial dispersion of $D(\omega,\mathbf{q})$ on the scale $q\sim\Lambda$ (Λ is a parameter of the order of the inverse interatomic distance) is of little interest; only the "anomalous" dispersion, determined by the scale ξ^{-1}, which can arise near the Anderson transition, is of interest. The quantity $B(\mathbf{q})$ does not contain anomalous dispersion, since it is determined by the function $\Delta G_{\mathbf{k}}(\mathbf{q})$, which is regular at the transition point, and the function $e^{(0)}_{\mathbf{k}}(\mathbf{q})$, which can be assumed to be constant (Sec. 5.4); this is also true of the quantity $C(\mathbf{q})$, relating, according to Eq. (54), two regular functions (see, however, Sec. 4). On the basis of what we have said above, the expression (38) assumes the form

$$\phi_{\mathbf{kk}'}(\mathbf{q})=\frac{if^{(0)}_{\mathbf{k}}(\mathbf{q})f^{(0)}_{\mathbf{k}'}(\mathbf{q})}{-i\omega+D(\omega,\mathbf{q})q^2}+\phi^{(1)}_{\mathbf{kk}'}(\mathbf{q}),\quad \phi^{(1)}_{\mathbf{kk}'}(\mathbf{q})\sim q^2. \qquad (56)$$

It follows from the relation (9) that $\phi_{\mathbf{kk}'}(\mathbf{q})$ contains a diffusion pole in the limit $\mathbf{k}+\mathbf{k}'\to0$, which can be separated from $\phi^{(1)}_{\mathbf{kk}'}(\mathbf{q})$:

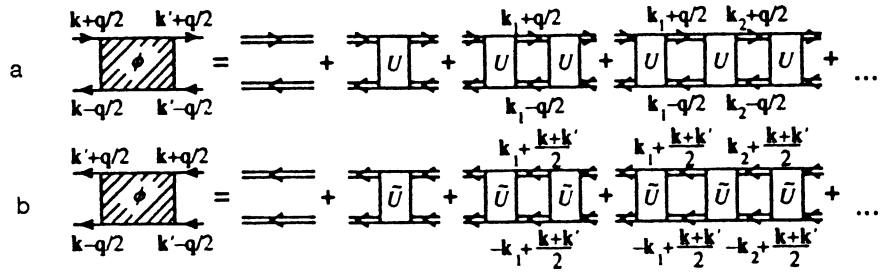

FIG. 2. a—Structure of the diagrammatic series for $\phi_{\mathbf{k}\mathbf{k}'}(\mathbf{q})$. b—same, with the upper G line expanded; the U and $\tilde{U}$ blocks are topologically equivalent, but they correspond to different values of the momenta.

$$\phi_{\mathbf{k}\mathbf{k}'}(\mathbf{q})=\frac{if^{(0)}_{\mathbf{k}}(\mathbf{q})f^{(0)}_{\mathbf{k}'}(\mathbf{q})}{-i\omega+D(\omega,\mathbf{q})q^2}+\frac{if^{(0}_{\mathbf{k}-\mathbf{k}'+\mathbf{q}/2}(\mathbf{k}+\mathbf{k}')f^{(0)}_{\mathbf{k}'-\mathbf{k}+\mathbf{q}/2}(\mathbf{k}+\mathbf{k}')}{-i\omega+D(\omega,\mathbf{k}+\mathbf{k}')(\mathbf{k}+\mathbf{k}')^2}+\phi^{\mathrm{reg}}_{\mathbf{k}\mathbf{k}'}(\mathbf{q}). \tag{57}$$

In diagrammatic language, the pole in the limit $q\to 0$ is related to the fact that in the diagrams containing two or more blocks U (Fig. 2a), the contour of integration in the integrals (58)

$$\int d^dk_i G^R_{\mathbf{k}_i+\mathbf{q}/2}G^A_{\mathbf{k}_i-\mathbf{q}/2} \tag{58}$$

is confined between the poles of two Green's functions. For small U the divergence in the expression (58) in the limit $\omega,q\to 0$ is limited only by the small damping Im Σ in the denominators of the G functions and compensates the smallness associated with the addition of an extra block U; all diagrams in Fig. 2a are found to be of the same order, and the series diverges, leading to a diffusion pole. For arbitrary U the divergence of the series in the limit $\omega,q\to 0$ is guaranteed by the Ward identity (13); it is important that the diffusion pole is determined by diagrams with a large number of blocks U. Since $G(\mathbf{r}_1,\ \mathbf{r}_2)=G(\mathbf{r}_2,\ \mathbf{r}_1)$ the result for $\phi_{\mathbf{k}\mathbf{k}'}(\mathbf{q})$ will remain unchanged, if in constructing the diagrams the upper G line is expanded; then the diagram contains blocks $\tilde{U}$ (Fig. 2b), topologically equivalent to the blocks U, but taken for other values of the momenta. Now the poles of the two G-functions converge toward one another as $\mathbf{k}+\mathbf{k}'\to 0$, giving a second diffusion pole in the expression (57). When the upper G line in Fig. 2a is expanded, the diagrams containing two or more U blocks become irreducible and enter into a $\tilde{U}$ block (Fig. 2b) and, conversely, expansion of the G line in the diagram with one U block generates the entire sequence of diagrams in Fig. 2b with more than two $\tilde{U}$ blocks. For this reason, the second pole term in the expression (57) is contained, with no changes, in $U_{\mathbf{k}\mathbf{k}'}(\mathbf{q})$, differing only by the contribution of the four terminal G lines. The result (12) with a function $F(\mathbf{k},\mathbf{k}',\mathbf{q})$ of the form

$$F(\mathbf{k},\mathbf{k}',\mathbf{q})=if^{(0)}_{(\mathbf{k}-\mathbf{k}'+\mathbf{q})/2}(\mathbf{k}+\mathbf{k}')f^{(0)}_{(\mathbf{k}'-\mathbf{k}+\mathbf{q})/2}(\mathbf{k}+\mathbf{k}')\times(G^R_{\mathbf{k}+\mathbf{q}/2}G^A_{\mathbf{k}-\mathbf{q}/2}G^R_{\mathbf{k}'+\mathbf{q}/2}G^A_{\mathbf{k}'-\mathbf{q}/2})^{-1} \tag{59}$$

is valid for $U_{\mathbf{k}\mathbf{k}'}(\mathbf{q})$. This proves the Vollhardt–Wölfle hypothesis.

4. BEREZINSKIĬ–GOR'KOV CRITERION AND ITS CONSEQUENCES

The spectral density (30) contains a singular contribution $\sim\delta(\omega)$, originating from terms with $s=s'$, that is finite in the localized phase and vanishes in the delocalized phase in the thermodynamic limit. This is the Berezinskiĭ–Gor'kov localization criterion.[24] The $\delta(\omega)$ singularity in $\langle\rho_E\rho_{E+\omega}\rangle_{\mathbf{q}}$ leads to, by virtue of the Eq. (32), a $1/\omega$ singularity in the function $\phi(\mathbf{q})$ (Ref. 10)

$$\phi(\mathbf{q})=\frac{2\pi N(E)}{-i\omega}A(\mathbf{q})+\phi_{\mathrm{reg}}(\mathbf{q}), \tag{60}$$

$$A(\mathbf{q})=\int d\mathbf{r}e^{-i\mathbf{q}\mathbf{r}}A(\mathbf{r}),$$

$$A(\mathbf{r})=\frac{1}{N(E)}\left\langle\sum_s|\psi_s(\mathbf{r})|^2|\psi_s(0)|^2\delta(E-\epsilon_s)\right\rangle. \tag{61}$$

A number of important consequences follow from the relation (60).

1. Comparing Eqs. (60) and (35) shows that in the localized phase $D(\omega,\mathbf{q})\sim\omega$. A slower dependence would destroy the $1/\omega$ singularity in Eq. (60) and a more rapid dependence would cause the dependence on $\mathbf{q}$ to vanish in the singular part (35); such a dependence obviously exists according to Eq. (60). This result, valid in the $D(\omega,\mathbf{q})$ gauge, in which the functions $\phi_{\mathrm{reg}}(\mathbf{q})$ in Eqs. (35) and (60) are identical, remains valid in any other gauge [see Eqs. (53) and (54)]. Therefore

$$D(\omega,\mathbf{q})=(-i\omega)d(q), \tag{62}$$

where it is assumed that the limit $\omega\to 0$ is taken in the function $d(q)$. Therefore it follows from the Berezinskiĭ–Gor'kov criterion that $D(0,\mathbf{q})$ vanishes for all $\mathbf{q}$. This completes the proof of all of the main localization criteria.[6,10] In view of Eq. (62), the second diffusion pole in Eq. (57) leads to the singularity $1/\omega$ in the sum over s in Eq. (51). To eliminate this singularity from $\phi_{\mathrm{reg}}(\mathbf{q})$ the expression (54) must be transformed with $C(\mathbf{q})\sim 1/\omega$, without destroying at the same time the proportionality of $D(\omega,\mathbf{q})$ and $\tilde{D}(\omega,\mathbf{q})$ to the frequency. The function $C(\mathbf{q})$ is determined, by virtue of Eq. (57), by the quantities $\Delta G_{\mathbf{k}}(\mathbf{q})$ and $e^{(0)}_{\mathbf{k}}(\mathbf{q})$, which are regular at the transition point and do not lead to anomalous dispersion, while the associated renormalization of $D(\omega,\mathbf{q})$ is small near the transition because of the divergence of $d(q)$ (see below).

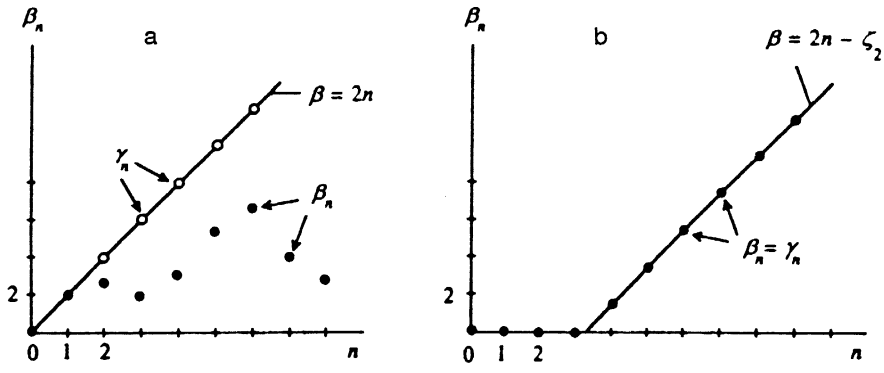

FIG. 3. Possible configurations of the exponents β_n and γ_n for $\zeta<0$ (a) and $\zeta>0$ (b).

2. A relation between the diffusion coefficient and the properties of the wave functions follows from Eqs. (35), (60), and (62):

$$\frac{1}{1+d(q)q^2}=A(\mathbf{q})=\int d\mathbf{r}e^{-i\mathbf{qr}}A(\mathbf{r}). \tag{63}$$

Exponential localization of the wave functions leads to exponential decay of $A(\mathbf{r})$ in the limit $r\to\infty$ [see Eq. (61)] and the finiteness of the coefficients in the expansion over $\mathbf{q}$ of the right-hand side of Eq. (63). Because of isotropy in the mean, there are no odd powers of $\mathbf{q}$ and $d(q)$ is a regular function of q^2; it is important that it does not contain noninteger powers of q, which arise naturally in the case of diffusion over fractal structures.[41] The reality and positiveness of $d(q)$ follow from the reality of $A(\mathbf{q})$ and the inequalities $0\leqslant A(\mathbf{q})\leqslant 1$.[10,24]

3. Restrictions on the form of the spatial dispersion of $D(\omega,\mathbf{q})$ follow from the relation (63). In the localized phase the spatial dispersion is determined by the expansion[2)]

$$1+d(q)q^2=\xi^{\beta_0}+\xi^{\beta_1}q^2+\xi^{\beta_2}q^4+\ldots+\xi^{\beta_n}q^{2n}+\ldots,$$
$$\beta_0=0, \tag{64}$$

where $\beta_n\geqslant 0$, since the contributions associated with the atomic scale Λ^{-1} and the corresponding $\beta_n=0$, obviously exist.

Different estimates show that the smoothed (over oscillations) behavior of the squared modulus of a typical wave function has the form

$$|\psi(\mathbf{r})|^2=\text{const}\begin{cases} r^{-b}, & \Lambda^{-1}\lesssim r\lesssim\xi \\ \exp(-r/\xi), & r\gtrsim\xi \end{cases}. \tag{65}$$

This behavior should actually be expected on the basis of Poincaré's theorem on the analytic dependence of the solution of a differential equation on a parameter: If the behavior of the wave function at the transition point is characterized by the exponent b $(0\leqslant b\leqslant\infty)$, $|\psi_c(\mathbf{r})|^2\sim r^{-b}$, then near the transition for large $\mathbf{r}$ we have $\psi(\mathbf{r})\approx\psi_c(\mathbf{r})$ on the basis of Poincaré's theorem; the theorem is valid only for a finite region, whose maximum size is determined by the scale ξ on which the exponential decrease of $\psi(\mathbf{r})$ starts. By virtue of Eq. (61) the function $A(\mathbf{r})$ has a similar behavior

$$A(\mathbf{r})=\text{const}\begin{cases} r^{-d-\zeta}, & \Lambda^{-1}\lesssim r\lesssim\xi \\ \exp(-r/\xi), & r\gtrsim\xi \end{cases}, \tag{66}$$

where const is chosen from the condition $A(q)=1$ at $q=0$. The series expansion of $A(q)$

$$A(\mathbf{q})=\xi^{\gamma_0}+\xi^{\gamma_1}q^2+\xi^{\gamma_2}q^4+\ldots+\xi^{\gamma_n}q^{2n}+\ldots, \quad \gamma_0=0 \tag{67}$$

and the estimate of the integrals arising in Eq. (61) show that only two variants are possible: (a) $\gamma_n=2n$ for $\zeta<0$ and (b) $\gamma_n=\max\{0,2n-\zeta\}$ for $\zeta>0$. Substituting the expressions (64) and (67) into Eq. (63) gives a relation between γ_n and β_n

$$\gamma_n=\max_{i+j+k+\cdots=n}\{\beta_i+\beta_j+\beta_k+\ldots\}, \tag{68}$$

leading to the two possibilities for the exponents β_n: $\beta_1=2$, $\beta_n\leqslant 2n$ for $\zeta<0$ (Fig. 3a) and $\beta_n=\max\{0,2n-\zeta\}$ for $\zeta>0$ (Fig. 3b). For these results to be valid it is important only that if the integral of $A(\mathbf{r})r^n$ diverges in the limit $\xi\to\infty$ as ξ^a, then the integral of $A(\mathbf{r})r^{n+m}$ should diverge as ξ^{a+m}, since it is determined by the region $r\sim\xi$. The specific approximation (66) is actually not used, but it is convenient for interpreting the results. To determine the localization length ξ from the known diffusion coefficient, in general, it is necessary to know all the exponents β_n. The result $D(\omega,0)\sim(-i\omega)\xi^2$ proposed in Refs. 18 and 10 is valid only for $\zeta<0$. From Eq. (25) we obtain for the permittivity

$$\epsilon(0,0)=1+4\pi e^2N(\epsilon_F)d(0)=\begin{cases} \sim\xi^2, & \zeta<0 \\ \sim\xi^{2-\zeta}, & 0<\zeta<2, \\ \sim 1, & \zeta>2 \end{cases} \tag{69}$$

i.e., on the basis of the general analysis, $\epsilon(0,0)$ can diverge according to a law that is different from ξ^2 (obtained by cutting off the metallic behavior of $\epsilon(0,\mathbf{q})\sim q^{-2}$ on the scale $q\sim\xi^{-1}$), and $\epsilon(0,0)$ can even be finite in the limit $\xi\to\infty$ (see the discussion in Refs. 23, 40, and 41).

5. BASIC STRUCTURE OF THE THEORY

It is convenient to begin the construction of the theory by analyzing the localized phase, obtaining the metallic state as a result of the instability of the localized phase.

5.1 Spectrum of the operator $\hat{L}$ in the localized phase

Let M be the set of values of the index s that enumerates the eigenvalues λ_s of the operator $\hat{L}$. We shall show that in the localized phase the decomposition

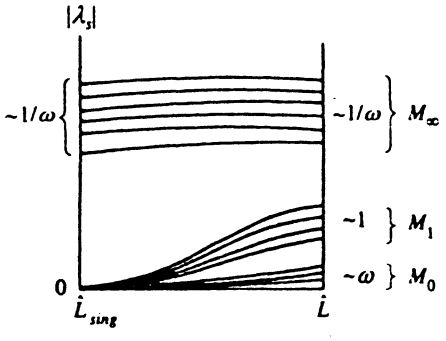

FIG. 4. Evolution of the spectrum of eigenvalues λ_s on transferring from $\hat{L}_{sing}$ to $\hat{L}$, i.e., with the "gradual switching on" of the operator $\hat{L}_{reg}$.

$$M = M_0 \oplus M_1 \oplus M_\infty \tag{70}$$

such that

$$\lambda_s = \begin{cases} \omega \nu_s, & s \in M_0 \\ \nu_s, & s \in M_1, \quad \nu_s \sim 1 \\ \nu_s/\omega, & s \in M_\infty \end{cases} \tag{71}$$

(Fig. 4) is valid. The set M_0 is not empty, since it contains the element $\lambda_0 \sim \omega$, related with the diffusion coefficient. We shall show that it is not the only element. According to Eq. (57) $\phi_{\mathbf{k}\mathbf{k}'}(\mathbf{q})$ contains the singularity $\sim 1/\omega$, associated with two diffusion poles. In Eq. (38) this feature originates from terms with $s \in M_0$. Comparing these two representations and taking into account the fact that the diffusion pole at $q=0$ corresponds to the term with $s=0$ in Eq. (38), we obtain

$$\frac{f^{(0)}_{\mathbf{k}-\mathbf{k}'+\mathbf{q}/2}(\mathbf{k}+\mathbf{k}') f^{(0)}_{\mathbf{k}'-\mathbf{k}+\mathbf{q}/2}(\mathbf{k}+\mathbf{k}')}{1+d(\mathbf{k}+\mathbf{k}')(\mathbf{k}+\mathbf{k}')^2} = i \sum_{s \in M_0'} \frac{f^{(s)}_{\mathbf{k}}(\mathbf{q}) f^{(s)}_{\mathbf{k}'}(\mathbf{q})}{1+\nu_s(\mathbf{q})}, \tag{72}$$

where M_0' is the set M_0 without the element $s=0$. Since $d(q)$ diverges as $\xi \to \infty$ [see Eq. (64)], the left-hand side of Eq. (72) contains a δ-function singularity in the limit $\mathbf{k}+\mathbf{k}' \to 0$, which terms of the form $f_{\mathbf{k}} f_{\mathbf{k}'}$ on the right-hand side of Eq. (72) with $\mathbf{k}' \to -\mathbf{k}$ cannot have at the point $\mathbf{k}$ which is in no way distinguished. The same is true of the sum of a finite number of such terms. The example of the Fourier expansion

$$\frac{1}{1+d(\mathbf{k}+\mathbf{k}')(\mathbf{k}+\mathbf{k}')^2} = \sum_{\mathbf{x}} A_{\mathbf{x}} e^{i(\mathbf{k}+\mathbf{k}')\mathbf{x}} \tag{73}$$

shows that the pole term in Eq. (72) can be reproduced by an infinite number of terms of the form $f_{\mathbf{k}} f_{\mathbf{k}'}$ and that this does not require a complete system of functions (eliminating from the sum in Eq. (73) terms with small $\mathbf{x}$ leads to the appearance of a smooth component, but it does not change the singularity in the limit $\mathbf{k}+\mathbf{k}' \to 0$). It is clear from what has been said above that the set M_0 contains an infinite number of elements, but generally speaking it does not coincide with the set M.

Sadovskiĭ[10,44] proposed a localization criterion according to which a nontrivial solution of the homogeneous Bethe–Solpeter equation appears in the limit $\omega \to 0$. A stronger assertion follows from what has been said above: An infinite number of such solutions appears at the transition point.

The following decomposition of the operator $\hat{L}$ follows from Eqs. (36) and (12):

$$\hat{L} = \hat{L}_{reg} + \hat{L}_{sing}, \tag{74}$$

$$\hat{L}_{reg}\psi_{\mathbf{k}} \equiv \hat{M}\psi_{\mathbf{k}} + \Delta\Sigma_{\mathbf{k}}(\mathbf{q})\psi_{\mathbf{k}} - \frac{1}{N}\sum_{\mathbf{k}'} U^{reg}_{\mathbf{k}\mathbf{k}'}(\mathbf{q}) \sqrt{\Delta G_{\mathbf{k}}(\mathbf{q}) \Delta G_{\mathbf{k}'}(\mathbf{q})}\,\psi_{\mathbf{k}'}, \tag{75}$$

$$\hat{L}_{sing}\psi_{\mathbf{k}} \equiv -\frac{1}{N}\sum_{\mathbf{k}'} U^{sing}_{\mathbf{k}\mathbf{k}'}(\mathbf{q}) \sqrt{\Delta G_{\mathbf{k}}(\mathbf{q}) \Delta G_{\mathbf{k}'}(\mathbf{q})}\,\psi_{\mathbf{k}'}. \tag{76}$$

In the localized phase the diffusion pole in U^{sing} gives a $1/\omega$ singularity:

$$\hat{L} = \hat{L}_{reg} + \hat{L}_1/\omega, \tag{77}$$

where the limit $\omega \to 0$ has been taken in the operator $\hat{L}_1$. Terms of higher order in ω are included in $\hat{L}_{reg}$. From Eqs. (76), (12), and (72) we obtain the following representation for $\hat{L}_1$

$$\hat{L}_1\psi_{\mathbf{k}} = \frac{1}{N}\sum_{\mathbf{k}'}\left(\sum_{s \in M_0'} \frac{g^{(s)}_{\mathbf{k}}(\mathbf{q}) g^{(s)}_{\mathbf{k}'}(\mathbf{q})}{1+\nu_s(\mathbf{q})}\right)\psi_{\mathbf{k}'},$$

$$g^{(s)}_{\mathbf{k}}(\mathbf{q}) = \frac{f^{(s)}_{\mathbf{k}}(\mathbf{q})\sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})}}{G^R_{\mathbf{k}+\mathbf{q}/2} G^A_{\mathbf{k}-\mathbf{q}/2}}, \tag{78}$$

whence it is clear that the eigenvectors of the operator $\hat{L}_1$, corresponding to nonzero eigenvalues, lie in the subspace constructed on the vectors $g^{(s)}_{\mathbf{k}}(\mathbf{q})$, and the number of eigenvalues is equal to the number of elements in M_0'. The nonzero eigenvalues of $\hat{L}_1$ correspond to the eigenvalues $\sim 1/\omega$ of the operator $\hat{L}_{sing}$.

The overall picture is as follows (Fig. 4). The operator $\hat{L}_{sing}$ has an infinite number of eigenvalues $\sim 1/\omega$ and an infinite number of eigenvalues equal to zero. When the operator $\hat{L}_{reg} \sim 1$ is added, the eigenvalues $\sim 1/\omega$ change very little and form the set M_∞ of the operator $\hat{L}$; the zero eigenvalues become, generally speaking, of order one, forming the set M_1, but some of them remain $\sim \omega$ and lie in the set M_0. The number of elements in M_∞ is equal to the number of elements in M_0'; no assertions can be made with respect to the set M_1, but this is not important for what follows.

5.2 Relation between $\hat{L}$ and $\hat{L}_{sing}$

We now introduce the spectral representation for the singular part of the operator $\hat{L}_{sing} = \hat{L}_1/\omega$

$$\hat{L}_1 = \sum_s |u_s\rangle \eta_s \langle u_s|, \quad \eta_s = \begin{cases} 0 & s \in M_0 \oplus M_1 \\ \sim 1 & s \in M_\infty \end{cases} \tag{79}$$

and find a relation between $\hat{L}$ and $\hat{L}_{sing}$, regarding $\hat{L}_{reg}$ as a perturbation. For $s \in M_\infty$ the ordinary perturbation theory can be used, since all differences of the eigenvalues $\sim 1/\omega$ and a regular expansion in powers of ω is obtained:

$$|e_s\rangle=|u_s\rangle+\omega\sum_{s'\neq s}\frac{\langle u_{s'}|\hat{L}_{\rm reg}|u_s\rangle}{\eta_s-\eta_{s'}}|u_{s'}\rangle,$$

$$\lambda_s=\frac{\eta_s}{\omega}+\langle u_s|\hat{L}_{\rm reg}|u_s\rangle,\quad s\in M_\infty. \tag{80}$$

For $s\in M_0\oplus M_1$ we seek the eigenfunctions of $\hat{L}$ in the form

$$|e\rangle=\sum_{s\in M_0\oplus M_1}C_s|u_s\rangle+\omega\sum_{s\in M_\infty}D_s|u_s\rangle, \tag{81}$$

where $C_s, D_s\sim 1$. Substituting for the eigenvalues in the equation, we obtain a system of equations for C_s and D_s, which can be solved by iterations in ω. Eliminating D_s, we obtain to first order in ω

$$\sum_{s'\in M_0\oplus M_1}(\lambda_s\delta_{ss'}-T_{ss'})C_{s'}=0,\quad s\in M_0\oplus M_1,$$

$$T_{ss'}=\langle u_s|\hat{L}_{\rm reg}|u_{s'}\rangle-\omega\sum_{s''\in M_\infty}\frac{\langle u_s|\hat{L}_{\rm reg}|u_{s''}\rangle\langle u_{s''}|\hat{L}_{\rm reg}|u_{s'}\rangle}{\eta_{s''}}, \tag{82}$$

i.e., an ordinary secular equation taking into account the first correction from transitions into states with $s\in M_\infty$.

5.3 Mechanism for satisfying the Ward identity

We now demonstrate the cancelling of the singular contribution $\sim 1/\omega$, associated with the diffusion pole in $U_{\mathbf{kk}'}(\mathbf{q})$, on the right-hand side of the Ward identity (13). The specific form of $\hat{L}_{\rm reg}$ was not used in Eqs. (5.1) and (5.2). To determine $\hat{L}_{\rm reg}$ in the form (75) with $\hat{M}\equiv 0$ we have

$$\frac{1}{N}\sum_{\mathbf{k}}\sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})}e_{\mathbf{k}}^{(s)}(\mathbf{q})=0,\quad s\in M_\infty, \tag{83}$$

since $\sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})}$ is the exact eigenfunction of $\hat{L}$ belonging to the set M_0. By virtue of the relation (80), the difference of $|u_s\rangle$ from $|e_s\rangle$ for $s\in M_\infty$ is of order ω for any $\hat{L}_{\rm reg}$, whence

$$\frac{1}{N}\sum_{\mathbf{k}}\sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})}u_{\mathbf{k}}^{(s)}(\mathbf{q})=O(\omega),\quad s\in M_\infty. \tag{84}$$

Comparing Eqs. (76) and (79), we have

$$-U_{\mathbf{kk}'}^{\rm sing}(\mathbf{q})\sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})\Delta G_{\mathbf{k}'}(\mathbf{q})}=\frac{1}{\omega}\sum_{s\in M_\infty}u_{\mathbf{k}}^{(s)}\times(\mathbf{q})\eta_s(\mathbf{q})u_{\mathbf{k}'}^{(s)}(\mathbf{q}), \tag{85}$$

so that the singular contribution on the right-hand side of Eq. (13), taking into account Eq. (84), has the form

$$\frac{1}{N}\sum_{\mathbf{k}'}U_{\mathbf{kk}'}^{\rm sing}\Delta G_{\mathbf{k}'}(\mathbf{q})=-\frac{1}{\omega}\sum_{s\in M_\infty}\frac{u_{\mathbf{k}}^{(s)}(\mathbf{q})\eta_s(\mathbf{q})}{\sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})}}\frac{1}{N}\times\sum_{\mathbf{k}'}\sqrt{\Delta G_{\mathbf{k}'}(\mathbf{q})}u_{\mathbf{k}'}^{(s)}(\mathbf{q})=\frac{O(\omega)}{\omega} \tag{86}$$

and the $1/\omega$ singularity cancels. For the same reason, there will be no singularities on the right-hand side of Eq. (13) as the transition into the metallic phase is approached. In this case the spectrum of the operator $\hat{L}$ has the same structure (Fig. 4) with ω replaced by D_0—the characteristic value of the diffusion coefficient (Sec. 6.2)—and we obtain on the right-hand side of Eq. (86) $O(D_0)/D_0$.

5.4 Symmetry approach

The symmetry of the system is clearly expressed in the properties of the operator $\hat{L}_{\rm sing}$:

(a) The spatial uniformity in the mean makes it possible to introduce the three-momentum notations (Fig. 1d) and to introduce the operator $\hat{L}$ in general and the operator $\hat{L}_{\rm sing}$ in particular.

(b) The isotropy in the mean, combined with time-reversal invariance, guarantees that $\hat{L}$ and $\hat{L}_{\rm sing}$ are self-conjugate and the existence of orthonormal bases of eigenvectors for them.

(c) As a result of time-reversal invariance, $\hat{L}_{\rm sing}$ has a high symmetry, manifested in the existence of an infinite number of zero modes.[3)]

The decomposition (77) represents the operator $\hat{L}$ as a sum of the operator $\hat{L}_{\rm sing}$ with a high degree of symmetry and a regular operator $\hat{L}_{\rm reg}$ of a general type. It is similar to the decomposition (1) and is convenient for symmetry analysis. The condition on the transition point will be determined below and the origin of the parameter τ will thereby be determined.

Following Sec. 1, we consider the response of the system to a perturbation $\delta\hat{L}_{\rm reg}$ of a general form. Not all changes in the system will be important. We decompose the change in the operator $\hat{L}$ into two parts

$$\delta\hat{L}=\delta\hat{L}_\lambda+\delta\hat{L}_e, \tag{87}$$

where $\delta\hat{L}_\lambda$ changes the eigenvalues and $\delta\hat{L}_e$ changes the eigenfunctions of $\hat{L}$. For an infinitesimal change $\delta\hat{L}$ such a decomposition is trivial—$\delta\hat{L}_\lambda$ and $\delta\hat{L}_e$ are the diagonal and off-diagonal parts of $\delta\hat{L}$ in the representation of the eigenvectors $|e_s\rangle$. Changes of the type $\delta\hat{L}_e$ do not change the eigenvalues of $\hat{L}$ and therefore the diffusion coefficient $D(\omega,\mathbf{q})$, directly related with $\lambda_0(\mathbf{q})$ and determining uniquely the location of the system—in a localized phase, in a metallic phase, or at the transition point. It is clear that the changes $\delta\hat{L}_e$ do not drive the system out of the transition point, they only displace the system along the critical surface.[1] Such displacements do not lead to nonanalyticity of the physical quantities[4)] and they can be ignored. The critical exponents obtained by motion along the normal to the critical surface are identical to the exponents obtained under an arbitrary nonzero angle to the tangent plane. Similarly, in perturbations of the type $\delta\hat{L}_\lambda$ the part corresponding toa change in λ_s with $s\in M_1\oplus M_\infty$ need not be considered.

Only the changes in the eigenvalues λ_s from the set M_0, whose response to a perturbation is indeed nontrivial, are important. Let the system lie deep in the localized phase. A small perturbation $\delta\hat{L}_{\rm reg}$ does not drive the system out of the state of localization and preserves the proportionality $\lambda_s\sim\omega$ for $s\in M_0$. On the other hand, a perturbation $\delta\hat{L}_{\rm reg}$ of a general type possesses nonzero matrix elements with the respect to the eigenvectors $|e_s\rangle$ of the subspace M_0 and

should lead to small but nonvanishing, in the limit $\omega \to 0$, values of λ_s. The resolution of this contradiction will lead to the self-consistency equation (Sec. 5.6).

5.5 "Rotation" of the singular operator

To formulate an adequate language for the further discussion, we shall examine the following problem of the "rotation" of a singular operator.

Let the decomposition (77), where $\omega \to 0$, be valid for the operator $\hat{L}$. The operator $\hat{L}_{\text{reg}}$ acts in the space Ω, while the operator $\hat{L}_1$ has nonzero eigenvalues ~ 1 in the subspace Ω_1, which is a part of $\Omega = \Omega_0 \oplus \Omega_1$. This justifies retaining in Eq. (77) two terms of different orders. Let $\delta\hat{L}_1$ be a perturbation of the operator $\hat{L}_1$. If this perturbation is of a general form, then the correction $\delta\hat{L}_1/\omega$ to the operator $\hat{L}$ can be studied by the standard perturbation theory and gives corrections $\sim 1/\omega$. Let the perturbation $\delta\hat{L}_1$ be such, however, that the operator $\hat{L}_1 + \delta\hat{L}_1$ has the same properties as the initial operator $\hat{L}_1$. Then the dimension of the subspace Ω_1 remains the same; only a "rotation" of the operator occurs (in this case $\delta\hat{L}_1$ has no nonzero matrix elements in Ω_0). It is required to determine the result of such a perturbation in the subspace Ω_0.

Let $\bar{\eta}_s$ and $|\bar{u}_s\rangle$ be the eigenvalues and eigenvectors of the initial operator $\hat{L}_1$. The operator $\hat{L}_{\text{reg}}$ can be neglected in the "upper" subspace Ω_1, and in the "lower" subspace Ω_0 a secular equation in the matrix elements $\langle \hat{u}_s|\hat{L}_{\text{reg}}|\bar{u}_{s'}\rangle$ must be formulated. The perturbation $\delta\hat{L}_1$ produces the change $\delta u_s \sim \delta\hat{L}_1$ of the eigenvectors $|u_s\rangle$ and the matrix of the secular equation is determined by the elements

$$\langle u_s|\hat{L}_{\text{reg}}|u_{s'}\rangle = \langle \bar{u}_s + \delta u_s|\hat{L}_{\text{reg}}|\bar{u}_{s'} + \delta u_{s'}\rangle \equiv \langle \bar{u}_s|\hat{L}_{\text{reg}} + \delta\hat{V}|\bar{u}_{s'}\rangle. \tag{88}$$

The qualitative result is that a limitation of the form of the operator $\delta\hat{L}_1$ weakens its action on the lower subspace: The effective perturbation $\delta\hat{V}$ is found to be $\sim \delta\hat{L}_1$ instead of $\delta\hat{L}_1/\omega$ for the operator of general form.

The change in $|u_s\rangle$ in the subspace Ω_1 can be calculated by the standard perturbation theory, since all differences of the eigenvalues ~ 1 and a series in the small parameter arises:

$$|u_s\rangle = |\bar{u}_s\rangle + \sum_{s' \neq s} \frac{\langle \bar{u}_{s'}|\delta\hat{L}_1|\bar{u}_s\rangle}{\bar{\eta}_s - \bar{\eta}_{s'}}|\bar{u}_{s'}\rangle, \quad s \in \Omega_1. \tag{89}$$

An arbitrary choice of $|u_s\rangle$ that is compatible with the orthogonality relations can be made, in view of degeneracy, in the subspace Ω_0. To first order in $\delta\hat{L}_1$ we can set

$$|u_s\rangle = |\bar{u}_s\rangle - \sum_{s' \in \Omega_1} \frac{\langle \bar{u}_s|\delta\hat{L}_1|\bar{u}_{s'}\rangle}{\bar{\eta}_{s'}}|\bar{u}_{s'}\rangle, \quad s \in \Omega_0. \tag{90}$$

Substituting the expression (90) into Eq. (88), we obtain for the matrix elements of the effective perturbation

$$\langle \bar{u}_s|\delta\hat{V}|\bar{u}_{s'}\rangle = -\sum_{s'' \in \Omega_1} \frac{\langle \bar{u}_s|\delta\hat{L}_1|\bar{u}_{s''}\rangle\langle \bar{u}_{s''}|\hat{L}_{\text{reg}}|\bar{u}_{s'}\rangle + \langle \bar{u}_{s'}|\delta\hat{L}_1|\bar{u}_{s''}\rangle\langle \bar{u}_{s''}|\hat{L}_{\text{reg}}|\bar{u}_s\rangle}{\bar{\eta}_{s''}}. \tag{91}$$

5.6 Self-consistency equation

It is now easy to understand how to resolve the contradiction stated in Sec. 5.4. The perturbation $\delta\hat{L}_{\text{reg}}$ produces the change $\delta d(q)$ in the diffusion coefficient (62), which in view of the relation

$$\hat{L}_{\text{sing}}\psi_{\mathbf{k}} = \frac{1}{N}\sum_{\mathbf{k}'} \frac{W(\mathbf{k},\mathbf{k}',\mathbf{q})\psi_{\mathbf{k}'}}{-i\omega + D(\omega,\mathbf{k}+\mathbf{k}')(\mathbf{k}+\mathbf{k}')^2} = \frac{1}{(-i\omega)}\int \frac{d^d\tilde{q}}{(2\pi)^d}\frac{W(\mathbf{k},-\mathbf{k}+\tilde{\mathbf{q}},\mathbf{q})\psi_{-\mathbf{k}+\tilde{\mathbf{q}}}}{1+d(\tilde{q})\tilde{q}^2} \equiv \frac{\hat{L}_1\psi_{\mathbf{k}}}{\omega} \tag{92}$$

gives the following change in $\hat{L}_1$:

$$\delta\hat{L}_1\psi_{\mathbf{k}} = (-i)\int \frac{d^d\tilde{q}}{(2\pi)^d}\frac{\tilde{q}^2\delta d(\tilde{q})}{[1+d(\tilde{q})\tilde{q}^2]^2} \times W(\mathbf{k},-\mathbf{k}+\tilde{\mathbf{q}},\mathbf{q})\psi_{-\mathbf{k}+\tilde{\mathbf{q}}}. \tag{93}$$

Rotation of the subspace M_∞ of the operator $\hat{L}_{\text{sing}}$ produces in the subspace M_0 the effective perturbation $\delta\hat{V}$, which in zeroth order in ω compensates $\delta\hat{L}_{\text{reg}}$.

Introducing into Eq. (82) the small changes $\delta\hat{L}_{\text{reg}}$ and $\delta\hat{L}_1$ [the latter enters via the change in the eigenfunctions (90)], we obtain for the matrix of the secular equation

$$T_{ss'} = \langle \bar{u}_s|\bar{T} + \delta\hat{L}_{\text{reg}} + \delta\hat{V}|\bar{u}_{s'}\rangle \quad s,s' \in M_0 \oplus M_1, \tag{94}$$

where the overbar denotes the unperturbed value, and $\bar{T}$ and $\delta\hat{V}$ are determined by the expressions (82) and (91) (with the substitution $\Omega_1 \to M_\infty$, $\Omega_0 \to M_0 \oplus M_1$). In the terms $\sim \omega$ we confine ourselves to zeroth order in the increments. The choice of the vectors $|\bar{u}_s\rangle$ in the subspace $M_0 \oplus M_1$ is arbitrary in view of the degeneracy. We choose them so as to diagonalize the matrix $\hat{\bar{T}}$—then, to zeroth order in ω, they are identical to the eigenvectors $|\bar{e}_s\rangle$ of the operator $\hat{\bar{L}}$ [see Eq. (81)]. Since the eigenvalues of the matrix $\hat{T}$ are identical to the eigenvalues of $\hat{L}$, we have

$$T_{ss'} = \bar{\lambda}_s\delta_{ss'} + \langle \bar{e}_s|\delta\hat{L}_{\text{reg}} + \delta\hat{V}|\bar{e}_{s'}\rangle, \quad s,s' \in M_0 \oplus M_1. \tag{95}$$

For infinitesimal $\delta\hat{L}_{\rm reg}$ and $\delta\hat{V}$, the diagonal elements of the matrix $\hat{T}$ determine the eigenvalues of the operator $\hat{L}$

$$\lambda_s=\bar{\lambda}_s+\langle\bar{e}_s|\,\delta\hat{L}_{\rm reg}+\delta\hat{V}|\bar{e}_s\rangle,\quad s\in M_0\oplus M_1 \tag{96}$$

and the off-diagonal elements determine the corrections to its eigenfunctions; the latter correspond to the perturbations of the type $\delta\hat{L}_e$ (Sec. 5.4) and can be dropped. For constant $|e_s\rangle$ it is possible to switch in Eq. (96) from infinitesimals to finite increments. Further, the changes in λ_s in the subspace M_1 can be ignored (Sec. 5.4). Finally, we note that fixing $\lambda_0(\mathbf{q})$ for all $\mathbf{q}$ means fixing the diffusion coefficient, which in turn determines all $\lambda_s(\mathbf{q})$ with $s\in M_0'$, which can be reconstructed according to the binary decomposition (72). Therefore if Eq. (96) is satisfied for $s=0$

$$-i[D(\omega,q)q^2-\bar{D}(\omega,q)q^2]=\langle\bar{e}_0|\,\delta\hat{L}_{\rm reg}|\bar{e}_0\rangle - 2\sum_{s''\in M_\infty}\frac{\langle\bar{e}_0|\,\delta\hat{L}_1|\bar{e}_{s''}\rangle\langle\bar{e}_{s''}|\hat{L}_{\rm reg}|\bar{e}_0\rangle}{\bar{\eta}_{s''}} \tag{97}$$

(Eqs. (55), (91), and (80) were employed), then it is automatically satisfied for all s from M_0. It is easy to show (see Appendix) that the expansion in $\mathbf{q}$ of the right-hand side of Eq. (97) contains only even powers of $\mathbf{q}$, and terms $\sim q^0$ are absent in each of the two terms. Setting

$$\langle\bar{e}_0|\,\delta\hat{L}_{\rm reg}|\bar{e}_0\rangle=-iq^2\delta f(q) \tag{98}$$

and substituting the expression (93) into Eq. (97), we obtain

$$D(\omega,q)-\bar{D}(\omega,q)=\delta f(q)+\hat{Q}\,\delta d(q), \tag{99}$$

$$\hat{Q}\,\delta d(q)=\int\frac{d^d\tilde{q}}{(2\pi)^d}\frac{B(q,\tilde{q})\tilde{q}^2}{[1+d(\tilde{q})\tilde{q}^2]^2}\,\delta d(\tilde{q}), \tag{100}$$

where $\delta f(q)$ and $B(q,\tilde{q})$ are regular functions of general form of the arguments q^2 and $\tilde{q}^2$. The quantity $W(\mathbf{k},\mathbf{k}',\mathbf{q})$ in Eq. (92) can be expressed, by virtue of Eqs. (76), (12), and (59), in terms of functions which are regular at the transition point. This makes it possible to vary only $d(q)$ on switching from Eq. (92) to Eq. (93). The equation (99) contains the diffusion coefficient on the right- and left-hand sides and replaces the self-consistency equation (16) of the Vollhardt–Wölfle theory.

5.7 Condition on the transition point

In the localized phase $D(\omega,q)$ and $\bar{D}(\omega,q)$ vanish at $\omega=0$ and Eq. (99) determines the change $\delta d(q)=-\hat{Q}^{-1}\delta f(q)$ for the prescribed perturbation $\delta\hat{L}_{\rm reg}$. Making the small changes $\delta\hat{L}_{\rm reg}$, we obtain the corresponding changes $\delta d(q)$, which preserve the proportionality $D(\omega,q)\sim(-i\omega)$. This situation remains as long as there exists an operator inverse to $\hat{Q}$, i.e. as long as all eigenvalues of $\hat{Q}$ are nonzero. Let a nonzero eigenvalue of the operator $\hat{Q}$ appear at some point in the course of the motion from the interior of the localized phase. As we shall see below, such a point corresponds to the physical notions of the Anderson transition.

The divergence of $d(q)$ in the limit $\xi\to\infty$ (Sec. 4) means [see Eq. (100)] that at the transition point the operator $\hat{Q}$ vanishes entirely or on some subspace. For this reason, it should be kept in mind in the analysis that many or even all eigenvalues μ_s of the operator $\hat{Q}$, for each of which it is convenient to introduce the critical exponent $\delta_s\geqslant 0$

$$\hat{Q}\phi_s(q)=\mu_s\phi_s(q),\quad \mu_s\sim\tau^{\delta_s} \tag{101}$$

can vanish simultaneously at the transition point. As $d(q)\to\infty$, the changes of the function $B(q,\tilde{q})$ cannot make the operator $\hat{Q}$ finite and therefore they do not drive the system out of the critical point; they only displace it along the critical surface and can be ignored. For the function $B(q,\tilde{q})$ which is independent of τ, the equality

$$\int\frac{d^dq}{(2\pi)^d}B(q,\tilde{q})\phi(q)=0 \tag{102}$$

cannot be satisfied for any function $\phi(q)$, since it corresponds to the presence of a zero mode for the transposed operator $\hat{Q}^T$ and therefore for the operator $\hat{Q}$ itself, not only at the transition point but also in an entire neighborhood of the transition point.

6. SOLUTION OF THE SELF-CONSISTENT EQUATION

6.1 Classification of the possible solutions

The self-consistent equation for the metallic phase can be derived only by making specific assumptions about the functional form $D(\omega,q)$. For this reason, it is convenient to examine several cases which exhaust all possibilities.

(a) Let there be among the exponents δ_s in Eq. (101) a maximum exponent (for definiteness, δ_0), i.e., among the set of soft modes, one mode is the softest. Then, as the transition is approached, the component $\mathrm{const}\phi_0(q)$, contained in $\delta f(q)$, will give rise to an anomalously large response $\mathrm{const}\tau^{-\delta_0}\phi_0(q)$ in the function $\delta d(q)$. For this reason, near the transition the solution can be sought in the form

$$D(\omega,q)=D_0[\phi_0(q)+\varphi(q)],\quad \varphi(q)\ll\phi_0(q). \tag{103}$$

For $\phi_0(q)$ to dominate for all values of q, it is necessary that $\phi_0(0)\neq 0$, which we shall assume is the case.

(b) Let several exponents have the maximum value $\delta_0=\delta_1=\ldots=\delta_p$, and let at least one of the functions $\phi_0(q)$, $\phi_1(q),\ldots,\phi_p(q)$ [for example $\phi_0(q)$] be different from zero at $q=0$. Then near the transition

$$D(\omega,q)=D_0[\phi_0(q)+C_1\phi_1(q)+\ldots + C_p\phi_p(q)+\varphi(q)], \tag{104}$$

where $C_1\sim C_2\sim\ldots\sim C_p\sim 1$, $\varphi(q)\ll\phi_0(q)$.

(c) If for the two eigenfunctions as $q\to 0$ we have $\phi_0(q)\sim q^{2n_0}$, $\phi_1(q)\sim q^{2n_1}$ and $n_0>n_1$, $\delta_0>\delta_1$, then in the expansion of $D(\omega,q)$ in $\phi_s(q)$ both functions must be retained. Although the coefficient of $\phi_0(q)$ grows more rapidly near a transition, the function $\phi_1(q)$ dominates for small values of q. In the general case, $d(q)$ must be sought in the form of the expansion (64) with arbitrary β_n.

Actually, as we shall see below, the case (b) is realized (Sec. 6.3), but the analysis of this case is virtually identical to the simpler case (a) (Sec. 6.2), which reproduces the solution of the self-consistent theory of localization.[18] Analysis of the case (c) requires a special mathematical apparatus

(Sec. 7), and it is sufficient to perform the analysis for the localized phase, since the solutions which differ from those reducing to (b) do not exist.

6.2 Case of a single dominant mode

We seek the solution in the form (103). The definition of the operators $\hat{L}_1$ and $\hat{Q}$ in Sec. 5 presumed that only the dependence on ω in the localized phase is investigated. To investigate the dependence on ω and τ it is necessary to take into account the fact that near the transition and in the metallic phase the magnitude of the diffusion denominator is determined by the parameter $D_0 \gg \omega$. Making the decomposition

$$\frac{1}{-i\omega+D(\omega,q)q^2}=\frac{1}{D_0}\left\{\frac{1}{\phi_0(q)q^2}-\frac{\varphi(q)-\dfrac{i\omega}{D_0}q^{-2}}{\phi_0(q)\left[-\dfrac{i\omega}{D_0}+\phi_0(q)q^2+\varphi(q)q^2\right]}\right\} \tag{105}$$

we write $\hat{L}_{sing}$ in the form

$$\hat{L}_{sing}=\frac{\hat{\hat{L}}_1+\delta\hat{L}_1}{D_0}, \tag{106}$$

where $\hat{\hat{L}}_1$ and $\delta\hat{L}_1$ correspond to the first and second terms in the braces in Eq. (105). Substituting $\delta\hat{L}_1$ into Eq. (97) gives, instead of Eq. (99), the equation

$$D(\omega,q)=\tau f(q)+\hat{Q}_R\varphi(q)-\frac{i\omega}{D_0}\hat{Q}_R q^{-2}, \tag{107}$$

$$\hat{Q}_R\varphi(q)=\int\frac{d^dq}{(2\pi)^d}\frac{B(q,\tilde{q})}{\phi_0(\tilde{q})\left[-\dfrac{i\omega}{D_0}+\phi_0(\tilde{q})\tilde{q}^2\right]}\varphi(\tilde{q}), \tag{108}$$

where we have neglected $\varphi(q)$ in the denominator of Eq. (105), and have written $\delta f(q)$ in the form $\tau f(q)$ and taken into account the fact that $\bar{D}(\omega,q)\equiv 0$, since the operator $\hat{\hat{L}}_1$ corresponds to the limit $\omega,\tau\to 0$ (see Fig. 5 below). For the decomposition (103) to be unique, we require that $\varphi(q)$ satisfy the condition

$$[\bar{\phi}_0(q),\varphi(q)]=0 \tag{109}$$

($\bar{\phi}_0(q)$ is an eigenfunction of $\hat{Q}_R^T$ that corresponds to the eigenvalue μ_0^R), expressing the requirement that $\varphi(q)$ "not contain in itself" the component const$\cdot\phi_0(q)$. Forming the scalar product of the expression (107) with $\bar{\phi}_0(q)$, we obtain

$$D_0(\bar{\phi}_0,\phi_0)=\tau(\bar{\phi}_0,f)-\frac{i\omega}{D_0}(\bar{\phi}_0,\hat{Q}_R q^{-2}), \tag{110}$$

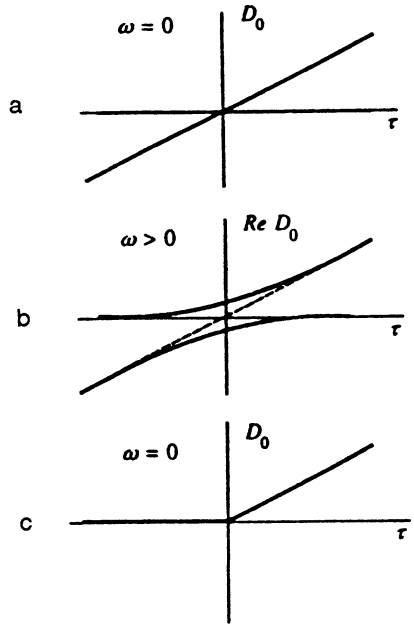

FIG. 5. Selection of the solutions of the self-consistency equation: a—Intersection of the terms with $\omega=0$; b—splitting into physical and unphysical branches for $\omega>0$; c—behavior of $D_0(\tau)$ for the physical branch with $\omega=0$.

where the last term is different from zero in view of the impossibility of Eq. (102), and the term with $\varphi(q)$ is absent since $(\bar{\phi}_0,\hat{Q}_R\varphi)=(\varphi,\hat{Q}_R^T\bar{\phi}_0)=\mu_0^R(\varphi,\bar{\phi}_0)=0$. Written out in detail Eq. (110) has the structure

$$D_0=A\tau-\frac{i\omega}{D_0}\int\frac{d^d\tilde{q}}{(2\pi)^d}\frac{B(\tilde{q})}{\phi_0(\tilde{q})\tilde{q}^2\left[-\dfrac{i\omega}{D_0}+\phi_0(\tilde{q})\tilde{q}^2\right]}\varphi(\tilde{q}). \tag{111}$$

For $d>4$ the integral is determined by large values of $\tilde{q}$ and $-i\omega/D_0$ in the denominator can be neglected. For $d<4$ the integral is determined by small $\tilde{q}$ and we can set $\tilde{q}=0$ in the slowly varying functions $B(\tilde{q})$ and $\phi_0(\tilde{q})$, and the dimensions of the region of integration can be made to go to infinity and the expressions can be made dimensionless. The result for both cases can be written in the unique form

$$D_0=A\tau+B(-i\omega/D_0)^{1/2\nu}, \tag{112}$$

introducing exponent ν according to Eq. (18). The equation (112) has two types of solutions: in the metallic phase $D_0=$ const $\neq 0$ as $\omega\to 0$ and Eq. (112) gives $D_0=A\tau$ in accordance with the value $s=1$ for the conductivity exponent (18); in the dielectric phase $D_0=(-i\omega)\xi^2$ (in the case at hand, the configuration of exponents β_n corresponds to the case Fig. 3a) and $\xi\sim\tau^{-\nu}$ in accordance with the definition of the exponent of the localization length. The equation (112) and the values of the indices s and ν are identical to those obtained in Ref. 18.

For the case $d>4$ Eq. (112) reduces to a quadratic equation and it is easy to trace how the solutions are selected (Fig. 5). For $\omega=0$ the terms $D_0=A\tau$ and $D_0=0$ (Fig. 5a); for finite ω, the degeneracy is removed by the amount $\sim\omega^{1/(2\nu+1)}$ (Fig. 5b), and of the two branches, only one

satisfies the condition Re $D(\omega,q)\geqslant 0$, following from Eqs. (31) and (35) and the non-negativity of $\langle \rho_E \rho_{E+\omega}\rangle_{\mathbf{q}}$. Choosing the indicated branch and passing to the limit $\omega\to 0$, we obtain finiteness of D_0 only on one side of the transition—for definiteness, for $\tau>0$ (Fig. 5c).

From Eq. (107) we have for the function $\varphi(q)$

$$\varphi(q)=\hat{Q}_R^{-1}\hat{P}_\perp\left(D_0\phi_0(q)-\tau f(q)+\frac{i\omega}{D_0}\hat{Q}_R q^{-2}\right), \quad (113)$$

where $\hat{P}_\perp$ is a projection operator onto the subspace which is orthogonal to $\bar{\phi}_0(q)$. Since $\hat{Q}_R\sim 1$ (see below), we obtain $\varphi(q)\sim\max\{|\tau|,\omega^{1/(2\nu+1)}\}$, which justifies the assumption $\varphi(q)\ll\phi_0(q)$. For $d>2$ the integral in Eq. (108) is determined by large values of $\tilde{q}$ for any regular function $\varphi(\tilde{q})$ and all eigenvalues μ_s^R of the operator $\hat{Q}_R$ are found to be of order unity. For the operator $\hat{Q}$ from Sec. 5 (which differs from $\hat{Q}_R$ in the localized phase by the factor ξ^{-2}) this means that all μ_s vanish according to the same law. Therefore the assumption that one mode predominates is not confirmed by the result and actually the case (b) of Sec. 6.1 obtains.

6.3 Case of several dominant modes

We seek $D(\omega,q)$ in the form (104), where the choice of the function $\varphi(q)$ is fixed by the conditions $(\bar{\phi}_0,\varphi)=0$, $C_i=\text{const}(\tau)$ as $\tau\to 0$ (if φ is required to be orthogonal to $\bar{\phi}_1,\ldots,\bar{\phi}_p$, then the coefficients C_i are functions of τ, and this leads to inconveniences in defining the operator $\hat{L}_1$ corresponding to the limit $\omega,\tau\to 0$ and not depending on τ). Using instead of ϕ_0 the "correct" linear combination $\phi_0+C_1\phi_1+\ldots+C_p\phi_p$ and repeating the arguments of Sec. 6.2, we arrive at equations of the type (107) and (108); forming the scalar product of the first equation with $\bar{\phi}_0$, we arrive at Eq. (112) with all consequences following from this. Once again, all eigenvalues of $\hat{Q}$ vanish according to the same law and the limit $p\to\infty$ must be taken in Eq. (104), i.e., all $\phi_s(q)$ must be included in the correct linear combination. Forming the scalar product of the analog of Eq. (107) with $\bar{\phi}_1,\bar{\phi}_2,\ldots$, we obtain a system of equations for C_i:

$$D_0 C_i(\bar{\phi}_i,\phi_i)=\tau(\bar{\phi}_i,f)+\mu_i^R(\bar{\phi}_i,\varphi) - \frac{i\omega}{D_0}(\bar{\phi}_i,\hat{Q}_R q^{-2}), \quad i=1,2,\ldots, \quad (114)$$

where $\mu_i^R\sim 1$. The function $\varphi(q)$ is found to be $\sim\tau$, and in the limit $\omega\to 0$, it has a discontinuity at $\tau=0$, i.e.,

$$\varphi(q)=\tau\begin{cases} B_1^M\phi_1(q)+B_2^M\phi_2(q)+\ldots, & \tau>0 \\ B_1^D\phi_1(q)+B_2^D\phi_2(q)+\ldots, & \tau<0 \end{cases}. \quad (115)$$

Substituting the expression (115) into Eq. (114), we obtain in the limit $\omega\to 0$ the equations

$$D_0 C_i(\bar{\phi}_i,\phi_i)=\tau(\bar{\phi}_i,f)+\tau\mu_i^R B_i^M(\bar{\phi}_i,\phi_i)$$

$$0=\tau(\bar{\phi}_i,f)+\tau\mu_i^R B_i^D(\bar{\phi}_i,\phi_i)+\xi^{-2}(\bar{\phi}_i,\hat{Q}_R q^{-2}) \quad (116)$$

for the metallic and dielectric phases, respectively. For any C_i the equations (116) can be satisfied by appropriate choice of B_i^M and B_i^D, i.e., the coefficients of the correct linear combination are completely arbitrary. The meaning of this arbitrariness will be explained in Sec. 8.

Finally, $D(\omega,q)$ near the transition has the form

$$D(\omega,q)=D_0\bar{d}(q),$$

$$D_0\sim\begin{cases} \tau, & \tau\gg\omega^{1/(2\nu+1)} \\ \omega^{1/(2\nu+1)}, & |\tau|\lesssim\omega^{1/(2\nu+1)} \\ (-i\omega)|\tau|^{-2\nu}, & -\tau\gg\omega^{1/(2\nu+1)} \end{cases}, \quad (116a)$$

where the function $\bar{d}(q)\equiv d(q)/d(0)$ varies on the scale $q\sim\Lambda$. This result, obtained for $D(\omega,q)$ defined as in Eq. (55), is also valid for the observed diffusion coefficient $D_{\text{obs}}(\omega,q)$, since the renormalizations associated with the functions $B(\mathbf{q})$ and $C(\mathbf{q})$ in Eqs. (53) and (54) either contain no anomalous dispersion or they are small.

7. UNIQUENESS OF THE SOLUTION

In this section the self-consistent equation (99) in the localized phase is investigated assuming for $d(q)$ an expansion of the general form (64).

7.1 Method of reference points

In what follows, integrals of the form

$$I_k=\int\frac{d^dq}{(2\pi)^d}\frac{q^{2k}}{\xi^{\alpha_0}+\xi^{a_1}q^2+\ldots+\xi^{\alpha_n}q^{2n}+\ldots}, \quad \alpha_n\geqslant 0 \quad (117)$$

play a large role. The asymptotic expressions of these integrals in the limit $\xi\to\infty$ are calculated by the method of "reference points." We now choose an appropriate scaling in order to make the integral dimensions, making the substitution $q=\xi^{-b}t$, and removing the common factor ξ^a from the denominator, as a result of which the indices α_s become $\alpha_s-a-2sb$. By choosing appropriate values of a and b the exponents in the first two terms of the denominator in Eq. (117) can be made to be zero and the remaining exponents become negative. Then

$$I_k=\xi^{-a-b(d+2k)}\int\frac{d^dt}{(2\pi)^d}\times\frac{t^{2k}}{t^{2s_1}+t^{2s_2}+\sum_{s\neq s_1,s_2}\xi^{\alpha_s-a-2sb}t^{2s}}\sim\xi^{-a-b(d+2k)}. \quad (118)$$

If s_1 and s_2 satisfy the condition $2s_1<d+2k<2s_2$, guaranteeing that the integral converges after the sum over s is dropped (to avoid indeterminacies, we assume that d is non-integral, and pass to the limit of integral d in the final results). The procedure described above admits a simple geometric interpretation.[5] We construct a plot of sequence α_n (Fig. 6a), mark on the abscissa axis the point $x_k=(d+2k)/2$, and construct an upper tangent at the point x_k to the set of points (n,α_n): If the points are graphically represented by nails, then this construction is made with the aid of "sticks" (solid line in Fig. 6a) and "rope" (dashed

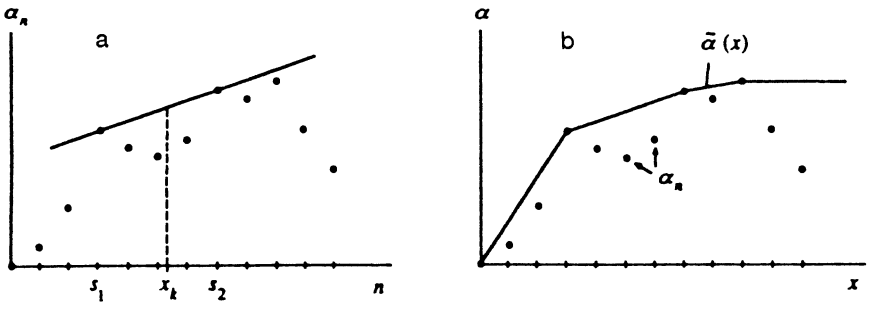

FIG. 6. a—Construction of the upper tangent at the point x_k to the set of points (n,α_n); b—convex envelope $\tilde{\alpha}(x)$ for the sequence α_n.

line). The numbers of the points, on which the upper tangent "lies," determine s_1 and s_2, and its equation $\alpha=a+2bn$ determines the parameters a and b.

Constructing the broken line, consisting of segments of the upper tangents (Fig. 6b), we obtain a convex envelope $\tilde{\alpha}(x)$, in terms of which the result (118) assumes the form

$$I_k\sim\xi^{-\tilde{\alpha}(x_k)},\quad x_k=\frac{d+2k}{2}. \tag{119}$$

By construction the function $\tilde{\alpha}(x)$ is increasing and convex (in the nonrigorous sense). For a bounded sequence α_n with maximum at $n=n_0$, $\tilde{\alpha}(x)$ is strictly increasing for $x<n_0$ and constant for $x>n_0$ (for $x_k>n_0$ the reference point s_2 lies at infinity). For a strictly increasing and strictly convex sequence α_n, the following inequalities follow from Eq. (119):

$$I_0\gg I_1\gg I_2\gg\ldots\gg I_k\gg I_{k+1}\gg\ldots, \tag{120}$$

$$I_{k_1}I_{k_2}\ll I_{k_1-1}I_{k_2+1}\ll I_{k_1-2}I_{k_2+2}\ll\ldots,\quad k_1\leqslant k_2. \tag{121}$$

For an arbitrary sequence α_n, some of the strong inequalities are replaced by weak inequalities. In what follows, for definiteness, we proceed from the strong inequalities in Eqs. (120) and (121), having in mind the fact that the results remain valid in order of magnitude in the general case.

7.2 Symmetrization of the operator $\hat{Q}$

We set in Eq. (99) $\delta f(q)\equiv\delta\tau f(q)$ and expanding all functions in series

$$f(q)=\sum_{k=0}^{\infty}f_kq^{2k},\quad \delta d(q)=\sum_{k=0}^{\infty}\delta d_kq^{2k},$$

$$B(q,\tilde{q})=\sum_{k,k'=0}^{\infty}B_{kk'}q^{2k}\tilde{q}^{2k'} \tag{122}$$

we obtain in the limit $\omega\to0$

$$-\delta\tau f_k=\sum_{k'=0}^{\infty}B_{kk'}\sum_{k''=0}^{\infty}I_{k'+k''}\delta d_{k''}, \tag{123}$$

where

$$I_k=\int\frac{d^dq}{(2\pi)^d}\frac{q^{2k+2}}{[1+d(q)q^2]^2}. \tag{124}$$

The matrix $\hat{B}=\|B_{kk'}\|$ and the column matrix $\hat{f}=\|f_k\|$ are of general form with elements ~1. The column matrix $\hat{\tilde{f}}=\hat{B}^{-1}\hat{f}$ has the same properties. Multiplying Eq. (123) by $\hat{B}^{-1}$ we obtain

$$-\delta\tau\begin{vmatrix}\tilde{f}_0\\ \tilde{f}_1\\ \tilde{f}_2\\ \vdots\end{vmatrix}=\begin{vmatrix}I_0 & I_1 & I_2 & \ldots\\ I_1 & I_2 & I_3 & \ldots\\ I_2 & I_3 & I_4 & \ldots\\ \ldots & \ldots & \ldots & \ldots\end{vmatrix}\begin{vmatrix}\delta d_0\\ \delta d_1\\ \delta d_2\\ \vdots\end{vmatrix}, \tag{125}$$

i.e., an equation of the type (99) but with the symmetrized matrix of the operator $\hat{Q}$.

Using the expansion (64) for $d(q)$, the integrals (124) acquire the form (117) with the exponents

$$\alpha_n=\max_{\{k_i\}}(\beta_{k_1}+\beta_{k_2})_{k_1+k_2=n} \tag{126}$$

and an extra q^2 in the numerator. The exponents β_n are nonnegative and increase more rapidly than $2n$ (Sec. 4). This guarantees the condition $\alpha_n\geqslant0$ and make is possible to construct a convex envelope.

7.3 Inversion of the operator $\hat{Q}$

Limiting the upper limit of the summation in Eq. (122) by some finite n, we obtain in Eq. (125) a system of equations of finite order that can be solved by Cramer's rule. The determinant of the matrix $\hat{Q}$ in Eq. (125) consists of all possible products of the form

$$I_{k_0}I_{k_1+1}I_{k_2+2}\ldots I_{k_n+n}, \tag{127}$$

where $k_0,k_1,\ldots,k_n$ is a permutation of 0, 1,..., n. We separate in Eq. (125) the pair $I_{k_s+s}I_{k_{s'}+s'}$ with $s<s'$. If $k_s>k_{s'}$, then it follows from Eqs. (120) and (121) that

$$I_{k_s'+s}I_{k_{s'}+s}\gg I_{k_s+s}I_{k_{s'}+s'} \tag{128}$$

and the product (127) can be increased by interchanging k_s and $k_{s'}$, without touching the other k_i. It is obvious that in the maximum product among the products (127), which determines the order of magnitude of the determinant Q, should have $k_0<k_1<\ldots<k_n$, whence $k_0=0$, $k_1=1,\ldots,$ $k_n=n$ and therefore

$$\det Q\sim I_0I_2I_4\ldots I_{2n}. \tag{129}$$

The minor Q_j^i of the matrix Q, obtained by crossing out the ith row and the jth column, consists of all possible products of the form

$$I_{k_0}I_{k_1+1}\ldots I_{k_{j-1}+(j-1)}I_{k_{j+1}+(j+1)}\ldots I_{k_n+n}, \tag{130}$$

where $k_0, k_1,\ldots,k_{j-1}, k_{j+1},\ldots,k_n$ is a permutation of 0, 1,..., $i-1, i+1,\ldots,n$. In the maximum product these two sequences are identical. It is easily verified that

$$Q_j^0 \ll Q_j^1 \ll \ldots \ll Q_j^n \sim I_0 I_2 \ldots I_{2j-2} I_{2j+1} I_{2j+3} \ldots I_{2n-1}. \tag{131}$$

Solving Eq. (125) by Cramer's rule and using Eqs. (129) and (131), we obtain

$$\delta d_k \sim \delta\tau \frac{1}{I_{2k}} \frac{I_{2k+1}I_{2k+3}\ldots I_{2n-1}}{I_{2k+2}I_{2k+4}\ldots I_{2n}}, \quad k=0,1,\ldots,n \tag{132}$$

and, using Eq. (119), the result can be expressed in terms of the sequence α_k. For a convex sequence β_k, we have from Eq. (126) $\alpha_{2m}=2\beta_m$, $\alpha_{2m+1}=\beta_m+\beta_{m+1}$, which can be written in the form $\alpha_k=2\beta_{k/2}$, if the sequence β_k is additionally defined at half-integral points by the relation $\beta_{k+1/2}=(\beta_k+\beta_{k+1})/2$. Since the values of $\beta_{k+1/2}$ lie on the convex envelope $\tilde{\beta}(x)$, for arbitrary x we obtain

$$\tilde{\alpha}(x)=2\tilde{\beta}(x/2). \tag{133}$$

This result remains valid for an arbitrary sequence β_k. To prove this, it is necessary to introduce an auxiliary convex sequence $\bar{\beta}_k=\tilde{\beta}(k)\geqslant\beta_k$ and note that replacing β_k by $\bar{\beta}_k$ does not change the value of the integrals I_k. For convex β_k we have

$$\tilde{\beta}(k+\varphi)=(1-\varphi)\beta_k+\varphi\beta_{k+1}, \quad 0\leqslant\varphi\leqslant 1, \tag{134}$$

which makes it possible to switch from the convex envelope directly to the values of β_k. In the general case Eq. (134) is correct with β_k replaced by $\bar{\beta}_k$. Setting

$$d\equiv 4m+4\psi, \quad m-\text{integer}, \quad 0\leqslant\psi\leqslant 1 \tag{135}$$

we obtain from Eqs. (132), (119), (133) and (134) in the limit $n\to\infty$

$$\delta d_k \sim \delta\tau\xi^{S(k)}, \quad S(k) = \begin{cases} (1-2\psi)\bar{\beta}_{m+k}+2\psi\bar{\beta}_{m+k+1}+\bar{\beta}_\infty, & 0\leqslant\psi\leqslant\frac{1}{2} \\ \bar{\beta}_{m+k+1}+\bar{\beta}_\infty, & \frac{1}{2}\leqslant\psi\leqslant 1 \end{cases}, \tag{136}$$

where the limit $\bar{\beta}_\infty=\lim_{k\to\infty}\bar{\beta}_k$ is assumed to be finite in accordance with the considerations of the next section.

7.4 Impossibility of unbounded growth of β_k

For an unbounded sequence β_k the convex envelope $\tilde{\beta}_k$ is strictly increasing and the hierarchy (120) continues to infinity. By virtue of Eq. (132), this means that δd_k diverges as $n\to\infty$. To clarify the reasons for the divergence, we note that the off-diagonal part of the matrix Q in Eq. (125) under the conditions (120) and (121) can be regarded as a perturbation. Its eigenvalues in leading order are equal to I_{2k} and they bunch up near zero in the limit $k\to\infty$. In the proof of Fredholm's theorem it is shown[46] that when the expansions (122) are truncated at the nth term, the $(n+1)$ maximum eigenvalues of the operator $\hat{Q}$ are reproduced; in the limit $n\to\infty$, arbitrarily small eigenvalues are reproduced and the response of the system to a small perturbation diverges. This situation occurs not only at the transition point but also in a neighborhood of the transition point (as long as $\xi\gg\Lambda^{-1}$); it is unphysical, since the system is unstable with respect to an infinitely small perturbation of a general form.

This result has important qualitative consequences, since it excludes the cases corresponding to Fig. 3b and proves the uniqueness of the results $D(\omega,0)\sim(i\omega)\xi^2$ and $\epsilon(0,0)\sim\xi^2$ in the localized phase.

7.5 Change in $d(q)$ as the transition is approached

Expanding the numerator in Eq. (92) in powers of $\tilde{q}^2$, we obtain integrals that can be calculated by the method of reference points and which are of order $\xi^{-\tilde{\beta}(x_0)}$, $\xi^{-\tilde{\beta}(x_1)}$, and so on. We set

$$\hat{L}_{\text{sing}}(\xi)=\frac{\hat{L}_1(\xi)}{(-i\omega)\xi^{\tilde{\beta}(x_0)}}=\frac{\hat{\hat{L}}_1+\xi^{-y_1}\hat{l}_1+\xi^{-y_2}\hat{l}_2+\ldots}{(-i\omega)\xi^{\tilde{\beta}(x_0)}}, \tag{137}$$

where the terms $\xi^{-y_k}\hat{l}_k$ arise from the higher order terms in the expansion in $\tilde{q}^2$ and from corrections to the main scaling in the method of reference points. Changing the definition of $\hat{L}_1$ from the definition (77) makes it possible to separate the main singularity as $\tau\to 0$ and introduce the operator $\hat{\hat{L}}_1$, corresponding to the limit $\omega,\tau\to 0$.

In Eq. (137) it was assumed that β_k is constant. Now, let the change $\delta\tau$ in the parameter τ generate the changes $\delta\xi$ and $\delta\beta_k$ in the quantities ξ and β_k. Then

$$\hat{L}_{\text{sing}}(\xi+\delta\xi)=\frac{\hat{L}_1(\xi+\delta\xi)}{(-i\omega)(\xi+\delta\xi)^{\tilde{\beta}(x_0)}} = \frac{\hat{\hat{L}}_1+(\xi+\delta\xi)^{\tilde{\beta}(x_0)}(-i)\delta\hat{L}_1+(\xi+\delta\xi)^{-y_1}\hat{l}_1}{(-i\omega)(\xi+\delta\xi)^{\tilde{\beta}(x_0)}}, \tag{138}$$

where only the term with the minimum index y_1 is retained, and $\delta\hat{L}_1$ is determined by the expression (93) with $\delta d(q)$ of the form

$$\delta d(q)=\sum_{k=0}^{\infty} q^{2k}\xi^{\beta_{k+1}}\ln\xi\delta\beta_{k+1}. \tag{139}$$

Using as $\delta\hat{L}_1$ in Eq. (97) the quantity $\hat{L}_1(\xi+\delta\xi)-\hat{L}_1(\xi)$ we obtain instead of Eq. (99)

$$\delta D(\omega,q)=\delta\tau f(q)+\xi^{-y_1-1}\delta\xi R(q)+\xi^{\tilde{\beta}(x_0)}\hat{Q}\,\delta d(q), \tag{140}$$

where $\delta D(\omega,q)\to 0$ as $\omega\to 0$. In the case of exact scaling, when $\delta d(q)\equiv 0$, the first two terms on the right-hand side cancel one another, whence $y_1=1/\nu$. In the general case,

they are of the same order of magnitude. Inverting the operator $\hat{Q}$ according to Sec. 7.3 and comparing with Eq. (139), we have

$$\delta d_k \sim \delta\tau \xi^{-\tilde{\beta}(x_0)+S(k)} \sim \xi^{\beta_{k+1}} \ln \xi \delta\beta_{k+1}, \tag{141}$$

whence

$$\delta\beta_k \sim \frac{\delta\tau}{\ln \xi} \xi^{\gamma_k}, \tag{142}$$

where it is convenient to write γ_k in the form

$$\gamma_k = \begin{cases} (\bar{\beta}_k - \beta_k) + (\bar{\beta}_\infty - \tilde{\beta}(x_0)) + (\bar{\beta}_{m+k} - \bar{\beta}_k), & \frac{1}{2} \leqslant \psi \leqslant 1 \\ (\bar{\beta}_k - \beta_k) + (\bar{\beta}_\infty - \tilde{\beta}(x_0)) + (1-2\psi)(\bar{\beta}_{m+k-1} - \bar{\beta}_k) + 2\psi(\bar{\beta}_{m+k} - \bar{\beta}_k), & 0 \leqslant \psi \leqslant \frac{1}{2}, \\ & m \geqslant 1, \\ (\bar{\beta}_k - \beta_k) + (\bar{\beta}_\infty - \bar{\beta}_k) + (1-2\psi)(\bar{\beta}_{k-1} - \bar{\beta}_0) + 2\psi(\bar{\beta}_k - \bar{\beta}_1), & 0 \leqslant \psi \leqslant \frac{1}{2}, \\ & m = 0 \end{cases} \tag{143}$$

All combinations in parentheses are non-negative and $\gamma_k \geqslant 0$. For fixed ξ the definition of the exponents β_k in Eq. (64) is not unique: the coefficient of q^{2k} can be written as $C_k \xi^{\beta k}$ and a change in β_k is equivalent to a change in C_k. The specific configuration of the exponents β_k make sense only if it remains unchanged when τ changes. According to Eq. (142), for $\gamma_k > 0$ a large change in the exponents occurs. The changes $\delta\beta_k \sim \delta\tau / \ln \xi$ can be included in the changes in C_k, which are small for a small change in τ, only for $\gamma_k \equiv 0$. The condition $\gamma_k \equiv 0$ requires that all combinations in parentheses in Eq. (142) vanish and it fixes the only configuration of exponents that is different for $d > 2$ and $d < 2$:

$$\begin{aligned} &\beta_1 = \beta_2 = \beta_3 = \ldots, \quad d > 2, \\ &\beta_0 = \beta_1 = \beta_2 = \ldots, \quad d < 2. \end{aligned} \tag{144}$$

By definition, $\beta_0 = 0$ and $d < 2$ all exponents are equal to zero. This means that $d(q)$ does not diverge and the localized phase remains for all τ.[21] For $d > 2$, all indices can be made equal to 2 in accordance with the requirement $\beta_1 = 2$ by defining ξ (Fig. 3a). All eigenvalues of $\hat{Q}$ vary according to the same law and we return to the case (b) of Sec. 6.1.

8. CHANGE IN SYMMETRY AT THE ANDERSON TRANSITION

A change in $\hat{L}_{\text{reg}}$ gives rise to a rotation of the subspace M_∞ of the operator $\hat{L}_{\text{sing}}$. This is analogous to a rotation of the magnetization vector **M** accompanying a change in the magnetic field **H** in a ferromagnet. This analogy is formalized in the form of Table I. We shall give some explanations.[6)]

The operators $\hat{L}_{\text{reg}}$ and $\hat{L}_{\text{sing}}$ have many degrees of freedom, many of which do not appear in the self-consistency equation. The important degrees of freedom are determined by the functions $f(q)$ and $\bar{d}(q)$ [see Eq. (116a)], whose expansion coefficients

$$\begin{aligned} f(q) &= 1 + f_1 q^2 + \ldots + f_n q^{2n} + \ldots, \\ \bar{d}(q) &= 1 + d_1 q^2 + \ldots + d_n q^{2n} + \ldots \end{aligned} \tag{145}$$

TABLE I. Analogy between a ferromagnet and a disordered system.

Ferromagnet	Disordered system
Orientation of the magnetic field **H**:	Operator $\hat{L}_{\text{reg}}$:
Components of the unit vector	Coefficients f_n
Orientation of the magnetization **M**:	Space M_∞ of the operator $\hat{L}_{\text{sing}}$
Components of the unit vector	Coefficients d_n
Squared modulus of the field H^2	Frequency ω
Squared modulus of the magnetization M^2	Diffusion coefficient D_0
Magnetic susceptibility tensor χ_{ij}	Operator $\hat{Q}^{-1}$
Paramagnetic phase	Localized phase
Ferromagnetic phase	Metallic phase
Curie point	Point of the Anderson transition
$T - T_c$	Distance to the transition τ

can be regarded as components of the unit vectors $\hat{\mathbf{H}}$ and $\hat{\mathbf{M}}$. In the localized phase small changes in them are related with the operator $\hat{Q}$, whose inverse is analogous to the magnetic susceptibility tensor χ_{ij}.

The finiteness of the frequency ω smears the transition, similarly to the finiteness of the magnetic field in a ferromagnet. In the localized phase $D_0 \sim \omega$ and in the metallic phase $D_0 = \text{const}\ (\omega)$, which is analogous to the appearance of spontaneous magnetization, i.e. the quantities D_0 and ω are analogous to $|\mathbf{M}|$ and $|\mathbf{H}|$. In view of the qualitative character of the analogy, this identification is not unique. For example, any monotonic function $F(|\mathbf{M}|)$, equal to zero for $|\mathbf{M}| = 0$, can be taken as the analog of D_0; as a function of the magnetization itself, it has the form $F(M^2)$, since a scalar must be formed from a vector. Finally, for small M^2, it can be expanded in a series, obtaining an analogy of D_0 to M^2. Similarly, H^2 is the only analog for ω.

In the analogy found, it is important that (a) the number of components of the vector **M** is infinite, since the number of expansion coefficients d_n is infinite, and (b) the ferromagnet is isotropic. The latter property is obvious from the fact that all eigenvalues of the "susceptibility tensor" $\hat{Q}^{-1}$ diverge at the transition point according to the same law and for small changes in d_n and f_n they can be made equal by a

τ-independent linear transformation. The analog of colinearity of $\mathbf{M}$ and $\mathbf{H}$ in an isotropic ferromagnet exists with the following stipulation: The problem is that in the case of the Anderson transition the "vector $|\mathbf{H}|$" and the "vector $|\mathbf{M}|$" lie in different subspaces and there is no natural method for establishing the mutual orientation of these subspaces. For this reason, for a fixed function $f(q)$ the choice of $d(q)$ is arbitrary (Sec. 6.3) in accordance with the arbitrariness in the choice of bases in the two subspaces. For the special choice $B_i^M \equiv 0$ in Eq. (115) we have $f(q) \equiv \bar{d}(q)$, which corresponds to the choice of the "correct" mutual orientation of the bases.

The model of an isotropic ferromagnet with the number of components $n \to \infty$ is well known in the theory of phase transitions and is the basis for the $1/n$ expansion.[1] Its critical exponents are known exactly. Specifically, for the magnetization exponent $M \sim \tau^{\beta}$ and the correlation length exponent $\xi \sim \tau^{-\nu}$ we have

$$\beta = \frac{1}{2}, \quad d > 2; \quad \nu = \begin{cases} \dfrac{1}{d-2}, & 2 < d < 4 \\ \dfrac{1}{2}, & d > 4 \end{cases}, \tag{146}$$

which, since $s = 2\beta$, corresponds exactly to Eq. (18).

9. CONCLUSIONS

The approach to the theory of localization based on the formalism of σ-models[22,47,48] is currently considered to be the most rigorous approach. However, its rigor should not be overestimated. First, the degree to which the approximations employed in the derivation of the σ-models retain exact invariance under time reversal of the initial disordered system and the satisfaction of the Ward identities (13), which are important for reconstructing the pole structure of $U_{\mathbf{k}\mathbf{k}'}(\mathbf{q})$, is not clear. Second, to take into account the spatial dispersion of $D(\omega, \mathbf{q})$, it is necessary to introduce into the Lagrangian of the σ-model additional gradient vertices which grow anomalously at the initial stage of the renormalization group transformations.[49] The analog of such growth can be obtained from Eqs. (142) and (143), assuming for the initial configuration of exponents $\beta_1 = 2$, $\beta_k = 0$ $(k \geqslant 2:)$

$$\frac{\partial \beta_k}{\partial \tau} \sim \frac{1}{\ln \xi} \xi^{2-\beta_k}, \quad k = 1, 2, \ldots . \tag{147}$$

Growth of β_k with $k \geqslant 2$ indicates intensification of spatial dispersion of $D(\omega, \mathbf{q})$ as the transition is approached, and in the language of the magnetic analogy (Sec. 8) a transformation of a uniaxial ferromagnet into an isotropic ferromagnet.[7)] Apparently, the renormalization group transformations transform analogously the zero-component σ-model into an infinite component model. These difficulties apparently are not important in low orders in $\epsilon = d - 2$, since for small ϵ Anderson's transition falls in the region of weak disorder, for which the derivation of the σ-model is indeed substantiated.

We now discuss the possible reasons for the disagreement of Eq. (18) with the result of Ref. 28 for the exponent s. The result of Ref. 28 for the permittivity $\epsilon(0,0) \sim \xi$ corresponds to the case $\zeta = 1$ of the Sec. 4. This is also indicated by the expression given in Ref. 28 for the function $A(\mathbf{r})$ from Eq. (66). For $\zeta = 1$ the exponents β_k increase linearly with k. It is clear from Sec. 7.4 that if the result of Ref. 28 corresponds to the exact solution of some idealized model, then this model is unphysical in view of the instability with respect to an infinitesimal perturbation of a general form. More likely, the approximations employed in the derivation of the σ-model and the selection of diagrams in Ref. 28 destroy the pole structure of $U_{\mathbf{k}\mathbf{k}'}(\mathbf{q})$. In this connection, the fact that the results for models in which the time-reversal invariance is and is not destroyed agree with one another is suspicious. Finally, in the derivation of the σ-model for a large d it is necessary to introduce a construction consisting of weakly coupled granules, for which, because of the presence of artificial small parameters, the critical region can narrow anomalously and, as a result of the approximations, contract into a point. The results of Ref. 28 could correspond to some intermediate asymptotic behavior.

There are some objections to the arguments of Efetov[43] that the diagrammatic approach in principle does not "feel" the noncompactness, which in his opinion determines the main difference between the theory of disordered systems and the theory of phase transitions. One can agree with the last assertion: Noncompactness is a consequence of the added imaginary terms $\pm i\delta$, determining the type of Green's function, which lead to nonperturbative contributions giving rise to the difference between the two indicated theories.[17] However, the nonperturbative contributions can be obtained from the diagrammatic technique.[16] The added terms $\pm i\delta$ play an important role in the separation of the diffusion poles, since as a result of these additions, the integration contour in Eq. (58) is confined between the poles of two Green's functions.

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APPENDIX

Expansion of the self-consistency equation in powers of q

If we have for the operator $\hat{L}$

$$\hat{L}(\mathbf{q})\psi_{\mathbf{k}} \equiv \frac{1}{N}\sum_{\mathbf{k}'} L_{\mathbf{k}\mathbf{k}'}(\mathbf{q})\psi_{\mathbf{k}'},$$

$$L_{\mathbf{k}\mathbf{k}'}(\mathbf{q}) = L_{\mathbf{k}'\mathbf{k}}(\mathbf{q}) = L_{-\mathbf{k},-\mathbf{k}'}(-\mathbf{q}), \tag{A1}$$

then it is easy to prove that: (a) the eigenvalues of $\hat{L}$ are even as a function of $\mathbf{q}$, $\lambda_s(\mathbf{q}) = \lambda_s(-\mathbf{q})$; (b) the eigenfunctions $e_{\mathbf{k}}^{(s)}(\mathbf{q})$ can be chosen so that $e_{\mathbf{k}}^{(s)}(\mathbf{q}) = e_{-\mathbf{k}}^{(s)}(-\mathbf{q})$; and, (c) if there are several operators of the type (A1), then the matrix elements of one operator with respect to the eigenfunctions

of the other operator are even functions of $\mathbf{q}$. The operators $\hat{L}$, $\hat{L}_{\rm reg}$, and $\hat{L}_1$ are of the form (A1) and, by virtue of (a–c), the right-hand side of Eq. (97) is even in $\mathbf{q}$ and can be expanded in powers of q^2.

Since the operator $\hat{L}_1$ in Sec. 5.1 is independent of ω, we obtain that in Eq. (86) $O(\omega)\equiv 0$ and the definitions (75) and (76) are equivalent to the following definitions:

$$\hat{L}_{\rm reg}\psi_{\mathbf{k}}=(\epsilon_{\mathbf{k}+\mathbf{q}/2}-\epsilon_{\mathbf{k}-\mathbf{q}/2})\psi_{\mathbf{k}}+\frac{1}{N}\sum_{\mathbf{k}'}U^{\rm reg}_{\mathbf{k}\mathbf{k}'}(\mathbf{q})\times[\Delta G_{\mathbf{k}'}(\mathbf{q})\psi_{\mathbf{k}}-\sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})\Delta G_{\mathbf{k}'}(\mathbf{q})}\psi_{\mathbf{k}'}], \quad \text{(A2)}$$

$$\hat{L}_{\rm sing}\psi_{\mathbf{k}}=\frac{1}{N}\sum_{\mathbf{k}'}U^{\rm sing}_{\mathbf{k}\mathbf{k}'}(\mathbf{q})[\Delta G_{\mathbf{k}'}(\mathbf{q})\psi_{\mathbf{k}}-\sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})\Delta G_{\mathbf{k}'}(\mathbf{q})}\psi_{\mathbf{k}'}]. \quad \text{(A3)}$$

For the definition (A2) we have, on account of Eq. (39),

$$\hat{L}_{\rm reg}|e_0\rangle=\hat{L}_{\rm reg}\{\text{const}\sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})}+O(\mathbf{q})\}=\text{const}(\epsilon_{\mathbf{k}+\mathbf{q}/2}-\epsilon_{\mathbf{k}-\mathbf{q}/2})\sqrt{\Delta G_{\mathbf{k}}(\mathbf{q})}+O(\mathbf{q})=O(\mathbf{q})$$

and the contribution $O(q^0)$ is absent in each term on the right-hand side of Eq. (97).

[1)]For definiteness, all quantities refer to the same spin projection. For purely potential scattering the spin subsystems are independent and the number of spin components can be easily taken into account in the final results.

[2)]In expansions of the type (64) the coefficients are assumed to be arbitrary. Taking them into account falls outside the limits of accuracy of the present analysis.

[3)]Their presence (see Sec. 5.1) is associated with the existence of a diffusion pole for $\mathbf{k}+\mathbf{k}'\to 0$, which follows from Eq. (9) (see Sec. 3). A magentic field or magnetic impurities eliminate this pole and cause all $\lambda_s(\mathbf{q})$ in the set M_0' [Eq. (72)] to become finite in the limit $\omega\to 0$, as a result of which $\hat{L}_{\rm sing}$ is left with only one null mode corresponding to $\lambda_0(\mathbf{q})$.

[4)]Singularities associated with a change in the type of phase transition—for example, a second-order phase transition into a first-order phase transition—can occur on the critical surface. We assume that the system is far away from such singularities.

[5)]Similar constructions arise in the investigation of Burgers equation.[45]

[6)]A similar, but not identical, analogy was discussed in Ref. 41.

[7)]A detailed investigation of the evolution of β_k requires a knowledge of the proportionality coefficients in Eq. (64) and (142).

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