Nuclear-structure corrections to the deuterium hyperfine structure and Lamb shift

A. I. Mil'shteĭn and I. B. Khriplovich

G. I. Budker Nuclear Physics Institute, Siberian Branch of the Russian Academy of Sciences, 630090 Novosibirsk, Russia

S. S. Petrosyan

Novosibirsk State University, 630090 Novosibirsk, Russia Novosibirsk, Russia

(Submitted 24 August 1995)
Zh. Éksp. Teor. Fiz. **109**, 1146–1158 (April 1996)

The low-energy theorem for the forward Compton scattering is generalized to the case of arbitrary target spin. The generalization is used to calculate the corresponding contribution to the deuterium hyperfine structure. Nuclear-structure corrections prove to be important due to the deuteron's large size. The calculated corrections of this type remove the discrepancy between the theoretical and experimental values of the deuterium hyperfine splitting. An explicit analytical result is also obtained for the deuteron polarizability contribution to the Lamb shift. © *1996 American Institute of Physics.* [S1063-7761(96)00404-0]

1. INTRODUCTION

The hyperfine (hf) splitting in the deuterium ground state has been measured with high accuracy. The most accurate experimental result was obtained[1] with an atomic deuterium maser:

$$\nu_{\rm exp}=327\,384.352\,522\,2(17)\ \text{kHz}. \tag{1}$$

On the other hand, theoretical predictions allowing for higher-order QED corrections yield

$$\nu_{\rm QED}=327\,339.27(7)\ \text{kHz}. \tag{2}$$

This value was obtained by using the theoretical result for hydrogen hf splitting from Ref. 2,

$$1\,420\,451.95(14)\ \text{kHz},$$

which does not allow for proton structure and the recoil radiative correction, and by employing the theoretical hydrogen-to-deuterium hf constants ratio from Ref. 3, equal to 4.339 387 6(8), which allows for the nuclear magnetic moments ratio and the reduced mass effect in $|\psi(0)|^2$.

In the present paper the discrepancy[1)]

$$\nu_{\rm exp}-\nu_{\rm QED}=45\ \text{kHz} \tag{3}$$

is removed by allowing for effects associated with the deuteron's finite size, which have a stronger influence in deuterium than in hydrogen. One nuclear-structure contribution to the deuterium hf splitting was obtained fairly long ago[4] by intuitive arguments and was later discussed in greater details in Refs. 5–7. We threat the deuteron finite-size effects systematically. Here we not only reproduce the old result but obtain new corrections, including those generated by the deuteron's electric and magnetic form factors.

To calculate some of the contributions to the deuterium hf structure we generalize the low-energy theorem for Compton scattering by a target with arbitrary spin.

Another topic considered is the contribution of deuteron polarizability to the deuteron Lamb shift. The fact that the corresponding correction is close to the accuracy attained in experiments was pointed out in Refs. 8 and 9, where the effect was calculated in the square-well approximation for the nuclear potential. The correction was also calculated in Ref. 10 for separable potentials. In the present paper we obtain a closed analytical result in the zero-range approximation for the deuteron polarizability contribution to the Lamb shift.

2. THE LOW-ENERGY THEOREM FOR FORWARD COMPTON SCATTERING BY A TARGET WITH ARBITRARY SPIN

According to the well-known theorem for Compton scattering by a spin-$\frac{1}{2}$ hadron,[11,12] the scattering amplitude is described by a Feynman pole diagram. Here we are interested not in (spin-independent) Thomson scattering, which is of zeroth order in the photon frequency ω, but in the next, spin-dependent, term in the ω- expansion. The result can also easily be obtained directly in the nonrelativistic approximation.[13] In this approximation an electromagnetic vertex can be written immediately for arbitrary spin $\mathbf{s}$:

$$\frac{e}{2m_p}\left\{\frac{Z}{A}(2\mathbf{p}+\mathbf{k})+\frac{\mu}{s}i[\mathbf{sk}]\right\}. \tag{4}$$

Here Z is the hadron charge, and the g-factor is related to the magnetic moment μ (measured nuclear magneton $e/2m_p$ units) as follows:

$$g=\mu/s.$$

In forward scattering, when the hadron is at rest ($\mathbf{p}=0$) and the initial and final photons have physical transverse polarizations $((\mathbf{ke})=(\mathbf{ke}')=0)$, such vertices are reduced to pure spin interaction. The nonrelativistic pole scattering amplitude generated by this interaction is

$$M_1=M_{1mn}e'_me_n=\left(\frac{e}{2m_p}\right)^2\omega g^2 i(\mathbf{s}[\mathbf{e}'\mathbf{e}]). \tag{5}$$

This expression is incomplete, however. If we apply it to scattering by a proton, the amplitude does not agree with the well-known result of Refs. 11 and 12, according to which the forward-scattering amplitude must be proportional to $(g-2)^2$. An explanation was given in Ref. 13; namely, the nonrelativistic pole amplitude must be supplemented by a contact term generated by spin–orbit coupling, which restores the agreement with the classical result.[11,12]

The contact term can easily be obtained for the case of arbitrary spin (as well as the nonrelativistic pole contribution (5)). We take the equation of motion for the spin in an external electric field **E** in the lowest nonvanishing order in v/c:

$$\frac{d\mathbf{s}}{dt}=\frac{e}{2m_p}\left(g-\frac{Z}{A}\right)[\mathbf{s}[\mathbf{Ev}]]. \tag{6}$$

Here A is the target mass measured in units of m_p (i.e., for a nucleus, which is the case we are mainly interested in here, A the atomic number). Obviously, the interaction Hamiltonian generating Eq. (6) is

$$H_s=-\frac{e}{2m_p}\left(g-\frac{Z}{A}\right)[\mathbf{s}[\mathbf{Ev}]]. \tag{7}$$

Equations (6) and (7) differ only slightly from formulas in the Berestetskiĭ *et al.*, book.[14] After substituting

$$\mathbf{v}=\frac{\mathbf{p}-Ze\mathbf{A}}{Am_p}$$

into the Hamiltonian (7) we arrive at the following contact interaction:

$$V_c=\left(\frac{e}{2m_p}\right)^2\frac{2Z}{A}\left(g-\frac{Z}{A}\right)(\mathbf{s}[\mathbf{EA}]). \tag{8}$$

It produces an additional contribution to the scattering amplitude:

$$M_2=M_{2mn}e'_m e_n=\left(\frac{e}{2m_p}\right)^2\omega\left(-\frac{4Z}{A}\right)\left(g-\frac{Z}{A}\right)i(\mathbf{s}[\mathbf{e}'\mathbf{e}]). \tag{9}$$

The sum of (5) and (9) yields the total spin part of the low-energy forward scattering amplitude:

$$M=\left(\frac{e}{2m_p}\right)^2\omega\left(g-2\frac{Z}{A}\right)^2 i(\mathbf{s}[\mathbf{e}'\mathbf{e}]). \tag{10}$$

This result is the generalization of the low-energy theorem we are looking for.

In the case of a proton $s=\frac{1}{2}$ and $Z=A=1$) the above formula becomes the result obtained in Refs. 11 and 12.

3. THE LOW-ENERGY THEOREM AND THE DEUTERIUM HYPERFINE STRUCTURE

Since this low-energy amplitude depends on the nuclear spin, there is an additional contribution to the hf structure. Calculating this amplitude, however, requires modifying the amplitude. The reason is that both photons exchanged by the electron and nucleus are off the mass shell, i.e., where $\omega \neq |\mathbf{k}|$. Moreover, virtual photons have extra polarizations. Here we use the gauge $A_0=0$, in which the photon propagator is

$$D_{im}(\omega,\mathbf{k})=\frac{d_{im}}{\omega^2-\mathbf{k}^2},\quad d_{im}=\delta_{im}-\frac{k_ik_m}{\omega_2},$$
$$D_{00}=D_{0m}=0. \tag{11}$$

First, the magnetic moment contribution M_{1mn} to the pole diagram is replaced by

$$\widetilde{M}_{1mn}=\left(\frac{e}{2m_p}\right)^2 g^2 i\epsilon_{mnk}k_k(\mathbf{ks})\frac{1}{\omega}. \tag{12}$$

Second, the convection current, proportional to $\pm\mathbf{k}$ for a nucleus at rest, yields a nonzero contribution to the spin-dependent forward scattering amplitude:

$$M_{3mn}=-\left(\frac{e}{2m}\right)^2 2\frac{Z}{A}gi(k_m\epsilon_{nrs}k_rs_s-k_n\epsilon_{mrs}k_rs_s)\frac{1}{\omega}. \tag{13}$$

We can now write the expression for the nuclear-spin-dependent electron–nucleus scattering amplitude generated by the two-photon exchange with the deuteron intermediate state:

$$T_{el}=4\pi\alpha i\int\frac{d^4k}{(2\pi)^4}\frac{d_{im}d_{jn}}{k^4}\frac{\gamma_i(\hat{l}-\hat{k}+m_e)\gamma_j}{k^2-2lk}$$
$$\times\times(\widetilde{M}_{1mn}+M_{2mn}+M_{3mn}). \tag{14}$$

Here $l_\mu=(m_e,0,0,0)$ is the electron momentum. The product $\gamma_i(\hat{l}-\hat{k}+m_e)\gamma_j$ is reduced to $-i\omega\epsilon_{ijl}\sigma_l$, where $\boldsymbol{\sigma}$ is the electron spin. We calculate the Feynman integral with logarithmic accuracy. Two points must be mentioned here. The singularity at $\omega=0$, originating from $1/\omega^2$ in the projection operator (11), must be interpreted in the principal-value sense. Moreover, calculations involve the integral

$$\int_0\frac{d|\mathbf{k}|}{|\mathbf{k}|^2},$$

which diverges algebraically as $|\mathbf{k}|\to 0$. Regularizing this integral requires introducing a nonzero electron velocity v. This leads to the well-known Coulomb wave-function correction $\pi\alpha/v$, which must be ignored since it is in no way related to our problem. We also discard the nonlogarithmic terms that emerge in calculating this integral.

It is convenient to write the result in the following way. The spin-dependent Born contribution to the electron–nucleus scattering amplitude is

$$T_0=-\frac{2\pi\alpha}{3m_em_p}g(\boldsymbol{\sigma}\,\mathbf{s}), \tag{15}$$

which is the Fourier transform of the lowest-order contact hf interaction multiplied by minus one. Hence the ratio $\Delta_{el}=T_{el}/T_0$ is simply the relative value of the discussed correction to the hf structure. The result for the integral (14) can be written as

$$\Delta_{el}=\frac{3\alpha}{8\pi}\frac{m_e}{m_p}\ln\frac{\Lambda}{m_e}\frac{1}{g}\left(g^2-4g\frac{Z}{A}-12\frac{Z^2}{A^2}\right). \tag{16}$$

For $s=\frac{1}{2}$ and $A=Z=1$ this agrees with the corresponding results of Refs. 15 and 16 for muonium (where the muon mass serves as the effective cutoff parameter Λ) and hydrogen (where the integral is cut off at the typical hadronic scale m_ρ).

In the case of deuterium ($s=1$, $g=\mu_d=0.857$, $A=2$, and $Z=1$) we are chiefly interested in the range of integration over the momentum transfer k, limited by the inverse deuteron size, $\kappa=45.7$ MeV. This leads to the following result for the relative correction in deuterium:

$$\Delta_{el}=\frac{3\alpha}{8\pi}\frac{m_e}{m_p}\ln\frac{\kappa}{m_e}\left(\mu_d-2-\frac{3}{\mu_d}\right). \tag{17}$$

For higher momentum transfers, $k>\kappa$, the amplitude of Compton scattering by a deuteron is the coherent sum of the amplitudes for scattering by a free proton and neutron. This contribution to the hf structure can easily be obtained from the above formula. Since both nucleons in the deuteron are in the triplet state, $\mathbf{s}/2$ can be substituted for $\mathbf{s}_p$ and $\mathbf{s}_n$. Cutting off the logarithmic integral at the usual hadronic scale $m_\rho = 770$ MeV, we get

$$\Delta_{in}=\frac{3\alpha}{4\pi}\frac{m_e}{m_p}\ln\frac{m_\rho}{\kappa}\frac{1}{\mu_d}(\mu_p^2-2\mu_p-3+\mu_n^2). \tag{18}$$

Here $\mu_p=2.79$ and $\mu_n=-1.91$ are the proton and neutron magnetic moments.

Note also the strong numerical cancellation of Δ_{el} and Δ_{in}.

4. CONTRIBUTION OF DEUTERON VIRTUAL EXCITATIONS

The low-energy Compton amplitude discussed above is the linear term in the expansion in powers of ω (and also linear in $|\mathbf{k}|^2/\omega$ for virtual photons) of the total amplitude. One could expect that for deuterium with its low binding energy this approximation is invalid even for the atomic problem considered here. However, as we will shortly see, the deuteron virtual excitations play an important role in our problem and dominate the present effect. Since the contribution of high momentum transfer $k>\kappa$ has been calculated earlier (see Eq. (18)), we limit ourselves to the region where $k<\kappa$. All calculations are carried out in the zero-range approximation, which makes it possible to achieve results in a closed analytical form.

We start with transitions induced solely by spin interaction. The corresponding scattering amplitude is

$$M_{4mn}=-\left(\frac{e}{2m_p}\right)^2\sum_n\left\{\frac{\langle 0|[\mathbf{kS}]_m|n\rangle\langle n|[\mathbf{kS}^\dagger]_n|0\rangle}{\omega-E_n-I}-\frac{\langle 0|[\mathbf{kS}^\dagger]_n|n\rangle\langle n|[\mathbf{kS}]_m|0\rangle}{\omega+E_n+I}\right\}. \tag{19}$$

Here $I=\kappa^2/m_p$ is the deuteron binding energy, $E_n=p^2/m_p$ is the energy of the intermediate state $|n\rangle$ (all intermediate states belong to the continuous spectrum), and

$$\mathbf{S}=\mu_p\boldsymbol{\sigma}_p e^{i\mathbf{kr}/2}+\mu_n\boldsymbol{\sigma}_n e^{-i\mathbf{kr}/2},$$

where $\boldsymbol{\sigma}_{p(n)}$ is the proton (neutron) spin operator.

In calculating this contribution we retain only the terms logarithmic in the parameter $\epsilon=I/\kappa=\kappa/m_p\ll 1$. The logarithm emerges as a result of integration with respect to k. Obtaining it requires only setting the exponentials in $\mathbf{S}$ equal to unity. The operator $\mathbf{S}$ can induce only $M1$ transitions. In the zero-range approximation the ground deuteron state is purely ${}^3S_1$, from which an $M1$ transition is possible only into S states.

Since the total spin operator $\mathbf{s}=(\boldsymbol{\sigma}_p+\boldsymbol{\sigma}_n)/2$ does not induce triplet–singlet transitions, we can reduce $\mathbf{S}$ to

$$\mathbf{S}\rightarrow(\mu_p-\mu_n)\frac{1}{2}(\boldsymbol{\sigma}_p-\boldsymbol{\sigma}_n).$$

Our problem of the hf structure requires that the tensor (19) have an antisymmetric part that is linear in the deuteron spin $\mathbf{s}$. This part has the following form:

$$M^1_{mn}=-\left(\frac{e}{2m_p}\right)^2(\mu_p-\mu_n)^2 i\epsilon_{mnk}k_k(\mathbf{ks})\omega\times\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{|\langle{}^1S_0,p|{}^3S_1\rangle|^2}{\omega^2-(p^2+\kappa^2)^2/m_p^2}, \tag{20}$$

where $\langle{}^1S_0,p|{}^3S_1\rangle$ is the overlap integral of the ground-state wave function in the zero-range approximation,

$$\psi_0=\sqrt{\frac{\kappa}{2\pi}}\frac{e^{-\kappa r}}{r}, \tag{21}$$

and the singlet function with momentum p.

The corresponding contribution to the electron–deuteron scattering amplitude,

$$T^1_{in}=4\pi\alpha i\int\frac{d^4k}{(2\pi)^4}\frac{d_{im}d_{jn}}{k^4}\frac{\gamma_i(\hat{l}-\hat{k}+m_e)\gamma_j}{k^2-2lk}M^1_{mn} \tag{22}$$

can easily be calculated with logarithmic accuracy. Indeed, to this accuracy the energy denominator in Eq. (20) can be simplified, so that we have

$$\frac{1}{\omega^2-\kappa^4/m_p^2}.$$

After this, integration with respect to $\mathbf{p}$ is done by employing the completeness relation. The resulting relative correction to the deuterium hf structure is

$$\Delta^1_{in}=\frac{3\alpha}{8\pi}\frac{m_e}{m_p}\ln\frac{m_p}{\kappa}\frac{(\mu_p-\mu_n)^2}{\mu_d}. \tag{23}$$

Finally, let us now consider the inelastic contribution induced by the combined action of the convection and spin currents. Since the convection current is spin-independent, all the intermediate states are triplets. Hence we can replace the operator $\mathbf{S}$ with

$$\mathbf{S}\rightarrow\mathbf{s}(\mu_p e^{i\mathbf{kr}/2}+\mu_n e^{-i\mathbf{kr}/2}).$$

In accordance with the common selection rules, there is no way by which the convection current can excite the ground state to ${}^3S_1$ states. But in the zero-range approximation all states with $l\neq 0$ are free, which means that for the intermediate states we can select plane waves, the eigenfunctions of the momentum operator. Then the only matrix element in the amplitude is

$$\langle 0|e^{\pm i\mathbf{kr}/2}|\mathbf{p}\rangle = \frac{\sqrt{8\pi\kappa}}{(\mathbf{p}\pm\mathbf{k}/2)^2+\kappa^2}. \tag{24}$$

Thus, the amplitude is

$$M_{mn}^2 = \left(\frac{e}{2m_p}\right)^2 2\kappa\omega \int \frac{d\mathbf{p}}{\pi^2}\left\{\frac{\mu_p}{[(\mathbf{p}-\mathbf{k}/2)^2+\kappa^2]^2} + \frac{\mu_n}{[(\mathbf{p}-\mathbf{k}/2)^2+\kappa^2][(\mathbf{p}+\mathbf{k}/2)^2+\kappa^2]}\right\} \times \frac{(2p-k/2)_m i\epsilon_{nrs}k_r s_s - (2p-k/2)_n i\epsilon_{mrs}k_r s_s}{\omega^2-(p^2+\kappa^2)^2/m_p^2}. \tag{25}$$

The integrals that emerge in the calculation of the corresponding part of the electron–deuteron scattering amplitude,

$$T_{in}^2 = 4\pi\alpha i \int \frac{d^4k}{(2\pi)^4}\frac{d_{im}d_{jn}}{k^4}\frac{\gamma_i(\hat{l}-\hat{k}+m_e)\gamma_j}{k^2-2lk}M_{mn}^2, \tag{26}$$

are fairly complicated even if we only retain terms singular in the parameter $\epsilon=\kappa/m_p \ll 1$, i.e., $1/\epsilon$ and $\ln\epsilon$. The relative correction to the hf structure in this approximation is

$$\Delta_{in}^2 = \alpha\frac{m_e}{2\kappa}\frac{\mu_p-\mu_n}{\mu_d} - \frac{3\alpha}{\pi}\frac{m_e}{m_p}\ln\frac{m_p}{\kappa}\frac{\mu_p-\mu_n}{\mu_d}. \tag{27}$$

The term

$$-\alpha\frac{m_e}{2\kappa}\frac{\mu_n}{\mu_d},$$

was obtained and discussed in Refs. 4–7. But numerically the new term

$$\alpha\frac{m_e}{2\kappa}\frac{\mu_p}{\mu_d},$$

is found to be greater.

5. CORRECTIONS DUE TO THE FINITE DISTRIBUTION OF THE DEUTERON CHARGE DENSITY AND MAGNETIC MOMENT

For hydrogen the current problem was considered long ago by Zemach.[17] Obviously, in deuterium these corrections must be larger. In the zero-range approximation the problem can be solved analytically.

We start with the second-order electron–deuteron scattering amplitude induced by the deuteron charge and magnetic moment. The nucleus is treated in the static limit. The distributions of the charge and magnetic moment in the nucleus are taken into account by introducing the corresponding form factors, F_{ch} and F_m. For the amplitude we have

$$V = -(4\pi\alpha)^2\frac{\mu_d}{2m_p}\int\frac{d\mathbf{q}}{(2\pi)^3}i[\mathbf{sq}] \times \frac{F_{ch}(\mathbf{q}^2)F_m(\mathbf{q}^2)}{\mathbf{q}^4}\frac{\gamma_0(\hat{l}+\hat{q}+m_e)\gamma-\gamma(\hat{l}+\hat{q}+m_e)\gamma_0}{(l+q)^2-m_e^2}. \tag{28}$$

Here again $l_\mu=(m_e,0,0,0)$, and $q_\mu=(0,\mathbf{q})$. It is convenient to transform this expression into

$$V = \frac{8m_e\alpha}{\pi}\int\frac{dq}{q^2}F_{ch}(q^2)F_m(q^2)T_0, \tag{29}$$

where T_0 is the momentum-independent magnetic Born amplitude (15).

The effect we are interested in vanishes if both form factors are replaced by unity. Hence the corresponding relative correction to the Born amplitude T_0 and to the hf splitting is

$$\Delta_f = \frac{8m_e\alpha}{\pi}\int\frac{dq}{q^2}[F_{ch}(q^2)F_m(q^2)-1]. \tag{30}$$

In the zero-range approximation the two form factors have the simple form

$$F_{ch}(q^2) = F_m(q^2) = \langle 0|e^{i\mathbf{qr}/2}|0\rangle = \frac{4\kappa}{q}\arctan\frac{q}{4\kappa}. \tag{31}$$

Substituting this into (3), we arrive at the following explicit expression for the correction:

$$\Delta_f = -\alpha\frac{m_e}{3\kappa}(1+2\ln 2). \tag{32}$$

Two closely related features of the effect characteristic not only of deuterium should be mentioned. The correction is of the first order (rather than of the second) in the ratio of the nuclear size to the Bohr radius, $m_e\alpha/\kappa$. Moreover, contrary to possible naïve expectations, the contributions of the electric and magnetic form factors are not additive. Both features can be traced to the fact that the characteristic momentum providing the principal contribution to the integral (30) are of the nuclear scale rather than of the atomic.

6. HYPERFINE SPLITTING IN DEUTERIUM. DISCUSSION OF RESULTS

The final result for the nuclear-structure correction to the hf splitting in deuterium is the sum of the corrections (17), (18), (23), (27), and (32):

$$\Delta = \alpha\frac{m_e}{2\kappa}\left\{\frac{\mu_p-\mu_n}{\mu_d} - \frac{2}{3}(1+2\ln 2)\right\} + \frac{3\alpha}{8\pi}\frac{m_e}{m_p}\ln\frac{m_p}{\kappa}\frac{(\mu_p-\mu_n)^2}{\mu_d} - \frac{3\alpha}{\pi}\frac{m_e}{m_p}\ln\frac{m_p}{\kappa}\frac{\mu_p-\mu_n}{\mu_d} + \frac{3\alpha}{8\pi}\frac{m_e}{m_p}\ln\frac{\kappa}{m_e}\frac{1}{\mu_d}(\mu_d^2-2\mu_d-3) + \frac{3\alpha}{4\pi}\frac{m_e}{m_p}\ln\frac{m_p}{\kappa}\frac{1}{\mu_d}(\mu_d^2-2\mu_d-3+\mu_n^2). \tag{33}$$

Numerically this correction to the deuterium hf structure is

$$\Delta\nu = 43 \quad \text{kHz}. \tag{34}$$

This value must be compared with the discrepancy (3), which amounts to 45 kHz. Bearing in mind the above approximations, especially the crude nuclear model (the zero-range approximation) and the fact that nonlogarithmic contributions are ignored, we believe that the agreement is satisfactory. In particular, inclusion of the corrections caused by the finiteness of the effective interaction radius r_0 into the normalization of the deuteron's ground-state wave function (see the details in Sec. 7) would enhance some contributions.

Clearly, the discussed nuclear effects are responsible for the bulk of the discrepancy between the results of purely QED calculations and the experimental data on the deuterium hf structure. Calculations of the hf corrections that incorporate an accurate treatment of nuclear effects may serve as another verification of more detailed models of the deuteron structure.

7. NUCLEAR POLARIZABILITY AND THE LAMB SHIFT IN DEUTERIUM

The contribution of nuclear polarizability to the Lamb shift in deuterium has recently been studied in Refs. 8–10. Here we calculate analytically the effect and arrive at a result in closed form. The zero-range approximation we use is applicable if the region in which the wave function is localized is much larger than the interaction region. The same condition is necessary if the true interaction is replaced with the crude approximation of a square-well potential, as is done in Refs. 8 and 9.

The effect of the electric polarizability, which we are now interested in, is caused by the photon–deuteron scattering amplitude induced only by convection currents. As we will shortly see, in our problem of the nuclear polarizability contribution to the Lamb shift the characteristic values of the photon 4-momenta are

$$\omega,|\mathbf{k}|\leqslant I=\frac{\kappa^2}{m_p}\ll\kappa\sim|\mathbf{p}|. \tag{35}$$

Hence we can neglect the dependence on $\mathbf{k}$ in the Compton amplitude. As in the amplitude M^2_{mn}, here all the intermediate states have $l\neq 0$ and can be described by plane waves. We again employ the matrix element (25), but this time set $\mathbf{k}=0$. Finally, here we are interested in the scalar part of the scattering amplitude, which can be reduced to

$$-\left(\frac{e}{2m_p}\right)^2\frac{4}{3}\delta_{mn}k\int\frac{d\mathbf{p}}{\pi^2}\frac{p^2}{(p^2+\kappa^2)^2}\left\{\frac{1}{\omega-(p^2+\kappa^2)/m_p}-\frac{1}{\omega+(p^2+\kappa^2)/m_p}\right\}.$$

From the expression in the braces we subtract

$$-2\frac{m_p}{p^2+\kappa^2}.$$

After integration with respect to $\mathbf{p}$, this term, being added to the Thomson scattering amplitude $-e^2/m_p$, reproduces the correct Thomson scattering amplitude for the deuteron, $-e^2/2m_p$. Employing the identity

$$\frac{1}{\omega-u}-\frac{1}{\omega+u}+\frac{2}{u}=\frac{2\omega^2}{(\omega^2-u^2)u},$$

we arrive at the following expression for the desired photon–deuteron scattering amplitude:

$$M^3_{mn}=-\left(\frac{e}{2m_p}\right)^2\frac{8}{3}\delta_{mn}\kappa\omega^2m_p\times\int\frac{d\mathbf{p}}{\pi^2}\frac{p^2}{(p^2+\kappa^2)^3\{\omega^2-(p^2+\kappa^2)^2/m_p^2\}}. \tag{36}$$

Its contribution to the electron–deuteron scattering amplitude,

$$T^3_{in}=4\pi\alpha i\int\frac{d^4k}{(2\pi)^4}\frac{d_{im}d_{jn}}{k^4}\frac{\gamma_i(\hat{l}-\hat{k}+m_e)\gamma_j}{k^2-2lk}M^3_{mn}, \tag{37}$$

can be transformed into

$$T^3_{in}=\frac{32\pi^2\alpha^2\kappa}{3m_p}\int\frac{d\mathbf{p}}{\pi^2}\frac{p^2}{(p^2+\kappa^2)^3}i\int\frac{d^4k}{(2\pi)^4}\times\left\{\frac{1}{\omega(k^2-2lk)}+\frac{2\omega^3}{k^4(k^2-2lk)}\right\}\frac{1}{\omega^2-(p^2+\kappa^2)^2/m_p^2}. \tag{38}$$

The first term in the braces contains no photon propagators, either $\propto 1/k^2$ nor $\propto 1/k^4$. In other words, it corresponds to the instantaneous Coulomb interaction. The second term corresponds to exchange by three-dimensional transverse photons, i.e., the magnetic interaction of convection currents.

Probably the most convenient procedure for integrating (38) is as follows: Wick rotation, transformation of the integral over the Euclidean ω to the integral over the interval $(0,\infty)$, substitution of $\mathbf{k}\omega$ for $\mathbf{k}$, integration over ω, integration over $\mathbf{k}$ (in the last two stages it becomes evident that the effective values of ω and $\mathbf{k}$ belong to the interval (35)), and, finally, integration with respect to p. The following relationship also proves useful:

$$\int_0^1 dx(1-x)^{a-1}x^{b-1}\ln x=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}[\psi(b)-\psi(a+b)],$$

$$\psi(b)=\frac{d}{db}\ln\Gamma(b).$$

The effective electron–nucleus operator (equal to $-T^3_{in}$) can finally be written in the coordinate representation:

$$V_{le}=-\alpha m_e\alpha_d(0)5\left(\ln\frac{8I}{m_e}+\frac{1}{20}\right)\delta(\mathbf{r}). \tag{39}$$

Here $\alpha_d(0)$ is the static value of the deuteron electric polarizability defined, as usual, by the following relationship:

$$\alpha_d(\omega)=4\pi\alpha\frac{2}{3}\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{p^2+\kappa^2}{m_p}\frac{\langle 0|\mathbf{r}|n\rangle\langle n|\mathbf{r}|0\rangle}{(p^2+\kappa^2)^2/m_p^2-\omega^2}. \tag{40}$$

Large distances provide the greatest contribution to the matrix elements in (40). In this asymptotic region the naïve zero-range-approximation expression (21) for the deuteron ground-state wave function must be augmented by a correction factor $(1-r_0\kappa)^{-1/2}$ accounting for the finiteness of the effective interaction radius r_0 (see Refs. 14 and 18). Thus, we arrive at the following result for the static electric polarizability:

$$\alpha_d(0)=\frac{\alpha}{32(1-r_0\kappa)}\frac{m_p}{\kappa^4}=0.64\ \text{fm}^3. \tag{41}$$

This is close to the experimental value[19] of 0.70(5) fm^3, as well as to the values 0.613, 0.623, and 0.625 fm^3 obtained in Ref. 10 with different separable nuclear potentials and to 0.635 fm^3 obtained in Ref. 8 with a square-well potential.

The final result (39) contains two contributions of different physical origin. The instantaneous Coulomb interaction is dominant. Its contribution to the numerical factor

$$-5\left(\ln\frac{8I}{m_e}+\frac{1}{20}\right)$$

in (39) amounts to

$$-4\left(\ln\frac{8I}{m_e}+\frac{5}{12}\right).$$

The magnetic interaction contribution to this factor is

$$-\left(\ln\frac{8I}{m_e}-\frac{17}{12}\right).$$

The level shift of the deuterium ground state produced by interaction (39) amcunts to -22.3 kHz. The Coulomb and magnetic contributions to it are -19.7 and -2.6 kHz, respectively. The results are close to those of Refs. 8–10.

It is natural that the Coulomb contribution is negative: it is a true second-order (in the electron–nucleus static interaction) correction to the ground state of the system consisting of an electron at rest and a nucleus in its ground state. There is no way in which the sign of the magnetic contribution can be fixed in a similar way: in terms of the common noncovariant perturbation theory this is a fourth-order correction: second-order in the photon–electron interaction and second-order in the photon–nucleus interaction.

One more contribution to the Lamb shift in deuterium, also considered earlier in Ref. 10, is related to the deuteron magnetic polarizability. It is determined by the scalar part of the amplitude (19). Calculations simplify because of the following. First, the numerator d_{im} of the photon propagator is reduced in this case to δ_{im}. Integration with respect to $\mathbf{k}$ is spherically symmetric. Hence in our case the scalar part of amplitude (19) becomes

$$M_{mn}^4=-4\pi\alpha\frac{(\mu_p-\mu_n)^2\kappa(\kappa+\kappa_1)^2}{9m_p^3}\delta_{mn}\mathbf{k}^2 \times\int\frac{d\mathbf{p}}{\pi^2}\frac{1}{(p^2+\kappa^2)(p^2+\kappa_1^2)[\omega^2-(p^2+\kappa^2)^2/m_p^2]} \tag{42}$$

Here we used the explicit form of the $^1S_0$–state wave function in the zero–range approximation,

$$\psi_s=\frac{\sin(pr+\delta)}{\sqrt{2}\pi r}, \tag{43}$$

where

$$\cot\delta=\frac{\kappa_1}{p},\quad \kappa_1=7.9\ \text{MeV}.$$

Its overlap with the ground-state zero-range-approximation wave function (21) is

$$\langle{}^1S_0|{}^3S_1\rangle=\frac{\sqrt{8\pi\kappa}(\kappa+\kappa_1)}{(p^2+\kappa^2)\sqrt{p^2+\kappa_1^2}}. \tag{44}$$

Further calculations are similar to those done when the electric polarizability was taken into account, the only difference being that here integration with respect to p for the nonlogarithmic contribution is done numerically. The resulting effective electron–nucleus interaction operator can be written as

$$V_{im}=\alpha m_e\beta_d(0)\left(\ln\frac{8I}{m_e}-1.24\right)\delta(\mathbf{r}). \tag{45}$$

Here $\beta_d(0)$ is the static value of the deuteron magnetic polarizability:

$$\beta_d(0)=\frac{\alpha(\mu_p-\mu_n)^2}{8m_p\kappa^2}\frac{1+\kappa_1/3\kappa}{1+\kappa_1/\kappa}. \tag{46}$$

The corresponding contribution to the Lamb shift of the deuterium ground state amounts to 0.31 kHz, which is very close to the result of Ref. 10.

The authors are grateful to M. I. Éĭdes, H. Grotch, and M. I. Strikman for useful discussions. The work was supported by the Universities of Russia program (Grant No. 94-6.7-2053).

Note added in proof on January 9, 1996. Recently, while the present paper was in press, we read a paper by J. Martorell, D. W. L. Sprung, and D. C. Zheng [Phys. Rev. C **51**, 1127 (1995)], who arrived at an analytical expression for the contribution of the deuteron's electric polarizability to the Lamb shift in the zero-range approximation. Their result agrees with ours.

[1)] This discrepancy has been known from the late 1940s, long before the maser experiment of Wineland and Ramsey,[1] which had a record-breaking accuracy. Earlier measurements are cited in Ref. 1.

[1] D. J. Wineland and N. F. Ramsey, Phys. Rev. A **5**, 821 (1972).
[2] G. T. Bodwin and D. R. Yennie, Phys. Rev. D **37**, 498 (1988).
[3] N. F. Ramsey, in *Quantum Electrodynamics*, edited by T. Kinoshita, World Scientific, Singapore (1990).
[4] A. Bohr, Phys. Rev. **73**, 1109 (1948).
[5] F. E. Low, Phys. Rev. **77**, 361 (1950).
[6] F. E. Low and E. E. Salpeter, Phys. Rev. **83**, 478 (1951).
[7] D. A. Greenberg and H. M. Foley, Phys. Rev. **120**, 1684 (1960).
[8] K. Pachucki, D. Leibfried, and T. W. Hänsch, Phys. Rev. A **48**, R1 (1993).
[9] K. Pachucki, M. Weitz, and T. W. Hänsch, Phys. Rev. A **49**, 2255 (1994).
[10] Yang Li and R. Rosenfelder, Phys. Lett. B **319**, 7 (1993); **333**, 564 (1994).
[11] F. E. Low, Phys. Rev. **96**, 1428 (1954).
[12] M. Gell-Mann and M. L. Goldberger, Phys. Rev. **96**, 1433 (1954).
[13] A. I. Mil'shteĭn, Sibirsk. Fiz. Zh. No. 1, 43 (1995).
[14] V. B. Berestetskiĭ, E. M. Lifshitz, and L. P. Pitaevskiĭ, *Quantum Electrodynamics*, 2nd ed., Pergamon Press, Oxford (1982).
[15] R. Arnowitt, Phys. Rev. **92**, 1002 (1953).
[16] H. Grotch and D. R. Yennie, Rev. Mod. Phys. **41**, 350 (1969).
[17] A. C. Zemach, Phys. Rev. **104**, 1771 (1956).
[18] J. L. Friar and S. Fallieros, Phys. Rev. C **29**, 232 (1984).
[19] N. L. Rodning, L. D. Knutson, W. G. Lynch, and H. B. Tsang, Phys. Rev. Lett. **49**, 909 (1982).

Translated by Eugene Yankovsky