Possible enhancement of superconductivity by a magnetic field (in the gapless region)

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A simple model of a two-dimensional superconductor is considered in which superconductivity is suppressed by ordinary impurities, with the appearance of a gapless region. Equations are derived for the order parameter in a magnetic field in the Ginzburg–Landau approximation near the critical impurity concentration at absolute zero and finite (low) temperatures. A characteristic feature of these equations is the coupling of the different components of the order parameter. Without a magnetic field there remains only one component, a situation that leads to the appearance of an isotropic gap in the Fermi spectrum. Because of such coupling, there is a rise in the transition temperature near the critical concentration and emergence of superconductivity in the supercritical region in the presence of a magnetic field. This enhancement and stimulation of superconductivity by a magnetic field occur within a narrow range of impurity concentrations: estimates set this range at about 1% of the critical concentration. © *1996 American Institute of Physics.* [S1063-7761(96)01803-1]

1. INTRODUCTION

Paramagnetic impurities, which lead to electron scattering with spin flip, are known to suppress superconductivity in ordinary superconductors (with S-pairing).[1] The same is true of ordinary impurities if Cooper pairs are formed with nonzero moments (see, e.g., Ref. 2). One of the most interesting properties of such systems is the emergence of a gapless region near the critical impurity concentration.[1]

In high-T_c superconductors such phenomena as superconductivity suppression and the gapless mode probably also occur. Experiments have shown[3,4] that these phenomena do not depend on whether the impurity has spin. If this is so, it is natural to assume that Cooper pairs are formed in a state with a nonzero moment.

An interesting question in this connection is what other consequences, in addition to those just mentioned, are possible and how they be manifested in experiments. This requires adopting a certain model and examining the results that the model produces. It appears that new properties for nonzero moments do exist.

In this paper we adopt a model discussed in Ref. 5. The model contains polarons with attraction that emerges because of the presence of a bipolar quasilevel. What is important is that there exists a one-component system of Fermi particles (a single projection of the "spin") or that only one component undergoes a superconducting transition, while the origin of the attraction is unimportant.

Thus, we are dealing with a one-component system. In such a system in the two-dimensional case, a Cooper pair must have an odd projection of the moment on the normal to the plane (e.g., ± 1, which is assumed from now on). The reaction of this system to ordinary impurities is a drop in the transition temperature and the appearance of a gapless region. This may be the simplest model that works with the "boundary" conditions specified by the experiment.[3,4]

The present paper is a study of how the gapless region behaves in an external magnetic field for a fairly small order parameter. The conditions are such that the Ginzburg–Landau approximation is applicable. The study is done for the case of absolute zero and for finite (low) temperatures. The results differ dramatically from those of Abrikosov and Gor'kov:[1]

1. There is coupling between the ± 1-components in the equations of the Ginzburg–Landau type, and this coupling cannot generally be ignored despite the fact that without the magnetic field there remains only one component.

2. The kinetic energy (at fairly low temperatures) contains contributions of the $\mathscr{P}^2\ln(1/\mathscr{P}^2)$ type in addition to the ordinary contribution $\sim\mathscr{P}^2$, where $\mathscr{P}$ is the momentum of a Cooper pair as a whole.

3. In addition to the ordinary contribution $\sim\mathscr{P}$, the expression for the flux density contains a contribution $\sim[\mathscr{P}\mathbf{n}]$, where $\mathbf{n}$ is the unit vector normal to the plane, and the value of this contribution depends on the phase difference between the two components.

4. Finally, an external magnetic field raises the transition temperature for impurity concentrations close to the critical value (enhancement of superconductivity); more than that, at higher concentrations, when there is no superconductivity in the absence of a magnetic field, the magnetic field induces superconductivity (stimulation of superconductivity). These effects can also be explained by the coupling of the ± 1-components.

The latter features, enhancement and stimulation of superconductivity by a magnetic field, point to experimental possibilities, although they emerge over a relatively narrow range of impurity concentrations (approximately 1% of the critical concentration).

2. THE MODEL

This section is of an auxiliary nature. In addition to specifying the model, we discuss the definitions and relationships needed for deriving the equations.

The Hamiltonian of the model is

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$$\hat{H}=\hat{H}_0+\hat{H}_1, \tag{2.1}$$

$$\hat{H}_0=\int d\mathbf{r}\left\{\psi^+(\mathbf{r})\frac{(-i\nabla-\mathbf{A})^2}{2m}\psi(\mathbf{r})+\psi^+(\mathbf{r})W(\mathbf{r})\psi(\mathbf{r})\right\},$$

$$\hat{H}_1=\frac{1}{2}\int d\mathbf{r}_1 d\mathbf{r}_2\psi^+(\mathbf{r}_1)\psi^+(\mathbf{r}_2)U(\mathbf{r}_1-\mathbf{r}_2)\psi(\mathbf{r}_2)\psi(\mathbf{r}_1),$$

where $\mathbf{A}$ is the vector potential multiplied by e/c, and $W(\mathbf{r})$ is the potential of the interaction with impurities:

$$W(\mathbf{r})=\sum_n u(\mathbf{r}-\mathbf{R}_n), \tag{2.2}$$

with $u(\mathbf{r}-\mathbf{R}_n)$ representing the interaction with the nth impurity positioned at the point $\mathbf{R}_n$. It is postulated that there is attraction U between the particles, but the origin of this attraction is assumed unimportant. We also assume that this interaction is weak, and we allow for it only to the extent that it leads to superconductivity. This means that $\hat{H}_1$ can be written as

$$\hat{H}_1\to\frac{1}{2}\int d\mathbf{r}_1 d\mathbf{r}_2\{\psi^+(\mathbf{r}_1)\psi^+(\mathbf{r}_2)\Delta(\mathbf{r}_2,\mathbf{r}_1)+\text{h.c.}\} \tag{2.3}$$

(to within an unimportant constant term). Here by definition

$$\Delta(\mathbf{r}_2,\mathbf{r}_1)=U(\mathbf{r}_1-\mathbf{r}_2)\langle\langle\psi(\mathbf{r}_2)\psi(\mathbf{r}_1)\rangle\rangle. \tag{2.4}$$

This quantity is the order parameter. The averaging in (2.4) consists in Gibbs averaging (over the ground state at absolute zero) and averaging over the impurity positions.[6] The notation (2.3) for $\hat{H}_1$ stems from the self-consistent field approximation, which produces accurate results in our case.

We start with the case of absolute zero (the temperature correction will also be found) and employ the appropriate diagrammatic technique.[7] As usual, we introduce the functions

$$\begin{aligned}G(x,x')&=-i\langle T\psi(x)\psi^+(x')\rangle,\\ F^+(x,x')&=\langle T\psi^+(x)\psi^+(x')\rangle,\\ F(x,x')&=\langle T\psi(x)\psi(x')\rangle,\end{aligned} \tag{2.5}$$

where $x\equiv(\mathbf{r},t)$ and $x'=(\mathbf{r}',t')$. Here we assume that there is ground-state averaging.

The equations for these functions have the ordinary form. In the frequency representation we have

$$G(\omega;1,2)=G^{(0)}(\omega;1,2)+i\int d\mathbf{r}_3 d\mathbf{r}_4 G^{(0)}\times(\omega;1,3)\Delta(3,4)F^+(\omega;4,2), \tag{2.6}$$

$$F^+(\omega;1,2)=-i\int d\mathbf{r}_3 d\mathbf{r}_4 G^{(0)}(-\omega;3,1)\Delta^*(3,4)G(\omega;4,2), \tag{2.7}$$

where $G^{(0)}$ is the normal-state Green's function, and the numbers stand for the respective coordinates; for example,

$$G(\omega;1,2)\equiv G(\omega;\mathbf{r}_1,\mathbf{r}_2). \tag{2.8}$$

Equations (2.6) and (2.7) are the input equations. They form a complete system if we add the definition (2.4), which for the conjugate quantity Δ^* can be written as follows:

$$\Delta^*(1,2)=-U(\mathbf{r}_1-\mathbf{r}_2)\int\frac{d\omega}{2\pi}\langle F^+(\omega;1,2)\rangle. \tag{2.9}$$

The angle brackets stand for averaging over the impurity positions (the notation coincides with that used in (2.5), but this should not lead to a mix-up).

What happens in the homogeneous case (without a magnetic field)? The corresponding equations after averaging over the impurity positions are given in Ref. 7. Our analysis follows that of Abrikosov and Gor'kov.[1] The only difference (unimportant here) is that the order parameter exhibits an angular dependence. In the homogeneous case the order parameter $\Delta^*(1,2)$ depends only on the relative coordinate $\boldsymbol{\rho}=\mathbf{r}_1-\mathbf{r}_2$, and we can write the following expression for the Fourier component $\Delta^+(\mathbf{p})$ for the ±1 moment:

$$\Delta^+(\mathbf{p})=\Delta_1 e^{i\varphi}+\Delta_2 e^{-i\varphi}, \tag{2.10}$$

where φ is the angle of the vector $\mathbf{p}$ in the plane, and Δ_1 and Δ_2 are independent of the direction of $\mathbf{p}$. In the homogeneous case only one harmonic, Δ_1 or Δ_2, remains, so that the gap in the spectrum is isotropic. All results for the order parameter and the gap in the spectrum are the same as in Ref. 1, but the mean free time with spin flip, τ_s, of Ref. 1 must be replaced by 2τ, where τ is the mean free time. For isotropic scattering by impurities in the Born approximation for the two-dimensional case we have the following expression for τ:

$$\frac{1}{\tau}=mnu_0^2, \tag{2.11}$$

where n is the impurity concentration, and u_0 is the zeroth Fourier harmonic of the impurity potential,

$$u_0=\int d\mathbf{r}u(\mathbf{r}). \tag{2.12}$$

Here are some results. The gapless region lies in the interval

$$\tau_c<\tau<\tau_0. \tag{2.13}$$

For smaller values of τ there is no superconductivity. At τ_0 and τ_c we have

$$\frac{1}{\tau_0}=2\Delta_0\exp(-\pi/4)(2|\Delta|\tau_0=1),\quad \frac{1}{\tau_c}=\Delta_0, \tag{2.14}$$

where Δ_0 is the gap width in the Fermi spectrum for a pure superconductor at absolute zero. Near τ_c we have the following expression for the order parameter:

$$\frac{1}{3}\left|\frac{\Delta}{\Delta_0}\right|^2=\frac{\tau-\tau_c}{\tau_c} \tag{2.15}$$

(provided that $2|\Delta|\tau\ll1$).

What happens when there is angle-dependent scattering? In this case we must replace τ specified by Eq. (2.11) by the transport time τ_{tr}:

$$\frac{1}{\tau_{tr}}=nm\int\frac{d\varphi}{2\pi}|u(\varphi)|^2(1-\cos\varphi).$$

This would seem to be a natural result. Note, however, that the fact that $\cos\varphi$ appears in the definition of the relaxation

time in the final result is closely linked to the angular dependence $\exp(\pm i\varphi)$ of the order parameter; if there were another angular dependence $\exp(i\nu\varphi)$ ($\nu=\pm 3,\ldots$), this would involve another time variable in which $\cos\varphi$ would be replaced by $\cos\nu\varphi$, rather than the transport time, as expected.

Now let us turn to the problem of allowing for a magnetic field in deriving the equations. On the whole this is done in same way as in Ref. 6, but in our case the situation is somewhat more complicated, since the order parameter depends on two coordinates. For this reason a brief discussion is in order.

In the presence of a magnetic field the equation for the normal-state Green's function $G^{(0)}$ has the form

$$\left\{\frac{1}{2m}[-i\nabla_1-\mathbf{A}(\mathbf{r}_1)]^2-\omega-\mu\right\}G^{(0)}(\omega;1,2)$$
$$+W(\mathbf{r}_1)G^{(0)}(\omega;1,2)=-\delta(\mathbf{r}_1-\mathbf{r}_2), \tag{2.16}$$

where μ is the Fermi energy, and the coordinate $\mathbf{r}_2$ acts as a parameter.

For a slowly varying weak magnetic field the approach is as follows. We expand the vector potential near the point $\mathbf{r}_2$:

$$\mathbf{A}(\mathbf{r}_1)\approx\mathbf{A}_2+(\mathbf{r}_{12}\nabla_2)\mathbf{A}_2, \tag{2.17}$$

where $\mathbf{r}_{12}\equiv\mathbf{r}_2-\mathbf{r}_1$, and $\mathbf{A}_2\equiv\mathbf{A}(\mathbf{r}_2)$. It is now convenient to "introduce" a gauge in the vector potential and include all that is possible in a phase factor, i.e.,

$$G^{(0)}(1,2)=e^{i\Phi_{12}}G'(1,2), \tag{2.18}$$

with a phase

$$\Phi_{12}=\mathbf{r}_{12}\mathbf{A}_2+\frac{1}{2}\left\{x_{12}^2\frac{\partial A_{2x}}{\partial x_2}+y_{12}^2\frac{\partial A_{2y}}{\partial y_2}\right.$$
$$\left.+x_{12}y_{12}\left(\frac{\partial A_{2y}}{\partial x_2}+\frac{\partial A_{2x}}{\partial y_2}\right)\right\}, \tag{2.19}$$

where $\mathbf{r}_{12}\equiv(x_{12},y_{12})$, and $\mathbf{r}_2\equiv(x_2,y_2)$. The equation for G' has the same form (2.16) as for $G^{(0)}$ but with the following substitution:

$$\mathbf{A}\rightarrow\mathbf{A}'=\tfrac{1}{2}[\mathbf{H}_2\mathbf{r}_{12}],$$

where $\mathbf{H}_2$ is the magnetic field at the point $\mathbf{r}_2$. In view of the weakness of the magnetic field, the contribution of $\mathbf{A}'$ can be taken into account by perturbation-theory techniques as an addition to G_0, the normal-state Green's function without a magnetic field. However, it was found that this addition has no effect on the equation for the order parameter. Therefore, in (2.18) instead of G' we can simply put G_0, the normal-state Green's function without a magnetic field, so that the final expression is

$$G^{(0)}(1,2)\rightarrow e^{i\Phi_{12}}G_0(1,2). \tag{2.20}$$

The contribution with the magnetic field (that appears through $\mathbf{A}'$) could manifest itself as the energy of the intrinsic degrees of freedom (the intrinsic magnetic moment) of a Cooper pair in a magnetic field. The fact that this possibility is not realized means that this energy is low (in comparison to the energy of the motion as a whole) and is not felt in the employed approximations.

We also write the equation for the order parameter in a pure superconductor at absolute zero:

$$\Delta^+(\mathbf{p})=-\int\frac{d^2\mathbf{p}}{(2\pi)^2}U_{\mathbf{p}-\mathbf{p}'}\frac{\Delta^+(\mathbf{p}')}{2\sqrt{\chi_{p'}^2+\Delta_0^2}} \tag{2.21}$$

where $U_{\mathbf{q}}$ is the Fourier component of the interaction potential, ξ_p is the energy of a particle in the normal state measured from the Fermi surface, and Δ_0 is the width of the gap in the spectrum (the gap is isotropic). This equation is for one component (see Eq. (2.10)), while the other component is zero.

3. THE EQUATIONS

We start with the linear part of the equations, for which Eqs. (2.9) and (2.7) (with $G^{(0)}$ substituted for G) yield

$$\Delta^*(1,2)\rightarrow iU(\mathbf{r}_1-\mathbf{r}_2)\int\frac{d\omega}{2\pi}\int d\mathbf{r}_3d\mathbf{r}_4\Delta^*(3,4)$$
$$\times\langle G_-^{(0)}(3,1)G^{(0)}(4,2)\rangle, \tag{3.1}$$

where $G_-^{(0)}\equiv G^{(0)}(-\omega)$. We assume averaging over the impurity positions. First we discuss the dependence on the magnetic field.

In Eq. (3.1) we introduce new coordinates, the relative position vector $\boldsymbol{\rho}$ and the center-of-mass vector $\mathbf{R}$:

$$\Delta^*(\mathbf{r}_1,\mathbf{r}_2)\rightarrow\Delta^*(\mathbf{R},\boldsymbol{\rho}),$$
$$\mathbf{R}=\frac{\mathbf{r}_1+\mathbf{r}_2}{2},\quad\boldsymbol{\rho}=\mathbf{r}_1-\mathbf{r}_2. \tag{3.2}$$

According to (2.20), the phase factor in Eq. (3.1) contains the phase

$$\Phi=\Phi_{31}+\Phi_{42}. \tag{3.3}$$

In terms of the new coordinates ($\mathbf{R}'$ and $\boldsymbol{\rho}'$ for the $\mathbf{r}_3$, $\mathbf{r}_4$ pair) we have the following expression for Φ:

$$\Phi=(\mathbf{R}'-\mathbf{R})[\mathbf{A}(\mathbf{R})+\mathbf{A}(\mathbf{R}')]+\Phi', \tag{3.4}$$

where Φ' contains various bilinear combinations of the components of $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$ multiplied by derivatives of the vector-potential components. This part of the phase is unimportant since it contributes nothing to the equation in the employed approximations, and we will not write the corresponding expression.

As a result Eq. (3.1) assumes the form

$$\Delta^*(\mathbf{R},\boldsymbol{\rho})\rightarrow iU(\boldsymbol{\rho})\int\frac{d\omega}{2\pi}\int d\mathbf{R}'d\boldsymbol{\rho}'\Delta^*(\mathbf{R}',\boldsymbol{\rho}')$$
$$\times\langle G_0^-(3,1)G_0(4,2)\rangle\exp\{i(\mathbf{R}'-\mathbf{R})$$
$$\times[\mathbf{A}(\mathbf{R})+\mathbf{A}(\mathbf{R}')]\}. \tag{3.5}$$

Note the gauge invariance of (3.5). Indeed, if we perform the substitution

$$\mathbf{A}\to\mathbf{A}+\frac{\partial}{\partial\mathbf{R}}\chi(\mathbf{R}),$$

with $\chi(\mathbf{R})$ an arbitrary function, then the addition $\delta\Phi$ to the phase in (3.5) is

$$\delta\Phi\approx 2[\chi(\mathbf{R}')-\chi(\mathbf{R})]$$

to within second-order terms in the difference of coordinates, which is the accuracy with which the phase was taken into account. Thus, the order parameter is transformed as follows:

$$\Delta^*\to\Delta^* e^{-2i\chi},$$

after which the equation assumes its initial form.

Averaging over the impurity positions is done as usual.[7] Here it is convenient to employ the momentum representation

$$G_0(\omega;\mathbf{r}_1,\mathbf{r}_2)=\frac{1}{S}\sum_{\mathbf{p}_1,\mathbf{p}_2} G_0(\omega;\mathbf{p}_1,\mathbf{p}_2)e^{i(\mathbf{p}_1\mathbf{r}_1-\mathbf{p}_2\mathbf{r}_2)} \tag{3.6}$$

(S is the surface area of the system). As is known, the average of a single Green's function has the following form:

$$\begin{aligned}&\langle G_0(\omega;\mathbf{p},\mathbf{p}')\rangle=\delta_{\mathbf{p},\mathbf{p}'}\mathscr{G}(\omega,\mathbf{p}),\\ &\mathscr{G}(\omega,\mathbf{p})=\left[\omega-\xi_p+\frac{i}{2\tau}\operatorname{sign}\omega\right]^{-1},\end{aligned} \tag{3.7}$$

where ξ_p is the energy measured from the Fermi surface.

The average of the product of two Green's functions has the form

$$\begin{aligned}\langle G_0^-(1,2)G_0(3,4)\rangle=&\,\delta_{\mathbf{p}_1,\mathbf{p}_2}\delta_{\mathbf{p}_3,\mathbf{p}_4}\mathscr{G}_-(1)\mathscr{G}(3)\\ &+\delta_{\mathbf{p}_1+\mathbf{p}_3,\mathbf{p}_2+\mathbf{p}_4}\mathscr{G}_-(1)\mathscr{G}_-(2)\mathscr{G}(3)\mathscr{G}\\ &\times(4)\frac{n}{S}\Gamma(1,2),\end{aligned} \tag{3.8}$$

where $\Gamma(1,2)$ is calculated in the ladder approximation and can be found by solving the following equation:

$$\begin{aligned}\Gamma(\mathbf{p}_1,\mathbf{p}_2)=&|u(\mathbf{p}_1-\mathbf{p}_2)|^2+\frac{n}{S}\sum_{\mathbf{p}}|u(\mathbf{p}_1\\ &-\mathbf{p})|^2\mathscr{G}_-(\mathbf{p})\mathscr{G}(\mathscr{P}-\mathbf{p})\Gamma(\mathbf{p},\mathbf{p}_2).\end{aligned} \tag{3.9}$$

Here $u(\mathbf{q})$ is the Fourier component of the impurity potential, and $\mathscr{P}\equiv\mathbf{p}_1+\mathbf{p}_3$ is the total momentum. The numbers in (3.8) stand for the respective momenta (rather than for the position coordinates, as in (2.8)).

In what follows we assume that scattering by an impurity is isotropic. Then Γ depends only on the total momentum. We are interested in the expression for Γ in the limit where

$$\mathscr{P}l\ll 1, \tag{3.10}$$

with $l=v\tau$ the mean free path (v is the Fermi velocity). In this limit

$$\Gamma\approx u_0^2\frac{(|\omega|+i/2\tau)^3}{|\omega|(|\omega|+i/2\tau)^2-i/16\tau(v\mathscr{P})^2}. \tag{3.11}$$

According to (3.8), the average of the product of two Green's functions consists of two parts, and the contributions of these parts were found to differ considerably. For this reason we study them separately.

The contribution of the first term on the right-hand side of Eq. (3.8) is analyzed as in Ref. 6. We write the order parameter in the general case as a function of the center-of-mass vector:

$$\Delta^+(\mathbf{R},\mathbf{p})=\Delta_1(\mathbf{R},p)e^{i\varphi}+\Delta_2(\mathbf{R},p)e^{-i\varphi} \tag{3.12}$$

(see Eq. (2.10)). The quantities $\Delta_1(\mathbf{R}')$ and $\Delta_2(\mathbf{R}')$ in the integrand in (3.5) can be expanded near the point $\mathbf{R}$ up to the second order inclusive, and the same can be done with the phase factor.

The zeroth term in the expansion can be transformed by a calculation procedure involving Eq. (2.2). After this for Δ^+ we have an integral equation (in the relative position vector) with a kernel as in (2.21) plus an inhomogeneous term, to which contributions are provided also by the second term in (3.8) and certain nonlinear terms discussed below. The inhomogeneous term can be calculated explicitly and contains the value of Δ^+ (together with derivatives with respect to $\mathbf{R}$) with the relative momentum on the Fermi surface, so that only integration over the angles is needed. After integrating it is convenient to write $U_{\mathbf{p}-\mathbf{p}'}$ in the form of a harmonic expansion (in the angle between $\mathbf{p}$ and $\mathbf{p}'$). The above-mentioned integral equation can be solved only if the inhomogeneous term is zero. This provides the sought equation for the order parameter, which we write in the form

$$(\mathscr{H}_1+\mathscr{H}_2+V)\Delta^+=0, \tag{3.13}$$

where $\mathscr{H}_1\Delta^+$ stands for what remains in the inhomogeneous term from the first term on the right-hand side of Eq. (3.8), $\mathscr{H}_2\Delta^+$ is the contribution of the second term, and $V\Delta_+$ is the contribution of the inhomogeneous term.

As it turns out, the harmonics Δ_1 and Δ_2 are coupled. It proves convenient to write the order parameter as a column vector

$$\Delta^+=\begin{pmatrix}\Delta_1\\ \Delta_2\end{pmatrix}, \tag{3.14}$$

and $\mathscr{H}_1$, $\mathscr{H}_2$, and V in the form of 2-by-2 matrices that act on this vector.

As a result for $\mathscr{H}_1$ we can write

$$\mathscr{H}_1=\left(\frac{l}{2}\right)^2\begin{pmatrix}\hat{\mathbf{p}}^2 & \frac{1}{2}\hat{p}_-^2\\ \frac{1}{2}\hat{p}_+^2 & \hat{\mathbf{p}}^2\end{pmatrix}-\frac{\tau-\tau_c}{\tau_c}I, \tag{3.15}$$

where I is the identity matrix. Here we have used the notation

$$\hat{\mathbf{p}}=-i\frac{\partial}{\partial\mathbf{R}}+2\mathbf{A}(\mathbf{R}),\quad \hat{p}_\pm=\hat{p}_x\pm i\hat{p}_y. \tag{3.16}$$

The result (3.15) is obtained after separating out an insignificant common factor in the inhomogeneous term,

$$\frac{m}{(2\pi)^2}\int d\alpha\cos\alpha U(\cos\alpha), \tag{3.17}$$

where α is the angle between $\mathbf{p}$ and $\mathbf{p}'$.

The harmonics are coupled because the isotropy of the system breaks down if the order parameter is coordinate-dependent. The coupling proceeds through two intervals, a fact formally related to the second order of the above expansion (the first order contributes nothing). Hence theoretically the harmonics $\exp(\pm 3i\varphi)$ do appear, but they enter with a small coupling constant. The physical reason for the coupling is obvious: it corresponds to the coupling of components with the same total angular momenta if, as usual, we assume that the total momentum is the sum of the orbital angular momentum (in $\mathbf{R}$) and the intrinsic angular momentum (in $\boldsymbol{\rho}$). This can be verified directly if we introduce cylindrical coordinates (in the absence of a magnetic field).

Now let us study the contribution of the second term in (3.8). Gor'kov's approach[6] is inapplicable because of the logarithmic divergence that appears in integration with respect to frequency (see Eq. (3.11)). Therefore, a different approach is necessary.

We start with the plane-wave case without a magnetic field:

$$\Delta^+(\mathbf{R}) \to e^{i\mathscr{P}\mathbf{R}}\Delta^+(\mathscr{P}). \tag{3.18}$$

After performing simple calculations and extracting the common factor (3.17) we get

$$\mathscr{H}_2 = \left(\frac{l\mathscr{P}}{2}\right)^2\left(L-\frac{3}{2}\right)\begin{pmatrix} 1 & e^{-2i\varphi_0} \\ e^{2i\varphi_0} & 1 \end{pmatrix}, \tag{3.19}$$

$$L \equiv \ln\frac{2}{(l\mathscr{P})^2},$$

where φ_0 is the angle of the vector $\mathscr{P}$ in the plane. If we return to position coordinates, we can write

$$\mathscr{H}_2 = \left(\frac{l}{2}\right)^2\left(\hat{L}-\frac{3}{2}\right)\begin{pmatrix} \mathscr{P}^2 & \hat{\mathscr{P}}_-^2 \\ \hat{\mathscr{P}}_+^2 & \mathscr{P}^2 \end{pmatrix}, \quad L \equiv \ln\frac{2}{l^2\mathscr{P}^2}, \tag{3.20}$$

where

$$\hat{\mathscr{P}} = -i\frac{\partial}{\partial \mathbf{R}}, \quad \hat{\mathscr{P}}_\pm = \hat{\mathscr{P}}_x \pm i\hat{\mathscr{P}}_y. \tag{3.21}$$

The expression (3.20) is valid if condition (3.10) is met. The meaning of the operator form of (3.20) is clear without further explanation.

Generalization to the case of a magnetic field by going from operators (3.21) to operators (3.16), which appears natural due to the gauge invariance of (3.5) noted earlier, is difficult because of the logarithm, since the different components of the operator $\hat{\mathbf{p}}$ are not commutative. Another approach might be to leave $\mathscr{H}_2\Delta^+$ in integral form, but this also fails. Hence in what follows we use the above form of $\mathscr{H}_2$ and its possible generalizations—all the more so, as not many alternative generalizations exist, and all yield approximately the same results, with the results coinciding in some cases.

What is unusual in (3.19) is the appearance of a logarithm, caused by the low-frequency behavior of Γ [Eq. (3.11)]. This is quite natural, since if pairing with a zero moment (spins $\uparrow$ and $\downarrow$) is possible, ordinary impurities have no effect (in the sense of suppressing Cooper instability) and hence the logarithmic divergence at $T=0$ remains, which is evident in the behavior of Γ. Here the divergence would be in the leading contribution to the equation (at $\mathscr{P}=0$), while in our case due to the angular dependence of the order parameter the logarithm appears only when $\mathscr{P} \neq 0$.

Finally, some remarks about nonlinear contributions to the equations are appropriate. The contributions can be found by following the approach developed in Refs. 6 and 1. After separating the factor (3.17), the result can be written as

$$V = \begin{pmatrix} a_{12} & 0 \\ 0 & a_{21} \end{pmatrix}, \quad a_{12} = \tfrac{1}{3}\tau^2(|\Delta_1|^2 + 4|\Delta_2|^2). \tag{3.22}$$

We note in passing that the form of this contribution implies that without a magnetic field the state with only one component corresponds to the ground state. The results agree with (2.15).

Thus, the equations at absolute zero have been obtained to within the uncertainty that emerges in connection with the generalization of (3.20). We return to this problem in Sec. 4. No such uncertainty exists at finite temperatures (and fairly weak magnetic fields). This case is studied here.

Fluctuation effects can present certain difficulties for a two-dimensional system. However, in real high-T_c superconductors at low temperatures this, apparently, is not true because of the coupling between the planes and the effective three-dimensional system. We do not study this problem here, but we assume that the appropriate conditions are met and that the self-consistent approach still works (as it does at absolute zero).

We consider low temperatures:

$$T\tau_c \ll 1. \tag{3.23}$$

Temperature corrections are essential in two places: we must establish what replaces $(\tau-\tau_c)/\tau_c$ in (3.15) and what replaces the logarithm in (3.19).

Here we use the temperature diagrammatic technique.[7] Without going into details of the corresponding calculations, we restrict our discussion to the example of modifying (3.15).

The integral in the equation (3.1) for the order parameter is replaced by a sum s over the frequencies, which for the first contribution has the form

$$s = T\sum_n \mathscr{G}(-\omega_n, p)\mathscr{G}(\omega_n, p), \tag{3.24}$$

where $\omega_n = \pi T(2n+1)$. We use the ordinary rules to transform this sum into integrals along contours in the complex ω plane encompassing the poles of $\tanh(\omega/2T)$ and then into integrals along the real axis, so that

$$s \to 2\int_{-\infty}^{\infty}\frac{d\omega}{4\pi i}\tanh\frac{\omega}{2T}\mathscr{G}_R(\omega,p)\mathscr{G}_A(-\omega,p), \tag{3.25}$$

where $\mathscr{G}_R$ and $\mathscr{G}_A$ are the retarded and advanced Green's functions. Using the well-known relationship between these functions and the causal function (3.7), we find that

$$\mathscr{G}_R(\omega)=\left(\omega-\xi+\frac{i}{2\tau}\right)^{-1},$$
$$\mathscr{G}_A(\omega)=\left(\omega-\xi-\frac{i}{2\tau}\right)^{-1}. \tag{3.26}$$

We substitute (3.26) into (3.25) and write $\tanh(\omega/2T)$ as an identity:

$$\tanh\frac{\omega}{2T}=\operatorname{sign}\omega+\left(\tanh\frac{\omega}{2T}-\operatorname{sign}\omega\right). \tag{3.27}$$

The integral (3.25) with the first term on the right-hand side of Eq. (3.27) yields the former result (at $T=0$), while with the second term on the right-hand side of Eq. (3.27) it yields a temperature correction, which can easily be calculated if the condition (3.23) is met. As a result we arrive at a generalization of (3.15) that reduces to the substitution

$$\frac{\tau-\tau_c}{\tau_c}\to\frac{\tau-\tau_c}{\tau_c}-\frac{2\pi^2}{3}(T\tau)^2. \tag{3.28}$$

We can proceed in the same manner in the generalization of (3.19). As a result we arrive at the same expression but with the substitution

$$L\to\ln\frac{1}{2T\tau}-\int_0^\infty dx\frac{e^x\ln(x^2+A^2)}{(e^x+1)^2}, \tag{3.29}$$
$$A\equiv\frac{(l\mathscr{P})^2}{4T\tau}.$$

For $A\gg1$ this leads to the former result with small corrections, and for $A\ll1$ we have

$$L\to\mathscr{L}=\ln\frac{1}{2T\tau}-2\int_1^\infty\frac{dx}{(x+1)^2}\ln\ln x\approx\ln\frac{1}{2T\tau}+0.126. \tag{3.30}$$

Here we used the value of the integral given in Ref. 8 (p. 585).

Now, since instead of the operator $\hat{L}$ in Eq. (3.20) we have a constant, $\mathscr{L}$, there is no problem generalizing to a nonzero magnetic field. Collecting all the results, we arrive at an equation for the order parameter at a finite (low) temperature and a fairly weak magnetic field (the precise meaning of this is given below), which we write as

$$(\mathscr{H}+V)\Delta^+=E\Delta^+, \tag{3.31}$$
$$\mathscr{H}=\left(\frac{l}{2}\right)^2\begin{pmatrix}(\mathscr{L}-\frac{1}{2})\hat{\mathbf{p}}^2 & (\mathscr{L}-1)\hat{p}_-^2\\ (\mathscr{L}-1)\hat{p}_+^2 & (\mathscr{L}-\frac{1}{2})\hat{\mathbf{p}}^2\end{pmatrix},$$
$$E=\frac{\tau-\tau_c}{\tau_c}-\frac{2\pi^2}{3}(T\tau)^2,$$

where $\mathscr{L}$ is specified by (3.30) and V by (3.22). The condition that the magnetic field be weak, which corresponds to the condition $A\ll1$ for which (3.30) is valid, can be written as

$$|H|l^2\ll T\tau. \tag{3.32}$$

Below we state why precisely this condition is required.

4. THE PROPERTIES

We restrict our analysis to a linear equation for the order parameter; this proves sufficient, say, to establish the effect of the magnetic field H on the transition temperature T_c. We set $V=0$ in Eq. (3.31). We are interested in the lowest level $E_{\min}$ of the equation

$$\mathscr{H}\Delta^+=E\Delta^+, \tag{4.1}$$

since the transition point is determined by the relationship

$$E_{\min}=\frac{\tau-\tau_c}{\tau_c}-\frac{2\pi^2}{3}(T\tau)^2. \tag{4.2}$$

We use a gauge in which

$$\mathbf{A}=(-Hy,0) \tag{4.3}$$

and look for the solution of Eq. (4.1) in the form of a plane wave in x:

$$\Delta^+\sim\exp(ikx).$$

After this it is convenient to introduce the creation and annihilation operators a^+ and a of the Bose type by the ordinary formulas for a harmonic oscillator:

$$a=\frac{1}{2|H|^{1/2}}\left\{\hat{\mathscr{P}}_y-i2|H|\left(y-\frac{k}{2H}\right)\right\}. \tag{4.4}$$

We can express $\mathscr{H}$ in terms of these operators as follows:

$$\mathscr{H}\to l^2|H|\begin{pmatrix}(\mathscr{L}-\frac{1}{2})(a^+a+\frac{1}{2}) & -(\mathscr{L}-1)aa\\ -(\mathscr{L}-1)a^+a^+ & (\mathscr{L}-\frac{1}{2})(a^+a+\frac{1}{2})\end{pmatrix}. \tag{4.5}$$

This is true for $H>0$, while for $H<0$ we must simply interchange the off-diagonal elements. As expected, symmetry is present: a change in sign in the magnetic field interchanges the role of the components Δ_1 and Δ_2.

Equation (4.5) shows that two oscillator states are linked. The minimum energy is achieved for states with numbers 0 and 2. For these numbers we have

$$\frac{E_{\min}}{l^2|H|}=\frac{3}{2}\left(\mathscr{L}-\frac{1}{2}\right)-\sqrt{\left(\mathscr{L}-\frac{1}{2}\right)^2+2(\mathscr{L}-1)^2}. \tag{4.6}$$

Interestingly, for large values of $\mathscr{L}$ this level is negative, i.e., for

$$\mathscr{L}>\frac{11+2\sqrt{10}}{6}\approx2.887,$$

which corresponds to

$$T\tau<0.0316. \tag{4.7}$$

This means that according to (4.2), in a weak magnetic field the transition temperature grows to such values (enhancement of superconductivity). We also see that the transition temperature T_c can be much higher than the transition temperature in the absence of a magnetic field, $T_c^{(0)}$ [the reader will recall the applicability condition (3.32)]. In addition, a transition occurs in the supercritical region $\tau<\tau_c$ (stimulation of superconductivity).

The explanation of all this lies in the coupling of the components Δ_1 and Δ_2. As in any two-level system, because of the coupling the minimum value of the energy decreases, but no one could anticipate that this value would become negative, which is what matters. This is the property of the model employed in the present investigation. For other multicomponent models this effect can remain (and even get stronger), but it can also disappear—there is no way to know beforehand. In any case, this is a situation in which an experiment could be the decisive factor in choosing the model.

Thus, a weak magnetic field raises the transition temperature to the value

$$T_c^{(0)}\tau_c \approx 0.0316 \tag{4.8}$$

(see the condition (4.7)), which corresponds to a mean free time

$$\frac{\tau-\tau_c}{\tau_c} \approx 6.57\times 10^{-3} \tag{4.9}$$

(we have used (4.2) at $E_{\min}=0$), and at this point a weak magnetic field has no effect on the transition temperature.

Note that the smallness factors in (4.8) and (4.9) are numerical rather than parametric (no parameters correspond to them); the same was the case (to a lesser extent) in defining the width of the gapless region.

It would be interesting to establish the limit values $\tau<\tau_c$ at which stimulation of superconductivity is still possible. For this we take the limit opposite to (3.32):

$$l^2|H| \gg T\tau, \tag{4.10}$$

where the logarithm is an operator rather than a number. The equation has the same form (4.1) but with another "Hamiltonian":

$$\hat{\mathscr{H}}\Delta^+ = E\Delta^+, \tag{4.11}$$

where $\hat{\mathscr{H}}$ differs from $\mathscr{H}$ in that $\mathscr{L}$ is replaced by an operator $\hat{\mathscr{L}}$, which according to the results of Sec. 3 has the form

$$\hat{\mathscr{L}} = \ln \frac{2}{l^2\hat{\mathbf{p}}^2}, \tag{4.12}$$

where some ordering scheme for the operators must be chosen.

If we do not touch the logarithm in (4.12), the following alternatives are possible. For the diagonal elements in $\hat{\mathscr{H}}$,

$$\hat{\mathbf{p}}\hat{\mathscr{L}}\hat{\mathbf{p}} \quad \text{or} \quad \hat{\mathbf{p}}^2\hat{\mathscr{L}}. \tag{4.13}$$

For the off-diagonal elements, say, for the second element in the first row,

$$\hat{p}_-\hat{\mathscr{L}}\hat{p}_- \quad \text{or} \quad \tfrac{1}{2}\{\hat{p}_-^2\hat{\mathscr{L}}+\hat{\mathscr{L}}\hat{p}_-^2\}. \tag{4.14}$$

Here we have chosen the ordering that satisfies the above-mentioned symmetry property when H is replaced by $-H$. Of course, linear combinations of these variants are also possible.

After we have selected an operator-ordering scheme, we proceed as in the previous case, i.e., we express all quantities in terms of the creation and annihilation operators a^+ and a. For instance, for the operator $\hat{\mathscr{L}}$ of (4.12) we have

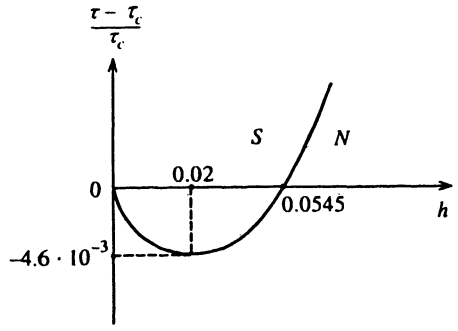

FIG. 1.

$$\hat{\mathscr{L}} = \ln \frac{1}{2l^2|H|(a^+a+\frac{1}{2})}. \tag{4.15}$$

It appears that the different alternatives lead to roughly the same results, since the logarithm is a rather slowly varying function. Here is the result for the first variants in (4.13) and (4.14) (as before, for oscillatory states with numbers 0 and 2). For $E_{\min}$ we have

$$E_{\min}/h = -\frac{3}{4}[\ln(7h)+\ln(3h)+1] - \left\{\left[\frac{3\ln(7h)+\ln(3h)+2}{4}\right]^2 + 2[\ln(3h)+1]^2\right\}^{1/2}, \tag{4.16}$$

where $h\equiv l^2|H|$ is the dimensionless magnetic field strength (the ratio of squares of two lengths, the mean free path and the magnetic length), with $h\ll 1$.

In the logarithmic approximation $|\ln h|\gg 1$ this equation yields

$$E_{\min}/h \to \left(\frac{3}{2}-\sqrt{3}\right)\ln\frac{1}{h}.$$

In this limit, we have a constant instead of the operator (4.15), and no problem of operator ordering in $\hat{\mathscr{H}}$ arises, so that there cannot be the slightest doubt in this result.

Figure 1 depicts a phase diagram in the τ,h plane at absolute zero (Eq. (4.2) at $T=0$). The curve represents the h-dependence of $E_{\min}$ specified by (4.16). The superconductive state is above the curve. Clearly, for $\tau<\tau_c$ there are two critical values of the magnetic field, and only between these values is there superconducting ordering (presumably as an Abrikosov lattice). The lowest boundary (in τ) of this region is

$$\frac{\tau_c-\tau}{\tau_c} \approx 4.6\times 10^{-3}. \tag{4.17}$$

Together with (4.9) this provides the total interval of enhancement and stimulation of superconductivity by a magnetic field, and in impurity concentration the interval amounts to about 1% of the critical concentration.

Note that the second variants in (4.13) and (4.14) do not lead to dramatic changes in the results. For instance, the point of intersection of the curve in Fig. 1 with the horizontal axis shifts to the right by only 10%.

Let us estimate the transition temperature T_c at $\tau=\tau_c$. This quantity depends on the magnetic field strength. The peak value of T_c, obtained from (4.2) and (4.16), is reached at $h\approx 0.02$ and amounts to

$$T_c\tau_c\approx 2.64\times 10^{-2}. \tag{4.18}$$

This is only an estimate, since the value is obtained at the limit of applicability of the result (4.16), which is valid if the condition (4.10) is met.

This ends the discussion of the properties of the model. Some aspects, for instance items 2 and 3 in the Introduction, have been left out. Together with an estimate of the contribution of the intrinsic degrees of freedom, these aspects require a separate study.

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