Finite scaling of the effective conductivity in percolation systems with nonzero ratio of the phase conductivities

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A study is made of the critical behavior of the effective conductivity $\{\sigma_e\}$ and effective resistivity $\{\rho_e\}$ averaged over the realizations in percolation systems in the fractal region for length scales L of the system less than the correlation length. The proposed model of the percolation structure is used to determine the dependences of $\{\sigma_e\}$ and $\{\rho_e\}$ on the size of the system, including the case of nonzero ratio of the phase conductivities. It is shown that for a nonzero ratio of the phase conductivities $\{\sigma_e\}=\sigma_1(\tau_L+\tau/2\tau_L)(L/a_0)^{-t/\nu}+\sigma_2(\tau_L-\tau/2\tau_L)(L/a_0)^{q/\nu}$, $\{\rho_e\}=\rho_1(\tau_L+\tau/2\tau_L)(L/a_0)^{t/\nu}+\rho_2(\tau_L-\tau/2\tau_L)(L/a_0)^{-q/\nu}$, where $\tau=(p-p_c)/p_c$ is the proximity to the percolation threshold p_c, σ_i are the phase conductivities ($\sigma_2/\sigma_1\ll 1$), and $\tau_L\propto L^{-1/\nu}$. For a definite set of parameters, crossover of the quantities $\{\sigma_e\}$ and $\{\rho_2\}$ can be observed, namely, interchange of the critical behaviors when L is changed. Scaling relations are proposed for the conductivity $\{\sigma_e\}$ and resistivity $\{\rho_e\}$ averaged over the realizations. © *1996 American Institute of Physics.* [S1063-7761(96)02402-5]

1. INTRODUCTION

It is well known[1] that near the percolation threshold the effective conductivity of two-phase randomly inhomogeneous media exhibits critical behavior. Below the percolation threshold, the effective conductivity σ_e diverges as a power as the percolation threshold p_c is approached:

$$\sigma_e=\sigma_2(p_c-p)^{-q},$$

where p is the concentration of the well-conducting phase with conductivity σ_1, σ_2 is the conductivity of the poorly conducting phase, and it is assumed that the resistivity of the well-conducting phase can be ignored ($\rho_1=1/\sigma_1=0$). Above the percolation threshold, the effective resistivity $\rho_e\doteq 1/\sigma_e=\sigma_1(p-p_c)^{-t}$ diverges, and it is now assumed that $\sigma_2=0$. In the case of infinitely large inhomogeneity ($\sigma_2=0$ or $\rho_1=0$), such behavior is valid only for a large system of length scale greater than the correlation length, which tends to infinity as the percolation threshold is approached.

In the experimental study of percolation systems, the length of the sample in one or several directions may be less than the correlation length. Such a situation is possible, for example, in thick-film resistors, which are widely used in practice (see the bibliography of studies of the electrical conductivity of thick-film resistors.[2]) In this case, the behavior of σ_e and ρ_e can be radically changed. In a system with lengths in all directions less than the correlation length, the power-law divergences associated with the change in the concentration disappear, but a power dependence on the size of the system, called finite scaling, appears.

In this paper, we consider the behavior of the effective conductivity and effective resistivity in percolation systems with dimensions less than the correlation length. The model constructed here makes it possible to describe in a unified manner the standard case of zero ratio of the phase conductivities as well as the case of a nonzero ratio: $\sigma_2/\sigma_1\neq 0$.

Above and below the percolation threshold, the effective conductivity for a nonzero ratio of the phase conductivities has the form[3]

$$\sigma_e=\sigma_1\tau^t(A_0+A_1h\tau^{-\varphi}+\dots),\quad p>p_c,$$

$$\sigma_e=\sigma_2|\tau|^{-q}(B_0+B_1h|\tau|^{-\varphi}+\dots),\quad p<p_c,\tag{1}$$

where $\tau=(p-p_c)/p_c$, in which p_c is the concentration of the well-conducting phase $\sigma_1(h=\sigma_2/\sigma_1\ll 1)$, $\varphi=t+q$, where t and q are the critical exponents of the conductivity, and A_i and B_i are constants of the order of unity. In the region of smearing Δ (the analog of the region of smearing of a second-order phase transition), the effective conductivity is practically independent of the concentration:

$$\sigma_e=(\sigma_1^q\sigma_2^t)^{1/\varphi}(D_0+D_1\tau h^{-1/\varphi}+\dots),\tag{2}$$

this expression holding for $|\tau|\leq\Delta$, where $\Delta=h^{1/\varphi}$; D_i are constants of order unity. The expressions (1) and (2) are valid for samples with $L>\xi$, where $\xi=a_0|\tau|^{-\nu}$ is the correlation length, ν is its critical exponent, and a_0 is the minimum scale in the system—the bond length in the case of lattice models. For lengths $L>\xi$, self-averaging of the effective conductivity occurs—the fluctuations of σ_e can be ignored.

For L less than or of the order of the correlation length, the fluctuations of the effective conductivity are appreciable, and, as a rule, one does not consider σ_e of the actual realization of the random structure, which is an ill-defined quantity, but its average $\{\sigma_e\}$ over a large number of realizations. The same can be said for the resistivity $\rho_e = 1/\sigma_e$; moreover, in the region $L<\xi$ (in what follows, we shall call this the fractal region) we have in general $\{\sigma_e\} \neq 1/\{\rho_r\}$. The mean values $\{\sigma_e\}$ and $\{\rho_e\}$ depend on L and for L greater than ξ go over into (1) and (2).

In the limiting case $h=0$ of strong inhomogeneity, $\sigma_e = \sigma_1 \tau^t$ above the percolation threshold, and there exists a device that makes it possible to obtain $\{\sigma_e\}$ in the fractal region. For this, using the relationship between the correlation length ξ and the concentration τ, $\xi = a_0 |\tau|^{-\nu}$, one substitutes in place of τ in σ_e the quantity $\tau = (L/a_0)^{-1/\nu}$, from which[4]

$$\{\sigma_e\} = \sigma_1 (L/a_0)^{-t/\nu}, \quad \tau > 0. \tag{3}$$

We note immediately that the analogous operation with $\rho_e = \rho_1 \tau^{-t}$ cannot be done, since when ρ_e is averaged realizations occur without percolation paths and since for $h=0$ and $p>p_c$ if follows that $\sigma_2 = 0$ and $\{\rho_e\} = \infty$. The mean values of ρ_e below the percolation threshold are found similarly:

$$\{\rho_e\} = \rho_2 (L/a_0)^{q/\nu}, \quad \{\sigma_e\} = \infty. \tag{4}$$

The relations (3) and (4) are called finite scaling, since in this case there is a linear dependence between $\ln\{\sigma_e\}$ (or $\ln\{\rho_e\}$) and $\ln L$ over lengths $L<\xi$:

$$\ln\{\sigma_e\} \propto -\frac{t}{\nu} \ln L, \quad \ln\{\rho_e\} \propto \frac{q}{\nu} \ln L \tag{5}$$

and $\ln\{\sigma_e\}$ and $\ln\{\rho_e\}$ do not depend on L at large lengths. Similar dependences hold for the quantities characterized by fractal dimension, for example, for the density of a fractal cluster (see, for example, Ref. 5). The use of (5) is a convenient computational device for determining the critical exponents t and q of the conductivity. One proceeds similarly in the numerical determination of the critical exponents of other physical quantities, for example, the relative spectral density of $1/f$ noise.[6]

It will be shown that this simple device—the substitution $\tau \to (L/a_0)^{-1/\nu}$—is not satisfactory for all sets of values of h, τ, and L/a_0. We shall construct a model of a conducting percolation structure that makes it possible to determine the finite scaling of $\{\sigma_e\}$ and $\{\rho_e\}$ for all parameter sets.

2. MODEL OF CONDUCTING PERCOLATION STRUCTURE IN THE FRACTAL REGION

Both above and below the percolation threshold, percolating and nonpercolating structures will both be encountered at scales $L \leqslant \xi$ (Fig. 1). It can be assumed that the former (region I in Fig. 1) are above the percolation threshold and the latter (region II in Fig. 1) below. The distribution function of the percolation threshold is determined in Ref. 7, where it is shown that to within the errors of the calculation this distribution is Gaussian. However, for simplicity we shall replace it by a uniform distribution function with spread $2\tau_L$, where τ_L is the mean square fluctuation of the Gaussian function. As is shown in Refs. 7 and 1, to within an unimportant factor, it can be represented in the form

$$\tau_L = (L/a_0)^{-1/\nu}, \tag{6}$$

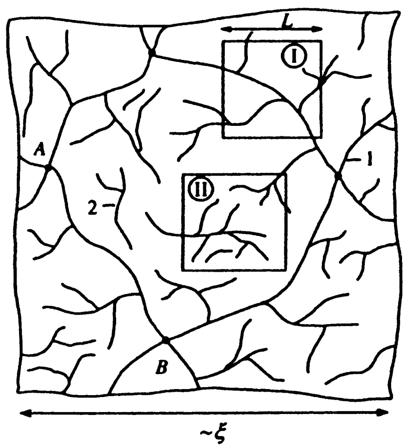

FIG. 1. Schematic representation of a percolation system above the percolation threshold. The heavy lines represent the well-conducting phase. *1*—Single-vein macroscopic network ($L_{AB} \propto \tau^{-\nu}$); 2—finite cluster; I is a region of length scale L above the percolation threshold and II is the same below the threshold.

and τ_L is much greater than the shift of the maximum of the distribution function of the thresholds from the percolation threshold of the "infinite" ($L \gg \xi$) system, so that this shift can be ignored.

We now consider an ensemble of systems of scale L ($L<\xi$) with given value of τ. As can be seen from Fig. 2, some of these systems are above the percolation threshold and some below, i.e., for a random sample from the ensemble the system is above the percolation threshold (the hatched region in Fig. 2) with probability P_L and below it with probability $1-P_L$. Here

$$P_L = \frac{\tau_L + \tau}{2\tau_L}, \quad 1 - P_L = \frac{\tau_L - \tau}{2\tau_L}. \tag{7}$$

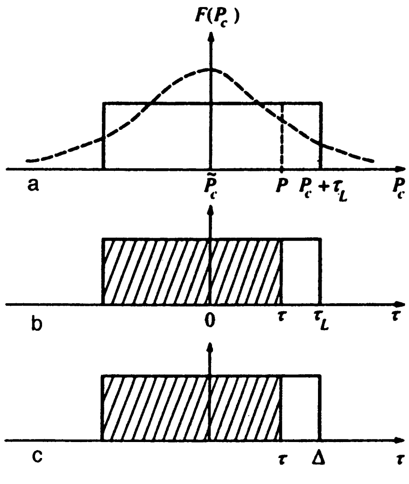

FIG. 2. Distribution function $F(p_c)$ of percolation thresholds; $\tilde{p}_c$ is the percolation threshold of the system with $L>\xi$, $\tau_L = (L/a_0)^{-1/\nu}$. a) The dashed curve shows the Gaussian distribution, the solid line the homogeneous approximation; b) a more convenient axis; the hatched region gives the probability that the system is above the percolation threshold; c) distribution of the probabilities of a system with $L>\xi$ in the region of smearing; the hatched region gives the probability of encountering a bridge in the correlation volume—a structural element corresponding to percolation above the percolation threshold.

This ensemble of systems can be regarded as obtained from one large system by cutting out parts with scales L.

A similar situation also occurs when the system is in the region of smearing for $L>\xi$. Indeed, if for $L>\xi$ the system is in the region of smearing ($|\tau|<\Delta$), then with a corresponding probability we can find systems both with and without percolating structures. Of course, $h\neq 0$ is assumed here, for otherwise $\xi=\infty$ holds and a region of smearing will not exist at all: $\Delta=0$.

To describe the kinetic properties of percolation systems in the region of smearing, a model that extends the hierarchical weak-link model[8] to this region was developed.[9,10] Since this model will be applied to finite scaling, we briefly describe it.

In the first approximation in h, the percolation medium in the region of smearing can be represented as a medium in which in different correlation volumes L^d (d is the number of dimensions) one can encounter, with different probabilities, both a structure associated with percolation above the percolation threshold—a bridge with resistance R_1—and below—an interlayer with resistance R_2:

$$R_1=r_1N_1,\quad R_2=r_2/N_2, \tag{8}$$

where $r_1=1/(\sigma_1 a_0^{d-2})$, $r_2=1/(\sigma_2 a_0^{d-2})$ are the resistances of the well and poorly conducting bonds, and N_1 and N_2 are the number of so-called single connected bonds (SCB) and single disconnected bonds (SDCB):

$$N_1=|\tau|^{-\alpha_1},\quad N_2=|\tau|^{-\alpha_2}. \tag{9}$$

In node–line–blob (NLB) models, the exponent $\alpha_1=1$ (Refs. 4 and 11–13); in accordance with Refs. 14 and 15, $\alpha_2=1$ holds too. In models of weak-link type,

$$\alpha_1=t-\nu(d-2),\quad \alpha_2=q+\nu(d-2). \tag{10}$$

Within the region of smearing $|\tau|\leqslant\Delta$, the length of the bridge and the area of the interlayer are constant; they change only outside this region. The probability with which one encounters the one or the other structure (bridge or interlayer; SCB or SDCB) is determined by the concentration—by the value of τ (Fig. 2c); in addition, since we are considering the region of smearing, $-\Delta<\tau<\Delta$, it is necessary in N_1 to set $\tau=\Delta$, and in N_2 to set $\tau=-\Delta$. Both situations can be considered together by introducing into the circuit of the percolation structure an additional resistance r_m (Fig. 3), which with probability P_Δ takes the value r_1 and with probability $1-P_\Delta$ the value r_2:

$$P_\Delta=\frac{\Delta+\tau}{2\Delta},\quad 1-P_\Delta=\frac{\Delta-\tau}{2\Delta}. \tag{11}$$

For r_m-r_1 (Fig. 3), the system takes the form of a bridge and an interlayer connected in parallel, for $r_m=r_2$ (r_2 is, as it were, connected to the interlayer) the form of a bridge and interlayer connected in series. It is readily seen that such a scheme leads to the standard expression for the effective conductivity of two-phase systems in the region of smearing (2). If we ignore the small terms of the form $D_1\tau h^{-1/\varphi}$ (2), then to good accuracy we can suppose that for $r_m=r_1$ the resistance of the system is determined by the bridge, $R_{1e}=R_1=r_1N_1$, and for $r_m=r_2$ by the interlayer: $R_{2e}=R_2=r_2/N_2$. In the two extreme cases $\tau\to\pm\Delta$ (here the resistance R_e of the complete system must now be calculated with allowance for both R_1 and R_2) it is necessary to substitute in place of Δ in N_1 and N_2 the quantity τ—the model goes over into the weak-link hierarchical model and gives the expressions (1) for σ_e outside the region of smearing.

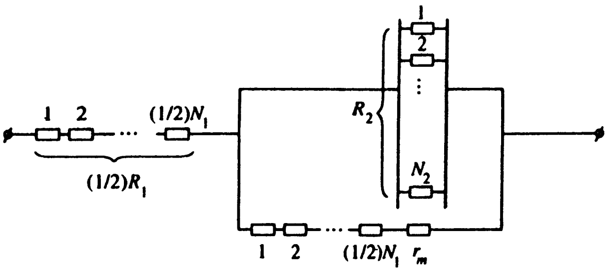

FIG. 3. Electrical circuit of the percolation structure in the region of smearing.

We now consider systems with $L<\xi$. In accordance with the above, the effective conductivities and resistivities are calculated using the circuit of Fig. 3, in which τ_L now plays the role of the region of smearing Δ. With probabilities P_L and $1-P_L$, respectively, the resistances of the system are now

$$R_1=r_1\left(\frac{L}{a_0}\right)^{\alpha_1/\nu},\quad R_2=r_2\left(\frac{L}{a_0}\right)^{-\alpha_2/\nu}, \tag{12}$$

where, as before, we have restricted ourselves to the accuracy with which one can take into account in the first case only the bridge and in the second only the interlayer.

From (12) and (7) one can obtain both $\{\sigma_e\}$ and $\{\rho_e\}$ over lengths $L<\xi$; in the first case, it is necessary to calculate the mean conductance

$$\{G\}=\{\sigma_e\}L^{d-2}=P_LR_1^{-1}+(1-P_L)R_2^{-1},$$

and in the second the mean resistance $\{R\}=\{\rho_r\}/L^{d-2}=P_LR_1+(1-P_L)R_2$:

$$\{\sigma_e\}=\sigma_1\frac{\tau_L+\tau}{2\tau_L}\left(\frac{L}{a_0}\right)^{-t/\nu}+\sigma_2\frac{\tau_L-\tau}{2\tau_L}\left(\frac{L}{a_0}\right)^{q/\nu}.$$

$$\{\rho_e\}=\rho_1\frac{\tau_L+\tau}{2\tau_L}\left(\frac{L}{a_0}\right)^{t/\nu}+\rho_2\frac{\tau_L-\tau}{2\tau_L}\left(\frac{L}{a_0}\right)^{-q/\nu}, \tag{13}$$

where τ_L is defined in (6). We note immediately that the well-known expressions for the special case $h=0$ follow from (13). For example, for $\tau>0$ the case $h=0$ means that σ_1 is finite and $\sigma_2=0$; then from (13)

$$\{\sigma_e\}=\sigma_1\frac{\tau_L+\tau}{2\tau_L}\left(\frac{L}{a_0}\right)^{-t/\nu},\quad \{\rho_e\}=\infty, \tag{14}$$

and at the same time it must be borne in mind that $0<\tau<\tau_L$ in $\{\sigma_e\}$; then $\{\sigma_e\}$ for $\tau=0$ and $\{\sigma_e\}$ for $\tau=\tau_L$ differ only by an unimportant factor 1/2. The situation is analogous for $\tau<0$ too:

$$\{\sigma_e\}=\infty,\quad \{\rho_e\}=\rho_2\frac{\tau_L-\tau}{2\tau_L}\left(\frac{L}{a_0}\right)^{-q/\nu}. \tag{15}$$

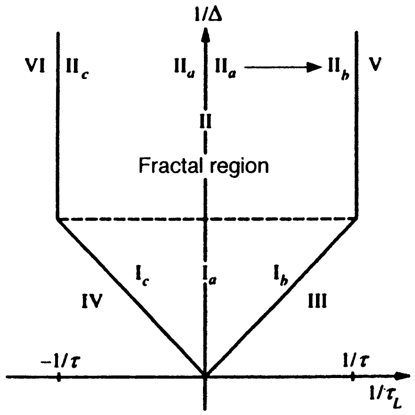

FIG. 4. The space $1/\tau_L - 1/\Delta$. The first quadrant corresponds to $\tau>0$, the second to $\tau<0$. The region between the continuous lines I and II corresponds to fractal behavior. The region below the dashed line is the region of smearing I.

3. ANALYSIS OF THE RESULTS

To analyze the expressions obtained for $\{\sigma_e\}$ and $\{\rho_e\}$, it is convenient to use $1/\tau_L - 1/\Delta$ space (Fig. 4), since the variable $1/\tau_L$ is directly proportional to the length scale of the system. It is necessary to exclude from consideration, first, the region directly next to the $1/\tau_L$ axis, since in it the condition $\sigma_2/\sigma_1 \ll 1$ is not satisfied, and, second, the region surrounding the $1/\Delta$ axis, since L must be greater than the characteristic minimal scale a_0 of the system. The parameters of the regions I and II correspond to the fractal regime. At the boundaries between the regions I and III and I and IV, $\{\sigma_e\}$ and $\{\rho_e\}$ go over to the standard expressions $\sigma_e = (\sigma_1^q \sigma_2^t)^{1/\varphi}$, $\rho_e = 1/\sigma_e$; on the boundary between II and V they go and over to $\sigma_e = \sigma_1 \tau^t$ and between II and VI to $\sigma_e = \sigma_2 |\tau|^{-q}$.

Within the fractal region, $\{\sigma_e\}$ and $\{\rho_e\}$ behave as follows. In the region I near the $1/\Delta$ axis, the region II_a, it follows from (13) ($1/\tau_L \ll 1/\tau$) that

$$\{\sigma_e\} = \sigma_1 \left(\frac{1}{\tau_L}\right)^{-t} \equiv \sigma_1 \left(\frac{L}{a_0}\right)^{-t/\nu}, \quad I_a,$$

$$\{\rho_e\} = \rho_2 \left(\frac{1}{\tau_L}\right)^{-q} \equiv \rho_2 \left(\frac{L}{a_0}\right)^{-q/\nu}, \quad I_a, \tag{16}$$

and in region I_a

$$\{\sigma_e\} \neq 1/\{\rho_e\}. \tag{17}$$

As L (or, what is the same thing, $1/\tau_L$) is increased, the expressions (16) remain valid as the lines $1/\tau_L = 1/\Delta$—the regions I_b and I_c—are approached; however, since $1/\tau_L \approx 1\Delta$ and, therefore, $\sigma_1(1/\Delta)^t \approx \sigma_2(1/\Delta)^q$, we now have

$$\{\sigma_e\} \approx 1/\{\rho_e\}, \quad \mathrm{I}_b, \mathrm{I}_c. \tag{18}$$

We now consider the region II far from the region of smearing. Near the $1/\Delta$ axis, where $1/\tau_L \ll 1/\tau$, the same expressions (16) for $\{\sigma_e\}$ and $\{\rho_e\}$ hold as in I_a. For example, for $\{\rho_e\}$ the relation (16) holds, since in this region

$$\left(\frac{1/\tau_L}{1/\Delta}\right)^{\varphi} \frac{1/\tau + 1/\tau_L}{1/\tau - 1/\tau_L} \ll 1.$$

However, with increasing $1/\tau_L$ the denominator on the left-hand side of this inequality tends to zero and, ultimately, it is replaced by the opposite inequality. Note that in the region of smearing such a situation is impossible. Thus, in II_b the mean value $\{\rho_e\}$ has the form

$$\{\rho_e\} = \rho_1 (1/\tau_L)^t = 1/\{\sigma_e\} \tag{19}$$

and, therefore, on the transition from II_a to II_b (the arrow in Fig. 4) crossover occurs—the critical behavior of $\{\rho_e\}$ changes:

$$\rho_2 \left(\frac{L}{a_0}\right)^{-q/\nu} \to \rho_1 \left(\frac{L}{a_0}\right)^{t/\nu}, \quad \mathrm{II}_a \to \mathrm{II}_b. \tag{20}$$

For $\{\sigma_e\}$, crossover occurs on the transition from II_a to II_c:

$$\sigma_1 \left(\frac{L}{a_0}\right)^{-t/\nu} \to \sigma_2 \left(\frac{L}{a_0}\right)^{q/\nu}, \quad \mathrm{II}_a \to \mathrm{II}_c. \tag{21}$$

4. CONCLUSIONS

Over scales L less than the correlation length, i.e., in the fractal region, the effective conductivity $\{\sigma_e\}$ and effective resistivity $\{\rho_e\}$ averaged over the realizations depend on L as powers. However, in contrast to the usually considered case of infinitely great inhomogeneity $h = \sigma_2/\sigma_1 = 0$, crossover—change of the critical behavior—can occur when L is increased, and it is only then, with further increase of L to ξ, that the system goes over into the homogeneous region, in which self-averaging of $\sigma_e = 1/\rho_e$ and σ_e, ρ_e occurs and there is no more dependence on the length scale of the system. It can be assumed that this crossover is described by the following scaling function, which is the same for $\{\sigma_e\}$ and $\{\rho_e\}$:

$$\{\sigma_e\} = \sigma_1 \left(\frac{L}{a_0}\right)^{-t/\nu} \frac{\tau_L + \tau}{2\tau_L} f\left(\frac{\tau_L - \tau}{\tau_L + \tau}\left(\frac{\Delta}{\tau_L}\right)^{\varphi}\right),$$

$$\{\rho_e\} = \rho_2 \left(\frac{L}{a_0}\right)^{-q/\nu} \frac{\tau_L - \tau}{2\tau_L} f\left(\frac{\tau_L + \tau}{\tau_L - \tau}\left(\frac{\Delta}{\tau_L}\right)^{\varphi}\right), \tag{22}$$

where the scaling function $f(z)$ has the asymptotic behaviors

$$f(z \to 0) \to 1, \quad f(z \to \infty) \to z. \tag{23}$$

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