General instability of a relativistically intense electromagnetic wave in matter

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It is shown on the basis of an analysis of the nonlinear wave equation that the filamentation and modulation instability of a plane wave must be considered together. The general instability growth rates $g(k_\perp, k_\|)$ of a relativistically intense plane wave in matter are calculated. It is shown that the instability region in the k plane is bounded. The energies of the individual peaks into which the original wave breaks up have a lower bound. © *1996 American Institute of Physics.* [S1063-7761(96)02903-4]

1. INTRODUCTION

An intense electromagnetic wave experiences filamentation and modulation instability in matter.[1] Filamentation instability is manifested by collapse of the radiation into one or more filaments. As a result of modulation instability, a laser pulse propagating along an individual filament is transformed into a sequence of peaks. This instability is sometimes called the longitudinal–transverse or general instability of a plane wave. The transverse and longitudinal instabilities are not separable in the general case and must be considered together, although this circumstance is usually ignored in model descriptions of nonlinear wave propagation.

The filamentation instability of a plane wave in a Kerr medium was considered by Bespalov and Talanov,[2] who employed the linear theory of the stability of small perturbations superposed on a plane wave and described the propagation of the latter using the nonlinear Schrödinger (NLS) equation. At high laser output powers the leading edge of a pulse ionizes atoms of the medium, and the bulk of the pulse propagates in a plasma consisting of multiply-charged ions and free electrons. The plasma exhibits significant relativistic–ponderomotive nonlinearity due to the increase in the mass of the free electrons, which oscillate with velocities close to that of light in the intense field, and to the ejection of electrons from the region of the intense field by ponderomotive forces. The filamentation instability of a plane wave in a medium with relativistic and relativistic–ponderomotive electron nonlinearity was studied by analyzing solutions of the corresponding NLS equation in Ref. 3.

Under fairly general assumptions the propagation of a short laser pulse in matter can be described by a nonlinear wave equation with relativistic–ponderomotive nonlinearity.[4] The NLS equation model is a special case of the nonlinear wave equation model, and is obtained by neglecting several second derivatives in the diffraction operator of the nonlinear wave equation in comoving variables, which corresponds to an incomplete description of the wave properties of the radiation along the propagation coordinate. As a consequence, the NLS equation correctly describes the wave properties of radiation in the transverse direction and incorrectly describes these properties in the longitudinal direction. Therefore, the NLS equation can be used to analyze filamentation instability, but this model is ill-suited to the analysis of modulation instability. Thus, the use of the NLS equation to describe the general instability of a plane wave is questionable.

In the present work we analyze the general instability of an intense electromagnetic plane wave in matter. Linear instability theory is employed to investigate solutions of the nonlinear wave equation with relativistic–ponderomotive electron nonlinearity. The growth rates of general instability are determined numerically. The analysis of general instability makes it possible to understand the essential features of the filamentation process alone and the temporal modulation of ultrashort laser pulses in matter.

2. BASIC EQUATION

The nonlinear wave equation describing the propagation of ultrashort intense laser pulses in a medium with relativistic–ponderomotive electron nonlinearity was considered in Refs. 4 and 5 and has the following form:

$$(v_g^{-1}\partial_t+\partial_z)a+\frac{i}{2k}[\Box+k_p^2(1-\gamma^{-1}(1+k_p^{-2}\Delta\gamma))]a=0. \tag{1}$$

Here $a(t,r)$ is the complex amplitude of the electromagnetic vector potential, $k^2=k_0^2-k_p^2$, $k_0=\omega_0/c$ is the wave number of laser output at frequency ω_0, $k_p=\omega_{p,0}/c$ is the reciprocal length of the skin layer, and $\omega_{p,0}=\sqrt{4\pi e^2 N_{e,0}/m_{e,0}}$ is the plasma frequency in the unperturbed plasma. As was shown in Ref. 4, the expressions for the relativistic γ factor of circularly and linearly polarized waves differ, but are close to the model expression $\gamma^2=1+|a|^2$. We shall use the model expression for the γ factor below. In Eq. (1) the d'Alembertian operator describes diffraction, and the following term describes refraction with consideration of the relativistic and ponderomotive effects.

Equation (1) is usually considered in the comoving variables

$$\xi=v_g t-z, \quad \tau=t. \tag{2}$$

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It then takes the following form:

$$v_g^{-1}\partial_\tau a=\frac{i}{2k}[\Delta_\perp a+(1-v_g^2c^{-2})\partial_{\xi\xi}^{-2}a-2v_gc^{-2}\partial_{\xi\tau}^2 a$$
$$-c^{-2}\partial_{\tau\tau}^2 a+k_p^2[1-\gamma^{-1}(1+k_p^{-2}\Delta\gamma)]a)=0. \quad (3)$$

We introduced the parameters

$$\alpha=1-v_g^2c^{-2}, \quad \beta=\sqrt{\alpha(1-\alpha)}=k_pk/k_0^2, \delta=\alpha/4=k_p^2/4k_0 \quad (4)$$

and the dimensionless constants

$$\tau_0=\frac{2k}{v_gk_p^2}, \quad r_0=\xi_0=k_p^{-1}. \quad (5)$$

In the dimensionless variables (we shall use the previous notations τ, r, and ξ) the starting nonlinear wave equation for subsequent analysis takes the following form:

$$\partial_\tau a+i[\Delta_\perp a+\alpha\partial_{\xi\xi}^2 a-\beta\partial_{\xi\tau}^2 a-\delta\partial_{\tau\tau}^2 a+(1-\gamma^{-1}(1$$
$$+\Delta\gamma))a]=0. \quad (6)$$

Equation (6) has a plane wave solution:

$$a=a_0e^{i(\omega_0\tau+\mathbf{k}_0\mathbf{r})}, \mathbf{r}=(\mathbf{r}_\perp,\xi), \quad (7)$$

where ω_0 and $\mathbf{k}_0$ satisfy the dispersion relation

$$\omega_0+(-k_{0,\perp}^2-\alpha k_{0,\parallel}^2+\beta k_{0,\parallel}+\delta\omega_0^2+(1-\gamma_0^{-1}))=0. \quad (8)$$

3. EQUATION FOR SMALL PERTURBATIONS

Let us consider the evolution of small perturbations superposed on the plane wave described by (7) and (8):

$$a=a_0e^{i(\omega_0\tau+\mathbf{k}_0\mathbf{r})}+\varphi(\tau,\mathbf{r}). \quad (9)$$

The substitution of (9) into (6), keeping only terms linear in φ, leads to the following linear equation, which describes the evolution of small perturbations superposed on the plane wave:

$$\partial_\tau\varphi+i(\Delta_\perp\varphi+\alpha\partial_{\xi\xi}^2\varphi-\beta\partial_{\xi\tau}^2\varphi-\delta\partial_{\tau\tau}^2\varphi-\mu\Delta(\varphi+\varphi^*)$$
$$+\nu(\varphi+\varphi^*))=0. \quad (10)$$

In deriving (10) we used the expansion

$$\gamma^{-1}(1+\Delta\gamma)\simeq\gamma_0^{-1}+\mu\Delta(\varphi^*+\varphi)-\nu(\varphi^*+\varphi) \quad (11)$$

and introduced the following notation:

$$\mu=\frac{a_0^2}{2(1+a_0^2)}, \quad \nu=\frac{a_0^2}{2(1+a_0^2)^{3/2}}. \quad (12)$$

Note that the parameters β and δ in (1) can be expressed in terms of α via (4); therefore, Eq. (10) contains only two external parameters: α, which is a characteristic of the medium, and a_0^2, which is the intensity of the external field.

The linear differential equation (10) can be solved by the Fourier–Laplace method. After application of the inverse Laplace transformation, the asymptotic solution for $t\to\infty$ takes the following form:

$$\varphi_{\mathbf{k}}(t\to\infty)=\sum_j R_je^{-i\omega_jt}, \quad (13)$$

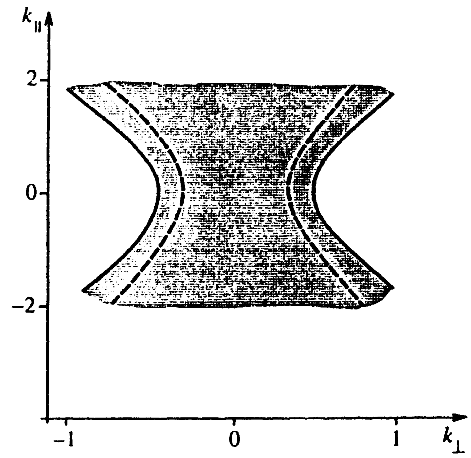

FIG. 1. Instability region of the NLS equation in the $(k_\perp,k_\parallel)$ plane and maximum growth rate curve (dashed line) for $k_\parallel$=const. The variables k are plotted in units of $2(\nu/(1-2\mu))^{1/2}$.

where the R_j are the residues at the poles and the ω_j are the roots of the dispersion equation

$$\omega^2-(\delta\omega^2-\alpha k_\parallel^2+\beta\omega k_\parallel-k_\perp^2)(\delta\omega^2-\alpha k_\parallel^2+\beta\omega k_\parallel-k_\perp^2$$
$$+2(\mu k^2+\nu))=0. \quad (14)$$

Thus, according to (13), the growth rate of small perturbations superposed on the plane wave at a given point $(k_\perp,k_\parallel)$ can be naturally understood to be equal to the maximum positive value of the imaginary part of the root of the dispersion equation (14).

4. CALCULATION OF THE GROWTH RATE OF GENERAL INSTABILITY

Modifications of the NLS equation, i.e., approximations of the nonlinear wave equation, have sometimes been considered in the literature. The choice of a particular approximation can be made by formally setting one or more of the constants α, β, and δ to zero.

4.1. NLS equation approximation with $\alpha=\beta=\delta=0$

The dispersion equation for the NLS equation takes the form[3]

$$\omega^2=k_\perp^2(k_\perp^2-2(\mu k^2+\nu)), \quad (15)$$

and its solution is

$$\omega_{1,2}=\pm k_\perp\sqrt{k_\perp^2(1-2\mu)-2(\mu k_\parallel^2+\nu)}. \quad (16)$$

The instability region can be determined by requiring that the expression under the radical be negative:

$$k_\perp^2<\frac{2\mu k_\parallel^2+2\nu}{1-2\mu}. \quad (17)$$

Since $1-2\mu=(1+a_0^2)^{-1}>0$, the characteristic region of instability for the NLS equation is open to ∞ and is located between two curves in the $(k_\perp,k_\parallel)$ plane (see Fig. 1).

When $k_\parallel$=const, the maximum growth rate is attained when

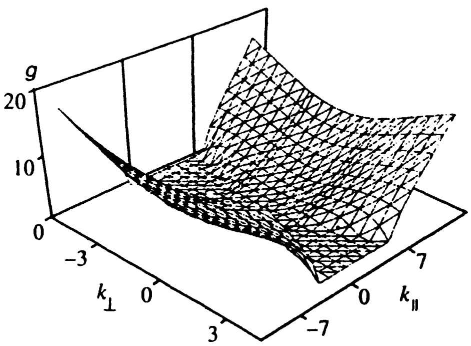

FIG. 2. Growth rate of the modified NLS equation ($\alpha \neq 0$, $\beta=\delta=0$) as a function of the two variables $k_\perp$ and $k_\parallel$ at a large intensity value $a_0=1.0$.

$$k_\perp^2=\frac{\mu k_\parallel^2+\nu}{1-2\mu}. \tag{18}$$

When $k_\perp=\text{const}$, the growth rate increases without bound as $|k_\parallel|$ increases. The curves corresponding to the condition (18) are shown in Fig. 1 in the form of dashed lines. Filamentation instability appears when $k_\parallel=0$. It corresponds to the maximum value of the growth rate, which is achieved when

$$k_\perp^2=\frac{\nu}{1-2\mu}. \tag{19}$$

The corresponding values of the growth rate and the threshold filamentation power were calculated in Ref. 3.

4.2. Modified NLS equation with $\alpha \neq 0$ and $\beta=\delta=0$

In this case the instability region is described by the following inequality:

$$k_\perp^2<\frac{(2\mu-\alpha)k_\parallel^2+2\nu}{1-2\mu}. \tag{20}$$

At small intensities, at which $2\mu-\alpha<0$, the boundary of the instability region is an ellipse extended along the $k_\parallel$ axis. At an intensity value which satisfies the condition $2\mu-\alpha=0$, the instability region transforms into a band parallel to the $k_\parallel$ axis. At high intensities $(2\mu-\alpha>0)$ the instability region is close to that found in the preceding section for the NLS equation. We note again that the unbounded increase in the growth rate for $|k_\parallel|\to\infty$ (see Fig. 2) is a consequence of the inadequacy of the NLS equation model for describing modulation instability.

4.3. Modified NLS equation with $\alpha \neq 0$, $\beta=0$, and $\delta \neq 0$

In this case the dispersion equation takes the form of a biquadratic equation with a positive coefficient of ω^4. In this case a sufficient condition for being in the instability region is a negative value for the free term in the biquadratic equation. This condition is equivalent to the condition (20) or (17) when $\alpha=0$. Thus, the instability region is no narrower than the instability region of the preceding two sections.

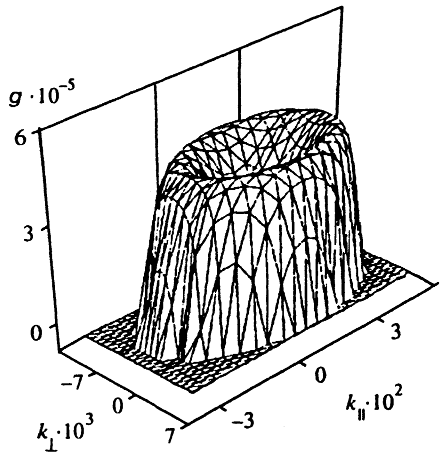

FIG. 3. Growth rate of the nonlinear wave equation ($\alpha \neq 0$, $\beta \neq 0$, $\delta \neq 0$) as a function of the two variables $k_\perp$ and $k_\parallel$ at a low intensity value $a_0=0.1$.

4.4. Modified NLS equation with $\alpha=\delta=0$ and $\beta \neq 0$

In this case the condition for being in the instability region is a negative value for the discriminant of the quadratic dispersion equation, which is equivalent to the condition

$$\beta^2\frac{k_\parallel^2}{k_\perp^2}(\mu k^2+\nu)+\frac{k_\perp^2}{\mu k^2+\nu}<2. \tag{21}$$

It follows from the condition (21) that the instability region is bounded.

4.5. Variant of the nonlinear wave equation with $\alpha \neq 0$, $\beta \neq 0$, and $\delta \neq 0$

Analysis of the complete problem is of the greatest interest. The growth rate can be determined numerically. The most common case for the roots of the dispersion equation (14) in the instability region is the situation with two real roots and two complex-conjugate roots, although a variant with two pairs of complex-conjugate roots is considered in the algorithm. Denoting the real roots by ω_1 and ω_2 and the imaginary roots by $\omega_3\pm ig$, we have, according to Vieta's theorem (comparison of the coefficients of ω^3 and the free term):

$$2\beta k_\parallel=-\delta^2(\omega_1+\omega_2+2\omega_3), \tag{22}$$

$$(\alpha k_\parallel^2+k_\perp^2)(\alpha k_\parallel^2+k_\perp^2-2(\mu k^2+\nu))=\omega_1\omega_2(\omega_3^2+g^2)\delta^2. \tag{23}$$

At low intensities the real roots equal $\omega_{1,2}=\pm\delta^{-1}$ to high accuracy, and therefore the values of the real part ω_3 of the complex-conjugate roots and the growth rate g can be obtained analytically from (22) and (23). At low intensities the expressions (22) and (23) (for example, when $\beta=0$) pass continuously into variant 4.2 with the instability region (20). The relations (22) and (23) can also be used at high intensities, but the values of $\omega_{1,2}$ are determined numerically. The results of the calculations of the growth rate for low and high intensity values are presented in Figs. 3 and 4.

At low intensities the maximum of the growth rate as a function of $k_\perp$ and $k_\parallel$ is achieved on a curve having the form of an ellipse. The ellipse extends along the $k_\parallel$ axis, and the

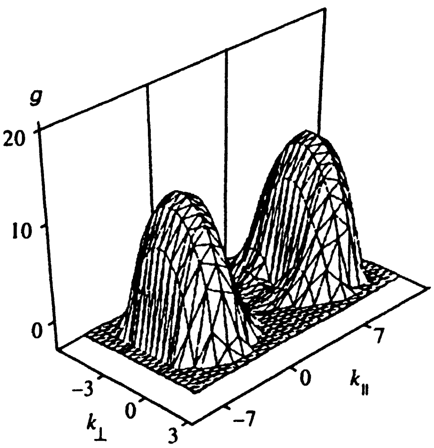

FIG. 4. Growth rate of the nonlinear wave equation ($\alpha \neq 0, \beta \neq 0, \delta \neq 0$) as a function of the two variables $k_\perp$ and $k_\parallel$ at a high intensity value $a_0 = 1.0$.

situation for the general case is very similar to variant 4.2. This agreement reflects the similarity between the properties of the solutions of the NLS equation and the nonlinear wave equation in the initial time segment. As the intensity increases, gaps appear along the $k_\perp$ axis, which separate the closed curve of the maximum growth rate values into two parts: upper and lower half-planes. The asymmetry introduced by the term with the coefficient β is scarcely noticeable.

One interesting feature of the solution obtained is the appearance of lacunae, which can be observed in Fig. 4. These lacunae are located on the $k_\parallel$ axis, i.e., at $k_\perp = 0$. The lacunae appear over a finite range of radiation intensities. Their presence signifies the possible physical existence of a longitudinally modulated plane wave propagating without damping and amplification in a nonlinear medium. This wave, incidentally, must be unstable against small perturbations.

One significant feature is the finiteness of the instability region of the nonlinear wave equation, which means that both longitudinal and transverse short-wavelength perturbations of a plane wave are stable. The type of growth rate obtained makes it possible to establish several special features of the general instability of a plane wave in matter. Instabilities with $k_\perp$ and $k_\parallel$ corresponding to the maximum growth rate values appear under the conditions of a real experiment. For example, if filamentation with a characteristic length of the transverse inhomogeneity $\lambda_\perp = 2\pi k_\perp^{-1}$ appears, the laser pulse propagating along the filament breaks up into longitudinal peaks with a length $\lambda_\parallel = 2\pi k_\parallel^{-1}$, where $k_\parallel$ corresponds to the value of the maximum growth rate along the $k_\perp = \text{const}$ line. Thus, the number of peaks into which the pulse breaks up is related to the size of the transverse inhomogeneity and to the radiation intensity.

The quasiperiodic evolution of the solution of the nonlinear wave equation[5,6] can be explained on the basis of the results obtained. Upon the appearance of the first focus, the radiation intensity reaches values at which modulation instability begins to develop, the focus breaks up into several peaks of lower intensity, and the focusing mechanism of the ordinary nonlinear wave equation model begins to operate again. Thus, the fundamental difference between the nonlinear wave equation and the NLS equation is the modulation, rather than the relativistic–refraction, mechanism for breakup of a focus.

One consequence of the fact that the maximum growth rate curve in the $k_\perp, k_\parallel$ plane does not pass through the origin is the lower bound imposed on the energy concentrated in a single peak. In fact, a qualitative analysis shows that the energy of the electromagnetic field concentrated in a parallelepiped with lateral edges $\lambda_\perp$ and $\lambda_\parallel$ reaches a minimum at certain values $\lambda_\perp^{\min}$ and $\lambda_\parallel^{\min}$. Therefore, the energy concentrated in a single peak cannot be arbitrarily low when general instability of a plane wave of prescribed intensity appears.

The growth rates $g(k_\perp)$ and $g(k_\parallel)$ of the general instability of a plane wave in a medium with a relativistic–ponderomotive nonlinearity have been investigated in this paper. It has been shown that the instability region is bounded at any intensity of the original wave. The distributions of the growth rate are different at low and high intensities. Filamentation and modulation instabilities are related, and they must be considered in concert. The general instability can be described in the nonlinear wave equation model and cannot be described in approximations of the NLS equation which do not take into account the temporal dispersion. A modulation mechanism for breaking up a focus into a finite number of peaks has been described for the self-channeling of a laser pulse in matter.

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