Properties of superfluid ^{3}He-*B* rotating in an annulus between two co-axial cylinders

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The properties of ^{3}He-*B* rotating in an annulus in the presence of a vortex cluster are considered. The various textures of the order parameter are calculated, and nonexponential time decay of the angular velocity of a freely suspended decelerating vessel with superfluid liquid is predicted. © *1996 American Institute of Physics.* [S1063-7761(96)01102-0]

Several states of the rotating *B* phase of superfluid ^{3}He have been identified and investigated experimentally. Along with an equilibrium rotation at angular velocity Ω supported by an array of quantized vortices[1] with 2D density $n_v=2\Omega/\Gamma_0$ (Γ_0 being the superfluid circulation quantum $h/2m$), other metastable rotating states are possible. The high critical velocity for nucleation of singular vortex lines in ^{3}He-*B* allows rotating states with macroscopic vortex-free regions characterized by large counterflows of normal and superfluid components. An interesting example of such metastable rotation is the vortex-cluster state[2] in a cylindrical vessel (of radius R) with a vortex bundle of radius $R_c<R$, surrounded by the counterflow region ($R_c<r<R$). The number of quantized vortices per unit area within the bundle is $2\Omega/\Gamma_0$ (as in equilibrium), but this rotating state is characterized by a deficiency ΔN_v in the total number of vortices as compared to the state of full rotating equilibrium ($\Delta N_v=\pi(R^2-R_c^2)(2\Omega/\Gamma_0)$).

The orienting action of a dense system of vortices on the order parameter of ^{3}He-*B* within the vortex bundle has been the subject of extensive experimental and theoretical investigations. The case of full equilibrium (where the vortex bundle fills the whole volume of the rotating cylindrical vessel) has been investigated in detail (see Ref. 1), as well as the metastable vortex-cluster carrying state[2,3] and totally vortex-free counterflow state.[4]

The spatial distribution of the order parameter (texture) can be efficiently explored by NMR techniques. Observation of the *cw* NMR frequency spectrum allows one to identify various textural transitions driven by large superfluid counterflows in the vortex-free and vortex-cluster carrying states of ^{3}He-*B* rotating in a cylindrical vessel.[2,4]

It is interesting to analyze possible states of ^{3}He-*B* rotating in an annulus (confined between two coaxial cylinders of radius R_1, and $R_2>R_1$). This double-connected geometry is expected to be quite different from the singly-connected cylindrical case. An obvious difference between these two geometries is connected with the difference in boundary conditions imposed on the order parameter of the *B* phase at the lateral walls of the container. This question was considered in Ref. 5, where various textures in a rotating annulus were computed for a vortex-controlled equilibrium state, as well as for the case of metastable vortex-free rotation with large superfluid counterflows. It was shown, in particular, that an annular geometry permits accurate measurement of the vortex parameters λ and κ characterizing the average interaction of vortices with the *B* phase order parameter.[6,7]

In what follows we shall consider vortex-cluster carrying states of ^{3}He-*B* rotating in an annulus. In this case, the topological distinction between the annular and hollow cylindrical geometry is of primary importance. In the first place one has to recognize that for the case of an annular geometry the superfluid circulation $\Gamma_\Omega(r)$ is the result of the combined action of real and virtual vortices. This last contribution is connected to superflows trapped around the inner cylinder (of radius R_1). For the case of a vortex-controlled rotating equilibrium at $\Omega=\Omega_v$ the circulation about the circumference of the outer cylindrical wall is $\Gamma_\Omega(R_2)=2\pi R_2\Omega_v R_2=\tilde{N}_v\Gamma_0$, where $\tilde{N}_v=N_v+N_v^*$ is the total number of the real (N_v) and the virtual (N_v^*) vortices.

Consider now what happens when the rotation is accelerated ($\Omega>\Omega_v$) and new vortices are not generated. It is expected that in this situation a macroscopic irrotational counterflow region of width $\Delta R=R_2-R_c$ is formed as in the case of a cylindrical vessel. The superfluid circulation around the boundary of the vortex cluster is $\Gamma_\Omega(R_c)=2\pi\Omega R_c^2=\tilde{N}_v^{(c)}\Gamma_0$, and for $\tilde{N}_v^{(c)}=\tilde{N}_v$ we conclude that the outer radius of the vortex cluster is $R_c(\Omega)=R_2\sqrt{\Omega_v/\Omega}<R_2$. It is important to notice that, in contrast to the case of cylindrical geometry, in an annulus the number of real vortices is not conserved: when Ω increases, singular vortex lines are pushed across the inner wall under the action of the Magnus force, with the corresponding circulation being trapped around the cylinder of radius R_1. At $\Omega=\Omega_1^*=(R_2/R_1)^2\Omega_v$ the vortex cluster disappears. The counterflow velocity field $w(r)=v_n(r)-v_s(r)$ in the case of a vortex cluster state in the rotating annulus is given by

$$w(r)=\begin{cases}0, & R_1\leqslant r\leqslant R_c,\\ \Omega R_c(r/R_c-R_c/r), & R_c<r\leqslant R_2.\end{cases}\tag{1}$$

In the case of deceleration ($\Omega<\Omega_v$), when the trapped superfluid circulation around the inner wall of the annulus persists, a counterflow is established at the inner wall:

$$w(R_1)=\Omega R_1-\frac{\Gamma_{\Omega_v}(R_1)}{2\pi R_1}=(\Omega-\Omega_v)R_1,\tag{2}$$

and a counterflow-carrying irrotational region of width $\Delta R=R_c-R_1$ with $R_c(\Omega)=R_1\sqrt{\Omega_v/\Omega}>R_1$ is formed. The number of real vortices in the cluster again is not conserved,

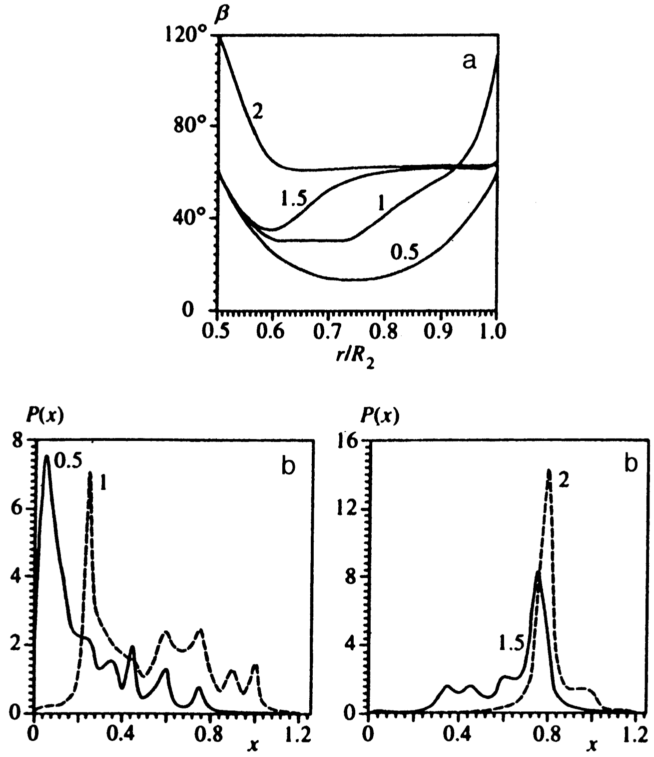

FIG. 1. Radial distributions $\beta = \beta(r/R_2)$ for various Ω in the case $\Omega > 0$ and $\Omega_v = 0.5$ (a) and corresponding transverse NMR spectrum $P(x)$ (b); here $x = (\omega - \omega_L)/(\Omega_B^2/2\omega_L), H = 300\,\mathrm{Oe}, P = 30\,\mathrm{bar}, T/T_c = 0.7, R_2 = 4\,\mathrm{mm}$, $R_1 = 2\,\mathrm{mm}$.

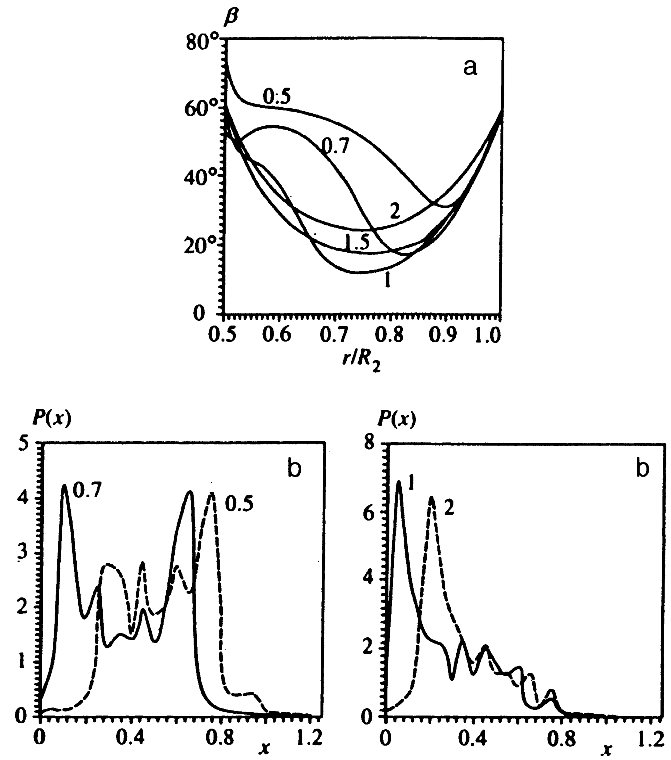

FIG. 2. Radial distributions $\beta = \beta(r/R_2)$ for various $\Omega(\dot{\Omega} < 0, \Omega_v = 2)$ (a) and corresponding transverse NMR spectrum $P(x)$ (b).

since they escape across the outer wall of the annulus. At $\Omega = \Omega^* = (R_1/R_2)^2(\Omega)_v$ the vortex cluster disappears completely. In the case of deceleration

$$w(r) = \begin{cases} \Omega R_c(r/R_c - R_c/r), & R_1 \leq r \leq R_c, \\ 0, & R_c < r \leq R_2. \end{cases} \tag{3}$$

The vortex cluster appears inthe case of deceleration because new vortices cannot enter the volume from the inner cylinder (while at the same time, part of the vortices leave the annulus through the outer cylinder). Note that a vortex cluster cannot exist for $\Omega < \Omega_v$ in the case of a hollow cylinder. From (3) one can conclude that $w(r) \neq 0$ holds even at $\Omega = 0$. In this case we have $v_n(r) = 0$, but $v_s = \Omega_v R_1^2/r$.

In order to determine the texture of the order parameter of ^{3}H-B in the presence of a vortex cluster in an annular vessel one has to minimize the bulk free energy

$$\begin{aligned} F_B = & \int \{f_{an}(\hat{n}) + f_{\mathrm{grad}}(\nabla_i n_j)\} d^3 r \\ & - dH^2 \int_{R=R_1} dS(\hat{h}\mathbf{R}(\hat{n})\mathbf{s})^2 \\ & - dH^2 \int_{R=R_2} dS(\hat{h}\mathbf{R}(\hat{n})\mathbf{s})^2. \end{aligned} \tag{4}$$

In (4) $f_{an}(\hat{n})$ describes the anisotropic (orienting) part of the free energy density stemming from the combined action of the applied magnetic field $\mathbf{H} = H\hat{h}$, dipole effects, macroscopic superfluid counterflow, and vortices:

$$f_{an}(\hat{n}) = -aH^2 \begin{cases} (\hat{n}\hat{h})^2 + \dfrac{2}{5}\left(\dfrac{w(r)}{v_D}\right)^2 (\hat{h}\mathbf{R}(\hat{n})\hat{\varphi})^2, \\ (\hat{n}\hat{h})^2 - \dfrac{2}{5}\lambda(\hat{h}\mathbf{R}(\hat{n})\hat{z})^2, \end{cases} \tag{5}$$

and f_{grad} stands for the contribution of the energy density inhomogeneity. The first line in (5) refers to the counterflow region and the second line to the vortex cluster domain.

The coefficient in Eqs. (5) satisfies $a \simeq 10^{-12}$; the order parameter $R(\hat{n})$ describes relative spin-orbital rotation around $\hat{n}$ by the Leggett angle $\vartheta_0 = \arccos(-1/4)$, $w(r)$ is given by (1) [or (3)], v_D is the dipole velocity and λ describes the average contribution to the anisotropy energy of quantized vortices within the vortex cluster.

The last two terms in (4) describe the surface magnetic energy, the separate minimization of which gives the following values for the polar and azimuthal angles of the directrices $\mathbf{n}$ at the surfaces:

$$\beta = 63.4^\circ, \quad \text{and} \quad \alpha = 60^\circ, \text{ or } \beta = 116.6^\circ \quad \text{and} \quad \alpha = 120^\circ, \tag{6}$$

however, it should be noted that since $d/a = 2$ mm is comparable with the cylinder radii, one has to minimize the whole functional (4) in order to determine the equilibrium texture. For this reason, at relatively low angular velocities the orientation of $\mathbf{n}$ on the surfaces is not determined exactly by the conditions (6), but is close to them.

The results of numerical computation of the spatial distribution of the order parameter axis $\mathbf{n}$ across an annular container are shown in Figs. 1 and 2 for the cases of acceleration ($\Omega > \Omega_v$) and deceleration ($\Omega < \Omega_v$).

When the angular velocity Ω_0 increases, textural phase transitions take place near the cylinder surfaces (for more details see Refs. 8 and 9). For $\Omega > 0$ (Fig. 1) we first have the so called simple texture with $\beta_S \approx 63°$. At $\Omega=1$ rad/s an extended texture ($\beta(R_2) \approx 116°$) develops near the outer cylinder. For $\Omega>1.4$ **n** is "stripped" from the external surface ($\beta(R_2) \approx 63°$, $\alpha(R_2) = 150°$), and the direction of **n** is determined by minimization of the orientational contribution of the counterflow to the free energy. At $\Omega=2$ rad/s the transition to an extended texture occurs near R_1.

For $\dot{\Omega}<0$ the textural transitions take place near the inner cylinder. For $\Omega>0.6$ we have a simple texture, while for $\Omega<0.5$ an extended texture develops.

The presence of a vortex cluster in ^{3}He-B rotating in an annulus affects the behavior of a freely suspended container. The time decay of the angular velocity $\Omega(t)$ of a decelerating annular vessel filled with a superfluid liquid is governed by the equation

$$\frac{dL}{dt}=T, \tag{7}$$

where L is the z-component of the total (liquid + container) angular momentum and $T = -\gamma\Omega$ describes an external frictional torque acting on the rotating vessel.

The angular momentum generated by the mass flow $\mathbf{j} = \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s$ is calculated as

$$L_{\text{liq}}=2\pi\int [\mathbf{jr}]_z r\,dr=\frac{1}{2}\pi\left\{[\rho_n(R_2^4-R_1^4)+\rho_s(R_2^4-R_c^4)]\Omega+\rho_s\frac{\Gamma_\Omega(R_c)}{\pi}(R_c^2-R_1^2)\right\}. \tag{8}$$

Equation (7) gives

$$I_{\text{eff}}(\Omega)\frac{d\Omega}{dt}=-\gamma\Omega(t), \tag{9}$$

where the effective (Ω-dependent) moment of inertia is

$$I_{\text{eff}}(\Omega)=I_0+\frac{1}{2}\pi\rho_s[R_1^4-R_c^4(\Omega)], \tag{10}$$

and

$$I_0=I_{\text{vessel}}+\frac{1}{2}\rho(R_2^4-R_1^4).$$

The solution of Eq. (9) is given implicitly by

$$\int_{\Omega_v}^{\Omega(t)} I_{\text{eff}}(\Omega')\frac{d\Omega'}{\Omega'}=-\gamma t, \tag{11}$$

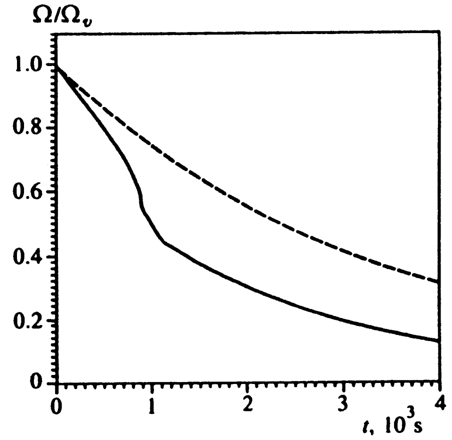

FIG. 3. The time decay of $\Omega(t)$ for a decelerating annular vessel with superfluid ^{4}He for $\rho_s/\rho = 0.9$ in the presence of a vortex cluster. The dashed line represents the time decay for the case of a vortex-controlled rotating equilibrium (the frictional coefficient γ describes the deceleration of the rotating vessel in a helium gas environment at the saturation pressure).

from which we obtain

$$I_1\ln\left[\frac{\Omega(t)}{\Omega_0}\right]+\frac{1}{4}\rho_s\{R_c^4[\Omega(t)-R_c^4(\Omega_0)]\}=-\gamma t \tag{12}$$

with $I_1=I_0+\pi\rho_s R_1^4/2$.

By inspection of Eq.(12) we conclude that the Ω-dependence of R_c makes the frictional decay of $\Omega(t)$ nonexponential until $t \leqslant t_*$, where t_* is the time at which the vortex cluster disappears completely ($\Omega(t_*) = \Omega_2^* = (R_1/R_2)^2\Omega_v$). After $t > t_*$ we have the exponential dependence $\Omega(t) = \Omega_2^*\exp[-(\gamma/I_0)(t-t_*)]$ (Fig. 3).

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