Anomalous dimensions in the Burger–Kardar–Parisi–Zhang model

É. V. Teodorovich*

Institute for Problems of Mechanics, Russian Academy of Sciences, 117526 Moscow, Russia

(Submitted 27 March 1995)
Zh. Éksp. Teor. Fiz. **109**, 506–523 (February 1996)

The exponents of the scale transformations determining the large-scale behavior of the statistical solutions of the nonlinear diffusion equation, known under the name of the Burgers–Kardar–Parisi–Zhang equation, are calculated by means of the renormalization approach in the two-loop approximation. The value found for the dynamic exponent and the roughness exponent, $z=1.73$ and $\chi=0.27$, respectively, are in satisfactory agreement with results obtained by other theoretical methods and by numerical analysis of the problem. [S1063-7761(96)01402-9]

1. INTRODUCTION

At present, the nonlinear diffusion equation in the presence of an external random source, known by the name of the Burgers–Kardar–Parisi–Zhang (BKPZ) equation,[1] is the subject of intense study. The interest in this equation is due to the fact that it describes a wide variety of physical and physico-chemical processes such as fluctuations of the shape of the boundary surface of a growing solid body or the interfaces of two-liquid flows, and the motion of domain walls and clusters.[2–4] This equation turns out to be equivalent to the equation for the distribution function of directed polymers, dislocations, and vortices in a random field. The BKPZ equation governs the velocity potential of liquid flow described by the Burgers equation,[5,6] the large-scale asymptotic behavior of the propagation of flame fronts (the Kuramoto–Sivashinski equation),[7] and diffusion in a randomly inhomogeneous medium.[8] This wide range of applicability of the BKPZ equation is due to the fact that this equation is the simplest nonlinear generalization of the diffusion equation.

As the unknown function, one considers the scalar quantity $h(\mathbf{r},t)$, whose dimensionality and physical meaning can be quite various. The radius vector $\mathbf{r}$ is defined in a space of dimension d, which can be equal to the dimension of real space or to the dimension of a hypersurface in $(d+1)$-dimensional space. The quantity $h(\mathbf{r},t)$ can have the meaning of a velocity potential, or perturbations of the profile of a surface moving in the direction perpendicular to itself, or be proportional to the logarithm of the concentration distribution function in a randomly inhomogeneous medium.

A characteristic feature of all the above systems is that the absence of a characteristic length scale causes the large-scale fluctuations of h to display universal scaling properties, i.e., for the pairwise correlation function $C(\mathbf{r},t)$ the following relation holds[9,10]

$$C(\mathbf{r},t)=\langle[h(\mathbf{r}+\mathbf{r}_0,t+t_0)-h(\mathbf{r}_0,t_0)]^2\rangle=r^{2\chi}f(t/r^z), \tag{1.1}$$

where the universal (depending only on the dimension d) values χ and z are called respectively the roughness exponent and the dynamic exponent and the function $f(x)$ in the limit $x\to\infty$ behaves like $x^{2\chi/z}$, and in the limit $x\to 0$ approaches a constant value. One of the main problems in the study of the BKPZ equation is to find the values of these universal exponents. Note that the presence in the system of some symmetry property (Galilean invariance) leads to a relation between the two exponents of the form[2]

$$\chi+z=2, \tag{1.2}$$

as a result of which, only one of the exponents is independent.

The BKPZ equation in the presence of an external random source η and a regular source $\tilde{f}$ has the form

$$\frac{\partial h}{\partial t}=\nu_0\Delta h+\frac{1}{2}\lambda_0(\nabla h)^2+\eta+\tilde{f}, \tag{1.3}$$

where, depending on the specific physical meaning of the quantity h, the parameters ν_0 and λ_0 have a different meaning and dimensionality. When h defines the potential of a velocity field (the Burgers equation), the parameter ν_0 has the meaning of a viscosity coefficient, and the nonlinear interaction constant λ_0 is dimensionless and equal to unity. When h has the dimension of length and describes the fluctuations of the shape of a moving interface, the parameters ν_0 and λ_0 correspond to the coefficient of surface tension at the interface and the translational velocity of the boundary. (With the Burgers equation in mind, in what follows we will call the parameter ν_0 in Eq. (1.3) the viscosity coefficient.) The Langevin noise source η is assumed to satisfy Gaussian statistics and it is assumed that it is δ-correlated in space and time ("white noise"):

$$\langle\eta(\mathbf{r},t)\eta(\mathbf{r}',t')\rangle=2D_0\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')=D^{(0)}(\mathbf{r}-\mathbf{r}',t-t') \tag{1.4}$$

("colored noise" correlations $\eta(\mathbf{r},t)$ are considered in Ref. 6).

The exponents for the asymptotic power-law dependences in the absence of a characteristic scale can sometimes be found from dimensional arguments. However, such arguments are insufficient if divergent integrals arise in the theory. To regularize the divergent integrals by truncating them in the large-scale region (for IR divergences) or the small-scale region (for UV divergences), one introduces, either explicitly or implicitly, an additional spatial scale Λ^{-1}, which must be taken into account when using dimen-

 1063-7761/96/020268-10$10.00

sional arguments. The presence of this scale causes the values of the exponents of the scale transformations to differ from the values that follow from simple dimensional arguments, and the corresponding correction carries the name of the anomalous dimension.[11] The value of the dimension of the space at which the divergences appear is called the critical (crossover) dimension. At the critical dimension the divergence is weak (logarithmic), and the behavior of the power-law exponents on both sides of the critical dimension is different.

Usually, the problem of calculating the power-law exponents is solved by using the renormalization-group (RG) method. The RG method, which first appeared in quantum field theory and was later modified and successfully applied to the theory of critical phenomena, may be considered as a way of studying substantially multimode systems, in which the absence of an isolated characteristic scale makes the modes of all scales equally important for understanding the processes taking place in the system.[12]

Two somewhat different formulations of the RG method are known. In the so-called field-theoretic formulation, the property of renormalization invariance, lying at the basis of the method, consists in the independence of the total perturbation-theory series on the manner in which the equation (or the corresponding Lagrangian) is divided into an unperturbed part and the perturbation.[13,14] This property means that by knowing the lower approximations of the perturbation theory one can predict the form of the subsequent terms of the series and sum up this series or some infinite subsequence of it. The second formulation, which was developed and successfully applied by Wilson in the theory of critical phenomena,[15,16] is based on the idea of successively decreasing the number of considered modes by an iterative partial averaging over the small-scale (fast) modes in the equation (or distribution function) for the slow modes (the Kadanoff procedure) in combination with a scale transformation defining the interface between the slow modes remaining after the transformation and the fast modes excluded from consideration by the Kadanoff procedure. The asymptotic behavior of the system is determined by the properties and nature of the stability of the fixed points of the RG-transformation, which reduces to a combination of the Kadanoff transformation and the scale transformation.

The RG approach was first applied to the problem under consideration in Ref. 17, whose authors proposed a generalization of the one-dimensional Burgers equation to the case of a space of arbitrary dimension and used the RG method in the Wilson formulation to examine the dependence of the large-scale asymptotic behavior of the statistical solutions on the dimension of the space. They showed that for $d=1$ the property of Galilean invariance in combination with the fluctuation–dissipation theorem allows one to obtain the values $\chi=1/2$ and $z=3/2$ for the exponents. In the region $1<d<2$ there are no divergences and as a consequence of this it should be possible to determine the exponents by dimensional arguments, which leads to the values $z=(d+2)/2$ and $\chi=(2-d)/2$ (Refs. 1 and 18). In fact, in the absence of divergences the RG approach reproduces the results of a dimensional analysis in analogy with the situation that obtains in the RG description of turbulence.[19] In the theory of turbulence, corrections to the Kolmogorov scaling exponents arise when one takes account, within the RG approach, of spatially localized wave numbers of the intermode interactions, which lead to the logarithmic IR divergences.[20] An estimate of the exponents was made in Ref. 21 for the BKPZ equation in the linear approximation, where the value $z=2$ was found for the dynamic exponent, and the value $\chi=(2-d)/2$ for the roughness exponent. The failure of relation (1.2) for $d\neq 2$ is due to violation of the Galilean invariance of Eq. (1.3) when neglecting the nonlinear term.

We are interested in the case $d=2$ corresponding to the critical dimension at which the trivial and nontrivial fixed points merge, and to find a new nontrivial fixed point it is necessary to use a higher perturbational approximation (the two-loop approximation). The BKPZ equation has been studied by means of the two-loop approximation in recent papers.[22,23]

2. CONSTRUCTION OF STATISTICAL SOLUTIONS

The BKPZ equation in its structure is reminiscent of the equations lying at the basis of the statistical description of fully developed turbulence, and in some respect it is even simpler since the unknown function is a scalar, and not a vector. It thus appears possible to formulate the problem in a way analogous to the statistical theory of turbulence.[24]

Following this approach, we write the BKPZ equation in the form proposed in Ref. 25:

$$-K^{(0)}(12)h(2)-\frac{\lambda_0}{2}V(1|23)h(2)h(3)+\eta(1)+\tilde{f}(1)=0. \tag{2.1}$$

Here we have introduced the notation $(\mathbf{r}_1,t_1)=1$, $h(\mathbf{r}_1,t_1)=h(1)$ and the integration is understood to be over repeating space-time variables. According to Eq. (1.3)

$$K^{(0)}(12)=\left(\frac{\partial}{\partial t}-\nu_0\Delta_1\right)\delta(1-2),$$

$$V(1|23)=(\nabla_2\nabla_3)\delta(1-2)\delta(1-3). \tag{2.2}$$

To describe the system statistically, it is convenient to use the method of the characteristic functional,[25–27] the latter being the functional Fourier transform of the density of the distribution of realizations of the field h for prescribed sources η and $\tilde{f}$:

$$W[f,\tilde{f}]=\langle\exp\{ih(1|\eta,\tilde{f})f(1)\}\rangle, \tag{2.3}$$

where the angular brackets denote the average over the ensemble of realizations, which reduces to averaging over the random sources η for fixed $\tilde{f}$. Knowing the characteristic functional allows one to find the cumulant means and average response functions from the relations

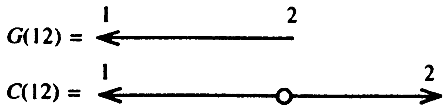

FIG. 1.

$$\langle h(1)\rangle = \frac{\delta \ln W[f,\tilde f]}{i\delta f(1)}\bigg|_{f=0},$$

$$C(12)=\langle h(1)h(2)\rangle-\langle h(1)\rangle\langle h(2)\rangle$$

$$=\frac{\delta^2 \ln W[f,\tilde f]}{i\delta f(1) i\delta f(2)}\bigg|_{f=0}, \qquad (2.4)$$

$$iG(12)=\langle h(1)\tilde h(2)\rangle=\frac{\delta^2 \ln W[f,\tilde f]}{i\delta f(1) i\delta \tilde f(2)}\bigg|_{f=0}.$$

In the development that follows, we will use the skeleton diagram technique to depict the Green's function G (propagator) and correlator C by thick lines with one and two oppositely directed arrows, respectively, as shown in Fig. 1. In the zeroth perturbational approximation they correspond to diagrams with thin lines, where

$$G^{(0)}(12)=[K^{(0)}(12)]^{-1},$$

$$C^{(0)}(12)=2D_0[K^{(0)}(11')]^{-1}[K^{(0)}(22')]^{-1}\delta(1'-2').$$

As is well known, the characteristic functional $W[f,\tilde f]$ can be represented in the form of a functional integral over the fields h and $\tilde h$[24–27]

$$W[f,\tilde f]=\int d[h]d[\tilde h]J^{-1}[h]\exp\{iS[h,\tilde h]+ihf+i\tilde h\tilde f\},$$

$$S[h,\tilde h]=-\tilde h(1)K^{(0)}(12)h(2)+\frac{i}{2}\tilde h(1)D^{(0)}(12)\tilde h(2)$$

$$-\frac{\lambda_0}{2}\tilde h(1)V(1|23)h(2)h(3), \qquad (2.5)$$

$$J[h]=\frac{\delta[K^{(0)}h+(\lambda_0/2)Vhh]}{\delta h}.$$

Usually such a representation is used to construct the perturbation-theory series by expanding the exponential inside the functional integral in a power series in $\lambda_0 V$. Thereby the possibility arises of finding the statistical moments and the response function in the prescribed perturbational approximation.[28] The construction of the series is facilitated by constructing the terms of the corresponding series of Feynman diagrams. Another way of constructing the perturbation-theory series for the statistical moments is to make direct use of the dynamic equation (1.3) and solve it iteratively. Then subsequent multiplication of the series and term-by-term averaging of the so-obtained series allows one to obtain an expression for the statistical moments. In this procedure one can also use the Feynman diagram technique,[29] where an analysis of these diagrams allows one to find equations for the statistical moments—the so-called Dyson (and Wild) equations for the skeletal diagrams.

On the other hand, the Dyson equations can be obtained outside the framework of perturbation theory from the formalism of the characteristic functional in a way analogous to that employed in the theory of turbulence.[24] Toward this end, we make use of the invariance property of the path integral with respect to the operation of a shift in the functional argument $\tilde h(1)\rightarrow\tilde h(1)+\varphi(1)$. Setting the functional derivative $\delta W[f,\tilde f]/\delta\varphi(1)$ equal to zero, we find an equation in the functional derivatives for the characteristic functional which is an analog of the Hopf equation in the theory of turbulence:[30]

$$\left\{-K^{(0)}(12)\frac{\delta}{i\delta f(2)}-\frac{\lambda_0}{2}V(1|23)\frac{\delta}{i\delta f(2)}\frac{\delta}{i\delta f(3)}\right.$$

$$\left.+iD^{(0)}(12)\frac{\delta}{i\delta\tilde f(2)}+\tilde f(1)\right\}W[f,\tilde f]=0. \qquad (2.6)$$

In order to obtain the Dyson equations for the correlator $C(12)$ and the Green's function (propagator) $G(12)$, we transform to new functional variables

$$H(1)=\frac{\delta \ln W[f,\tilde f]}{i\delta f(1)},\quad \tilde H(1)=\frac{\delta \ln W[f,\tilde f]}{i\delta\tilde f(1)}, \qquad (2.7)$$

which in the limit $f\rightarrow 0$ go over to the mean values of the fields h and $\tilde h$, respectively, where $\langle\tilde h\rangle=0$. This transformation is carried out by means of the Legendre functional transformation by introducing a new characteristic functional for the single-particle irreducible diagrams:

$$\Psi[H,\tilde H]=-\ln W[f,\tilde f]+ifH+i\tilde f\tilde H,$$

$$\frac{\delta\Psi}{i\delta H(1)}=f(1),\quad \frac{\delta\Psi}{i\delta\tilde H(1)}=\tilde f(1). \qquad (2.8)$$

Calculation of the mixed derivatives of Ψ with respect to the fields f and H leads to the relations

$$\frac{\delta^2\Psi}{\delta f(2)i\delta H(1)}=\delta(1-2)=\frac{\delta H(2')}{i\delta f(2)}\frac{\delta^2\Psi}{\delta H(2')\delta H(1)}$$

$$+\frac{\delta\tilde H(2')}{\delta f(2)}\frac{\delta^2\Psi}{i\delta\tilde H(2')\delta H(1)},$$

$$\frac{\delta^2\Psi}{\delta f(2)\delta\tilde H(1)}=-\frac{\delta H(2')}{i\delta f(2)}\frac{\delta^2\Psi}{\delta H(2')i\delta\tilde H(1)}$$

$$-\frac{\delta\tilde H(2')}{\delta f(2)}\frac{\delta^2\Psi}{i\delta\tilde H(2')i\delta\tilde H(1)}=0, \qquad (2.9)$$

from which the extremality conditions $\delta\Psi/i\delta H=f=0$ imply

$$\frac{\delta}{\delta H(1)}\frac{\delta}{i\delta\tilde H(2)}\Psi=G^{-1}(21),\quad \frac{\delta}{i\delta\tilde H(1)}\frac{\delta}{i\delta\tilde H(2)}\Psi$$

$$=-G^{-1}(11')G^{-1}(22')C(1'2')$$

$$=-D(12), \qquad (2.10)$$

where G^{-1} is the inverse Green's function, and it is natural to call the quantity $D(12)$ the correlation function of the effective random sources.

If we represent $\Psi[H,\tilde{H}]$ by a filled circle, then the action on Ψ of the operators $\delta/\delta H$ and $\delta/i\delta\tilde{H}$ will correspond to insertion of an arriving or departing arrow. We also introduce into consideration quantities corresponding in graphical language to single-particle irreducible vertices of three types: those containing two arriving and one departing arrow (conversion of two quanta into one), one arriving and two departing arrows (conversion of one quantum into two), and three departing arrows (creation of three quanta):

$$\Gamma(1|23)=\frac{\delta^3\Psi}{i\delta\tilde{H}(1)\delta H(2)\delta H(3)},$$

$$\Gamma(12|3)=\frac{\delta^3\Psi}{i\delta\tilde{H}(1)i\delta\tilde{H}(2)\delta H(3)},$$

$$\Gamma(123)=\frac{\delta^3\Psi}{i\delta\tilde{H}(1)i\delta\tilde{H}(2)i\delta\tilde{H}(3)}. \tag{2.11}$$

In the lower perturbational approximation

$$\Gamma(1|23)=\lambda_0 V(1|23),\quad \Gamma(12|3)=0,\quad \Gamma(123)=0.$$

Departing from Eqs. (2.5) and definition (2.7), it is not hard to obtain an equation in functional derivatives for the functional Ψ:

$$\frac{\delta\Psi}{i\delta\tilde{H}(1)}=K^{(0)}(12)H(2)+\frac{\lambda_0}{2}V(1|23)\left[H(2)H(3)+\frac{\delta H(3)}{i\delta f(2)}\right]-iD^{(0)}(12)\tilde{H}(2). \tag{2.12}$$

Operating on Eq. (2.12) with the operator $\delta/\delta H(2)$, we obtain

$$\frac{\delta^2\Psi}{\delta H(2)i\delta\tilde{H}(3)}=G^{-1}(12)=K^{(0)}(12)+\lambda_0 V(1|23)H(3)+\frac{\lambda_0}{2}V(1|34)\frac{\delta^2 H(3)}{\delta H(2)i\delta f(4)}. \tag{2.13}$$

Analogously, by operating on Eq. (2.12) with the operator $\delta/i\delta\tilde{H}(2)$, we obtain

$$-\frac{\delta^2\Psi}{i\delta\tilde{H}(2)i\delta\tilde{H}(1)}=D(12)=D^{(0)}(12)-\frac{\lambda_0}{2}V(1|34)\times\frac{\delta^2 H(3)}{i\delta\tilde{H}(2)i\delta f(4)}. \tag{2.14}$$

The functional derivatives entering into Eqs. (2.13) and (2.14), represented in a form containing only C and G and the single-particle irreducible diagrams are calculated by variational differentiation relations (2.9) with respect to the fields H and $\tilde{H}$, which can be carried out also with the help of the graphical correspondence rules.[24] As a result we have

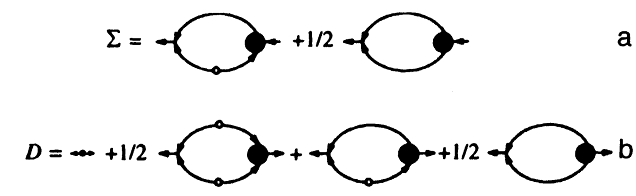

FIG. 2.

$$\frac{\delta^2 H(3)}{\delta H(2)i\delta f(4)}=-C(33')C(44')\Gamma(2|3'4')-2G(33')C(44')\Gamma(23'|4')-G(33')G(44')\Gamma(23'4'),$$

$$\frac{\delta^2 H(3)}{\delta\tilde{H}(2)\delta f(4)}=-2G(33')\Gamma(3'|24')C(4'4)-G(33')G(44')\Gamma(3'4'|2). \tag{2.15}$$

Substituting formulas (2.15) in Eqs. (2.13) and (2.14) gives

$$G^{-1}(12)=K^{(0)}(12)-\lambda_0 V(1|34)\times\left[G(33')C(44')\Gamma(3'|4'2)+\frac{1}{2}G(33')G(44')\Gamma(3'4'|2)\right]\equiv[G^{(0)}]^{-1}-\Sigma, \tag{2.16}$$

$$D(12)=D^{(0)}(12)+\lambda_0 V(1|34)\times\left[\frac{1}{2}C(33')C(44')\Gamma(2|3'4')+G(33')C(44')\Gamma(23'|4')+\frac{1}{2}G(33')G(44')\Gamma(23'4').\right. \tag{2.17}$$

The corresponding graphical representations for the self-energy operator Σ and correlator of the effective random forces D are shown in Figs. 2a and b.

Let us now derive the Ward–Takahashi identities, which follow from the invariance of Eq. (1.3) with respect to a transformation of Galilean type for the Burgers equation, and which are needed as a foundation of renormalizability. From the form of the action functional $S[h,\tilde{h}]$ we have the easily verified relation

$$S[h,\tilde{h}]=S[\hat{L}(\mathbf{v})h,\hat{L}(\mathbf{v})\tilde{h}], \tag{2.18}$$

where the operator $\hat{L}(\mathbf{v})$ is defined by the relations

$$\hat{L}(\mathbf{v})h(\mathbf{r},t)=h(\mathbf{r}+\lambda\mathbf{v}t,t)+\mathbf{v}\cdot\mathbf{r}+(\lambda/2)\mathbf{v}^2 t,$$

$$\hat{L}(\mathbf{v})\tilde{h}(\mathbf{r},t)=\tilde{h}(\mathbf{r}+\lambda\mathbf{v}t,t), \tag{2.19}$$

and $\mathbf{v}$ is an arbitrary vector parameter.

Following the development in Ref. 31, we substitute relation (2.18) into Eq. (2.5) and calculate the derivative of the characteristic functional with respect to the vector parameter $\mathbf{v}$ at the point $\mathbf{v}=0$. As a result, taking Eq. (2.6) into account, we find

$$\int d1\left\{\lambda t_1 f(1)\nabla\frac{\delta}{i\delta f(1)}+\lambda t_1\tilde{f}(1)\nabla\frac{\delta}{i\delta\tilde{f}(1)}+\mathbf{r}_1 f(1)\right\}W[f,\tilde{f}]=0. \tag{2.20}$$

After performing the Legendre transformation, we obtain the generating equation for the Ward–Takahashi equalities for the strongly coupled (single-particle irreducible) diagrams:

$$\int d1\left\{\lambda t_1\frac{\delta\Psi}{i\delta H(1)}\nabla H(1)+\lambda t_1\frac{\delta\Psi}{i\delta\tilde{H}(1)}\nabla\tilde{H}(1)+\mathbf{r}_1\frac{\delta\Psi}{i\delta H(1)}\right\}=0. \tag{2.21}$$

Operating on Eq. (2.21) in the first case with the operator $\delta/\delta H(2)\cdot\delta/i\delta\tilde{H}(3)$, and in the second, with the operator $[\delta/i\delta\tilde{H}(2)][\delta/i\delta\tilde{H}(3)]$, we obtain the two relations

$$\int d1\{\lambda t_2 G^{-1}(31)\nabla_1\delta(1-2)+\lambda t_3 G^{-1}(12)\times\nabla_1\delta(1-3)+\mathbf{r}_1\Gamma(3|21)\}=0,$$

$$\int d1\{\lambda t_2 D(13)\nabla_1\delta(1-2)+\lambda t_3 D(12)\nabla_1\delta(1-3)-\mathbf{r}_1\Gamma(32|1)\}=0. \tag{2.22}$$

Taking the Fourier transforms leads to the relations

$$\left.\frac{\partial\Gamma(\hat{k}+\hat{q}|\hat{k},\hat{q})}{\partial\mathbf{q}}\right|_{\hat{q}=0}=i\lambda\mathbf{k}\frac{\partial G^{-1}(\hat{k})}{\partial\omega},$$

$$\left.\frac{\partial\Gamma(\hat{k}+\hat{q},-\hat{k}|\hat{q})}{\partial\mathbf{q}}\right|_{\hat{q}=0}=-i\lambda\mathbf{k}\frac{\partial D(\hat{k})}{\partial\omega}, \tag{2.23}$$

$$\hat{k}=(\mathbf{k},\omega),\quad \hat{q}=(\mathbf{q},\omega').$$

From Eqs. (2.23) we can obtain a representation for the vertices when the frequency and wave number of one of the arriving arrows tend to zero (processes with absorption of zero quanta):[23]

$$\Gamma(\hat{k}+\hat{q}|\hat{k},\hat{q})|_{\hat{q}\to 0}=i\lambda\mathbf{k}\mathbf{q}\frac{\partial G^{-1}(\hat{k})}{\partial\omega},$$

$$\Gamma(\hat{k}+\hat{q},-\hat{k}|\hat{q})|_{\hat{q}\to 0}=-i\lambda\mathbf{k}\mathbf{q}\frac{\partial D(\hat{k})}{\partial\omega}. \tag{2.24}$$

Note that relations (2.23) and (2.24) are an exact consequence of the conditions for invariance of the BKPZ equation with respect to the transformation group $\hat{L}(\mathbf{v})$ and the assumptions of δ-correlatedness in time of the external random sources and the statistical homogeneity of the system; the specific form of the BKPZ equation was not employed in the derivation of these relations. It follows from relations (2.23) that eliminating the divergences in G and D by renormalization also eliminates the divergences of the vertices and decreases the number of independent constants of the renormalization.

3. RENORMALIZATION AND RENORMALIZATION-GROUP ANALYSIS

In the case of interest $d=2$ logarithmic UV divergences arise in the integrals over the wave numbers in the expressions for the self-energy operator Σ, the correlation function of the effective random forces D, and the vertex containing one departing arrow. To eliminate these divergences, it suffices to renormalize the viscosity coefficient by making the substitution $\nu_0\to\nu=Z_2\nu_0$, the amplitude of the random forces by the substitution $D_0\to D=Z_1D_0$, and the amplitude of the auxiliary field $\tilde{h}$ by the substitutions $\tilde{h}\to\tilde{h}^R=Z_3\tilde{h}$ and $\lambda_0\to\lambda=Z_3^{-1}\lambda_0$, including the divergences in the renormalization constants Z_i, which is the condition of renormalization theory.

Compensation for the effects of renormalization reduces to adding to the expression for the action $S[h,\tilde{h}]$ counterterms of the form

$$\delta S[h,\tilde{h}]=-\tilde{h}^R\left[(Z_3^{-1}-1)\frac{\partial h}{\partial t}-(Z_2^{-1}-1)Z_3^{-1}\nu\Delta h+\frac{i}{2}(Z_1^{-1}-1)Z_3^{-1}D\tilde{h}\right], \tag{3.1}$$

which must be added to the perturbation in the construction of the renormalization perturbation theory.

To eliminate ambiguity in the procedure of subtracting the infinite expressions, it is necessary to define normalization conditions.[13,14] In the solution of this problem by means of dimensional regularization in combination with the scheme of minimal subtractions[32] the additional parameter μ with dimensions of inverse length (the mass parameter) is introduced in such a way as to ensure invariance of the dimensionality of the actual expansion parameter with variation of d. A more suitable and physically more transparent subtraction scheme is one in which the dimension of the space does not vary, but the renormalization coefficients are determined by the requirement that at the chosen normalization point $k=\mu$, $\omega=0$ the corrections to the renormalized parameter values due to the interaction vanish, i.e., near the normalization point the Fourier transforms of the correlator and the propagator should have the same form as in the absence of the nonlinear interactions, but with renormalized parameter values. Thereby the large additive terms due to the interaction turn out to a significant degree to be taken into account by the renormalization, and the renormalized perturbation theory is then used to calculate the remaining small corrections. The renormalization constants in the above-mentioned subtraction schemes will then differ by a finite quantity.

From the Dyson equation (2.16) it follows that taking the nonlinear intermode coupling into account reduces to replacing the viscosity coefficient ν_0 in the expression for the Fourier transform of the Green's function by the effective viscosity as follows:

$$G^{-1}(\mathbf{k},\omega)=-i\omega+\tilde{\nu}(\mathbf{k},\omega)k^2=-i\omega+\nu_0k^2-\Sigma(\mathbf{k},\omega). \tag{3.2}$$

To find the dynamic exponent z it is sufficient to consider the effective viscosity in the static limit $\omega\to 0$, which is a function of the wave number k, the renormalized numerical parameters ν, D, λ, and the mass parameter μ, which determines the position of the normalization point. From dimensional arguments the static effective viscosity coefficient can be represented in the form

$$\tilde{\nu}(k)=\nu f_2\left(\frac{k}{\mu},\frac{D\lambda^2}{\nu^3}\right), \tag{3.3}$$

where the normalization condition $\tilde{\nu}(\mu)=\nu$ implies that $f_2(1,D\lambda^2/\nu^3)=1$.

The condition of renormalization invariance consists in the requirement that the result of the calculation of the Green's function be independent of the choice of the normalization point μ. From this condition it follows that

$$Z_3\left(\frac{\mu}{\Lambda},\frac{D\lambda^2}{\nu^3}\right)\nu f_2\left(\frac{k}{\mu},\frac{D\lambda^2}{\nu^3}\right)=Z_3\left(\frac{\mu_1}{\Lambda},\frac{D_1\lambda_1^2}{\nu_1^3}\right)\nu_1 f_2\left(\frac{k}{\mu_1},\frac{D_1\lambda_1^2}{\nu_1^3}\right). \tag{3.4}$$

Analogously for the correlation function of the effective random forces we can take

$$D(k)=2Df_1\left(\frac{k}{\mu},\frac{D\lambda^2}{\nu^3}\right),\quad f_1\left(1,\frac{D\lambda^2}{\nu^3}\right)=1, \tag{3.5}$$

and the requirement of renormalization invariance leads to the relation

$$Z_3^2\left(\frac{\mu}{\Lambda},\frac{D\lambda^2}{\nu^3}\right)Df_1\left(\frac{k}{\mu},\frac{D\lambda^2}{\nu^3}\right)=Z_3^2\left(\frac{\mu_1}{\Lambda},\frac{D_1\lambda_1^2}{\nu_1^3}\right)D_1 f_1\left(\frac{k}{\mu_1},\frac{D_1\lambda_1^2}{\nu_1^3}\right). \tag{3.6}$$

We introduce another function, f_3, defined by the relation

$$f_3\left(\frac{\mu_1}{\mu},\frac{D\lambda^2}{\nu^3}\right)=\frac{Z_3(\mu_1/\Lambda,D_1\lambda_1^2/\nu_1^3)}{Z_3(\mu/\Lambda,D\lambda^2/\nu^3)} \tag{3.7}$$

(the possibility of writing f_3 in a form independent of Λ is a consequence of the renormalizability of the theory). It follows from (3.7) the definition that this function satisfies the group composition law

$$f_3\left(\frac{\mu_2}{\mu},\frac{D\lambda^2}{\nu^3}\right)=f_3\left(\frac{\mu_2}{\mu_1},\frac{D_1\lambda_1^2}{\nu_1^3}\right)f_3\left(\frac{\mu_1}{\mu},\frac{D\lambda^2}{\nu^3}\right). \tag{3.8}$$

From Eqs. (3.4), (3.6), and (3.8), noting the relation $\lambda^2=f_3^2(\mu_1/\mu,D\lambda^2/\nu^3)\lambda_1^2$, we obtain

$$\frac{gf_1(k/\mu,g)}{f_2^3(k/\mu,g)f_3(k/\mu,g)}=\frac{g_1f_1(k/\mu_1,g_1)}{f_2^3(k/\mu_1,g_1)f_3(k/\mu_1,g_1)},$$

$$g=\frac{\lambda^2D}{\nu^3},\quad g_1=\frac{D_1\lambda_1^2}{\nu_1^3}. \tag{3.9}$$

Thus, the function

$$\tilde{g}(x,g)=\frac{gf_1(x,g)}{f_2^3(x,g)f_3(x,g)} \tag{3.10}$$

is an invariant of the RG transformation $\mu\to\mu_1$, $g\to g_1$ and is the actual wavenumber-dependent expansion parameter in the renormalized perturbation series (an analog of the invariant charge in quantum field theory[13]). The function $\tilde{g}(x,g)$ is normalized by the condition

$$\tilde{g}(1,g)=g, \tag{3.11}$$

from which together with Eqs. (3.9) it follows that it satisfies the functional RG equation

$$\tilde{g}(x,g)=\tilde{g}\left(\frac{x}{t},\tilde{g}(t,g)\right),\quad t=\frac{\mu_1}{\mu}. \tag{3.12}$$

From Eqs. (3.4), (3.6), and (3.8) also follow functional equations for the functions $f_i(x,g)$:

$$f_i(x,g)=f_i(t,g)f_i\left(\frac{x}{t},\tilde{g}(t,g)\right). \tag{3.13}$$

Differentiating the functional equations (3.12) and (3.13) with respect to t and then setting $t=1$, we find the differential RG equations

$$\left[-x\frac{\partial}{\partial x}+\beta(g)\frac{\partial}{\partial g}\right]\tilde{g}(x,g)=0,\quad \beta(g)=\left.\frac{\partial\tilde{g}(x,g)}{\partial x}\right|_{x=1}, \tag{3.14}$$

$$\left[-x\frac{\partial}{\partial x}+\beta(g)\frac{\partial}{\partial g}+\gamma_i(g)\right]f_i(x,g)=0,$$

$$\gamma_i(g)=\left.\frac{\partial f_i(x,g)}{\partial x}\right|_{x=1}. \tag{3.15}$$

4. USING THE RENORMALIZATION-GROUP METHOD

The RG equations (3.14) and (3.15) have a universal character, and the details of a specific system (the BKPZ equation) are included in the so-called RG functions $\beta(g)$ and $\gamma_i(g)$, also called the Gell-Mann–Lowe functions, or the Wilson functions. In its quantum-field-theory formulation the RG method consists in the proposal to use the renormalized perturbation theory to calculate them.[13] Knowing the RG functions to lower approximations allows one to solve Eqs. (3.14) and (3.15), which corresponds to summing some infinite subsequence of the complete perturbation-theory series.

Using perturbation theory to find the logarithmic divergences necessitates bringing an additional scale Λ into the theory. This additional scale is defined by the cutoff of the divergent integrals. In the absence of a characteristic scale in the problem, the functions f_i, calculated from perturbation theory, can only have the form

$$f_i=1+gA_i\ln\frac{k}{\Lambda}+g^2\left(B_i\ln\frac{k}{\Lambda}+C_i\ln^2\frac{k}{\Lambda}\right)+\dots \tag{4.1}$$

Allowing for the counter-terms leads to the addition of terms proportional to powers of $\ln(\mu/\Lambda)$, as a result of which the dependence of the renormalized expressions on the parameter Λ vanishes and the functions f_i transform to

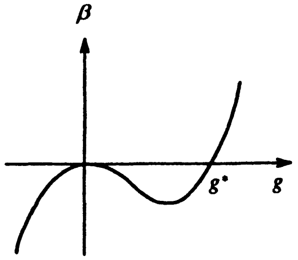

FIG. 3.

$$f_i\left(\frac{k}{\mu},g\right)=1+gA_i\ln\frac{k}{\mu}+g^2\left(B_i\ln\frac{k}{\mu}+C_i\ln^2\frac{k}{\mu}\right)+\ldots \tag{4.2}$$

It follows from Eqs. (3.14) and (3.15) that in the calculation of the RG functions only those terms contribute that are proportional to the first power of the logarithm

$$\gamma_i(g)=gA_i+g^2B_i, \tag{4.3}$$

while those terms proportional to higher powers of the logarithm do not contribute to the RG function, but are reproduced in the solutions of the RG equations, i.e., the values C_i and subsequent coefficients of the expansion are determined by the requirement of renormalization invariance, which corresponds to a "magic cancellation of the divergences."[14]

It follows from Eq. (3.10) that

$$\beta(g)=g(\gamma_1-3\gamma_2-\gamma_3)=g^2(A_1-3A_2-A_3)+g^3(B_1-3B_2-B_3)=g^2A+g^3B. \tag{4.4}$$

The asymptotic behavior of the system is determined by the fixed points of the RG transformation, found from the equation $\beta(g^*)=0$, where the condition for the existence of a fixed point is the relation $A/B<0$, and the condition for its stability in the IR limit is the requirement $A<0$. A graph of $\beta(g)$ corresponding to this situation is shown in Fig. 3.

The asymptotic solution of the differential equation (3.14) for small x can be found by standard methods[13,14,33] and has the form

$$\tilde{g}(x,g)\cong g^*+(g-g^*)x^{-Ag^*}. \tag{4.5}$$

The solution of differential equations (3.15) for $f_i(x,g)$ is given according to Ref. 13 by

$$f_i(x,g)=\exp\left\{\int_g^{\tilde{g}(x,g)}dg'\,\frac{\gamma_i(g')}{\beta(g')}\right\}\cong\exp\left\{\frac{\gamma_i(g^*)}{-Ag^*}\int_g^{\tilde{g}(x,g)}\frac{dg'}{g'-g^*}\right\}=x^{\gamma_i(g^*)}. \tag{4.6}$$

It follows from this formula that in order to find the dynamic exponent it is necessary to know the values of the coefficients A_i and B_i of the terms of the perturbation series for the functions $f_i(x,g)$ that are logarithmically divergent if the regularization is removed.

To find the coupling between the functions f_i and the previously introduced quantities $\Sigma(k,\omega)$ and $D(k,\omega)$, let us consider the power-series expansion of expression (3.2) in ω about the point $\omega=0$:

$$\begin{aligned}G^{-1}(k,\omega,\Lambda)&=Z_3\left(\frac{\mu}{\Lambda},g\right)[G^R(k,\omega;\mu)]^{-1}\cong Z_3\left(\frac{\mu}{\Lambda},g\right)\\&\times\left\{-i\omega\left[1-i\left.\frac{\partial\Sigma^R(k,\omega;\mu)}{\partial\omega}\right|_{\omega=0}\right]+\nu k^2\left[1-\frac{\Sigma^R(k,0,\mu)}{\nu k^2}\right]\right\}=Z_3\left(\frac{\mu_1}{\Lambda},g_1\right)\\&\times\left\{-i\omega\left[1-i\left.\frac{\partial\Sigma^R(k,\omega;\mu_1)}{\partial\omega}\right|_{\omega=0}\right]+\nu_1k^2\left[1-\frac{\Sigma^R(k,0;\mu_1)}{\nu_1k^2}\right]\right\}.\end{aligned} \tag{4.7}$$

Setting $k=\mu_1$ in Eq. (4.7) and making use of the normalization conditions

$$\Sigma^R(k,\omega;\mu)|_{NP}=0,\quad \left.\frac{\partial\Sigma^R(k,\omega;\mu)}{\partial\omega}\right|_{NP}=0,$$

$$D^R(k,\omega;\mu)|_{NP}=2D \tag{4.8}$$

(the values of the functions taken at the normalization point NP) and the relation $\nu f_2(\mu_1/\mu,g)=\nu_1f_3(\mu_1/\mu,g)$, which follows from Eq. (3.4), we find

$$[G^R(k,\omega;\mu)]^{-1}=-i\omega f_3\left(\frac{k}{\mu},g\right)+\nu k^2f_2\left(\frac{k}{\mu},g\right), \tag{4.9}$$

where

$$f_1=\frac{D^R(k,0;\mu)}{2D},\quad f_2=1-\frac{\Sigma^R(k,0;\mu)}{\nu k^2},$$

$$f_3=1-i\left.\frac{\partial\Sigma^R(k,\omega;\mu)}{\partial\omega}\right|_{\omega=0}, \tag{4.10}$$

and $\Sigma(k,\omega)$ and $D(k,\omega)$ are given by relations (2.16) and (2.17) after taking the Fourier transform, and the corresponding (exact) graphical representations are given in Fig. 2. The renormalized quantities are obtained by adding the counterterms, whose parameters are determined by the normalization conditions (4.8).

The formula for calculating the dynamic exponent

$$z=2+\gamma_2(g^*)-\gamma_3(g^*). \tag{4.11}$$

follows from Eqs. (4.6) and (4.9).

5. CALCULATION OF THE RENORMALIZATION-GROUP FUNCTIONS BY PERTURBATION THEORY AND FINDING THE EXPONENTS

The graphical representations for $\Sigma(k,\omega)$ and $D(k,\omega)$ in the second-order perturbation approximation are obtained by replacing the thick lines of the correlators and propagators by thin ones and the vertices by the bare vertices as prescribed by Eqs. (2.11) (see diagrams 1a and 1b in Fig. 4). Although

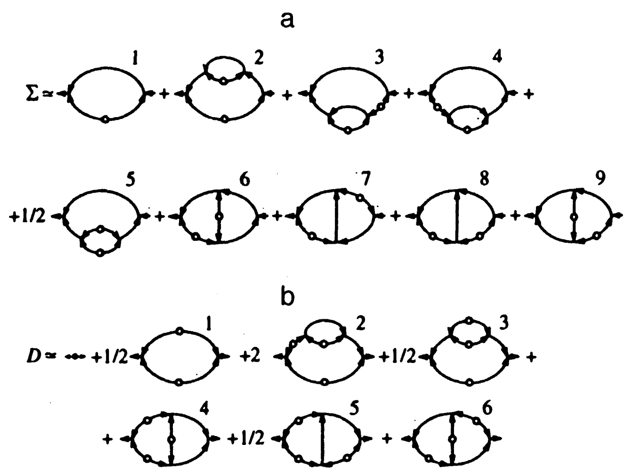

FIG. 4.

we are interested only in the terms that depend logarithmically on the cutoff Λ, we nevertheless present the complete results of the calculation of Σ and D to second order, since these elements enter into the form of the insertions in the fourth-order diagrams:

$$\Sigma_2^R(k,\omega)=-2D\lambda^2\int\frac{d\mathbf{q}}{(2\pi)^d}\frac{d\omega'}{2\pi}\times\frac{\mathbf{k}\cdot\mathbf{q}(\mathbf{k}\cdot\mathbf{q}-\mathbf{q}^2)}{[-i(\omega-\omega')+\nu(\mathbf{k}-\mathbf{q})^2][\omega'^2+\nu^2\mathbf{q}^4]}+ak^2+b\omega=-\frac{g}{2(4\pi)}\times\left[(-i\omega+\nu k^2)\ln\frac{-i\omega+\nu k^2}{-2i\omega+\nu k^2}-i\omega\right],\tag{5.1}$$

$$D_2^R(k,\omega)=\frac{4D^2\lambda^2}{2}\int\frac{d\mathbf{q}}{(2\pi)^d}\frac{d\omega'}{2\pi}\times\frac{(\mathbf{k}\cdot\mathbf{q}-\mathbf{q}^2)^2}{[(\omega-\omega')^2+\nu^2(\mathbf{k}-\mathbf{q})^4][\omega'^2+\nu^2\mathbf{q}^4]}+c=\frac{Dg^2}{8\pi}\left[-2\ln\frac{2k}{\mu}+\frac{-i\omega+\nu k^2}{\nu k^2}\times\ln\frac{-i\omega+\nu k^2}{-i\omega+\nu k^2/2}+\frac{i\omega+\nu k^2}{\nu k^2}\ln\frac{i\omega+\nu k^2}{i\omega+\nu k^2/2}\right].\tag{5.2}$$

It follows from Eqs. (5.1) and (5.2) that $A_1=-1/8\pi$ and $A_2=A_3=0$.

The fourth-order diagrams are obtained by inserting the second-order diagrams of the self-energy operator Σ_2 in the Green's-function line, inserting the correction D_2 in place of $2D$ in the correlation-function lines, and by using third-order diagrams for the vertices (the corresponding diagrams are shown in Fig. 4). Calculation of the fourth-order diagrams is more difficult. However, since we are interested only in the coefficients B_i multiplying $\ln(k/\Lambda)$, it is enough to consider the behavior of the integrand function near the upper cutoff in wave number, Λ, and set $k=0$, $\omega=0$ in the expression for $D(k,\omega)$ and in the expression for $\Sigma(k,\omega)$ after isolating the factors proportional to k^2 and ω. An alternative way of calculating the coefficients B_i is to use the method of dimensional regularization, borrowed from quantum field theory, in which the coefficients B_i are found as the residues of the poles in $\varepsilon=2-d$ at $\varepsilon=0$. This approach combined with the minimal subtraction scheme was used in Ref. 22. Although the minimal subtraction scheme is very simple and convenient at the one-loop level, it has the shortcoming that in the two-loop approximation in the presence of divergences, at each of the loops the corresponding expression contains second- and first-order pole singularities, while the poles of interest to us are the first-order poles which are obtained as the product of the pole singularity of the first loop with the finite part of the second loop. The finite part, found as the difference of two infinities, is not uniquely defined, and as a consequence of this the approach which we have used of normalizing by the mass scale μ is more reliable.

As a result, to first-order terms in $\ln(k/\mu)$ we have

$$\Sigma^R(k,\omega;\mu)=-\left(\frac{g}{4\pi}\right)^2\left[-\omega\,\frac{-1+\ln 2}{2}+\nu k^2\,\frac{\ln 2}{2}\right]\ln\frac{k}{\mu},\tag{5.3}$$

$$D^R(k,0;\mu)=2D\left[1-\frac{g}{8\pi}\ln\frac{k}{\mu}+\left(\frac{g}{4\pi}\right)^2\times\left(\frac{1}{2}-3\ln 2+\frac{5}{4}\ln 3\right)\ln\frac{k}{\mu}\right].\tag{5.4}$$

The result (5.3) coincides with the result obtained in Ref. 22, and the result (5.4) differs from the result obtained there by the presence of a term proportional to $g^2\ln(k/\mu)$ (in Ref. 22 the terms proportional to the square of the logarithm are also written out; we have omitted them since they do not contribute to the RG function).

It follows from Eqs. (5.3) and (5.4) that

$$B_1=\frac{1}{(4\pi)^2}\left(\frac{1}{2}-3\ln 2+\frac{5}{4}\ln 3\right),$$

$$B_2=-\frac{1}{(4\pi)^2}\frac{\ln 2}{2},\quad B_3=-\frac{1}{(4\pi)^2}\frac{1-\ln 2}{2},$$

$$\frac{g^*}{4\pi}=\frac{1}{(5/2)\ln 3-2\ln 2}=1.36,$$

$$z=2-\frac{1}{2}\left(\frac{g^*}{4\pi}\right)^2=1.73,\quad \chi=0.27.\tag{5.5}$$

6. CONCLUSION

The calculation of the scaling exponents which we have carried out here using the RG description gives results in satisfactory agreement with the results of approximate theoretical calculations based on other approaches and which predict values $z=1.67-1.70$ (Refs. 33 and 34). The problem of finding the exponents has been solved numerically by a number of authors, and their results in a number of cases differ substantially and lead to values of z from 1.6 (Refs. 35 and 36) to 1.8 (Ref. 37). The exponents were calculated with the help of the RG method at the two-loop level in Ref. 22, in which the authors obtained the values $z=1.84$ and $\chi=0.16$. The reason for the difference from our result (5.5) is that the authors of Ref. 22 did not allow for the contribution from the counter-terms arising in the regularization of the one-loop diagrams, which in the two-loop approximation give a finite contribution to the coefficients of the first powers of the logarithms.

APPENDIX

In the Appendices we present some technical details of the calculation and results for the Feynman diagrams presented in Fig. 4 and compare them with the results of Ref. 22.

a. Self-energy diagrams

From inspection of the diagram (1a in Fig. 4) of the self-energy operator in the second-order perturbational approximation [formula (5.1)] it follows that this diagram does not contain any divergences (Σ does not depend on Λ and Σ^R (the renormalized result) does not depend on μ). Nevertheless, to satisfy the normalization conditions (4.8) a finite counter-term having the form

$$\delta\Sigma(\hat{k})=-\frac{g}{8\pi}[-i\omega(1-\ln 2)+\nu k^2(-\ln 2)], \qquad (A1)$$

has been added to Σ.

Diagrams 2a–4a in Fig. 4 were obtained by inserting the second-order perturbational approximation to the self-energy operator in the Green's function $G^{(0)}$. This corresponds to adding the factor $G^{(0)}(\mathbf{q},\omega')\Sigma_2(\mathbf{q},\omega')$ to the integrand. After integrating the inner line of the large loop over the frequency ω' it turns out that in diagram 4a this factor is represented in the form $G^{(0)}(\mathbf{q},i\nu\mathbf{q}^2)=\text{const}$ and diagram 4a reduces to multiplication of this constant by Σ_2. This diagram does not contain any divergences (it does not depend on either Λ or μ) and does not contribute to the RG function. Analogously, after integrating over frequency, diagrams 2a and 3a reduce to the addition of the factor $G^{(0)}\times(\mathbf{q},\omega+i\nu(\mathbf{k}-\mathbf{q})^2)\Sigma_2(\mathbf{q},\omega+i\nu(k-\mathbf{q})^2)$, which is represented in the form of an expansion in ω and $\mathbf{k}\cdot\mathbf{q}/\mathbf{q}^2$ keeping enough terms to make out the factors ω and k^2 in front of the integral. In the expression so obtained, after integrating over the directions of the vector $\mathbf{q}$ one is left with an integral over q which diverges at the upper limit Λ. Thus, this allows one to find the coefficient of $\ln\Lambda$, a knowledge of which is necessary to calculate the RG function. The calculations give results that coincide with the coefficients of the poles in ε in Ref. 22. The additional contributions of the counter-terms $\delta\Sigma_2$ in diagrams 2a and 3a of Fig. 4 cancel each other out.

Diagram 5a of Fig. 4 is obtained by inserting the renormalized diagram $D_2^R(\mathbf{q},\omega')$ in the correlation function. In the calculation of diagram 5a it is necessary to allow for the fact that D_2 contains singularities in the upper as well as lower half-plane of complex values of ω' (the retarding part $D^r(\mathbf{q},\omega')$ and the advancing part $D^a(\mathbf{q},\omega')=D^r(\mathbf{q},-\omega')$), as a result of which after integrating over ω' we are left with expressions of the form $D_2^r(\mathbf{q},i\nu q^2)\sim\ln(q/\mu)+\text{const}$ and $D_2^r(\mathbf{q},\omega+i\nu(\mathbf{k}-\mathbf{q})^2)$. The latter expression expands into a series in ω and $\mathbf{k}\cdot\mathbf{q}/\mathbf{q}^2$, as in the previous case for insertion of the Σ diagram, which reproduces the result of Ref. 22 for the coefficient of the pole term in ε.

Diagrams 6a–8a of Fig. 4 are obtained by using the third-order perturbational approximation for the vertex in Fig. 2a corresponding to conversion of two quanta into one (three different diagrams). Although each of these diagrams contains divergences, their sum should be finite according to Ward's first identity (2.23). The calculation, after integrating over the intermediate frequency ω_p, gives the following representation for the vertex:

$$\Gamma(\hat{k}-\hat{q}|\hat{k},-\hat{q})=\frac{\lambda^3 D}{\nu}\int\frac{d\mathbf{p}}{(2\pi)^2}\times\frac{(\mathbf{p}-\mathbf{q})\cdot(\mathbf{k}-\mathbf{p})}{[-i(\omega-\omega')+\nu(\mathbf{k}-\mathbf{p})^2+\nu(\mathbf{p}-\mathbf{q})^2]}\times\left\{\frac{\mathbf{p}\cdot\mathbf{k}}{i\omega'+\nu\mathbf{p}^2+\nu(\mathbf{p}-\mathbf{q}^2)}\left[\frac{\mathbf{p}\cdot\mathbf{q}}{\mathbf{p}^2}-\frac{(\mathbf{p}-\mathbf{q})\cdot\mathbf{q}}{(\mathbf{p}-\mathbf{q})^2}\right]+\frac{\mathbf{p}\cdot\mathbf{k}}{-i\omega+\nu\mathbf{p}^2+\nu(\mathbf{p}-\mathbf{k})^2}\times\left[\frac{\mathbf{p}\cdot\mathbf{k}}{\mathbf{p}^2}-\frac{(\mathbf{p}-\mathbf{k})\cdot\mathbf{k}}{(\mathbf{p}-\mathbf{k})^2}\right]\right\}. \qquad (A2)$$

It is clear from Eq. (A2) that the integral over p does not contain any divergences in the region of large $\mathbf{p}$. Expanding Eq. (A2) in a series in ω and $\mathbf{k}\cdot\mathbf{q}/\mathbf{q}^2$, we obtain the contribution to the coefficient of $\ln\Lambda$ from the three diagrams 6a–8a.

Diagram 9a of Fig. 4 describes the contribution of the vertex corresponding to conversion of one quantum into two, calculated in the third-order perturbational approximation (two diagrams). The contribution of these diagrams to the RG function is calculated analogously.

It is interesting to note that the contributions of diagrams 2a–9a almost cancel each other out and only the contribution from the singular part of the insertion $D_2(\mathbf{q},\omega')$ in diagram 5a remains uncompensated. Whether this circumstance is coincidental or a regular result remains unclear.

b. Diagrams of the correlation function of the effective random forces

Diagram 2b of Fig. 4 is obtained by inserting the self-energy operator into one of the $G^{(0)}$ propagator lines (four such insertions) leading to D_2 (diagram 1b), while diagrams 4b and 5b are obtained by using the third-order perturbational approximation for the vertex corresponding to the conversion of two quanta into one. In the calculation of the parameters of the RG function γ_1 it is sufficient to restrict oneself to the case $\hat{k}\to 0$. The vertex with zero characteristics of the departing arrow is obtained from Eq. (A2) for $\hat{k}=\hat{q}$. As a result, it is possible to convince oneself of the validity of the relation

$$\Gamma_3(0|\hat{q};-\hat{q}) = -\lambda V(0|\hat{q};-\hat{q})[G^{(0)}(\hat{q})\Sigma_2(\hat{q}) + G^{(0)}(-\hat{q})\Sigma_2(-\hat{q})], \tag{A3}$$

from which it follows that the singular parts of diagrams 2b, 4b, and 5b cancel each other out exactly, which reproduces the result of Ref. 22. However, taking account of the contributions of the counter-term $\delta\Sigma_2(\hat{q})$ defined by Eq. A1 gives the correction to diagram 2b. The correction due to this contribution after renormalization and taking the weighting factor 1/2 into account has the form

$$\delta D_4^{(2b)} = 2D\left(\frac{g}{4\pi}\right)^2\left(\frac{1}{4} - \ln 2\right)\ln\frac{k}{\mu}. \tag{A4}$$

Calculation of diagram 6b of Fig. 4 leads to the coefficient of the logarithmically divergent term coinciding with the result obtained in Ref. 22:

$$D_4^{(6b)} = 2D\left(\frac{g}{4\pi}\right)^2\left(\frac{1}{6} - \frac{3}{4}\ln\frac{4}{3}\right)\ln\frac{k}{\mu}. \tag{A5}$$

Calculation of diagram 5b, taking the weighting factor 1/2 into account, gives

$$D_4^{(5b)} = 2D\left(\frac{g}{4\pi}\right)^2\left(\frac{7}{12} - \frac{1}{2}\ln\frac{4}{3}\right)\ln\frac{k}{\mu}, \tag{A6}$$

which differs from the result given in Ref. 22. After taking account of the contribution of the finite part of the counter-term $\delta D_2 = 2D(g/4\pi)(1-\ln 2)$ we find

$$D_4^{R(5b)} = 2D\left(\frac{g}{4\pi}\right)^2\left(\frac{1}{12} - \frac{1}{2}\ln\frac{2}{3}\right)\ln\frac{k}{\mu}. \tag{A7}$$

Summing expressions (A4), (A5), and (A6) gives the result written down in formula (5.4).

*E-mail: TEODOR@IPM.MSK.SU

[1] M. Kardar, G. Parisi, and Y.-C. Zhang, Phys. Rev. Lett. **56**, 889 (1986).
[2] M. Kardar and Y.-C. Zhang, Phys. Rev. Lett. **58**, 2087 (1987).
[3] L. B. Ioffe and V. M. Vinokur, J. Phys. C **20**, 6149 (1987).
[4] T. Nattermann and R. Lipowsky, Phys. Rev. Lett. **61**, 2508 (1988).
[5] J. M. Burgers, *The Nonlinear Diffusion Equation* (Reidel, Boston, 1974).
[6] E. Medina, T. Hwa, M. Kardar, and Y.-C. Zhang, Phys. Rev. A **39**, 3053 (1989).
[7] G. I. Sivashinski, Acta Astronautica **4**, 1177 (1977); Y. Kuramoto, Suppl. Prog. Theor. Phys. **64**, 346 (1978).
[8] H. van Beijeren, R. Kutner, and H. Spohn, Phys. Rev. Lett. **54**, 2026 (1985).
[9] F. Family and T. Viczek, J. Phys. A **18**, L75 (1989).
[10] F. Family, Physica A **168**, 561 (1990).
[11] K. G. Wilson, Phys. Rev. **179**, 1699 (1969).
[12] K. Wilson, Rev. Mod. Phys. **47**, 713 (1975).
[13] N. N. Bogolyubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Wiley Interscience, New York, 1959).
[14] J. C. Collins, *Renormalization: An Introduction to Renormalization, the Renormalization Group, and the Operator-Product Expansion* (Cambridge Univ. Press, New York, 1986).
[15] K. Wilson, Phys. Rev. B **4**, 3174 (1971).
[16] S.-K. Ma, *Modern Theory of Critical Phenomena* (Benjamin, Reading, Massachusetts, 1976).
[17] D. Forster, D. R. Nelson, and M. J. Stephen, Phys. Rev. A **16**, 732 (1977).
[18] V. V. Lebedev and V. S. L'vov, Pis'ma Zh. Éksp. Teor. Fiz. **58**, 301 (1993) [JETP Lett. **58**, 310 (1993)].
[19] C. De Dominicis and P. C. Martin, Phys. Rev. A **19**, 419 (1979).
[20] É. V. Teodorovich, Zh. Éksp. Teor. Fiz. **102**, 863 (1992) [Sov. Phys. JETP **75**, 472 (1992)].
[21] S. F. Edwards and D. R. Wilkinson, Proc. R. Soc. London A **381**, 17 (1982).
[22] T. Sun and M. Plischke, Phys. Rev. E **49**, 5046 (1994).
[23] E. Frey and U. C. Tauber, Phys. Rev. E **50**, 1024 (1994).
[24] É. V. Teodorovich, Usp. Mekh. **13**, 81 (1990); Teor. Mat. Fiz. **101**, 28 (1994).
[25] P. C. Martin, E. D. Siggia, and H. A. Rose, Phys. Rev. A **8**, 423 (1973).
[26] H. K. Janssen, Z. Phys. B **23**, 377 (1976).
[27] C. De Dominicis and L. Peliti, Phys. Rev. B **18**, 353 (1978).
[28] V. N. Popov, *Functional Integrals in Quantum Field Theory and Statistical Physics*, Dordrecht, Boston (1983).
[29] H. W. Wyld, Ann. Phys. **14**, 143 (1961).
[30] E. G. Hopf, J. Rat. Mech. Anal. **1**, 87 (1952).
[31] É. V. Teodorovich, Prikl. Mat. Mekh. **53**, 443 (1989).
[32] G. 't Hooft and M. Veltman, Nucl. Phys. B **44**, 189 (1972).
[33] D. J. Amit, *Field Theory, the Renormalization Group, and Critical Phenomena* (World Science Publishers, New York, 1984).
[34] J. Amar and F. Family, Phys. Rev. A **41**, 3399 (1990).
[35] B. Grossmann, H. Gio, and M. Grant, Phys. Rev. Lett. **64**, 1262 (1990); Phys. Rev. A **43**, 1727 (1991).
[36] L. H. Tang, B. M. Forest, and D. E. Wolf, Phys. Rev. A **45**, 7162 (1992).
[37] A. Chakrabarti and R. Toral, Phys. Rev. B **40**, 11419 (1989).

Translated by Paul F. Schippnick