An exact expression for magnetic oscillations in quantum electrodynamics

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One-loop corrections to the effective action in three- and four-dimensional electrodynamics in external magnetic field H at a chemical potential μ and zero temperature are discussed. Exact expressions for the oscillating component of the thermodynamic potential of relativistic electrons have been obtained in both cases at $H,\mu \neq 0$. Unlike the nonrelativistic case, the oscillation frequency is $\omega=(\mu^2-m^2)/(2eH)$, where m and e are the electron mass and charge. The exact solution allows us to calculate corrections to magnetic oscillations of higher orders in magnetic field. © *1996 American Institute of Physics.* [S1063-7761(96)01903-8]

1. INTRODUCTION

Studying condensed matter is an important branch of physics.[1] Most parameters of matter are weakly affected by changes in external agents, such as temperature, pressure, electromagnetic field, etc., when they vary within ranges far from extreme values, i.e., under laboratory conditions. Therefore, each event when a small change in external parameters leads to a considerable response is remarkable and means a discovery of a significant effect. We can mention as examples the quantum Hall effect[2] and oscillations of magnetic moment predicted in 1930 by Landau[3] and experimentally observed by de Haas.[4]

Sixty years have passed since the discovery of this effect, but to this day the problem of quantum oscillations solved in the nonrelativistic case (in the main asymptotic approximation)[3] has been discussed extensively.[5–9] We should note that most attention of researchers dealing with oscillations of magnetic moment is focused on the relativistic problem since results of these studies may be applied to cosmology and astrophysics.

In solving the relativistic problem some authors[5] have overlooked oscillations, whereas solutions derived by others[6] only contain formulas equivalent to the nonrelativistic oscillating solution,[3] and the specific nature of the relativistic limit had not been assessed. Previously[7,8] a strict mathematical procedure for calculating the thermodynamic potential of relativistic electron–positron gas in magnetic field has been developed to calculate the thermodynamic potential in all ranges of fields and temperatures. As a result, a new parameter of oscillations, $T/\sqrt{eH}$, was derived.[7] The structure of formulas in Refs. 7 and 8, however, is such that thermodynamic parameters are easily calculated only at $\mu\sim m$, where m is the electron mass, and if $\mu\gg m$, a power series in the parameter μ/m, which is not small, should be summed. This procedure is labor-consuming, therefore we proposed a new method in this paper to get around this difficulty.

In order to solve the problem in the four-dimensional quantum electrodynamics (QED), i.e., to obtain an exact oscillating solution at all magnetic fields and chemical potentials, we shall first consider the three-dimensional case. Thereby we shall demonstrate merits of low-dimensional models (earlier similar models were used in the Yang–Mills theory to construct solutions of field equations[10]). In the second section we shall obtain exact expressions for the one-loop correction to the effective action in QED_3 at zero temperature within the formal model to derive oscillations of magnetic moment at all μ and H. In the third section we shall derive from this solution an exact expression for the oscillating part of the effective action in QED_4 at $T=0$ and $\mu,H\neq 0$.

2. MAGNETIC MOMENT OSCILLATIONS IN QED_3

As we noted above, the calculation of the effective action in QED_3 at $T=0$ and nonzero chemical potential and magnetic field is an auxiliary problem in our study, nonetheless, this problem is interesting in itself because the QED_3, along with other three-dimensional theories, can be used to describe various planar systems. Therefore we consider in this section the formal QED_3 model to derive oscillations of magnetic moment of an ideal electron gas. The model Lagrangian in both three- and four-dimensional space-time has a form

$$L=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\bar{\psi}(i\hat{\partial}+m)\psi. \quad (1)$$

In the three-dimensional case we use a formalism with four-component Dirac's spinors ψ and

$$\gamma^\mu=\text{diag}(\tilde{\gamma}^\mu,-\tilde{\gamma}^\mu), \quad \tilde{\gamma}^0=\sigma_3,\tilde{\gamma}^{1,2}=i\sigma_{1,2}.$$

Here σ_k are the Pauli matrices, and the electromagnetic field tensor $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$.

Equation (1) has been discussed in detail.[7,11–13] In some previous studies QED_3 problems were investigated under nonzero magnetic field H,[12] temperature T, and chemical potential μ.[13] Earlier[7] we calculated the effective action in QED_3 in the one-loop approximation at $T,\mu,H\neq 0$. Now let us assume that the temperature is zero and investigate in detail the one-loop correction to the QED_3 effective action at $\mu,H\neq 0$, which can be derived from Eq. (17) in Ref. 7 by tending the temperature to zero. As a result, we have

$$\frac{1}{V}S_1(\mu,H)=L_1(H)-\Omega_3(\mu,H,m), \quad (2)$$

where

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$$\Omega_3(\mu,H,m)=-\frac{eH}{2\pi}\sum_{n=0}^{\infty}\alpha_n\Theta(\mu-\epsilon_n)(\mu-\epsilon_n) \qquad (3)$$

is the thermodynamic potential of the ideal relativistic electron gas under magnetic field at $\mu\neq 0$, $\alpha_n=2-\delta_{n0}$, $V=\int d^3x, \epsilon_n=\sqrt{m^2+2eHn}$, $\Theta(x)$ is the theta function, and

$$L_1(H)=-\frac{1}{4\pi^{3/2}}\int_0^\infty\frac{dx}{x^{5/2}}\exp(-xm^2)[eHx\cdot\coth(eHx)-1] \qquad (4)$$

is the one-loop correction at $\mu=0, H\neq 0$.[7,12]

Equation (2) is only convenient in the range of high magnetic field. Really, if $eH>\tilde{m}^2\equiv\mu^2-m^2$, only the first term in Eq. (2) is not zero, at $\tilde{m}^2/2<eH<\tilde{m}^2$ the second term should be added, and so on.

But in the range of small magnetic fields the parameter $\Omega_3(\mu,H,m)$ is an oscillating function of H,[14] which cannot be easily derived from Eq. (2). Equations (2) and (3) are not quite convenient to extract relevant physical information. In order to resolve this problem, let us apply Poisson's summation formula[3] to Eq. (2):

$$\sum_{n=0}^{\infty}\alpha_n\Phi(n)=2\sum_{k=0}^{\infty}\alpha_k\int_0^\infty\Phi(x)\cos(2\pi kx)dx. \qquad (5)$$

In this case $\Phi(n)=\Theta(\mu-\epsilon_n)(\mu-\epsilon_n)$, therefore integration in Eq. (5) is easy, and the thermodynamic potential takes the form

$$\Omega_3(\mu,H,m)=-L_1(\mu)-\frac{\Theta(\mu-m)}{\sqrt{2\pi}}\sum_{k=1}^{\infty}\left(\frac{eH}{\pi k}\right)^{3/2}\times\{\cos(\pi ku)[S(\pi kv)-S(\pi ku)]-\sin(\pi ku)[C(\pi kv)-C(\pi ku)]\}, \qquad (6)$$

where $u=m^2/(eH), v=\mu^2/(eH)$, $L_1(\mu)$ is the one-loop correction to the effective Lagrangian of the model at $H=0,\mu\neq 0$:

$$L_1(\mu)=-\frac{1}{6\pi}\Theta(\mu-m)(\mu-m)^2(\mu+2m),$$

and the functions $C(x)$ and $S(x)$ are Fresnel's integrals[15]:

$$\begin{pmatrix}C(x)\\ S(x)\end{pmatrix}=\frac{1}{\sqrt{2\pi}}\int_0^x x^{-1/2}\begin{pmatrix}\cos x\\ \sin x\end{pmatrix}dx. \qquad (7)$$

Then we use the following representations[15]:

$$C(x)=\frac{1}{2}+\sqrt{\frac{x}{2\pi}}[P(x)\sin x+Q(x)\cos x],$$
$$S(x)=\frac{1}{2}-\sqrt{\frac{x}{2\pi}}[P(x)\cos x-Q(x)\sin x], \qquad (8)$$

where the functions P and Q have known asymptotics at $x\to\infty$[15]:

$$P(x)=x^{-1}-\frac{3}{4}x^{-3}+\ldots,$$
$$Q(x)=-\frac{1}{2}x^{-2}+\frac{15}{8}x^{-4}+\ldots. \qquad (9)$$

Given the thermodynamic potential in Eq. (6), we can easily separate its oscillating component using Eq. (8):

$$\Omega_3^{\rm osc}=-eH\mu\Theta(\mu-m)\sum_{k=1}^{\infty}\left\{\frac{Q(\pi kv)}{2\pi^2k}\sin(2\pi k\omega)-\frac{P(\pi kv)}{2\pi^2k}\cos(2\pi k\omega)\right\}, \qquad (10)$$

which is a periodic function of the parameter $\omega\equiv(\mu^2-m^2)/(2eH)$ with the unit period. This is the exact relativistic formula for the oscillating component of the one-loop effective action valid at arbitrary μ and H. The first terms of the sum in Eq. (10) can be found using Eq. (9) at $eH\to 0$ and fixed μ and m:

$$\Omega_3^{\rm osc}=-\frac{\Theta(\mu-m)(eH)^2}{2\pi\mu}\left\{-B_2(\omega)-\frac{eH}{3\mu^2}B_3(\omega)+\frac{(eH)^2}{16\mu^4}B_4(\omega)+o((eH)^2)\right\}. \qquad (11)$$

Here $B_i(x)$ are periodic functions with the unit period presented on the interval $x\in[0,1]$ by Bernoulli polynomials[15]:

$$B_2(x)=x^2-x+1/6,\quad B_3(x)=x^3-3x^2/2+x/2;$$
$$B_4(x)=x^4-2x^3+x^2-1/30.$$

Note that Eq. (9) is identical to the respective equation in Ref. 7 at $\mu\sim m$. Finally, we present an exact expression for the monotonic part of the thermodynamic potential:

$$\Omega_3-\Omega_3^{\rm osc}=-L_1(\mu)-m\Theta(\mu-m)eH\times\sum_{k=1}^{\infty}P(\pi ku)/(2\pi^2k). \qquad (12)$$

Given the exact expression for $S_1(\mu,H)$, one can easily derive the magnetization $M\sim\partial S_1/\partial H$, which is obviously an oscillating function of ω (the van Alphen–de Haas effect).[1,3,4]

3. MAGNETIC OSCILLATIONS IN QED$_4$

Now we shall consider the one-loop correction to the effective action in four-dimensional electrodynamics. Let us temporarily introduce the temperature $T\equiv 1/\beta$, then we have (V is the system volume):

$$\frac{1}{\beta V}S_1(T,\mu,H)=L_1(H)-\Omega_4(\mu,T,H), \qquad (13)$$

where[16]

$$L_1(H)=-\frac{1}{8\pi^2}\int_0^\infty \frac{ds}{s^3}\left[esH\coth(eHs)-1-\frac{1}{3}(esH)^2\right]\exp(-m^2 s).$$

In Eq. (13) Ω_4 is the density of the thermodynamic potential of ideal relativistic electron–positron gas under magnetic field, which can be presented as[8,17]

$$\Omega_4(\mu,T,H)=-\frac{eHT}{2\pi^2}\int_0^\infty dp\sum_{n=0}^\infty \alpha_n \ln\{[1+\exp(-\beta(\epsilon_n-\mu))][1+\exp(-\beta(\epsilon_n+\mu))]\}. \tag{14}$$

Here $\varepsilon_n=\sqrt{m^2+p^2+2eHn}$, α_n are the same as in Eq. (3). By tending the temperature in Eq. (14) to zero, we obtain

$$\Omega_4(\mu,H)=-\frac{eH}{2\pi^2}\int_0^\infty dp\sum_{n=0}^\infty \alpha_n\Theta(\mu-\epsilon_n)(\mu-\epsilon_n)=\frac{1}{\pi}\int_0^\infty dp\,\Omega_3(\mu,H,\sqrt{m^2+p^2}), \tag{15}$$

where the thermodynamic potential Ω_3 is defined in Eq. (3). Our task is to derive from Eq. (15) an exact expression for the oscillating component of the thermodynamic potential at $eH\to 0$. With this end in view, we substitute Ω_3 from Eq. (6) to Eq. (15) and integrate with respect to the momentum taking into account Eq. (7). As a result, we have

$$\Omega_4(\mu,H)=\tilde{\Omega}_{\rm mon}+\tilde{\Omega}_{\rm osc}, \tag{16}$$

where

$$\tilde{\Omega}_{\rm mon}=-\frac{eH\Theta(\mu-m)}{2\pi^3}\int_m^\mu \frac{x\,dx}{\sqrt{x^2-m^2}}\left\{\frac{\pi}{3eH}(\mu-x)^2(\mu+2x)+x\sum_{k=1}^\infty \frac{1}{k}P(\pi kx/eH)\right\}, \tag{17}$$

$$\tilde{\Omega}_{\rm osc}=-\frac{\mu\Theta(\mu-m)}{(2\pi)^{3/2}}\sum_{k=1}^\infty\left(\frac{eH}{\pi k}\right)^{3/2}\{Q(\pi kv)\times[\sin(2\pi k\omega)C(2\pi k\omega)-\cos(2\pi k\omega)S(2\pi k\omega)]-P(\pi kv)[\cos(2\pi k\omega)C(2\pi k\omega)+\sin(2\pi k\omega)S(2\pi k\omega)]\}. \tag{18}$$

Note that $\tilde{\Omega}_{\rm osc}$ contains not only the oscillating component of the thermodynamic potential, but Eq. (18) also includes a monotonic component. It is evident that $\tilde{\Omega}_{\rm osc}$ in Eq. (17) does not oscillate at $eH\to 0$. In order to separate the oscillating component in Eq. (18), let us first separate in the explicit form oscillating components in Fresnel integrals $C(x)$ and $S(x)$ using Eq. (8). After substituting Eq. (8) into Eq. (18), we have

$$\Omega_4(\mu,H)=\Omega_{\rm osc}+\Omega_{\rm mon}. \tag{19}$$

Here

$$\Omega_{\rm mon}=\tilde{\Omega}_{\rm mon}-\frac{\mu eH\Theta(\mu-m)}{4\pi^3}\sqrt{\mu^2-m^2}\times\sum_{k=1}^\infty\frac{1}{k}\{Q(\pi kv)P(2\pi k\omega)-P(\pi kv)Q(2\pi k\omega)\}, \tag{20}$$

$$\Omega_{\rm osc}=\frac{\mu\Theta(\mu-m)}{4\pi^{3/2}}\sum_{k=1}^\infty\left(\frac{eH}{\pi k}\right)^{3/2}[Q(\pi kv)\cos(2\pi k\omega+\pi/4)+P(\pi kv)\cos(2\pi k\omega-\pi/4)]. \tag{21}$$

The parameters v and ω are the same as in Eqs. (6) and (10).

Equation (21) is an exact expression for oscillations of the magnetic moment of ideal electron gas generalized for the relativistic case at $T=0$. The main difference of the relativistic case from the nonrelativistic approximation is the oscillation frequency $\omega=(\mu^2-m^2)/(2eH)$ (in the nonrelativistic case the frequency is $\omega_0=m(\mu-m)/(eH)$). It was calculated at arbitrary μ and H and at $H\to 0$, $\mu\sim m$ it, naturally, coincides with the nonrelativistic expression.[3,6] Since properties of the functions $P(x)$ and $Q(x)$ in Eq. (21) are well known[15] (see Eq. (9)), Eq. (21) allows one to calculate corrections of higher orders in H to the oscillating part of the thermodynamic potential, unlike the cumbersome procedure proposed earlier[7,8] (but field corrections to the oscillating component at $T\neq 0$ can be calculated presently using only this procedure).

4. CONCLUSION

In our study we have derived and analyzed the thermodynamic potential of ideal electron gas under magnetic field at zero temperature in both three- and four-dimensional cases. In both cases exact expressions for the oscillating part of the thermodynamic potential at $eH\to 0$ have been obtained. The feature of our study is the analysis of the wide range of chemical potential μ, as a result we obtained the frequency of oscillations $\omega=(\mu^2-m^2)/(2eH)$, which is different from that derived in the nonrelativistic approximation, $\omega_0=m(\mu-m)/(eH)$, at $\mu\sim m$.[7,8]

In terms of QED, our results correspond to the one-loop correction to the effective action. Therefore the range of its applicability is defined by that of the one-loop approximation, which is determined by the condition that the two-loop correction, given in Refs. 18 and 19, is small. Note, however, that the oscillating component in Ref. 18 also refers to the case $\mu\sim m$.

Thus we can see that the exact calculation of the thermodynamic potential at zero temperature yields different frequencies of magnetic moment oscillations. Although the model considered in this paper is rather formal, the calculation technique and solution may be useful for studies of real physical systems.

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