Structure function of the velocity field in turbulent flows

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The exponents of the velocity structure function in turbulent flows are found on the basis of a semiempirical model. An assumption that these exponents are not self-similar with respect to the Reynolds number makes it possible to account for the disparities between the results of known experimental studies and to achieve good agreement between a model probability distribution of the dissipation breakdown coefficient and experiment. © *1996 American Institute of Physics.* [S1063-7761(96)03003-3]

1. INTRODUCTION

There is presently no correct mathematical formulation of the "problem of turbulence," i.e., the problem of obtaining a statistical description of developed turbulent flows. The need to adopt "closing" hypotheses arose in the theoretical study of the problem.[1] This is because the systems of exact (following from the equations of fluid mechanics) equations for the statistical characteristics of turbulence always contain a number of unknowns exceeding the number of equations. Most of the theories in statistical hydrodynamics are, therefore, phenomenological to some extent. Since it is generally difficult to directly verify closing hypotheses, the correctness of models is evaluated by comparing conclusions derived from them with a physical or numerical experiment.

The ideas of the theory of locally isotropic turbulence proposed by Kolmogorov and Obukhov[2,3] in 1941 have turned out to be extremely fruitful in the development of statistical hydrodynamics. As measurement techniques were subsequently perfected, experimental results were obtained which could be explained only after refinement of the classical theory.[2,3] Such refinements were needed primarily because of the extremely intermittent, temporally and spatially inhomogeneous structure of the small-scale fields in turbulent flows.[1] Small-scale intermittency can be characterized, for example, by the statistical moments of the velocity gradients. Experiments provide evidence[1] that such moments increase, apparently without bound, as the Reynolds number Re of the flow increases. Such a picture is typical of a field in which the relative volume of the region in space with a sufficiently high "intensity" of small scale turbulence tends to zero as Re increases.

Most of the known theories of intermittency, for example, the lognormal model,[4,5] the beta model,[6] the random beta model,[7] the alpha model,[8] the p model,[9] the log-stable model,[10] the log-gamma model,[11] and several other models refer to the case of asymptotically large Reynolds numbers. The characteristics of the intermittency in the inertial range of the spectrum, such as the probability distribution of the breakdown coefficient of the dissipation field, the intermittency exponents μ_q, and the exponents ζ_n of the structure function, are assumed to be universal (not dependent on the Reynolds number) in the models just enumerated. The experimental data of different investigators, which will be examined in Sec. 4, give values of ζ_n which differ from one another. The question of whether the exponents ζ_n are universal or not is presently unclear. In particular, it was postulated in Ref. 12 that the nonuniversality of the characteristics of small-scale turbulence is caused by the external intermittency.

It may be conjectured that the disparities between the experimental results just mentioned are not a consequence of measurement errors, but reflect a definite dependence, which is not taken into account in the known models of intermittency. In this paper the characteristics of small-scale turbulence are considered over a broad range of Re: from a certain critical value $\mathrm{Re}_{\mathrm{cr}}$ to $\mathrm{Re}\to\infty$. It is shown that rejection of the assumption that the intermittency exponents are self-similar with respect to the Reynolds number of the flow makes it possible to significantly improve the agreement between the previously proposed semiempirical model[13] and experiment, as well as to give a qualitative explanation for the possible nonuniversality of the exponents of the velocity structure function.

The relation between the velocity structure function and the probability distribution of the energy dissipation rate in turbulent flows is determined in Sec. 2. In Sec. 3 the exponents of the velocity structure function are found on the basis of a semiempirical model. A comparison with experiment is given in Sec. 4, and Sec. 5 contains the conclusions.

2. STRUCTURE FUNCTIONS OF THE VELOCITY FIELD

One possible approach to the description of small-scale turbulence is to study structure functions of the velocity field of different orders. A structure function of order n refers to the quantity $\langle(\delta u(\mathbf{x},\mathbf{r}))^n\rangle$, where the angle brackets denote statistical averaging, $\delta u(\mathbf{x},\mathbf{r})$ is the difference between the values of the flow velocity at points with the coordinates $\mathbf{x}+\mathbf{r}$ and $\mathbf{x}$:

$$\delta u(\mathbf{x},\mathbf{r})=|\mathbf{u}(\mathbf{x}+\mathbf{r})-\mathbf{u}(\mathbf{x})|.$$

The range of scales r which are such that $\eta\ll r\ll L$, where $\eta=\nu^{3/4}\langle\varepsilon\rangle^{-1/4}$ and $L=k^{3/2}/\langle\varepsilon\rangle$, i.e., the internal (Kolmogorov) and external (energy) scales of turbulence, respectively, has been termed the inertial range. In this range of scales the molecular viscosity has no appreciable influence on the dynamics of flow, and the motion of the fluid is

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determined mainly by the force of inertia. According to the picture of the cascade breakdown of eddies proposed by Richardson,[14] the characteristics of turbulence in the inertial range must "forget" the details of the initial and boundary conditions of a specific flow. On the basis of the qualitative conceptions in Ref. 14, it was theorized in Refs. 2 and 3 that the value of the structure function of the velocity field in the inertial range of the spectrum is determined only by the mean energy flux from larger to smaller scales along with the values of n and $r=|\mathbf{r}|$. In the case of equilibrium in the spectrum, the energy flux coincides with the dissipation rate of energy in the form of heat under the action of the molecular viscosity. Then, from dimensionality arguments for the structure function in the inertial range it follows that[2,3]

$$\langle(\delta u(\mathbf{x},\mathbf{r}))^n\rangle \propto r^{n/3}\langle\varepsilon\rangle^{n/3}. \tag{1}$$

If ζ_n denotes the scaling exponent of a structure function of order n in the inertial range, i.e., if

$$\langle(\delta u(\mathbf{x},\mathbf{r}))^n\rangle \propto r^{\zeta_n}, \tag{2}$$

$\zeta_n=n/3$ in the theory in Refs. 2 and 3.

The experimental data contradict the linearity of ζ_n as a function of n (Ref. 1; see also the comparison with experiment in Sec. 4 herein). The theory in Refs. 2 and 3 does not take into account the random, intermittent character of the energy dissipation rate field. To take into account the corrections caused by intermittency, a refinement of the theory of locally isotropic turbulence was proposed in Refs. 4 and 5. If the scale r falls in the inertial range, the probability distribution for $\delta u(\mathbf{x},\mathbf{r})$ is determined by the values of r and ε_r. Here ε_r denotes the turbulent energy dissipation rate ε averaged over a region with a characteristic linear dimension r. From dimensionality arguments

$$\langle(\delta u(\mathbf{x},\mathbf{r}))^n\rangle \propto r^{n/3}\langle\varepsilon_r^{n/3}\rangle. \tag{3}$$

In the scale-similarity range (see Sec. 3.1 below)

$$\langle\varepsilon_r^q\rangle \propto r^{-\mu_q}, \tag{4}$$

the relation (3) gives an expression which relates the exponents of the velocity structure function to the statistical dissipation moments in the scale-similarity range:

$$\zeta_n=n/3-\mu_{n/3}. \tag{5}$$

3. SEMIEMPIRICAL MODEL OF THE TURBULENT DISSIPATION PROBABILITY DISTRIBUTION

A phenomenological model of the evolution equation for the distribution density of the energy dissipation rate in turbulent flows of an incompressible liquid was proposed in Ref. 13. The model is based on the ideas of the improved theory of locally isotropic turbulence[4,5] and generalizes the familiar semiempirical $k-\varepsilon$ model. In this model the parameters determining the dissipation probability density are the mean turbulent energy k, the mean dissipation rate $\langle\varepsilon\rangle$, and the molecular viscosity ν. It is assumed that the energy dissipation probability distribution is a universal function of k, $\langle\varepsilon\rangle$, and ν, i.e., a function which does not depend on the special features of the flow. The model in Ref. 13 gives the following equation for the function $\Phi(q,x)$ describing the

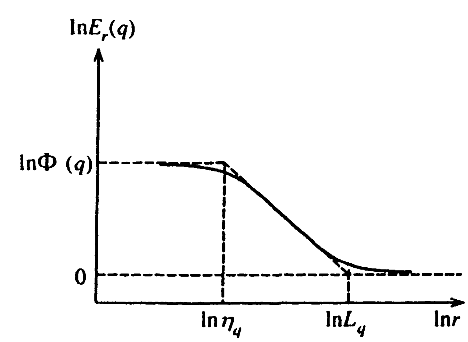

FIG. 1. Schematic dependence of the partially averaged dissipation moment on the scale.

statistical moments of the turbulent energy dissipation rate in developed equilibrium turbulent flows of an incompressible liquid:

$$\Phi''+2(\lambda_1+\lambda_{2q})\Phi'+\lambda_{3q}(q-1)\Phi=0, \tag{6}$$

where $\Phi(q,x)=\langle\varepsilon^q\rangle/\langle\varepsilon\rangle^q$, the prime denotes differentiation with respect to x, and $x=\ln[k^2/(\langle\varepsilon\rangle\nu)]$.

It was shown in Ref. 15 that Eq. (6) can be obtained by separating the "fast" and "slow" variables in the kinetic equation for the distribution density of ε in the more general case of nonequilibrium turbulence. In that case x is the logarithm of some effective Reynolds number Re. The shortcomings of the special scheme for closing the evolution equation for the dissipation distribution density previously proposed in Ref. 13 were also analyzed in Ref. 15.

3.1. Scale similarity of the breakdown coefficient of the dissipation field

In Ref. 16 Novikov considered the breakdown coefficient $q_{r,l}$ of the turbulent energy dissipation rate field:

$$q_{r,l}=\varepsilon_r/\varepsilon_l, \quad r\leqslant l. \tag{7}$$

If in the inertial range (when $\eta\ll r<m<l\ll L$), the scale-similarity conditions hold, i.e., 1) if the probability distribution of $q_{r,l}$ depends only on the ratio r/l and 2) if $q_{r,m}$ and $q_{m,l}$ are statistically independent, then[16]

$$\langle(q_{r,l})^q\rangle=(l/r)^{\mu_q}, \tag{8}$$

where the exponent μ_q does not depend on the scales r and l.

Let us consider the dependence of the normalized moment of order q of the turbulent energy dissipation rate $E_r(q)=\langle\varepsilon_r^q\rangle/\langle\varepsilon\rangle^q$ on the scale r. Clearly,

$$E_r(q)\rightarrow\Phi(q,x) \qquad \text{for } r\rightarrow 0,$$
$$E_r(q)\rightarrow 1 \qquad \text{for } r\rightarrow\infty,$$
$$E_r(q)=\kappa r^{-\mu_q} \qquad \text{for } \eta\ll r\ll L.$$

Figure 1 presents a schematic plot of the dependence of the logarithm of $E_r(q)$ on the logarithm of r. The scales L_q and η_q can be determined for a fixed value of q, so that

$$\Phi(q,x)=\kappa\eta_q^{-\mu_q},$$

$$1=\kappa L_q^{-\mu_q}.$$

The definitions of L_q and η_q, which are assigned by the points of intersection of the continuation of the linear (scaling) segment with the straight lines $\ln E_r(q)=0$ and $\ln E_r(q)=\ln \Phi(q,x)$, respectively (see Fig. 1), and Eq. (8) give

$$\Phi(q,x)=(L_q/\eta_q)^{\mu_q}. \tag{9}$$

The size of the smallest eddies in a turbulent flow coincides in order of magnitude with Kolmogorov's scale η, and the average of the energy scale L over the order scale can be assumed to coincide with the grand mean. Therefore, we assume that $L_q=a(q)L$ and $\eta_q=b(q)\eta$, where $a(q)$ and $b(q)$ are universal functions of q. Then

$$\Phi(q,x)=\left(\frac{a(q)L}{b(q)\eta}\right)^{\mu_q}. \tag{10}$$

3.2. Universal probability distribution of the breakdown coefficient

The solution of Eq. (6) was considered in Ref. 17 under the assumption that the exponents μ_q are universal with respect to the Reynolds number. The ratio of the external scale to the internal scale coincides with $\mathrm{Re}^{3/4}$; hence $L/\eta=\exp(3x/4)$. Plugging (10) into Eq. (6) gives

$$\Phi(q,x)=C(q)\exp(3\mu_q x/4), \tag{11}$$

where $C(q)=[a(q)/b(q)]^{\mu_q}$, and the exponents μ_q are determined by one of the solutions

$$\mu_q=\frac{4}{3}[-(\lambda_1+\lambda_2 q)\pm\sqrt{(\lambda_1+\lambda_2 q)^2-\lambda_3 q(q-1)}], \tag{12}$$

provided the coefficients λ_1, λ_2, and λ_3 in Eq. (6) are constants.

According to the data from the experimental work in Refs. 18–20, the probability distribution densities of the breakdown coefficient are continuous and unimodal. It was shown in Ref. 17 that the model probability distribution densities of the breakdown coefficient obtained from Eq. (12) contain no singular components only if the coefficients λ_2 and λ_3 in Eq. (12) are related by the expression

$$\lambda_2^2=\lambda_3. \tag{13}$$

The experimental data[18–20] indicate that the distribution of $q_{r,l}$ does not vanish for $0<q_{r,l}<l/r$. Hence it follows[17] that

$$\mu_q=q+o(q) \quad \text{for} \quad q\to\infty. \tag{14}$$

Then from Eqs. (12)–(14) we obtain

$$\lambda_2=-3/4, \quad \lambda_3=9/16. \tag{15}$$

Let us show that the infinite divisibility[21] of the distribution density of the logarithm of the dissipation breakdown coefficient follows from the scale-similarity condition.[16]

From the nonnegative value of the dissipation field it follows that[16]

$$0\leqslant q_{r,l}\leqslant l/r. \tag{16}$$

We introduce the notation $M(b)$ for the normalized breakdown coefficient from the inertial range: $M(b)=q_{r,l}/b$, where $b=l/r$. From the inequality (16) we obtain $0\leqslant M(b)\leqslant 1$, whence follows the nonnegative value of $Y(b)=-\ln M(b)$. We use $\varphi(q,b)$ to denote the Laplace transform of the probability distribution density $P(Y,b)$ in the variable Y:

$$\varphi(q,b)=\int_0^\infty P(Y,b)\exp(-qY)dY. \tag{17}$$

From Eq. (8) we obtain

$$\varphi(q,b)=b^{\mu_q-q}, \tag{18}$$

whence it follows directly that for any $n=1,2,\ldots$

$$\varphi^{1/n}(q,b)=\varphi(q,b^{1/n}). \tag{19}$$

In accordance with Eq. (19) the positive root of arbitrary order of the transform $\varphi(q,b)$ is the Laplace transform of a certain probability distribution, proving (see Sec. 7 in Chap. 13 of Feller's book[21]) the infinite divisibility of the distribution density $P(Y,b)$. The logarithmic infinite divisibility of the dissipation breakdown coefficient was utilized in Refs. 11, 17, and 22. It follows from the properties of infinitely divisible distributions[17] that the "slower" of the two solutions (12), i.e., the solution with a minus sign in front of the radical, must be chosen. The other solution, which was considered in Ref. 13, does not ensure a negative value for the distribution density and should be discarded. Then, using (13) and (15), for the intermittency exponents we obtain

$$\mu_q=q-\frac{4}{3}\left[\lambda+\sqrt{\lambda^2+q\left(\frac{9}{16}-\frac{3\lambda}{2}\right)}\right]. \tag{20}$$

Here and in the following the notation λ is used instead of λ_1. From the normalization condition $\mu_0=0$ (Ref. 16) we obtain $\lambda\leqslant 0$.

An explicit expression for the probability distribution density $P(M,b)$ of the breakdown coefficient $M(b)$ can be obtained[17] from Eq. (20):

$$P(M,b)=\frac{\rho\Delta}{\sqrt{2\pi}Y^{3/2}M}\exp\left\{-\frac{1}{2}\left[\frac{\rho\Delta}{\sqrt{Y}}+\frac{\lambda}{\Delta}\sqrt{Y}\right]^2\right\}, \tag{21}$$

where

$$\rho=\frac{4}{3}\ln b, \quad \Delta=\sqrt{\frac{9}{32}-\frac{3\gamma}{4}}, \quad Y=-\ln M.$$

It was assumed in Ref. 11 that the logarithm of the breakdown coefficient of the dissipation field has a gamma distribution. The distribution density of $M(b)$ in the log-gamma model has the form

$$P_{\mathrm{LG}}(M,b)=\frac{\beta^\omega}{\Gamma(\omega)}(-\ln M)^{\omega-1}M^{\beta-1}, \tag{22}$$

where Γ denotes the gamma function and

$$\omega=\frac{\ln b}{\ln[\beta/(\beta+1)]}.$$

For comparison, Fig. 2 presents the experimental distribution densities of $M(b)$ for $b=2$, 3, and 5 determined from

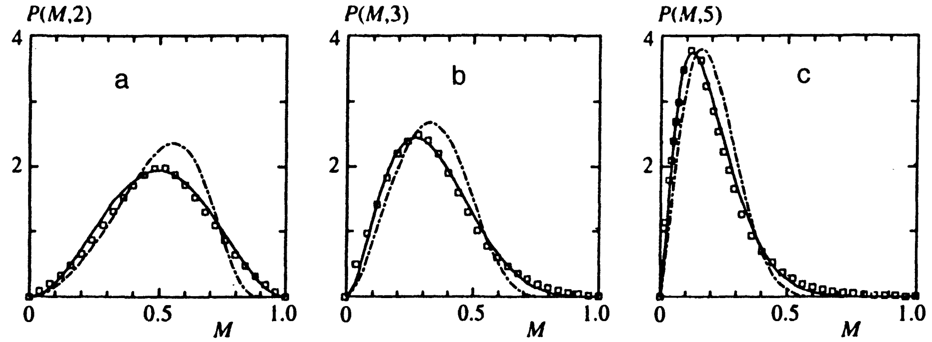

FIG. 2. Probability distribution densities of the breakdown coefficient $M(b)$ of the dissipation field for $b=2$, 3, and 5 (a, b, and c, respectively). The experimental results are based on data from Ref. 20, the solid curve is the log–gamma distribution (22) for $\beta=4$, and the dot-dashed curve was calculated from Eq. (21) for $\lambda=-4$.

the data in Ref. 20 and model densities obtained from Eq. (21) for $\lambda=-4$ and from (22) for $\beta=4$. It is seen from Fig. 2 that the logarithmic gamma distribution makes it possible to achieve good agreement with experiment. The model considered in the present work under the assumption that the exponents μ_q are universal can be reconciled with experiment only qualitatively: the model density (21), unlike the models in Refs. 6–9, does not contain atoms (singularities) and, unlike the models in Refs. 4, 5, and 10, has a restricted range of definition.

3.3. Nonuniversal probability distribution of the dissipation breakdown coefficient

It is not difficult to show that the main conclusions regarding the scaling dependence of the partially averaged dissipation moments also hold after some modification of the scale-similarity conditions.[16] It can be assumed that the distribution of the breakdown coefficient $q_{r,l}$ depends not only on the ratio r/l, but also on some other flow characteristics not associated with the scales r and l, for example, the ratio L/η. Under such an assumption the scaling dependence (8) remains valid, but the intermittency exponents μ_q are no longer universal functions of q. The proposed generalization of the scale-similarity condition admits a dependence of μ_q on the ratio between the scales η and L, i.e., on the Reynolds number.

It can be shown on the basis of infinitely divisible distributions[21] that even if the exponents μ_q are nonuniversal, only one of the two fundamental solutions of Eq. (6), viz., the solution which corresponds to the minus sign in Eq. (12), is of physical interest. The "distribution density" of the breakdown coefficient for the other fundamental solution is sign-alternating; therefore, this solution should be ruled out. The conditions (15) must hold for the limiting distribution density of the breakdown coefficient $M(b)$ not to contain singularities as $x\to\infty$ and not to vanish in the interval (0,1). The coefficient λ can be a function of the Reynolds number in the general case. Under the very simple assumption that λ is constant, we can obtain the following expression for the exponent μ_q of the dissipation moments:[23]

$$\mu_q=\frac{\ln C(q)+[-(\lambda-3q/4)-\sqrt{\lambda^2+3q(3/8-\lambda)/2}](x-x_0)}{\ln g(q)+3(x-x_0)/4}, \quad (23)$$

where $C(q)=\Phi(q,x_0)$, $g(q)=a(q)/b(q)$, and x_0 is some origin.

The relations (1)–(5), like the concept of an "inertial range," are applicable from a rigorous standpoint only in the limit $\mathrm{Re}\to\infty$. This is due to the fact that local isotropy and small-scale turbulence with statistical universality cannot occur at finite values of Re. If the Reynolds number is too low, there is no visible scaling range in the turbulence spectrum. Moreover, it was found in the experiments in Refs. 24 and 25 that the relations (1)–(5) can also be used at moderate values of the Reynolds number, provided Re is greater than a certain critical value $\mathrm{Re}_{\mathrm{cr}}$. In this case the turbulence in the inertial range is two-dimensional for $\mathrm{Re}=\mathrm{Re}_{\mathrm{cr}}$ and nearly three-dimensional (in agreement with the unimproved theory in Refs. 2 and 3) for $\mathrm{Re}\to\infty$. In Ref. 23 the dependence of the fractal dimension of the turbulence on the Reynolds number was studied on the basis of the model in Refs. 13 and 15. The generalized dimension D_q can be expressed in terms of the exponents μ_q (Ref. 26):

$$D_q=D+\mu_q/(1-q), \quad (24)$$

where D is the dimension of the space, and $D=3$ in the case under consideration.

If the turbulence in the scale-similarity range is two-dimensional, i.e., if $D_q=2$ for $q\geqslant 0$, then $\mu_q=q-1$. Setting $x_0=\ln \mathrm{Re}_{\mathrm{cr}}$, we obtain the relationship between $C(q)$ and $g(q)$ from Eq. (23):

$$\ln C(q)=(q-1)\ln g(q). \quad (25)$$

Substituting the expression (25) into Eq. (23), we obtain

$$\mu_q=q-1+\frac{[-\lambda+3/4-\sqrt{\lambda^2+3q(3/8-\lambda)/2}](x-x_0)}{\ln g(q)+3(x-x_0)/4}. \quad (26)$$

From Eq. (5) we find the expression for the exponents ζ_n:

$$\zeta_n=1+\frac{[\lambda-3/4+\sqrt{\lambda^2+n(3/8-\lambda)/2}](x-x_0)}{\ln g(n/3)+3(x-x_0)/4}. \quad (27)$$

3.4. "Tails" of the dissipation probability distribution

The dissipation moment function $\Phi(q,x)$ corresponding to the intermittency exponents assigned by Eq. (23) has the form

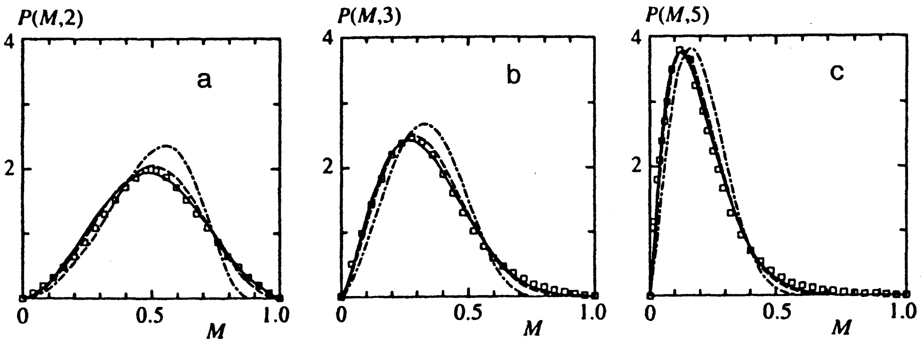

FIG. 3. Probability distribution densities of the breakdown coefficient $M(b)$ of the dissipation field for $b=2$, 3, and 5 (a, b, and c, respectively). The experimental results are based on data from Ref. 20, and the solid and dashed curve were obtained as a result of the reversion of Eq. (26) for values of $x-x_0$ equal to 2 and 3, $\lambda=-4$, and $\alpha=20$. The dot-dashed curve is the limiting distribution (21) for $x-x_0=\infty$.

$$\Phi(q,x)=C(q)\exp\left\{\left[\frac{3q}{4}-\lambda-\sqrt{\lambda^2+\frac{3q}{2}\left(\frac{3}{8}-\lambda\right)}\right](x-x_0)\right\}. \tag{28}$$

The expression (28) shows[27] that the distribution density $P(\zeta,x)$ for the normalized dissipation $\zeta=\varepsilon/\langle\varepsilon\rangle$ can be represented as a "convolution" of two functions:

$$P(\zeta,x)=\int_0^\infty \frac{1}{y}F_1\left(\frac{\zeta}{y}\right)F_2(y,x)dy. \tag{29}$$

Here $F_1(\zeta)$ is defined by the equation

$$C(q)=\int_0^\infty F_1(\zeta)\zeta^q d\zeta, \tag{30}$$

and $F_2(\zeta)$ has the form

$$F_2(\zeta)=\frac{h}{2\sqrt{\pi}}\frac{\zeta^{\gamma-1}}{\tau^{3/2}}\exp\left[\left(-\frac{3\gamma}{4}-\lambda\right)(x-x_0)-\frac{1}{4\tau}\right], \tag{31}$$

where

$$h=\frac{1}{(9/16-3\lambda/2)(x-x_0)^2},\quad \gamma=\frac{\lambda^2}{9/16-3\lambda/2},$$

$$\tau=h\left[\frac{3}{4}(x-x_0)-\ln\zeta\right].$$

The function $F_2(\zeta)$ is defined over a restricted region: it vanishes when $\zeta>\exp[3(x-x_0)/4]$. Therefore, it is clear that the tails of the dissipation distribution are determined by $F_1(\zeta)$, which is related to $C(q)$ by Eq. (30) and does not depend on the Reynolds number.

It was found in Ref. 28 that $P(\zeta)\propto\exp(-t\zeta^s)$, where $s\approx 1/2$, for $\zeta\gg 1$. Therefore, we assume that when $\mathrm{Re}=\mathrm{Re}_{cr}$, the dissipation distribution density can be represented in the form

$$P(\zeta,x_0)=A\zeta^\alpha\exp(-t\sqrt{\zeta}). \tag{32}$$

From the normalization conditions for $P(\zeta,x_0)$ we can obtain

$$t^2=(2\alpha+2)(2\alpha+3), \tag{33}$$

$$2A=t^{(2\alpha+2)}/\Gamma(2\alpha+2). \tag{34}$$

Then $C(q)$ has the form

$$C(q)=\frac{[(2\alpha+2)(2\alpha+3)]^{-q}\Gamma(2\alpha+2q+2)}{\Gamma(2\alpha+2)}, \tag{35}$$

whereupon for $\ln g(q)$ we obtain

$$\ln g(q)=\frac{1}{q-1}\ln\frac{[(2\alpha+2)(2\alpha+3)]^{-q}\Gamma(2\alpha+2q+2)}{\Gamma(2\alpha+2)}. \tag{36}$$

It was noted in Ref. 17 that the model probability distribution density (21) of the breakdown coefficient of the turbulent energy dissipation field, which was found under the assumption that the exponents μ_q are universal (not dependent on the Reynolds number) leads to results differing from experiment. The theoretical density tends to zero as $M\to 1$ more rapidly than does the experimental density (see Fig. 2); therefore, the values of the exponents of the velocity structure function of higher orders are overestimated. It is readily seen that better agreement with experiment can be attained by assuming that μ_q is not universal. When $q\gg 1$, the function $\ln C(q)$ has an asymptote:[29] $\ln C(q)\propto q\ln q[1+o(1)]$. From Eq. (25) we find that $\ln g(q)\propto\ln q[1+o(1)]$ at large q. Then the main term in the asymptote (for $n\to\infty$) of the exponents ζ_n which depend on the Reynolds number will be proportional to $\sqrt{n}/\ln n$, unlike the asymptote $\propto\sqrt{n}$ for a distribution which is self-similar with respect to Re.

4. COMPARISON WITH EXPERIMENT

From an analysis of the results in Refs. 24 and 25 it can be postulated that the ratio $\mathrm{Re}/\mathrm{Re}_{cr}$ varies in the range from 5–15 in the published experiments. Then the value of $x-x_0$ varies over the range from 1.6–2.7. Figure 3 presents probability distribution densities of the breakdown coefficient of the energy dissipation rate field obtained as a result of the numerical reversion of Eq. (26) for various values of $x-x_0$. The function $g(q)$ was given by Eq. (36), and the value $\alpha=20$ was chosen for α. It is seen from Fig. 3 that good agreement between these distributions and experiment can be attained under the assumption that the breakdown coefficient is not universal. Experimental plots of the exponents of the structure function versus the order based on the data in Refs. 30–36 are presented in Fig. 4. Despite the measurement errors, which are especially high in the determination of the moments of higher order, the disparity between the experimental results seems significant. Figure 5 presents theoretical plots of ζ_n calculated from Eq. (27) for several

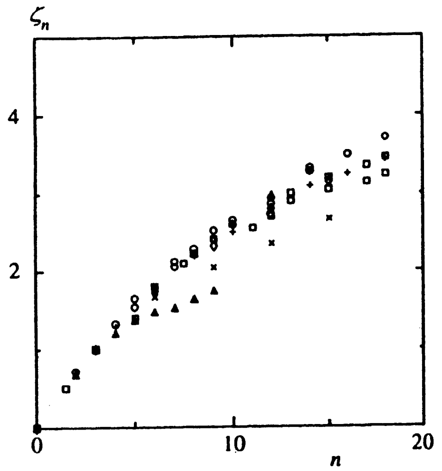

FIG. 4. Experimental plots of the exponent ζ_n of the structure function versus the order n. The circles, boxes, crosses, triangles, diamonds, filled triangles, and plus signs label data from Refs. 30–36, respectively.

values of $x-x_0$. The agreement between the experimental data (Fig. 4) and the model (Fig. 5) can be considered fully satisfactory. Nevertheless, it should be stressed that the results of the comparison of the experimental and model values of the exponents ζ_n should be regarded as only qualitative. This is due to the fact that a quantitative comparison of theory and experiment requires knowledge of the value of $\mathrm{Re}/\mathrm{Re}_{\mathrm{cr}}$. As the experiments in Refs. 24 and 25 show, the values of the critical Reynolds number $\mathrm{Re}_{\mathrm{cr}}$ for different flows are different: in a pipe $\mathrm{Re}_{\mathrm{cr}}\approx 2160$, downstream from a grid $\mathrm{Re}_{\mathrm{cr}}\approx 263$, and in the wake created by a cylinder $\mathrm{Re}_{\mathrm{cr}}\approx 165$. Therefore, a comparison of different experiments requires some caution: even if the Reynolds numbers in different flows coincide, the values of $\mathrm{Re}/\mathrm{Re}_{\mathrm{cr}}$ can be different. When experimental data and the model are compared quantitatively, the possible dependence of λ on the Reynolds number must also be taken into account. A lack of experimental data precludes definite conclusions regarding $\lambda(\mathrm{Re})$; therefore, a more complete analysis of the proposed model remains a matter for the future. The parameter λ (which may be a function of Re) is undefined in the proposed model, while the functions $C(q)$ and $g(q)$, as well as their interrelationship (25), are subject, at least in principle, to experimental verification. We note that the choice of the pair of values of λ and α considered in this work (-4 and 20, respectively) should not be considered definitive. It is not difficult to show that results at a similar level of agreement with experiment can be obtained for other combinations of these parameters. Here it is important to stress the fundamental possibility of better agreement with experiment for a model with intermittency exponents which depend on Re due to the different asymptote for the exponents of the velocity structure function.

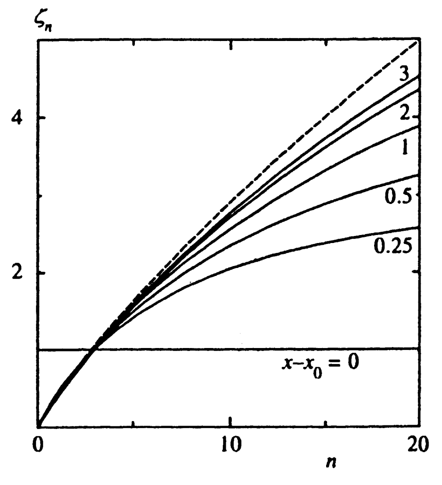

FIG. 5. Theoretical plots of the exponent ζ_n of the structure function versus the order n. Solid curves–calculation from Eq. (27) for $\lambda=-4$, $\alpha=20$, and $\ln(\mathrm{Re}/\mathrm{Re}_{\mathrm{cr}})=0$, 0.25, 0.5, 1, 2, 3. Dashed curve–limiting dependence for $\mathrm{Re}=\infty$.

5. CONCLUSIONS

The exponents of the velocity structure function in turbulent flows have been considered within a semiempirical model for the probability distribution density of the energy dissipation rate. The assumption that these exponents are not self-similar with respect to the Reynolds number in the scale-similarity range makes it possible to account for the disparity between the results of several experimental studies and to achieve good agreement between the theoretical probability distribution densities of the dissipation breakdown coefficient and experiment. Special attention has been focused in this work on the "tails" of the probability distribution of the turbulent energy dissipation and on the analysis of the dependence of the dimensionality of the turbulence on the Reynolds number.

We note that the results of the present work attest to the fact that modern experimental investigations of the intermittency of small-scale turbulence deal with the case of moderate values of Re, rather than an asymptotic regime of developed turbulence with $\mathrm{Re}\to\infty$: the value of the decisive parameter $\ln(\mathrm{Re}/\mathrm{Re}_{\mathrm{cr}})$, which appears in Eqs. (26) and (27), probably does not exceed 3.

The conclusions of this work should be regarded as preliminary, and further development of the proposed semiempirical model should be based on systematic experimentation. In particular, experimental determination of the distribution of the dissipation breakdown coefficient over a broad range of values of the Reynolds number of the flow would be of great interest.

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[1] A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics; Mechanics of Turbulence, Vols. 1 and 2*, MIT Press, Cambridge, Mass. (1971).
[2] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **30**, 299 (1941).
[3] A. M. Oboukhov, Dokl. Akad. Nauk SSSR **32**, 22 (1941).
[4] A. N. Kolmogorov, J. Fluid Mech. **13**, 82 (1962).
[5] A. M. Obukhov, J. Fluid Mech. **13**, 77 (1962).
[6] U. Frisch, P.-L. Sulem, and M. Nelkin, J. Fluid Mech. **87**, 719 (1978).
[7] R. Benzi, G. Paladin, G. Parisi, and A. Vulpiani, J. Phys. A **17**, 3521 (1984).
[8] D. Shertzer and S. Lovejoy, *Turbulent Shear Flows*, Springer, Berlin (1985), Vol. 4, p. 7.
[9] C. Meneveau and K. R. Sreenivasan, Phys. Rev. Lett. **59**, 1424 (1987).
[10] S. Kida, J. Phys. Soc. Jpn. **60**, 5 (1991).

[11] Y. Saito, J. Phys. Soc. Jpn. **61**, 403 (1992).
[12] V. R. Kuznetsov, A. A. Praskovskiĭ, and V. A. Sabel'nikov, Itogi Nauki Tekh., Ser.: Mekh. Zhidk. Gaza **6**, 51 (1988).
[13] P. L. Van'yan, Zh. Éksp. Teor. Fiz. **102**, 90 (1992) [Sov. Phys. JETP **75**, 47 (1992)].
[14] L. F. Richardson, Proc. R. Soc. London, Ser. A **110**, 709 (1926).
[15] P. L. Van'yan, Izv. Russ. Akad. Nauk, Fiz. Atmos. Okeana **31**, 767 (1995).
[16] E. A. Novikov, Prikl. Mat. Mekh. **35**, 266 (1971).
[17] P. L. Van'yan, JETP Lett. **58**, 25 (1993).
[18] M. Z. Kholmyanskiĭ, Izv. Akad. Nauk SSSR, Fiz. Atmos. Okeana **9**, 801 (1973).
[19] C. W. Van Atta and T. T. Yeh, J. Fluid Mech. **71**, 417 (1975).
[20] A. B. Chhabra and K. R. Sreenivasan, Phys. Rev. Lett. **68**, 2762 (1992).
[21] W. Feller, *An Introduction to Probability Theory and Its Applications*, Wiley, New York (1961).
[22] E. A. Novikov, Phys. Rev. A **50**, R3303 (1994).
[23] P. L. Van'yan, JETP Lett. **60**, 258 (1994).
[24] P. Tong and W. I. Goldburg, Phys. Fluids **31**, 2841 (1988).
[25] R. R. Prasad and K. R. Sreenivasan, Phys. Fluids A **2**, 792 (1990).
[26] H. G. E. Hentschel and I. Procaccia, Physica D **8**, 435 (1983).
[27] A. Erdélyi (ed.), *Bateman Manuscript Project. Tables of Integral Transforms, Vol. 1*, McGraw-Hill, New York (1954).
[28] P. Kailasnath, K. R. Sreenivasan, and G. Stolovitzky, Phys. Rev. Lett. **68**, 2766 (1992).
[29] M. Abramowitz and I. A. Stegun (editors), *Handbook of Mathematical Functions*, Dover, New York (1976).
[30] F. Anselmet, Y. Gagne, E. J. Hopfinger, and R. A. Antonia, J. Fluid Mech. **140**, 63 (1984).
[31] C. Meneveau and K. R. Sreenivasan, J. Fluid Mech. **224**, 429 (1991).
[32] M. Z. Kholmyanskiĭ, Izv. Akad. Nauk SSSR, Fiz. Atmos. Okeana **8**, 818 (1972).
[33] J. T. Park, Ph. D. Thesis, University of California, San Diego (1976).
[34] R. A. Antonia, B. R. Satyaprakash, and A. K. M. F. Hussain, J. Fluid Mech. **119**, 55 (1982).
[35] C. W. Van Atta and J. Park, in *Lecture Notes in Physics, Vol. 12: Statistical Models and Turbulence*, M. Rosenblatt and C. Van Atta (eds.), Springer-Verlag, Berlin (1972), p. 402.
[36] A. Vincent and M. Meneguzzi, J. Fluid Mech. **225**, 1 (1991).

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