Sum rules and the leading two-loop logarithms in the hydrogen-atom Lamb shift

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The leading two-loop logarithmic corrections to the Lamb shift of the np levels and the normalized difference of shifts of the s levels, $\Delta E_L(1s)-n^3\Delta E_L(ns)$, in hydrogen-like systems are found. The results are obtained by direct calculations involving diagrams and in the nonrelativistic setting via the sum rules for dipole matrix elements. New numerical values are given for the shift of the $2p_{1/2}$ level in the hydrogen atom and the normalized difference between the $1s$ and $2s$ levels in hydrogen and deuterium. © *1996 American Institute of Physics.* [S1063-7761(96)00303-3]

1. INTRODUCTION

Considerable progress has been achieved lately in experimental studies of the Lamb shift.[1–5] Successful comparison of the theoretical and experimental data requires finding two-loop contributions to this quantity with fairly high accuracy. Lamb-shift corrections of the second order in the fine-structure constant α can be written as a series:

$$\Delta E_L^{(\mathrm{II})}(nl_j)=\frac{\alpha^2 m}{\pi^2 n^3}\sum B_{ts}(nl_j)(Z\alpha)^t \ln^s\frac{1}{(Z\alpha)^2}, \qquad (1)$$

where, following tradition, we distinguish between the parameter α whose power indicates the number of loops and the parameter $Z\alpha$ related to atomic-physics effects, and use the relativistic system of units: $\hbar=c=1$.

Nonvanishing quantities in Eq. (1) are the coefficients B_{40}, B_{50} (see Refs. 6–10), B_{63} (see Ref. 11), B_{62}, and others. Note that for s levels only the last of the four explicitly named coefficients depends on the principal quantum number n. In states with nonzero values of the orbital angular momentum l the nonzero coefficients are B_{62} and the first coefficient, B_{40}, whose value is completely determined, for $l\neq 0$, by the contribution of the anomalous magnetic moment.

This paper is devoted to the discussion of the contribution of B_{62} into the shift of the energy levels with $l\neq 0$ and into the normalized difference of shifts for the s levels,

$$\Delta(n)=\Delta E_L(1s)-n^3\Delta E_L(ns). \qquad (2)$$

The latter could be of interest in processing experimental data (for more details see Refs. 12 and 13).

The two-loop correction is related to calculations of the self-energy of an electron in the Coulomb field of the nucleus, with the ten diagrams for this energy depicted in Fig. 1. In the Yennie gauge, the first of these diagrams, a, provides a cubic logarithmic contribution of order $\alpha^2(Z\alpha)^6 m$ to the shifts of the s levels[11] and a quadratic contribution to the difference (2). This diagram has been studied in detail in Refs. 12 and 14. The contribution to levels with nonzero orbital angular momentum can only be linear in $\ln(Z\alpha)$. The second diagram, b, is studied below. Here we only note that its imaginary part leads to a linear (in the logarithm) correction to the width of p levels[12,15,16] and, bearing in mind the standard derivation of the leading one-loop logarithm, we must expect doubly logarithmic corrections to their energy. The other diagrams cannot lead to corrections of the order discussed in this paper; nevertheless we note that for all quantities discussed earlier, diagrams with a vacuum polarization loop, d and e lead to corrections containing logarithms raised to a power one unit less and can serve as a good model problem.

The plan of the paper is as follows. We start by examining the logarithmic contributions of the diagrams d and e with vacuum polarization, in the low-energy formalism, which considers vacuum polarization as being an additional external field. In Sec. 3 the corresponding contributions are calculated directly from the diagrams, and the relationship between the different terms in the two derivations is discussed. In Sec. 4 we return to calculating the contributions of the diagrams a and b. Finally, in Sec. 5 we discuss the results.

2. SUM-RULE CALCULATIONS OF THE POLARIZATION LOGARITHMS

Here we take a model problem for the one-loop operator of the self-energy of an electron in an external field that is the sum of a Coulomb field V_C and a delta-like potential allowing for vacuum polarization,

$$V_{VP}(\mathbf{r})=-\frac{4\alpha(Z\alpha)}{15m^2}\delta(\mathbf{r}). \qquad (3)$$

It is easy to show that by using the standard method of calculating the low-momentum contribution[17,18] we arrive at the following expression:

$$\Delta E_L^{\log}(nl_j)=\mathscr{R}(nl)\frac{2}{3\pi}\ln\frac{1}{(Z\alpha)^2}, \qquad (4)$$

where

$$\mathscr{R}(\lambda)=\sum_{\lambda'}\omega_{\lambda'\lambda}^3|\langle\lambda'|\mathbf{r}|\lambda\rangle|^2 \qquad (5)$$

or

$$\mathscr{R}(\lambda)=\left\langle\lambda\left|\frac{1}{2m^2}\Delta V(r)\right|\lambda\right\rangle \qquad (6)$$

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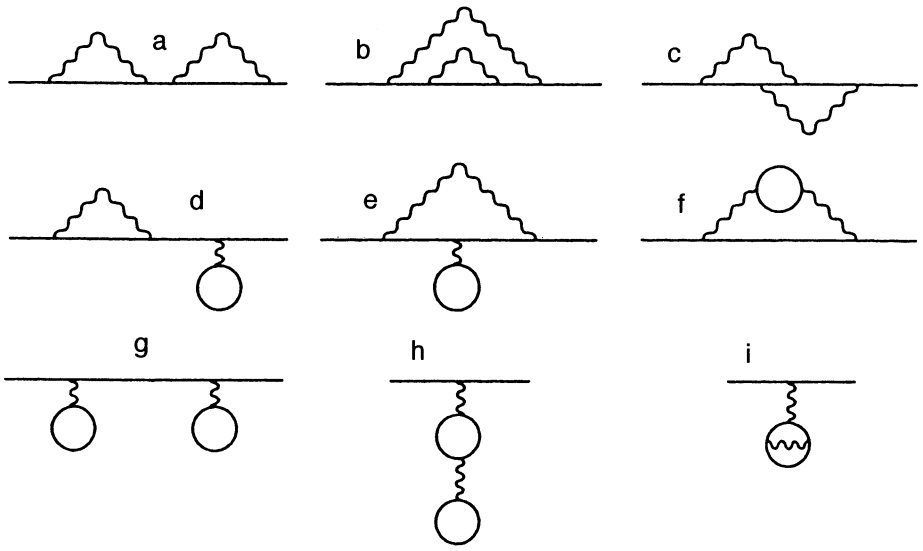

FIG. 1. The total two-loop operator of the self-energy of an electron in an external field.

for any external field.

The leading contribution of order $\alpha(Z\alpha)^4 m \ln(Z\alpha)$ is determined by the quantity (6) for a Coulomb field and Coulomb wave functions. There are two corrections related to vacuum polarization: the additional potential V_{VP} and the perturbation of wave functions.

The first contribution,

$$\delta\mathscr{B}^{(1)}(nl)=\left\langle nl\left|\frac{1}{2m^2}\Delta V_{VP}(r)\right|nl\right\rangle, \tag{7}$$

can easily be found by integrating by parts and substituting the explicit expressions for the potential (3) and the Coulomb wave functions. Clearly, for the s states there are infinities that emerge in terms outside the integral. These infinities correspond to small distances or high momenta,[1)] in which case the low-energy approximation (4)–(6) is inapplicable. However, for states with nonzero orbital momentum and for (2), the terms outside the integral are not infinite and, more than that, they are equal to zero, with corresponding results:

$$\delta\Delta^{(1)}(n)=-\frac{8}{45}\frac{n^2-1}{n^2}\frac{\alpha^2(Z\alpha)^6 m}{\pi^2}\ln\frac{1}{(Z\alpha)^2}, \tag{8}$$

$$\delta E_L^{(1)}(np)=-\frac{8}{135}\frac{n^2-1}{n^2}\frac{\alpha^2(Z\alpha)^6 m}{\pi^2 n^3}\ln\frac{1}{(Z\alpha)^2}. \tag{9}$$

At higher values of l the logarithmic correction (7) vanishes.

Let us now examine the contribution related to a perturbation of the wave functions in (6):

$$\delta\mathscr{B}^{(2)}(nl)=\frac{4(Z\alpha)^4 m}{n^3}\delta_{l0}\frac{\delta\psi_{ns}^{VP}(0)}{\psi_{ns}(0)}, \tag{10}$$

where $\psi(\mathbf{r})$ is the Schrödinger wave function of the hydrogen atom, and $\delta\psi^{VP}(\mathbf{r})$ is the correction to this function related to the potential (3). The perturbation of the values of the wave functions at zero can now be written as

$$\frac{\delta\psi_{ns}^{VP}(0)}{\psi_{ns}(0)}=-\frac{4\alpha(Z\alpha)}{15m^2}\sum_{q\neq n}\frac{\psi_{qs}^2(0)}{E_n-E_q}. \tag{11}$$

Clearly, integration over the states of the continuous spectrum leads to a divergence caused by relativistic intermediate states, but the difference (2) proves to be finite (for more details see Refs. 11, 12, and 14. The result is

$$\delta\Delta^{(2)}(n)=-\frac{32}{45}\frac{\alpha^2(Z\alpha)^6 m}{\pi^2}\ln\frac{1}{(Z\alpha)^2}\times\left(-\ln n+1-\frac{1}{n}+\sum_1^{n-1}\frac{1}{k}\right). \tag{12}$$

3. CALCULATIONS OF THE POLARIZATION LOGARITHMS FROM THE DIAGRAMS

As is known, the general expression for two-loop contributions has the form[19]

$$\delta E_L^{(\mathrm{II})}(nl_j)=\langle nl_j|\Sigma_1(E_{nl_j})\bar{G}_C(E_{nl_j})\Sigma_1(E_{nl_j})|nl_j\rangle+\langle nl_j|\Sigma_2(E_{nl_j})|nl_j\rangle+\langle nl_j|\Sigma_1(E_{nl_j})|nl_j\rangle\times\left\langle nl_j\left|\frac{\partial\Sigma_1(E)}{\partial E}\right|nl_j\right\rangle\Bigg|_{E=E_{nl_j}}, \tag{13}$$

where $\Sigma_r(E)$ is the r-loop single–particle irreducible operator of the self-energy of an electron in a Coulomb field, and $\bar{G}_C(E)$ is the reduced Coulomb Green's function.

In calculating the leading logarithmic contributions it has proved convenient to employ the Yennie gauge,[20,21] in which the diagrams have natural low-energy asymptotic behavior.[21–24] Within this gauge the leading polarization logarithms of order $\alpha^2(Z\alpha)^6 m$ emerge only from the first two terms in (13), which correspond to the diagrams d and e in Fig. 1, respectively.

Diagram d has been considered earlier[11,12,14] (more precisely, a similar diagram a). The logarithmic correction from the factorized diagram d introduces a quadratic logarithmic correction for the s levels, a linear correction for the difference (2), and a nonlogarithmic correction for levels with $l\neq 0$. For s states, in addition to the low-momentum contri-

bution discussed here, there is a correction of the preceding order $\alpha(Z\alpha)^5 m$ related to relativistic momenta, which leads to divergences, as in Sec. 2.

Calculating the correction to the energy is easy. The expression for the factorized diagram can be written as follows:

$$\delta E^{(2)}_{VP-SE}(nl) = 2\sum_{ql'm'\neq nlm} \frac{\langle nlm|V_{VP}|ql'm'\rangle\langle ql'm'|V_{SE}|nlm\rangle}{E_n - E_q}, \tag{14}$$

where the reduced Coulomb Green's function is represented as a sum over all states $ql'm'$ of the discrete and continuous spectra, and V_{VP} and V_{SE} are the one-loop diagrams emerging as a result of allowing for vacuum polarization of the self-energy of the electron. The first was discussed in Sec. 2,

$$\langle nlm|V_{VP}|ql'm'\rangle = -\frac{4\alpha(Z\alpha)^4 m}{15\pi}\delta_{l0}\delta_{ll'}\frac{\psi_{ns}(0)\varphi_{qs}(0)}{\psi_{1s}(0)\varphi_{1s}(0)},$$

and the self-energy contribution is

$$\langle nlm|V_{SE}|ql'm'\rangle = -\frac{4\alpha(Z\alpha)^4 m}{3\pi}\delta_{l0}\delta_{ll'}\ln\frac{m^2}{q^2} \times\frac{\psi_{ns}(0)\varphi_{qs}(0)}{\psi_{1s}(0)\varphi_{1s}(0)}.$$

Substituting these expressions into (14) yields

$$\delta E^{(2)}_{VP-SE}(nl) = -\frac{32\alpha^2(Z\alpha)^6 m}{45\pi^2 n^3}\delta_{l0} \times\sum_{q\neq n}\left(\frac{\psi_{qs}(0)}{\psi_{1s}(0)}\right)^2\frac{(Z\alpha)^2 m}{E_n - E_q}\ln\frac{1}{(Z\alpha)^2 q^2}, \tag{15}$$

where for the discrete spectrum the value of q is equal to the principal quantum number, while for continuous states, characterized by a wave number k, it is defined as $q = Z\alpha m/k$. The sum over states includes the contribution of high momenta ($k\sim m$) of the preceding order $\alpha^2(Z\alpha)^5 m$, the logarithmic integral over the continuous states with $(Z\alpha)m \ll k \ll m$, and the n-dependent contribution of atomic momenta $k\sim(Z\alpha)m$ and the states of the discrete spectrum. Each of these contributions is discussed in Refs. 11, 12, and 14.

The result is similar to the contribution $\mathscr{R}^{(2)}$, but in its derivation we did not use the sum rule, so that it is applicable not only to the difference of contributions of s states, which contain (12), but also for each state separately (containing the square of the logarithm).

The single-logarithmic term coincides with the corresponding expression (12), and the double-logarithmic correction is (cf. Ref. 11)

$$\delta E_{VP-SE}(ns) = \frac{8}{45}\frac{\alpha^2}{\pi^2}\frac{(Z\alpha)^6 m}{n^3}\ln^2\frac{1}{(Z\alpha)^2}. \tag{16}$$

Another diagram that also provides logarithmic contributions of order $\alpha^2(Z\alpha)^6 m$ is the vertex correction to Coulomb polarization (diagram e). At low momenta the correction potential has the form

$$\delta V_{VP-SE}(\mathbf{r}) = -\frac{\alpha}{3\pi}\frac{\Delta}{m^2}\frac{4\alpha(Z\alpha)}{15m^2}\delta(\mathbf{r})\ln\frac{m}{\kappa}. \tag{17}$$

After averaging over the wave functions and replacing the infrared cutoff κ by the characteristic binding energy (cf. Ref. 18) we again arrive at an expression for the contribution to the energy similar to the term with $\mathscr{R}^{(1)}$. The contribution of the s states again proves to be divergent at small distances, which corresponds to corrections of order $\alpha^2(Z\alpha)^5 m$. In the difference (2) the divergences cancel out. The contributions for the levels with $l \neq 0$ are finite from the start. The results coincide with (8) and (9).

Note that in spite of the agreement of results for the contributions of separate diagrams and the terms obtained from the sum rule in Sec. 2, no direct analogy exists. To verify this statement one should only compare the contributions of the diagram e for the s and p levels. The nonrelativistic interpretations of these levels are markedly different because intermediate states p (for the $ns_{1/2}$ levels) and s or d (for np_j) appear in the sum rule (5). But the delta-like potential (3) can perturb only s states and, consequently, for the corrections to the energy of the s states the intermediate states are not perturbed and the diagram e contributes nothing to (5).

This paradox is resolved in the following manner. The expression (5) is the total sum of the contributions of all diagrams, and in calculating separate diagrams additional intermediate states may emerge. The sum over physical states appears immediately only in the Coulomb gauge, in which, on the contrary, in calculating the diagrams there appears a number of additional contribution that cancel out in the process of summation. In covariant gauges there is a virtual-photon emission vertex, and this vertex is proportional to the matrix γ_0, which allows for an s state as an intermediate state. Here the correction to the energy of the $ns_{1/2}$ level incorporates the following quantity:

$$-\frac{2\alpha}{3\pi}\int\frac{d\Omega^+_{\mathbf{k}}}{2\pi}\int_0^\infty d\omega\sum_{qs}\langle ns|\exp(i\mathbf{k}\mathbf{r}_2)|qs\rangle \times\frac{\omega}{E_q - E + \omega}\langle qs|\exp(-i\mathbf{k}\mathbf{r}_1)|ns\rangle, \tag{18}$$

where integration is over all the angles of momentum $\mathbf{k}$ for which the scalar product $\mathbf{k}(\mathbf{r}_2 - \mathbf{r}_1)$ is positive; the absolute value of $\mathbf{k}$ is equal to the frequency ω. The expression for the contribution of the longitudinal terms of the photon propagator is more cumbersome, and we do not write it here. For the time being the quantity $E \approx E_n$ is kept as a free variable. In (18) we have left only the intermediate s states, which are sensitive to a delta-like perturbation.

In the logarithmic approximation, after renormalization we have

$$\frac{2\alpha}{3\pi}\ln\frac{m}{\langle\Delta E\rangle}\int\frac{d\Omega^+_{\mathbf{k}}}{2\pi}\sum_{qs}(E_q - E)\langle ns| \times\exp(i\mathbf{k}\mathbf{r}_2)|qs\rangle\langle qs|\exp(-i\mathbf{k}\mathbf{r}_1)|ns\rangle, \tag{19}$$

where now the photon wave number in the exponential corresponds to the difference $E_q - E$, and the logarithm con-

tains the characteristic energy difference. The order of magnitude of the scalar product $\mathbf{kr}$ is that of $Z\alpha$ because the radius in the matrix elements is of the order of atomic distances and the wave number of the photon corresponds to the characteristic atomic energy, so that the exponential can be expanded in series. The leading contribution, corresponding to the first term in the expansion, or unity, has the form

$$\frac{2\alpha}{3\pi}\ln\frac{m}{E_q-E}\sum_{qs}(E_q-E)\delta_{nq}, \tag{20}$$

where we have allowed for the fact that only one term contributes to the sum and, hence, the characteristic energy difference in the logarithm is determined by how far we are from the physical value. Thus, off the mass shell an infrared divergence emerges. The intermediate states coinciding with the initial state are examined in the Feynman gauge in Ref. 25.

The given divergence occurs in the absence of an external field, and in this case it corresponds to the well-known infrared divergence of the subtractive constant $\Sigma'(m)$ (see, e.g., Ref. 26). When the field is taken into account, in view of the Ward identity the divergence for the physical value of the energy, $E=E_n$, cancels out when the contributions of the free mass operator and the free vertex are added. In the Yennie gauge there are no infrared contributions in the separate diagrams either.[21-24]

Let us now discuss the further expansion of the exponentials.[2)] The product of the matrix elements of the linear term in the expansion,

$$\langle ns|i\mathbf{kr}_2|qs\rangle\langle qs|-i\mathbf{kr}_1|ns\rangle.$$

provides the standard contribution to the dipole width (equal to zero for intermediate s states). However, there is a contribution of the same order emerging from the product of the matrix elements of unity and the square of the radius:

$$-\langle qs\left|\frac{(\mathbf{kr}_2)^2}{2}\right|ns\rangle\langle ns|1|qs\rangle$$

$$-\langle qs|1|ns\rangle\langle ns\left|\frac{(\mathbf{kr}_1)^2}{2}\right|qs\rangle.$$

In the problem with one-loop self-energy this contribution vanishes in any external field. Indeed, the matrix element of unity leaves in the sum only the intermediate state coinciding with the initial state, for which the transition frequency is zero. Thus, in a pure Coulomb field and in a field with the perturbation (3) there is no such contribution. But in diagrammatic calculations the respective terms emerge at an intermediate stage. Indeed, the diagram d in Fig. 1 contains the perturbed initial state and unperturbed (Coulomb) intermediate states, while with the diagram e the intermediate states are perturbed and the initial state is a pure Coulomb state. Clearly, the scalar product of the perturbed state and the unperturbed states (i.e., the matrix element of unity) is finite for $q\neq n$ and is of order $\alpha(Z\alpha)^2$. The energy difference in the logarithm in (19) is determined by the characteristic quantity $(Z\alpha)^2m$, and the emerging contribution is of order $\alpha^2(Z\alpha)^6m\ln(Z\alpha)$. Thus, the intermediate s states in the Yennie gauge provide contributions in calculations of separate diagrams, and it is these states that are responsible for the correction to the energy of the s levels originating from the diagram e.

4. LEADING TWO-LOOP LOGARITHMS

The leading logarithmic corrections of order $\alpha(Z\alpha)^6m$ originate from diagrams a and b in Fig. 1 without vacuum polarization. As mentioned in the Introduction, the power of the logarithm they contain is one greater than the corresponding power in contributions of the diagrams d and e. The simplest way to obtain results is to use the diagrams directly (see Eq. (13)), similar to Sec. 3.

The leading contributions of the factorized diagram a were found earlier and have the following form in Ref. 11:

$$\delta E_{SE-SE}(ns_{1/2})=-\frac{8}{27}\frac{\alpha^2}{\pi^2}\frac{(Z\alpha)^6}{n^3}m\ln^3\frac{1}{(Z\alpha)^2} \tag{21}$$

and in Refs. 12 and 14:

$$\delta\Delta^{(2)}_{SE-SE}(1,n)=\frac{16}{9}\frac{\alpha^2(Z\alpha)^6}{\pi^2}m\ln^2\frac{1}{(Z\alpha)^2}\times\left(\ln n-1+\frac{1}{n}-\sum_1^{n-1}\frac{1}{q}\right). \tag{22}$$

The two-loop self-energy operator of the diagram b can be taken into account by expanding the Coulomb Green's function of the electron in a power series of the external field. The logarithmic contributions

$$\delta\Delta^{(1)}_{SE-SE}(1,n)=\frac{4}{9}\frac{n^2-1}{n^2}\frac{\alpha^2(Z\alpha)^6}{\pi^2}m\ln^2\frac{1}{(Z\alpha)^2} \tag{23}$$

and

$$\delta E^{(1)}_{SE-SE}(np_j)=\frac{4}{27}\frac{n^2-1}{n^2}\frac{\alpha^2}{\pi^2}\frac{(Z\alpha)^6}{n^3}m\ln^2\frac{1}{(Z\alpha)^2} \tag{24}$$

originate from the well-known low-energy asymptotic behavior of the electric form factor (see, e.g., Ref. 18), which leads to the potential

$$\delta V_{SE-SE}(\mathbf{r})=\frac{2}{9\pi}\alpha(Z\alpha)^2\ln^2\frac{1}{(Z\alpha)^2}\frac{\nabla^2}{m^2}\delta(\mathbf{r}). \tag{25}$$

DISCUSSION

There are several different definitions of a Lamb shift. In this paper we assume that the Lamb shift is the correction to the expression

$$E(nl_j)=m_R[f(n,j)-1]-\frac{m_R^2}{2(M+m)}[f(n,j)-1]^2, \tag{26}$$

where m_R is the reduced mass, M is the mass of the nucleus, and

$$f(n,j)=\left(1+\frac{(Z\alpha)^2}{[n-j-1/2+\sqrt{(j+1/2)^2-(Z\alpha)^2}]^2}\right)^{-1/2} \tag{27}$$

is the dimensionless energy of a Dirac electron in the Coulomb field of a nucleus of infinite mass.

The leading two-loop logarithmic corrections to the difference (2) [see Eqs. (22) and (23)],

$$\delta\Delta_{\log}(1,n)=\frac{\alpha^2(Z\alpha)^6}{\pi^2}\,m\,\ln^2\frac{1}{(Z\alpha)^2}\left\{\frac{16}{9}\left(\ln n-1+\frac{1}{n}-\sum_1^{n-1}\frac{1}{q}\right)+\frac{4}{9}\frac{n^2-1}{n^2}\right\}, \tag{28}$$

and to the Lamb shift of the p levels (see Eq. (24)),

$$\delta E_{\log}(np_j)=\frac{4}{27}\frac{n^2-1}{n^2}\frac{\alpha^2}{\pi^2}\frac{(Z\alpha)^6}{n^3}\,m\,\ln^2\frac{1}{(Z\alpha)^2}, \tag{29}$$

at $n=2$ amount to -10.7 kHz and 0.135 kHz, respectively, and lead to the following numerical results for hydrogen and deuterium:

$$\Delta(2,\mathrm{H})=-187.233(5)\ \ \mathrm{MHz}, \tag{30}$$

$$\Delta E_L(2p_{1,2},\mathrm{H})=-12.8360(1)\ \ \mathrm{MHz}, \tag{31}$$

and

$$\Delta(2,\mathrm{D})=-187.226(5)\ \ \mathrm{MHz}, \tag{32}$$

where we have allowed for the latest results for the corrections to recoil.[27–30] The main sources of the errors in (30)–(32) lie in the terms that follow the leading logarithmic corrections, and the contributions of these terms are estimated to be no greater than half the corresponding leading contributions. Another part in the errors is related to the interpolation[12,13] of the numerical data of Ref. 31. Both types are purely radiative and are independent of the mass of the nucleus, so that the isotopic shift in $\Delta(2)$ for hydrogen and deuterium amounts to 7.6 kHz with an error smaller than 0.1 kHz.

We did not include in the logarithmic corrections (28) and (29) the vacuum-polarization contributions,

$$\delta\Delta_{VP-SE}(n)=\frac{\alpha^2(Z\alpha)^6 m}{\pi^2}\ln\frac{1}{(Z\alpha)^2}\left\{\frac{32}{45}\left(-\ln n+1-\frac{1}{n}+\sum_1^{n-1}\frac{1}{k}\right)-\frac{8}{45}\frac{n^2-1}{n^2}\right\} \tag{33}$$

and

$$\delta E_{VP-SE}(np)=-\frac{8}{135}\frac{n^2-1}{n^2}\frac{\alpha^2(Z\alpha)^6 m}{\pi^2 n^3}\ln\frac{1}{(Z\alpha)^2}, \tag{34}$$

since the diagrams without polarization insertions contain unknown contributions of the same order. In addition to the calculations of polarization logarithms being useful from the methodological standpoint, Eqs. (33) and (34) may have practical applications. In a recent paper, Mitrushenkov *et al.*[32] obtained numerical results for the Lamb shift of the lowest levels of hydrogenlike ions at certain large values of the nuclear charge Z. If calculations with moderate values of Z are done and the results are compared with the known corrections of order $\alpha^2(Z\alpha)^5 m$ (see Refs. 6 and 33), then the logarithmic corrections (33) and (34) can be used to match the numerical results for different values of Z and, hence, to cross-check the calculations.

The contributions of diagrams without vacuum polarization of orders $\alpha^2(Z\alpha)^4 m$ and $\alpha(Z\alpha)^5 m$ are also known,[8–10] and the corrections (28) and (29) may also prove useful in checking the calculations for large and moderate values of Z.

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[1)]The appearance of such divergences is inevitable because there is the contribution of the preceding order $\alpha^2(Z\alpha)^5 m$.

[2)]Strictly speaking, we must first integrate over angles and then expand. However, it is easy to verify, by using the fact that the energy is real, that all the expansion terms discussed below appear in the more rigorous approach, too.

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