Dirac particle with anomalous magnetic moment in a circularly polarized wave

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(Submitted 8 September 1995)
Zh. Éksp. Teor. Fiz. **109**, 1234–1239 (April 1996)

The resonant frequencies in a circularly polarized wave propagating in the direction of a constant magnetic field are determined, and the wave functions of a charged Dirac particle with anomalous magnetic moment are determined in the linear approximation in the amplitude of the wave. In the absence of a magnetic field the wave functions of such particles are determined exactly. © *1996 American Institute of Physics.* [S1063-7761(96)01104-3]

1. INTRODUCTION

The wave function of a Dirac particle with anomalous magnetic moment satisfies the following modified Dirac equation:[1]

$$(\gamma_i\hat{k}_i+k_0)\psi=ig_1(e/4\,m_0c^2)F_{ij}\gamma_{ij}\psi,$$
$$\hat{k}_i=\frac{\partial}{\partial x_i}-\frac{ie}{\hbar c}\Phi_i,\quad k_0=\frac{m_0c}{\hbar},\quad F_{ij}=\frac{\partial\Phi_j}{\partial x_i}-\frac{\partial\Phi_i}{\partial x_j},\tag{1}$$

where e and m_0 are the charge and mass of the particle, g_1 is a dimensionless constant, γ_i is the Dirac matrix,[1,2] and Φ_i are the components of the 4-vector of the field, which in a circularly polarized wave with right circulation are given by

$$\Phi_x=\Phi\,\sin(\omega\eta),\quad \Phi_y=\Phi\,\cos(\omega\eta),\quad \Phi_z=0,\quad \Phi_t=0,\tag{2}$$

where $\omega\Phi$ is the field strength of the wave field, ωc its frequency, and $\eta=ct+z$. The solution of Eq. (1) based on the zero divisor

$$\Gamma=\tfrac{1}{4}(1+i\gamma_{12})(1+\gamma_4)\tag{3}$$

is sought in the form of the following sum:[2]

$$\psi=\exp(ik_0ct)(\psi_1+\psi_2\gamma_1+\psi_3\gamma_3+\psi_4\gamma_{31})\Gamma.\tag{4}$$

The components ψ_i of this sum depend only on η. In this case they satisfy the system of equations

$$\Phi_0e^{-i\omega\eta}(1+g)\psi_2-ig\Phi_0e^{-i\omega\eta}\psi_4+\frac{\partial}{\partial\eta}\psi_3+\left(2k_0-i\frac{\partial}{\partial\eta}\right)\psi_1=0,$$
$$-\Phi_0e^{i\omega\eta}(1+g)\psi_1-ig\Phi_0e^{i\omega\eta}\psi_3+\frac{\partial}{\partial\eta}\psi_4+i\frac{\partial}{\partial\eta}\psi_2=0,$$
$$-\Phi_0e^{i\omega\eta}(1-g)\psi_4+ig\Phi_0e^{-i\omega\eta}\psi_2+\frac{\partial}{\partial\eta}\psi_1+i\frac{\partial}{\partial\eta}\psi_3=0,\tag{5}$$
$$\Phi_0e^{i\omega\eta}(1-g)\psi_3+ig\Phi_0e^{-i\omega\eta}\psi_1+\frac{\partial}{\partial\eta}\psi_2+\left(2k_0-i\frac{\partial}{\partial\eta}\right)\psi_4=0,$$

where $\Phi_0=e\Phi/\hbar c$ and $g=g_1\omega/2k_0$. The solution of this system is

$$\psi_1=ic_1\,\exp\left[i\left(\lambda-\frac{\omega}{2}\right)\eta\right],\quad \psi_2=ic_2\,\exp\left[i\left(\lambda+\frac{\omega}{2}\right)\eta\right],$$
$$\psi_3=c_3\,\exp\left[i\left(\lambda-\frac{\omega}{2}\right)\eta\right],\quad \psi_4=c_4\,\exp\left[i\left(\lambda+\frac{\omega}{2}\right)\eta\right].\tag{6}$$

The constant coefficients c_i and the eigenvalue λ are given by

$$c_1=\Phi_0(-\omega-2g\Phi_0\pm\sqrt{\omega^2+4g^2\Phi_0^2}),$$
$$c_2=(2k_0+\Phi_0)\omega+(2k_0-\Phi_0)(2g\Phi_0\mp\sqrt{\omega^2+4g^2\Phi_0^2}),$$
$$c_3=-(2k_0+\Phi_0)\omega+(2k_0+\Phi_0)(2g\Phi_0\mp\sqrt{\omega^2+4g^2\Phi_0^2}),\tag{7}$$
$$c_4=\Phi_0(\omega-2g\Phi_0\pm\sqrt{\omega^2+4g^2\Phi_0^2}),$$
$$\lambda=\Phi_0^2/2k_0\pm\sqrt{\omega^2+4g^2\Phi_0^2}/2.$$

For the components of the velocity of the particle we have[2]

$$v_x=2ic\mathrm{Im}(\psi_1^*\psi_2+\psi_3^*\psi_4)/\psi_i^*\psi_1,$$
$$v_y=-2c\mathrm{Re}(\psi_1^*\psi_2+\psi_3^*\psi_4)/\psi_i^*\psi_1,\tag{8}$$
$$v_z=2ic\mathrm{Im}(\psi_1^*\psi_3-\psi_2^*\psi_4)/\psi_i^*\psi_1.$$

from which it follows that

$$v_\parallel=v_x\,\cos(\omega\eta)-v_y\,\sin(\omega\eta)=0,$$
$$v_\perp=v_x\,\sin(\omega\eta)+v_y\,\cos(\omega\eta)=-c(\Phi_0/k_0)/(1+\Phi_0^2/2k_0^2),\tag{9}$$
$$v_z=-c(\Phi_0^2/2k_0^2)/(1+\Phi_0^2/2k_0^2),$$

where the subscripts $\parallel$ and $\perp$ denote the velocity components parallel and perpendicular to the electric field of the wave. Equations (9) show that the velocity of the Dirac particle is independent of its anomalous magnetic moment. Like a Dirac particle without an anomalous magnetic moment, it moves in a circularly polarized wave like a classical particle.

In the case where a constant magnetic field **H** acts on the Dirac particle with anomalous magnetic moment, moving in a circularly polarized wave, where this field is aligned with the direction of propagation of the wave, the functions ψ_i in the sum (4) are defined in the rotating coordinate system in which one of the two azimuthal axes is aligned with the electric field and the other, with the magnetic field of the wave and satisfy the following system of equations:

$$e^{-i\omega\eta}[(\hat{k}_+ + \Phi_0)\psi_2 + g\Phi_0(\psi_2 - i\psi_4)] + \left(\frac{\partial}{\partial z} + \omega\frac{\partial}{\partial\varphi}\right)\psi_3 + \left[k_0(1+\kappa) - i\omega\frac{\partial}{\partial\varphi} + \frac{1}{ic}\frac{\partial}{\partial t}\right]\psi_1 = 0,$$

$$e^{i\omega\eta}[(\hat{k}_- - \Phi_0)\psi_1 - g\Phi_0(\psi_1 + i\psi_3)] + \left(\frac{\partial}{\partial z} + \omega\frac{\partial}{\partial\varphi}\right)\psi_4 + \left[k_0(1-\kappa) + i\omega\frac{\partial}{\partial\varphi} - \frac{1}{ic}\frac{\partial}{\partial t}\right]\psi_2 = 0,$$

$$e^{-i\omega\eta}[-(\hat{k}_+ + \Phi_0)\psi_4 + g\Phi_0(\psi_4 + i\psi_2)] + \left(\frac{\partial}{\partial z} + \omega\frac{\partial}{\partial\varphi}\right)\psi_1 + \left[k_0(1-\kappa) + i\omega\frac{\partial}{\partial\varphi} - \frac{1}{ic}\frac{\partial}{\partial t}\right]\psi_3 = 0, \tag{10}$$

$$e^{i\omega\eta}[-(\hat{k}_- - \Phi_0)\psi_3 - g\Phi_0(\psi_3 - i\psi_1)] + \left(\frac{\partial}{\partial z} + \omega\frac{\partial}{\partial\varphi}\right)\psi_2 + \left[k_0(1+\kappa) - i\omega\frac{\partial}{\partial\varphi} + \frac{1}{ic}\frac{\partial}{\partial t}\right]\psi_4 = 0,$$

where

$$\hat{k}_\pm = \left(\frac{\partial}{\partial x_\parallel} - \frac{i\alpha x_\perp}{2}\right) \pm i\left(\frac{\partial}{\partial x_\perp} + \frac{i\alpha x_\parallel}{2}\right),$$

$$\frac{\partial}{\partial x} = x_\parallel\frac{\partial}{\partial x_{nt}} - x_\perp\frac{\partial}{\partial x_\parallel}, \tag{11}$$

$$g = g_1\frac{\omega}{2k_0}, \quad \kappa = g_1\frac{\alpha}{2k_0^2}, \quad \alpha = \frac{eH}{\hbar c}.$$

The solution of the system of equations (10) is given by

$$\psi_1 = i(a_1^+ + a_1^-)\exp\left[i\left(\lambda - \frac{\omega}{2}\right)\eta\right],$$

$$\psi_2 = i(a_2^+ + a_2^-)\exp\left[i\left(\lambda + \frac{\omega}{2}\right)\eta\right],$$

$$\psi_3 = i(a_1^+ - a_1^-)\exp\left[i\left(\lambda - \frac{\omega}{2}\right)\eta\right], \tag{12}$$

$$\psi_4 = i(a_2^+ - a_2^-)\exp\left[i\left(\lambda + \frac{\omega}{2}\right)\eta\right],$$

where λ is a constant, and a_i^+ are functions that depend on $x_\perp$ and $x_\parallel$ and satisfy the system of equations

$$\left[\frac{\hat{k}_+\hat{k}_-}{k_0} + 2\lambda + \kappa(2-\kappa)k_0 - \omega - 2i\omega\frac{\partial}{\partial\varphi}\right]a_1^+ + 2\kappa\hat{k}_+ a_2^- = \frac{\Phi_0}{k_0}[-2(\kappa+g)k_0 a_2^- + (\hat{k}_+ - \hat{k}_-)a_1^+], \tag{13}$$

$$\left[\frac{\hat{k}_-\hat{k}_+}{k_0} + 2\lambda - \kappa(2+\kappa)k_0 + \omega - 2i\omega\frac{\partial}{\partial\varphi}\right]a_2^- - 2\kappa\hat{k}_- a_1^+ = \frac{\Phi_0}{k_0}[-2(\kappa+g)k_0 a_1^+ + (\hat{k}_+ - \hat{k}_-)a_2^-]$$

and the relations

$$a_2^+ = (1+\kappa)a_2^- + (1/k_0)(\hat{k}_- - \Phi_0)a_1^+,$$
$$a_1^- = (1+\kappa)a_1^+ + (1/k_0)(\hat{k}_+ - \Phi_0)a_2^-. \tag{14}$$

For $\Phi_0 = 0$ the solution of system of equations (13) is given by

$$a_1^+ = \psi, \quad a_2^- = -\hat{k}_-\psi/k_0(2+\gamma), \tag{15}$$

or

$$a_1^+ = -\hat{k}_+\psi/k_0(2+\gamma), \quad a_2^- = \psi, \tag{16}$$

where ψ is the eigenfunction of the equation

$$(\hat{k}_+\hat{k}_- + \alpha)\psi = (\hat{k}_-\hat{k}_+ - \alpha)\psi = \varepsilon_n\psi, \tag{17}$$

and the constant γ is defined in terms of the eigenvalue ε_n by the formula

$$1+\gamma = \sqrt{1-(\varepsilon_n \mp \alpha)/k_0^2} = \sqrt{1+2n|\alpha|/k_0^2}, \tag{18}$$

where n is a positive integer. The function ψ coincides with the eigenfunction of the corresponding Schrödinger equation (time-dependent), which rotates with angular velocity ωc about the symmetry axis. Therefore

$$\psi = \psi_0 e^{im\omega ct}, \tag{19}$$

where m is the azimuthal quantum number in ψ_0. If the right-hand side of Eqs. (13) is treated as a perturbation, then to first order in Φ_0 the solution of system of equations (13) has the following form:

$$a_1^+ = \psi + \frac{\Phi_0}{2k_0}\left(\frac{r_{-+}}{R_{-+}}\hat{k}_+ + \left[\frac{r_{+-}}{R_{+-}} - \frac{\eta}{(2+\gamma)R_{+-}}\right]\hat{k}_-\right)\psi,$$
$$a_2^- = -\frac{\hat{k}_-\psi}{k_0(2+\gamma)} + \frac{\Phi_0}{2}\left(\frac{\gamma r_{-+} + \eta}{R_{-+}} - \frac{r_{+-}}{(2+\gamma)R_{+-}}\frac{\hat{k}_-^2}{k_0^2}\right)\psi \tag{20}$$

or

$$a_1^+ = -\frac{\hat{k}_+ \psi}{k_0(2+\gamma)} + \frac{\Phi_0}{2}\left(\frac{\gamma r_{--}+\eta}{R_{--}} - \frac{r_{++}}{(2+\gamma)R_{++}}\frac{\hat{k}_+^2}{k_0^2}\right)\psi, \tag{21}$$

$$a_2^- = \psi + \frac{\Phi_0}{2k_0}\left(\left[\frac{2_{++}}{R_{++}} + \frac{\eta}{(2+\gamma)R_{++}}\right]\hat{k}_+ + \frac{r_{--}}{R_{--}}\hat{k}_-\right)\psi,$$

where the constants $r_{\pm\pm}$, $R_{\pm\pm}$, and η are given by

$$r_{\pm\pm} = \alpha/k_0 + \omega \pm 2\kappa(1+\gamma)k_0 \pm 2\kappa(g+\kappa)k_0,$$

$$R_{\pm\pm} = (\alpha/k_0+\omega)^2 \pm 2\kappa(1+\gamma)k_0(\alpha/k_0+\omega) \pm 2\kappa^2\alpha, \tag{22}$$

$$\eta = 2g(2\alpha/k_0+\omega).$$

The remaining components of the wave function are given in terms of a_1^+ and a_2^- by formulas (14). To first order in Φ_0 the first wave function gives the following expressions for the components of the average velocity of the particle:

$$\frac{\bar{v}_\perp}{c} = \frac{2\Phi_0}{1+(1+\gamma-\kappa)^2}\left\{-\frac{2}{k_0}\frac{1+\gamma}{2+\gamma} + \left(\frac{\alpha r_{-+}}{k_0^2} - \frac{2\kappa}{2+\gamma}\eta\right)\times\frac{1}{R_{-+}} + \gamma\left[(1+\gamma-\kappa)\left(\frac{r_{+-}}{R_{+-}} - \frac{r_{-+}}{R_{-+}}\right) - \left(1-\frac{2\kappa}{2+\gamma}\right)\eta\left(\frac{1}{R_{+-}} - \frac{1}{R_{-+}}\right)\right]\right\}, \tag{23}$$

$$\bar{v}_\parallel = 0,$$

$$\frac{\bar{v}_z}{c} = \frac{1-(1+\gamma-\kappa)^2}{1+(1+\gamma-\kappa)^2},$$

and the second gives

$$\frac{\bar{v}_\perp}{c} = \frac{2\Phi_0}{1+(1+\gamma+\kappa)^2}\left\{-\frac{2}{k_0}\frac{1+\gamma}{2+\gamma} + \left(\frac{\alpha r_{--}}{k_0^2} + \frac{2\kappa}{2+\gamma}\eta\right)\times\frac{1}{R_{--}} - \gamma\left[(1+\gamma+\kappa)\left(\frac{r_{++}}{R_{++}} - \frac{r_{--}}{R_{--}}\right) + \left(1+\frac{2\kappa}{2+\gamma}\right)\eta\left(\frac{1}{R_{++}} - \frac{1}{R_{--}}\right)\right]\right\}, \tag{24}$$

$$\bar{v}_\parallel = 0,$$

$$\frac{\bar{v}_z}{c} = \frac{1-(1+\gamma+\kappa)^2}{1+(1+\gamma+\kappa)^2}.$$

In the coordinate system in which the z coordinate of the velocity is equal to zero, the velocity of the particle points in the same direction as the magnetic field of the wave and is given by

$$\frac{v}{c} = \Phi_0\left\{-\frac{2}{k_0}\frac{1+\gamma}{2+\gamma} + \left(\frac{\alpha r'_{-+}}{k_0^2} - \frac{2\kappa}{2+\gamma}\eta\right)\frac{1}{R'_{-+}} + \gamma\left[(1+\gamma-\kappa)\left(\frac{r'_{+-}}{R'_{+-}} - \frac{r'_{-+}}{R_{-+}}\right) - \left(1-\frac{2\kappa}{2+\gamma}\right)\times\eta\left(\frac{1}{R'_{+-}} - \frac{1}{R'_{-+}}\right)\right]\right\},$$

$$\frac{v}{c} = \Phi_0\left\{-\frac{2}{k_0}\frac{1+\gamma}{2+\gamma} + \left(\frac{\alpha r'_{--}}{k_0^2} + \frac{2\kappa}{2+\gamma}\eta\right)\frac{1}{R'_{--}} - \gamma\left[(1+\gamma+\kappa)\left(\frac{r'_{++}}{R'_{++}} - \frac{r'_{--}}{R_{--}}\right) + \left(1+\frac{2\kappa}{2+\gamma}\right)\eta\left(\frac{1}{R'_{++}} - \frac{1}{R'_{--}}\right)\right]\right\}, \tag{25}$$

where r'_{-+}, r'_{+-}, R'_{-+}, and R'_{+-} are obtained respectively from r_{-+}, r_{+-}, R_{-+}, and R_{+-} by replacing ω in them by $(1+\gamma-\kappa)\omega$, and r'_{--}, r'_{++}, R'_{--}, and R'_{++} are obtained from r_{--}, r_{++}, R_{--}, and R_{++} by replacing ω in them by $(1+\gamma+\kappa)\omega$. The resonance frequencies in formulas (25) are the roots of the quadratic equations $R'_{\pm\pm}(\omega) = 0$ and are given by

$$\omega_\pm^{(1)} = -\omega_H\frac{1+(g_1/2(\sqrt{1+2n|\omega_H/\omega_0|} \pm \sqrt{1+2(n+1)|\omega_H/\omega_0|})}{\sqrt{1+2n|\omega_H/\omega_0|} - (g_1/2)\omega_H/\omega_0},$$

$$\omega_\pm^{(2)} = -\omega_H\frac{1-(g_1/2(\sqrt{1+2n|\omega_H/\omega_0|} \pm \sqrt{1+2(n-1)|\omega_H/\omega_0|})}{\sqrt{1+2n|\omega_H/\omega_0|} - (g_1/2)\omega_H/\omega_0},$$

$$\omega_\pm^{(3)} = -\omega_H\frac{1-(g_1/2(\sqrt{1+2n|\omega_H/\omega_0|} \pm \sqrt{1+2(n+1)|\omega_H/\omega_0|})}{\sqrt{1+2n|\omega_H/\omega_0|} + (g_1/2)\omega_H/\omega_0},$$

$$\omega_\pm^{(4)} = -\omega_H\frac{1+(g_1/2(\sqrt{1+2n|\omega_H/\omega_0|} \pm \sqrt{1+2(n-1)|\omega_H/\omega_0|})}{\sqrt{1+2n|\omega_H/\omega_0|} + (g_1/2)\omega_H/\omega_0}, \tag{26}$$

where $\omega_H = eH/m_0c$ is the cyclotron frequency, $\omega_0 = m_0c^2/\hbar$ is the eigenfrequency of the particle, and n is a positive integer. The negative sign in $\omega_\pm^{(i)}$ means that the resonance occurs for a wave with left circulation at the frequency $|\omega_\pm^{(i)}|$. As follows from formulas (23)–(26), the velocity of a Dirac particle with an anomalous magnetic moment in an electromagnetic wave in a longitudinal magnetic field, in contrast to the velocity of a Dirac particle without an anomalous magnetic moment[4] does not coincide with the velocity of the classical particle.

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Translated by Paul F. Schippnick