Supersonic flow past finite-length bodies in dispersive hydrodynamics

A. V. Gurevich

Theoretical Section of the P. N. Lebedev Physics Institute, Russan Academy of Sciences, 117924 Moscow, Russia

A. L. Krylov, and V. V. Khodorovskiĭ,

O. Yu. Schmidt Institute of Physics of the Earth, Russian Academy of Sciences, 123810 Moscow, Russia.

G. A. Él'

Institute of Terrestrial Magnetism, The Ionosphere, and the Propagation of Radio Waves, Russian Academy of Sciences, 142092 Troitsk, Moscow Province, Russia

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Using the Gurevich–Pitaevskiĭ approach, an analytic study is made of two-dimensional supersonic flow past slender bodies of finite length in nondissipative dispersive hydrodynamics. The problem of recovering the shape of the body from data on the wake at infinity is solved. It is shown that under conditions when the KdV approximation is valid the nonlinearity and dispersion do not affect the macroscopic properties of the flow—the drag and lift.

1. INTRODUCTION

The present paper is a continuation of Ref. 1 and is devoted to the study of supersonic flow of a nondissipative weakly dispersive fluid past pointed bodies. If the bodies are slender, then the general equations of steady two-dimensional dispersive hydrodynamics reduce to the Korteweg–de Vries (KdV) equation, in which the role of the time is played by the spatial coordinate at right angles to the direction of flow. The initial data for the KdV equation are determined by an arbitrary function $y(x)$, which specifies the profile of the body.[1,2]

The most important feature of supersonic dispersive flow past bodies is the formation of a nondissipative shock wave, a wedge shaped region of space occupied by small-scale nonlinear oscillations described by modulated elliptic functions. The oscillations have the shape of solitons on the front facing the oncoming flow and the shape of harmonic oscillations of infinitesimally small amplitude on the opposite front toward the body. In Ref. 1, the Gurevich–Pitaevskiĭ approach[3] was used to make an analytic study of the structure of the nondissipative shock waves that arise in the case of flow past infinite wedge-shaped bodies with $y''(x) \geqslant 0$. Such profiles correspond to monatomic initial data in the Gurevich–Pitaevskiĭ problem (the case of a nondissipative shock wave with intensity that does not decrease with time). However, it is important that by virtue of the supersonic nature of the motion the expressions obtained in Ref. 1 have finite "domains of influence" and can be used in the description of the flow past finite sections of bodies of a more complicated shape.

In this paper, we consider flow past thin pointed bodies possessing in profile a section (of finite or infinite extent) with $y''(x) < 0$, in particular we consider flow past finite-length bodies. Such a change in the geometry of the body, which would appear to be a minor one (compared with Ref. 1), leads to a significant modification in the solution to the problem. Difficulties arise manily because of the nonmonotonicity of the initial data in the corresponding Gurevich–Pitaevskiĭ evolution problem. In addition, it is often the case that the typical shapes of the bodies around which the flow takes place (see, for example, Fig. 1c) correspond to initial data that are not at all characteristic of the Gurevich–Pitaevskiĭ problem (Fig. 1d). Finally, flow past bounded bodies is accompanied by the formation of two nondissipative shock waves that possess different asymptotic properties. At the same time, it is clear that precisely these cases are the ones of greatest interest from the point of view of applications.

We note also that, since the supersonic nature of the flow makes it possible to study the flow in the upper half-plane independently, our problem can be interpreted as the problem of the flow past a convex (or concave) inhomogeneity on the bottom of a flat channel.

As in Ref. 1, to describe the rapidly oscillating region of the nondissipative shock wave we use Whitham's method of averaging.[4] Whitham's system for the KdV equation with the Gurevich–Pitaevskiĭ matching conditions is integrated by means of a generalized hodograph transformation[5,6] and the "scalar potential" technique,[7–9] but by virtue of the nonmonotonicity of the initial data the corresponding transforms have a "two-sheeted" nature.[10–12] The relatively simple asymptotic behavior of the solution for localized initial data makes it possible to efficiently solve the problem of recovering the shape of the body in the flow from data on the wake at infinity (recall that we are studying the purely nondissipative situation). We show that in the case of flow past a slender body under conditions for which the KdV approximation is valid the nonlinearity and the dispersion do not affect the drag and lift, which are the most important macroscopic characteristics of the flow (Ref. 13, § 125).

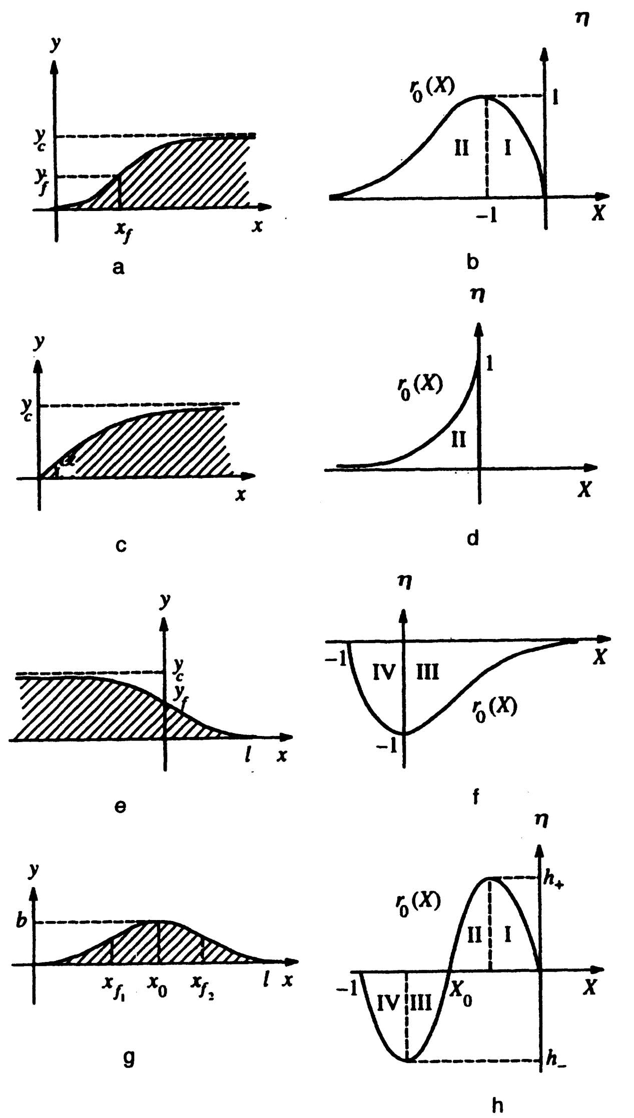

FIG. 1. Typical profiles of bodies (a, c, e, g) and the initial data corresponding to them (b, d, f, h) in the evolution problem.

2. BASIC EQUATIONS. FORMULATION OF THE PROBLEM

For convenience, we give some results of Ref. 1 needed for the subsequent exposition.

The problem of supersonic two-dimensional (x,y) flow past a slender pointed body with nonzero angle of attack by a nondissipative weakly dispersive fluid that has velocity $M\mathbf{e}_x$ $(M>1)$ at infinity reduces to the following initial-value problem for the KdV equation with small dispersion parameter:[1,2]

$$\partial_T\eta+\eta\partial_X\eta+\varepsilon^2\partial_{XXX}^3\eta=0, \quad \varepsilon\ll 1, \tag{1}$$

$$\eta(X,0)=\begin{cases} r_0(X)\equiv F'(-X) & \text{for } X\leqslant 0,\\ 0 & \text{for } X>0. \end{cases} \tag{2}$$

Here the function

$$\Phi(x,y)=y-bF(x/l)=0,$$

$$x\geqslant 0, \quad y\geqslant 0, \quad F(0)=0, \quad F(X)=O(1) \tag{3}$$

determines the shape of the section of the body (an infinite "sharp" cylinder with generators parallel to the z axis) in the upper half-plane (by virtue of the supersonic nature of the flow, the motions in the upper and lower half-planes are independent), b and l are the effective thickness and length of the profile, respectively, $\delta=b/l\ll 1$ is a small nonlinearity parameter, and ε is the effective dissipation parameter. In addition, in the derivation of the KdV equation the flow near the surface of the body is assumed to be smooth (absence of boundary-layer effects). For example, in a two-temperature plasma[1] this condition is ensured by the absence of free charges on the surface of the (nonconducting) body. The relationships between the quantities that occur in the KdV equation and the original flow parameters are given by

$$X=\frac{\sqrt{M^2-1}\,y-x}{l}, \quad T=\frac{M^4(\gamma+1)}{2(M^2-1)}\,\delta\,\frac{y}{l}, \tag{4a}$$

$$\eta=\frac{\sqrt{M^2-1}}{M}\,u_1, \quad \varepsilon^2=\frac{2\sqrt{M^2-1}\,\beta^2}{(\gamma+1)\,\delta}. \tag{4b}$$

Here γ is the adiabatic exponent of the corresponding ideal hydrodynamics, $\beta=D/l\ll 1$ (more precisely, $\delta^2\ll\beta^2\ll\delta$), where D is the characteristic dispersion scale of the medium (for example, the Debye radius in a two-temperature plasma[1]), and u_1 is the first correction to the horizontal component of the flow velocity (for more details, see Ref. 1):

$$u_x=M+\delta u_1+O(\delta^2),$$

$$u_y=\delta v_1+O(\delta^2),$$

$$n=1+\delta n_1+O(\delta^2),$$

$$n_1=-Mu_1, \quad v_1=-\sqrt{M^2-1}\,u_1. \tag{5}$$

Note that δ has the order of the maximum of the generating function that specifies the profile of the body in the flow.

The evolution described by (1)–(2) leads to the formation of a nondissipative shock wave described by a quasisteady modulated solution of the KdV equation.[1,3,6] This solution is characterized by three parameters $r_i(X,T)$: $r_3\geqslant r_2\geqslant r_1$. The finding of the modulation parameters $r_i(X,T)$ that ensure continuous matching of the exterior smooth flow described by the Hopf equation (the nondissipative limit of the KdV equation) to the average flow in the oscillating region of the nondissipative shock wave for different profiles (3) is the main task in the theory of supersonic flow past slender bodies in dispersive hydrodynamics.

The functions $r_i(X,T)$ are Riemann invariants of the Whitham modulation system[4,3,6]

$$\partial_T r_i+V_i(\mathbf{r})\partial_X r_i=0, \quad i=1,2,3, \tag{6}$$

where[7,8]

$$V_i(\mathbf{r})=\left[1-\frac{L}{\partial_i L}\,\partial_i\right]U, \quad \partial_i\equiv\frac{\partial}{\partial r_i}, \quad U=\frac{1}{3}(r_1+r_2+r_3),$$

$$L=\int_{r_1}^{r_2}\frac{d\tau}{\sqrt{(\tau-r_1)(\tau-r_2)(\tau-r_3)}}=\frac{2K(m)}{\sqrt{r_3-r_1}}. \tag{7}$$

Here U is the phase velocity, $\sqrt{6}\varepsilon L$ is the wavelength, $K(m)$ is the complete elliptic integral of the first kind,[14] and $m=(r_2-r_1)/(r_3-r_1)$ is the parameter of the elliptic function: $m=0$ on the trailing (linear) edge of the nondissipative shock wave $X^-(T)$ and $m=1$ on the leading (soliton) edge $X^+(T)$.

In this paper, we shall consider only quasisimple nondissipative shock waves[10–12] with $r_1\equiv 0$ (or with $r_3\equiv 0$). The conditions for the formation of such waves in the problem of flow past bodies are considered in detail in Ref. 1 and consist essentially of the absence of points on the profile of the body of points with vanishing third derivative. The required modulation **r** in the quasisimple nondissipative shock wave must satisfy the Gurevich–Pitaevskiĭ matching conditions on the boundaries $X^{\pm}(T)$:

$$r_3(X^-,T)=r(X^-,T),\quad r_2(X^-,T)=0,$$
$$r_2(X^+,T)=r_3(X^+,T), \tag{8}$$

where $r(X,T)$ is the solution of the Hopf equation with initial data (2) given by the implicit formula

$$x-rT=W(r), \tag{9}$$

where the function $W(r)$ is the inverse of $r_0(X)$.

The general solution of the system (6)–(7) is given by the hodograph transformation in generalized form:[5–9]

$$X-V_i(\mathbf{r})T=\left(-\frac{L}{\partial_i L}\partial_i\right)f,\quad \mathbf{r}\equiv(0,r_2,r_3),\quad i=2,3, \tag{10}$$

where the scalar potential f satisfies the Euler–Poisson equation

$$2(r_3-r_2)\partial_{32}^2 f=\partial_3 f-\partial_2 f. \tag{11}$$

Then the nonlinear Gurevich–Pitaevskiĭ conditions (8) on the unknown boundaries go over into a very simple linear condition on the coordinate axis $r_2=0$ (in the hodograph plane):[7]

$$f(0,r_3)=\frac{1}{2}\frac{1}{\sqrt{r_3}}\int_0^{r_3}\frac{1}{\sqrt{z}}W(z)dz \tag{12}$$

together with a certain condition whose form depends on the monotonicity properties of the initial data $r_0(X)$ (and, accordingly, on the behavior of the second derivative of the function that specifies the profile of the body). In the case of monotonic initial data (absence of points of inflection on the profile of the body), this condition reduces to the requirement that the solution be bounded on the diagonal $r_2=r_3$.

The geometrical boundaries of the nondissipative shock wave are the common caustics of the two families of characteristics for $m=0$ and $m=1$ on the family of solutions of (10), namely

$$\frac{dX^+}{dT}=V_2(0,r_3^+,r_3^+)=V_3(0,r_3^+,r_3^+)=\frac{2}{3}r_3^+, \tag{13a}$$

$$\frac{dX^-}{dT}=V_1(0,0,r_3^-)=V_2(0,0,r_3^-)=-r_3^-, \tag{13b}$$

where $r_3^{\pm}$ are the values of the invariant r_3 on the fronts that parametrize the equations of the boundaries. The functions $X^{\pm}(T)$ are transformed to the physical variables x and y by means of the linear transformations (4a).

3. FLOW PAST THE LEADING EDGE OF A THIN INFINITE WING

In our previous study of Ref. 1, we constructed solutions of the modulation equations corresponding to profiles of bodies (3) that become thicker monotonically, with $y''(x)\geqslant 0$, corresponding to monotonically decreasing initial data in the Gurevich-Pitaevskiĭ evolution problem. As we noted in the previous section, the required potential f, besides satisfying the condition (12), must be bounded on the leading edge $r_2=r_3$ of the wave, ensuring that the matching (8) can occur at finite times. These two requirements distinguish a unique solution of the Euler–Poisson equation (11):

$$f_{\rm I}(0,r_3)=\int_{r_2}^{r_3}\frac{\phi_{\rm I}(\tau)d\tau}{\sqrt{\tau(r_3-\tau)(\tau-r_2)}}, \tag{14}$$

$$\phi_{\rm I}(\tau)=\frac{1}{2\pi}\int_0^{\tau}\frac{W_{\rm I}(x)dx}{\sqrt{\tau-x}}. \tag{15}$$

Here, the index I denotes the solution corresponding to a monotonically decreasing section of the initial data $r_0(x)$ (2) [since this problem is hyperbolic, finite sections of the profile of the body have finite (for $T\sim 1$) domains of influence, which are bounded by the characteristics that emanate from the corresponding points of the x axis, onto which the profile of the body is "carried"[1]].

We now consider flow past the sharp leading edge of an infinite thin wing whose profile is given by a function $y(x)$ having a point of inflection at some point (x_f,y_f) (Fig. 1a); more precisely,

$$y(0)=0,\quad y'(0)=0,\quad y'(x_f)=\delta,$$
$$y''(x)\begin{cases}\to\infty & \text{as } x\to 0,\\ \geqslant 0 & \text{for } 0<x\leqslant x_f,\\ <0 & \text{for } x>x_f,\end{cases}$$
$$y(x)\to\text{const}\quad\text{as } x\to\infty. \tag{16}$$

The condition for the applicability of the KdV approximation has the form

$$y'(x_f)\ll 1, \tag{16a}$$

and as characteristic length it is natural to choose $l=x_f$.

Such a contour corresponds to a localized initial profile $r_0(X)$ (Fig. 1b), where $r_{0\,\max}=h_+$ [for the above choice of l and δ, $r_{0\,\max}=r_0(-1)$, $h_+=1$]. The fact that the second derivative of the contour function becomes infinite at the point of sharpening [$y(x)\propto x^q$, $1<q<2$ as $x\to 0$] ensures the condition for formation of a quasisimple nondissipative shock wave[1,10,11] (breaking in the corresponding Gurevich–Pitaevskiĭ problem occurs on the boundary with the homogeneous flow), and the point of formation of the nondissipative shock wave coincides with the origin. For $q=2$, breaking occurs at a certain time $T>0$, indicating separation of the nondissipative shock wave from the edge of the body.[1] Nevertheless, all the expressions that are obtained remain valid apart from a shift with respect to T. The case $q>2$ is more complicated, but the structure of the solution is not changed.[1]

Since the function $W(r)$ that is the inverse of $r_0(X)$ is two-valued, it is necessary to introduce a so-called two-sheeted hodograph transformation $(r_1,r_2)\to(X,T)$ (Refs. 10 and 11). This transformation is possible because Eqs. (6) remains hyperbolic when the independent variables are replaced by the dependent variables [we note that in the resulting Euler–Poisson equation (11) the characteristics are lines parallel to the coordinate axes r_2,r_3]. This makes it possible to distinguish on the hodograph plane the domains of influence of the monotonic sections of the initial data—the "sheets"—and then match the corresponding solutions along the characteristic $r_3=h_+$. As a result, the solution (10) of the Gurevich–Pitaevskiĭ problem on different sheets of the hodograph plane is determined by two different functions $f_{\mathrm{I,II}}$, which satisfy the Euler–Poisson equation and the boundary conditions

$$f_{\mathrm{I,II}}(0,r_3)=\frac{1}{3}\frac{1}{\sqrt{r_3}}\int_0^{r_3}\frac{i}{\sqrt{z}}W_{\mathrm{I,II}}(z)dz,$$

$$f_{\mathrm{I}}(r_3,r_3)\ \text{is bounded},$$

$$f_{\mathrm{II}}(r_2,h_+)=f_{\mathrm{I}}(r_2,h_+). \tag{17}$$

Here $W_{\mathrm{I,II}}(r)$ are monotonic branches of the inverse function $W(r)$. The function $f_{\mathrm{I}}(r_2,r_3)$ is given by the expressions (14) and (15). The solution $f_{\mathrm{II}}(r_2,r_3)$ has the form[11]

$$f_{\mathrm{II}}(r_2,r_3)=\int_{r_2}^{r_3}\frac{\phi_{\mathrm{II}}(\tau)d\tau}{\sqrt{\tau(r_3-\tau)(\tau-r_2)}}+\int_0^{r_2}\frac{\psi(\tau)d\tau}{\sqrt{\tau(r_3-\tau)(\tau-r_2)}}, \tag{18}$$

where

$$\phi_{\mathrm{II}}(\tau)=\frac{1}{2\pi}\int_0^{\tau}\frac{W_{\mathrm{II}}(x)}{\sqrt{\tau-x}}dx,$$

$$\psi(\tau)=\frac{1}{2\pi}\int_{\tau}^{h_+}\frac{D(x)dx}{\sqrt{x-\tau}},\quad D(x)=W_{\mathrm{I}}(x)-W_{\mathrm{II}}(x). \tag{19}$$

Changing the order of integration in (14) and (18) gives a convenient representation in the form of simple integrals:

$$f_{\mathrm{I}}(r_2,r_3)=\frac{1}{\pi\sqrt{r_3-r_2}}\int_{r_2}^{r_3}\frac{W_{\mathrm{I}}(x)}{\sqrt{x}}K(z)dx+\frac{1}{\pi\sqrt{r_2}}\int_0^{r_2}\frac{W_{\mathrm{I}}(x)}{\sqrt{r_3-x}}K\left(\frac{1}{z}\right)dx, \tag{20a}$$

$$f_{\mathrm{II}}(r_2,r_3)=f_{\mathrm{I}}(r_2,r_3)+\frac{1}{\pi\sqrt{r_3}}\int_{r_3}^{h_+}\frac{D(x)}{\sqrt{x-r_2}}K(z_1)dx, \tag{20b}$$

where

$$z=\frac{r_2(r_3-x)}{x(r_3-r_2)},\quad z_1=\frac{r_2(x-r_3)}{r_3(x-r_2)}.$$

We now find the boundaries of the nondissipative shock wave, which are determined by Eqs. (13), in the family of solutions (10), (20). On sheet I, we have for the trailing edge the parametric formulas[1]

$$X_{\mathrm{I}}(r)=W_{\mathrm{I}}(r)-\frac{\sqrt{r}}{2}\int_0^r\frac{W_{\mathrm{I}}'(x)}{\sqrt{x}}dx,$$

$$T_{\mathrm{I}}(r)=-\frac{1}{2\sqrt{r}}\int_0^r\frac{W_{\mathrm{I}}'(x)}{\sqrt{x}}dx. \tag{21}$$

For the leading edge,

$$X_{\mathrm{I}}^+(r)=\frac{1}{2\sqrt{r}}\int_0^r\frac{W_{\mathrm{I}}(x)-xW_{\mathrm{I}}'(x)}{\sqrt{r-x}}dx,$$

$$T_{\mathrm{I}}^+(r)=-\frac{3}{4r^{3/2}}\int_0^r\frac{xW_{\mathrm{I}}'(x)}{\sqrt{r-x}}dx. \tag{22}$$

The subsequent behavior of the trailing edge (sheet II) is described by the expressions

$$X_{\mathrm{II}}^-(r)=X_{\mathrm{I}}^-(r)-D(r)-\frac{\sqrt{r}}{2}\int_r^{h_+}\frac{D'(x)}{\sqrt{x}}dx,$$

$$T_{\mathrm{II}}^-(r)=T_{\mathrm{I}}^-(r)-\frac{1}{2\sqrt{r}}\int_r^{h_+}\frac{D'(x)}{\sqrt{x}}dx. \tag{23}$$

The leading edge of the nondissipative shock wave on the second sheet goes over asymptotically to the straight line

$$X_{\mathrm{II}}^+=\frac{2}{3}h_+T_{\mathrm{II}}^+, \tag{24}$$

which corresponds to the motion of a bow soliton (with fixed amplitude $2h_+$). We recall that all the obtained expressions are converted to the physical variables x and y by means of the linear transformations (4a).

The geometrical boundaries of the nondissipative shock wave in the xy plane are shown qualitatively in Fig. 2(a). The characteristic $r_3=h_+$ emanating from the point of inflection separates the regions corresponding to the first and second sheets of the hodograph plane.

The intensity of the nondissipative shock wave (understood as the jump of the hydrodynamic variables across the wave) increases monotonically in the region of sheet I up to the section $y=y^*$, where it takes its maximum value. With further motion along the y axis, the intensity decreases, tending to zero at infinity. Nevertheless, the integrated energy of the oscillations in each section $y=$const remains finite.

At large distances from the body, the nondissipative shock wave is transformed into a soliton wave—a train of a large number of noninteracting solitons whose amplitudes vary regularly.[10,11] The distance between the individual solitons increases with distance from the body. The soliton train can be described by two functions, which are conveniently

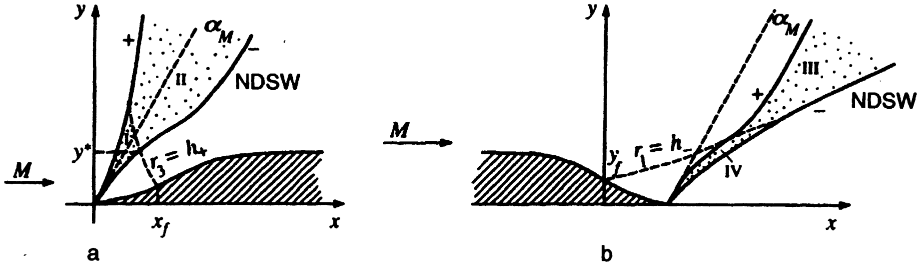

FIG. 2. a) Flow past the leading edge of the wing; b) flow past the trailing edge (NDSW denotes the nondissipative shock wave).

taken to be the distributions of the amplitude $a=2r_2$ (see [1]) and the wave number $k=2\pi/L$ [see (7)]. The solutions (10), (20) give in the limit $T\to\infty$ the following asymptotic expressions:[11]

$$m=\frac{r_2}{r_3}\to 1,$$

$$\varepsilon k(X,T)\simeq-\frac{\sqrt{6}}{4T}\int_{\sigma}^{h_+}\frac{D'(r)dr}{\sqrt{r-\sigma}},\quad \sigma=\frac{3}{2}\frac{X}{T},$$

$$a(X,T)\simeq 2\sigma-\frac{X_0(\sigma)}{T},\tag{25a}$$

$$X_0(\sigma)=\frac{3}{2\pi\sqrt{\sigma}}\int_{\sigma}^{h_+}\frac{\ln\left(\frac{r-\sigma}{r}\right)D(r)}{\sqrt{r-\sigma}}dr+\frac{3}{2\sqrt{\sigma}}\int_0^{\sigma}\frac{W_{\rm I}(r)dr}{\sqrt{\sigma-r}}.\tag{25b}$$

The expressions (25) establish a unique relationship between the parameters of the soliton train and the initial data $r_0(X)$ in the Gurevich–Pitaevskiĭ problem. The problem of recovering the shape of the body in the flow from data on the wake at infinity will be considered in Sec. 6.

The general expressions (18)–(25) simplify considerably if the body is not infinitely sharp but begins with a certain small angle δ [cf. (16)]:

$$y(0)=0,\quad y'(0)=\delta,\quad y''(0)<0\quad \text{for } x\geqslant 0,$$
$$y(x)\to\text{const}\quad \text{as } x\to\infty.\tag{26}$$

To this profile (Fig. 1c) correspond Gurevich–Pitaevskiĭ initial data in the form of a sawtooth pulse (Fig. 1d). For the solution on the first sheet, this case is degenerate (since $W_{\rm I}\equiv 0$, $W_{\rm II}\equiv W$), and therefore the hodograph transformation (10) is essentially single sheeted, and the solution is completely determined by its values on the second sheet and has the form

$$f=-\frac{1}{\pi\sqrt{r_3}}\int_{r_3}^{h_+}\frac{W(x)}{\sqrt{x-r_2}}K\left(\frac{r_2(x-r_3)}{r_3(x-r_2)}\right)dx.\tag{27}$$

The solution (27) is not bounded as $r_2\to r_3$ ($m\to 1$). This means that the solitons in such a nondissipative shock wave are realized only asymptotically as $T\to\infty$ (i.e., at infinite distance from the body). The boundaries of the nondissipative shock wave are now given by

$$X^-(r)=W(r)+rT^-(r),$$

$$T^-(r)=\frac{1}{2\sqrt{r}}\int_r^{h_+}\frac{W'(x)}{\sqrt{x}}dx,$$

$$X^+=\frac{2}{3}h_+T^+.\tag{28}$$

4. FLOW PAST THE TRAILING EDGE OF A THIN INFINITE WING

We now consider the geometrically opposite situation—the flow past the trailing edge of the wing (Fig. 1e). The flow behind the wing is unperturbed. At the same time, in the case of flow past a decreasing section of the profile, as in ordinary hydrodynamics, a rarefaction wave is formed, leading to the collision of flows of different densities in the region behind the wing. Thus, here too a shock wave arises, and in dispersive hydrodynamics it is a nondissipative shock wave, i.e., it has an oscillating structure and expands with increasing distance from the body.

Like the one considered in the previous section, this nondissipative shock wave is in the general case "two-sheeted": Its intensity first increases up to a certain section that corresponds to the characteristic which arrives from the point of inflection, and it then decreases, tending to zero at infinity. However, the asymptotic form of this nondissipative shock wave is completely different: It is a "solitonless" wave and at large distances from the body degenerates into a linear modulated wave of vanishingly small amplitude. At the same time, like the one considered earlier, this "solitonless" nondissipative shock wave carries away a finite energy (more precisely, an energy of order δ), and its asymptotic modulation also carries complete information about the profile (which is now a decreasing one).

It should also be noted that despite the "solitonless" nature of the wave nonlinear oscillations are realized at its leading edge as an intermediate asymptotic behavior, and these oscillations have a profile close to solitons whose amplitude decreases gradually with increasing distance from the wing.

For convenience in what follows in the analytic description of the flow past the trailing edge, we place the point of inflection at the origin ($x_f=0$) and take the coordinate of the end of the profile of the wing at l (Fig. 1e). The corresponding initial data of the Gurevich–Pitaevski problem are given in Fig. 1f [if we take $\delta=-y'(x_f)$, then $h_-=1$]. The evolution of a localized solitonless perturbation of this kind was investigated in Ref. 12 (the corresponding results for rapidly decreasing potentials can be found in Refs. 15 and 16, and the generalization of the Lax–Levermore theory[17] to the case of a nonvanishing reflection coefficient is given in Ref. 18).

In the present case, breaking at the point $(-1;0)$ leads to the formation of a quasisimple nondissipative shock wave with $r_3\equiv 0$ and varying $r_1,r_2<0$. The required modulation is also given by a two-sheeted hodograph transformation [cf. (10)]:

$$1+X-V_i(\mathbf{r})T=\left(1-\frac{L}{\partial_i L}\,\partial_i\right)f,$$

$$\mathbf{r}\equiv(r_1,r_2,0),\quad i=1,2, \tag{29}$$

where the scalar potential f is defined on the sheets III and IV (we retain the notation of the sheets I and II to describe the qualitatively different nondissipative shock wave considered in the previous section) corresponding to the monotonic slopes of the "well" (see Fig. 1f):

$$f_{\rm IV}(r_1,r_2)=\frac{1}{\pi\sqrt{r_2-r_1}}\int_{r_1}^{r_2}\frac{W_{\rm IV}(x)}{\sqrt{-x}}K(z)dx$$

$$+\frac{1}{\pi\sqrt{-r_2}}\int_{r_z}^{0}\frac{W_{\rm IV}(x)}{\sqrt{x-r_1}}K\left(\frac{1}{z}\right)dx, \tag{30a}$$

$$f_{\rm III}(r_1,r_2)=f_{\rm IV}(r_1,r_2)$$

$$+\frac{1}{\pi\sqrt{-r_1}}\int_{h_-}^{r_1}\frac{D-(x)}{\sqrt{r_2-x}}K(z_1)dx. \tag{30b}$$

Here

$$z=\frac{r_2(x-r_1)}{x(r_2-r_1)},\quad z_1=\frac{r_2(r_1-x)}{r_1(r_2-x)},$$

$$D_-(x)=W_{\rm IV}(x)-W_{\rm III}(x),\quad D_-(h_-)=0,$$

where $W_{\rm III}$, $W_{\rm IV}$ are the functions that are the inverses of the monotonic branches of the initial perturbation (Fig. 1f).

The solution (29)–(30) ensures the matching [cf. (8)]

$$r_1(X^+,T)=r(X^+,T),\quad r_2(X^+,T)=0,$$

$$r_2(X^-,T)=r_1(X^-,T). \tag{31}$$

In this case too, the boundaries of the nondissipative shock wave are found as the common caustics of the two families of characteristics [cf. (13)]

$$\frac{dX^+}{dT}=V_2(r_1^+,0,0)=V_3(r_1^+,0,0)=\frac{1}{3}r_1^+,$$

$$\frac{dX^-}{dT}=V_1(r_1^-,r_1^-,0)=V_2(r_1^-,r_1^-,0)=2r_1^- \tag{32}$$

on the family of solutions (29), (30). Here $r_1^\pm$ are the values of the invariant r_1 on the fronts that parametrize the equations of the boundaries. As a result, for the leading edge on sheet IV we obtain

$$X_{\rm IV}^+(r)=\frac{3}{4\sqrt{-r}}\int_r^0\frac{W_{\rm IV}(x)}{\sqrt{-x}}dx-\frac{W_{\rm IV}(r)}{2},$$

$$T_{\rm IV}^+(r)=-\frac{3}{4(-r)^{3/2}}\int_r^0\frac{W_{\rm IV}(x)}{\sqrt{-x}}dx-\frac{3W_{\rm IV}(r)}{2r}. \tag{33}$$

For the trailing edge, we have

$$X_{\rm IV}(r)=\sqrt{-r}\int_r^0\frac{xW_{\rm IV}''(x)}{\sqrt{x-r}}dx-1,$$

$$T_{\rm IV}^-(r)=-\frac{1}{4(-r)^{1/2}}\int_r^0\frac{W_{\rm IV}'(x)+2rW_{\rm IV}''(x)}{(x-r)^{1/2}}dx. \tag{34}$$

The further behavior of the leading edge (sheet III) is described by the expressions

$$X_{\rm III}^+(r)=X_{\rm IV}^+(r)+\frac{D_-(r)}{2}+\frac{3}{4\sqrt{-r}}\int_{h_-}^{r}\frac{D_-(x)}{\sqrt{-x}}dx,$$

$$T_{\rm III}^+(r)=T_{\rm IV}^+(r)+\frac{3D_-(r)}{2r}\,\frac{-3}{4(-r)^{3/2}}\int_{h_-}^{r}\frac{D_-(x)}{\sqrt{-x}}dx. \tag{35}$$

The trailing edge goes over asymptotically to the straight line

$$X_{\rm III}^-=2h_-T_{\rm III}^- \tag{36}$$

As we have already noted, in the case at hand the nondissipative shock wave at large distances from the wing is converted into a linear wave packet. The corresponding asymptotic behavior of the solution (29), (30) as $T\to\infty$ has the form[12]

$$m=\frac{r_2-r_1}{-r_1}\to 0,\quad r_2\to r_1\to r^*,$$

$$X=2r^*T+2(-r^*)^{1/4}T^{1/2}A^{1/2}(r^*)+O(T^{1/4}), \tag{37}$$

where

$$A(y)=-\int_{h_-}^{y}\frac{D_-(r)}{\sqrt{y-r}}dr.$$

For the amplitude and wave number we have the asymptotic behavior

$$a(X,T)=2(r_2-r_1)=\frac{4}{\sqrt{T}}\,\tau^{1/4}A^{1/2}(-\tau)+O\left(\frac{1}{T^{3/4}}\right), \tag{38a}$$

$$\varepsilon^2k^2(X,T)=\frac{2}{3}\tau-\frac{1}{6}a(X,T)+O\left(\frac{1}{T}\right), \tag{38b}$$

$$\tau=-\frac{X}{2T}.$$

The nature of the decrease of the amplitude in (38) reflects the law of conservation of the energy in the linear medium. The first term in the expansion (38b) corresponds to the motion of a linear wave packet with group velocity $\omega'(k)=3\varepsilon^2k^2$.

Like the asymptotic expressions (25), the expressions (38) establish a connection between the asymptotic modulation of the nondissipative shock wave and the profile of the body in the flow. However, it should be noted that this connection is single valued only for a wing having a trailing edge with small (but finite) edge angle [see (26)]. At the same time, $W_{\rm IV}\equiv-1$, $\hat{D}_-=-1-W_{\rm III}$. The general expressions that make it possible to determine both branches of the inverse function W are rather lengthy, and we shall not give them here.

The geometrical boundaries of the nondissipative shock wave in the xy plane are shown in Fig. 2b. The characteristic $r_1=h$ emanating from the point of inflection separates the regions corresponding to sheets III and IV of the hodograph plane.

5. FLOW PAST A FINITE BODY (WING WITH POINTED EDGES)

We now turn to a more realistic example—two-dimensional flow past finite bodies. We consider a fairly general case in which the finite profile has one point of inflection on each side of the maximum and vanishing derivatives at the ends (Fig. 1g).

The analytic description of the flow past such a wing reduces to the solution of the Gurevich–Pitaevskiĭ problem with initial perturbation of nonconstant sign (Fig. 1h). As effective length l, it is convenient to choose the width of the wing, and as the effective thickness b its maximum thickness $y(x_0)$. Then the thin-wing approximation $\delta=b/l\ll 1$ will be valid everywhere if

$$y'(x)\lesssim\delta \quad \text{for } 0<x<l.$$

In the chosen normalization, the initial data (2), (3) for the KdV equation have the form (Fig. 1h)

$$\eta(X,0)=\begin{cases}0 & \text{for } X<-1,\\ r_0(X)\equiv F'(-X) & \text{for } -1\leqslant X\leqslant 0,\\ 0 & \text{for } X>0\end{cases} \qquad (39a)$$

and

$$F(0)=F(-1)=0 \quad \text{(zero angle of attack)},$$

$$r_{0\,\max}=h_+\simeq\frac{\alpha_0}{\delta}>0,\quad r_{0\,\min}=h_-\simeq-\frac{\alpha_l}{\delta}<0, \qquad (39b)$$

where α_0 and α_1 are the small angles of inclination of the profile of the body at the points of inflection (Fig. 3a).

For convenience, we also assume that $r_0'\to-\infty$, $r_0'(-1)\to-\infty$, this ensuring that the nondissipative shock wave does not separate from the edge of the body. An obvious property of the function (39a) is the vanishing of the integral

$$\int_{-1}^{0} r_0(x)dx=0, \qquad (40)$$

which imposes an important restriction on the possible form of the Cauchy data in problems involving flow past finite bodies [we note that the expression (40) is valid only in the case of vanishing angle of attack].

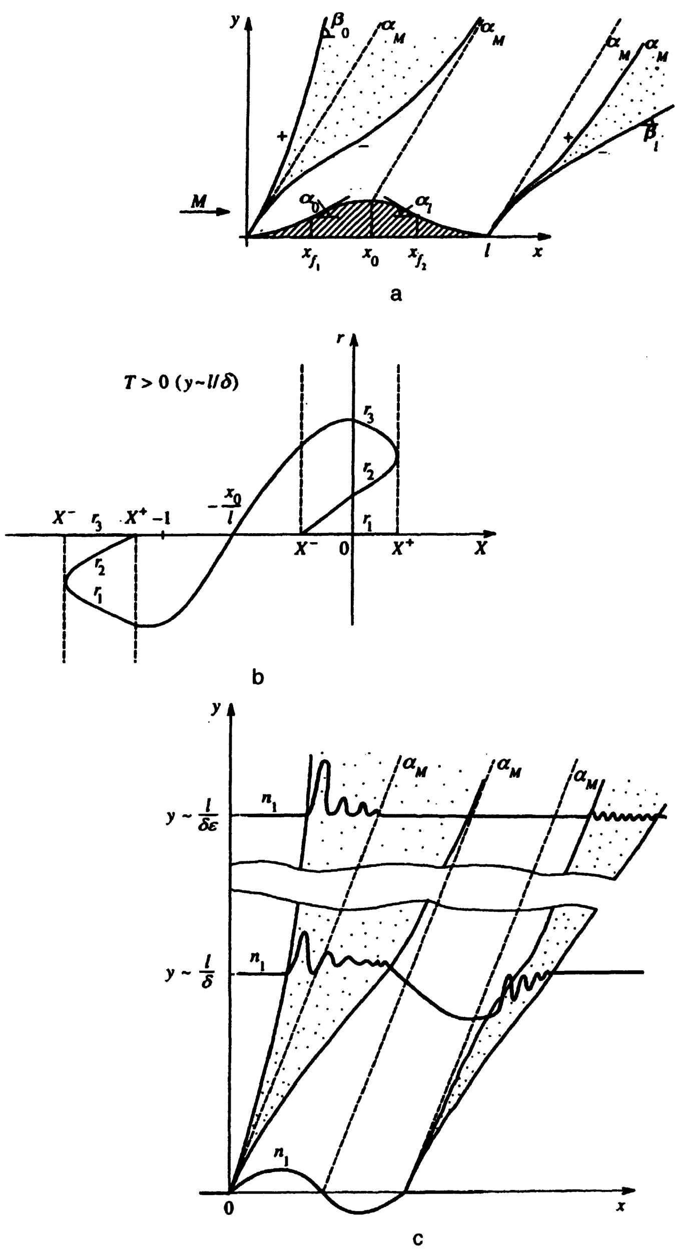

FIG. 3. Flow past a body of finite size: a) qualitative picture of the flow; b) behavior of the Riemann invariants. The right-hand part of the figure shows the bow nondissipative shock wave; the trailing wave is shown in the left; c) spatial evolution of the perturbation of the density n_1.

The evolution of the initial perturbation (39a) of nonconstant sign in the KdV hydrodynamics leads to the formation of two nondissipative shock waves at the points $X=0$ (bow wave) and $X=-1$ (trailing wave).

It is important that these waves not intersect anywhere, since the domains of influence of the sections of the body before and after the maximum are separated by the Mach line $x=x_0+\sqrt{M^2-1}y$ ($X=-x_0/l$ in terms of the evolution problem), which is now the asymptote of the trailing edge of the bow nondissipative shock wave (Fig. 3).

Thus, the analytic description of the bow nondissipative shock wave has been given in Sec. 3 and that of the trailing wave in Sec. 4, and the under consideration general case is simply a formal combination of the two previous ones. The complete solution is now defined on all four sheets [Fig. 1h, the expressions (10), (20) and (29), (30)]. The behavior of the Riemann invariants in the case of flow past a finite body is shown in Fig. 3b.

The qualitative picture of the distribution of the oscillations in the case of flow past a wing with pointed edges is shown in Fig. 3c. It is remarkable that, in contrast to the linearized theory (Ref. 13, § 125), in accordance with which the perturbation introduced by the body into the flow is concentrated in the region bounded by the Mach lines that emanate from the ends of the body, in dispersive hydrodynamics the entire perturbation is asymptotically outside this region, and the flow within it is homogeneous. The physical reason for this is the dispersive spreading of the perturbation at large distances from the body, so that the energy density of the oscillations tends to zero at infinity. At the same time, the interior region bounded by the Mach lines remains finite, and the total energy of the oscillations within this region also vanishes.

Indeed, it is easy to show that both boundaries of the trailing nondissipative shock wave lie to the right of the "Mach" region. With regard to the right-hand boundary of the bow nondissipative shock wave, we find that although as calculated from the solution of Whitham's equations it lies to the right of the Mach line emanating from the leading edge (see Figs. 3a and 3c) asymptotically the main perturbation is concentrated to the left of this line and consists of solitons of increasing amplitude, a soliton of infinitesimally small amplitude lying directly on the Mach line. In contrast, the oscillations to the right of this line represent a continuous spectrum whose energy contribution for the original potential in the semiclassical approximation is exponentially small[15,16] at large ($\sim l/\delta\varepsilon$) distances from the body.

6. RECOVERY OF THE PROFILE FROM THE ASYMPTOTIC MODULATION OF THE NONDISSIPATIVE SHOCK WAVE

From the analytic point of view, the problem of determining the shape of a body from data on the wake at infinity reduces to inversion of the asymptotic expressions (25) and (38). However, in order to use them directly, we must have data on the dimensions l and δ of the body, but these are not known in advance and occur in the transformation $(X,T)\leftrightarrow(x,y)$ (4a).

For simplicity, in this section we ignore the effects associated with the infinite sharpness of the edges of the wing (separation of the wave from the edge of the body, two-sheeted structure of the solution), which significantly complicate the asymptotic expressions but essentially do not introduce fundamental corrections to the determination of the body shape. The reason for this is that the domains of influence of the infinitely sharp edges of the body (sheets I and IV in the xy plane, see Fig. 2) are finite, whereas the domains of influence of the section of the profile between the points of inflection (sheets II and III) expand and reach to infinity.

Thus, we shall assume that we know the asymptotic distribution of the oscillations at large distance from the body; in the bow nondissipative shock wave the oscillations have the form of solitons, while in the trailing wave they have sinusoidal form. We note first of all that by virtue of the rectilinearity of the (centered) characteristics of the noninteracting solitons in the soliton train the origin can be defined as the point of intersection of the lines of equal amplitude in the bow nondissipative shock wave.

The problem consists of determining the function $y(x)$ [or $F(x)$, see (2)] that defines the profile of the body from known asymptotic wave parameters (for example, the distributions of the wave number and amplitude) far from the body around which the flow occurs. We also assume that we know the asymptotic position of the outer fronts of the nondissipative shock wave, i.e., the angles β_0 and β_l of inclination of the boundaries to the direction of the flow past the body with velocity M (see Fig. 3a). By means of (24), (36), and (4a), we find the small angles α_0 and α_l to which the leading edges of the wing are pointed (in the general case, these are the angles of inclination of the lines of the profile at the points of inflection):

$$\alpha_0\simeq\frac{3(M^2-1)}{M^2(\gamma+1)}(\beta_0-\alpha_M),$$

$$\alpha_l\simeq\frac{M^2-1}{M^2(\gamma+1)}(\alpha_M-\beta_l), \tag{41}$$

where α_M is the Mach angle, and $\tan\alpha_M = 1/\sqrt{M^2-1}$

The independence of the flows in the regions separated by the Mach line that emanates from the point x_0, at which the wing profile function takes its maximum value, implies that the contours $y_0(x)$ and $y_l(x)$ of the bow and trailing parts of the profile can be determined independently.

We take the nonlinearity parameter δ in the bow nondissipative shock wave equal to α_0 [see (39b)]. Then the length of the bow part of the body can be found in order of magnitude from the following qualitative considerations. The well-developed nonlinear oscillating structure in the nondissipative shock wave is formed at distances $y\sim y_{\text{nonlin}}\sim l/\delta$. The soliton train is formed in the region $y\sim y_{\text{sol}}\sim l/\delta\varepsilon$ (see Fig. 3c). We recall that the effective dispersion parameter is $\varepsilon\simeq(D/l)/\sqrt{8}$, where D, the dimensional dispersion scale of the medium, is the characteristic dimension of a soliton in the soliton train. Then by means of (4b) we find the (effective) length of the bow part of the contour:

$$l_0\simeq\frac{\sqrt{2}(M^2-1)^{1/4}}{\sqrt{\gamma+1}}\chi_0\frac{D}{\sqrt{\alpha_0}}, \tag{42}$$

where $\chi_0=y_{\text{sol}}/y_{\text{nonlin}}$.

We can similarly determine the length of the trailing part of the contour (with allowance for $\delta=\alpha_l$, $\chi_l=y_{\text{lin}}/y_{\text{nonlin}}$, where y_{lin} is the characteristic distance over which the sinusoidal shape of the wave is established).

Now, knowing the geometrical data of the body, we can use (4a) to go over from the known (measured) asymptotic wave parameters $a(x,y),k(x,y)$ to the functions $a(X,T)$ and $k(X,T)$ that occur in the expressions (25) and (38).

It can be seen from the asymptotic expressions (25) that the function kT at large T depends only on the similarity variable $\sigma=3X/2T$. Introducing the new function

$\kappa_0 \equiv k(X,T)T$, we find from the (Abel) equation (25a) for the function that determines the bow part of the body profile

$$F_0(X) = \int_{-X}^{0} W_{\mathrm{II}}^{-1}(z)dz, \tag{43}$$

where

$$W_{\mathrm{II}}(r) = -\frac{4\varepsilon}{\pi\sqrt{6}} \int_r^1 \frac{\kappa_0(\xi)}{\sqrt{\xi - r}}\, d\xi.$$

Similarly, for the function that determines the trailing part of the body profile, we obtain from (39a) [introducing $\kappa_l(\tau) \equiv a^2(X,T)T$, $\tau = -X/2T$]

$$F_l(X) = -\int_X^1 W_{\mathrm{III}}^{-1}(-z)dz, \tag{44}$$

where

$$W_{\mathrm{III}}(r) = \frac{1}{16\pi} \int_{-1}^{r} \frac{d}{d\xi}\left(\frac{\kappa_l(-\xi)}{\sqrt{-\xi}}\right) \frac{d\xi}{\sqrt{r-\xi}} - 1.$$

7. DRAG AND LIFT OF SLENDER BODIES IN DISPERSIVE HYDRODYNAMICS

The drag resulting from gas in steady supersonic flow over a pointed body in dispersive hydrodynamics is due to conversion of part of the kinetic energy of the body in the frame of reference moving with the oncoming flow into energy of waves emitted by the body.

In ideal hydrodynamics, these are nondispersive sound waves, and the drag can be calculated for any shape of the body section (Ref. 13, §§ 123, 125). In the present case of two-dimensional weakly nonlinear dispersive flow, the drag on the body is the x component of the momentum that is carried away by the nondissipative shock wave in unit time (as was shown in Sec. 5, at very large distances from the body, the entire perturbation is concentrated in the regions of the nondissipative shock wave, and there are no sound waves in the intermediate interval). As control surface, we choose a horizontal plane $y=\text{const}$ at a certain distance from the wing. The flux density of the x component of the momentum through this surface is

$$\Pi_{xy} = nV_xV_y = nc_s^2u_xu_y, \tag{45}$$

where n is the gas density, and c_s is the speed of sound.

Taking into account the expansion (5), we rewrite the expression (45) in the form

$$\Pi_{xy} = V_0c_snu_y + \delta^2 n_0c_s^2u_1v_1 + o(\delta^2), \tag{46}$$

where n_0 and V_0 are the density and velocity of the homogeneous flow at infinity.

The drag on the body per unit length in the direction perpendicular to the flow past the body is

$$F_x = -\int_{-\infty}^{\infty} \Pi_{xy}dx. \tag{47}$$

When integrated, the first term in the expansion (46) gives the total flux of mass through the control surface, which is equal to zero. Then with allowance for (5) and (4), we have

$$F_x \simeq \frac{n_0V_0l\delta^2}{\sqrt{M^2-1}} \int_{-\infty}^{\infty} \eta^2 dX. \tag{48}$$

Since η^2 is a conserved quantity in the KdV approximation, there is no need to calculate the integral (48) in the asymptotic regions, as is done in Ref. 13 (§ 123). Using the initial data (39) and their relationship to the wing profile $y(x)$, we have

$$F_x = \frac{n_0V_0^2}{\sqrt{M^2-1}} \int_0^l [y'(x)]^2 dx. \tag{49}$$

The drag coefficient is

$$C_x = \frac{F_x}{(1/2)n_0V_0^2l} = \frac{2}{\sqrt{M^2-1}\,l} \int_0^l [y'(x)]^2 dx. \tag{50}$$

The expression (50) is identical to the one obtained in Ref. 13 (§ 125) in the approximation of linearized nondispersive hydrodynamics. A similar result holds for the lift, which is equal to the difference of the pressure forces acting on the lower and upper surfaces of the wing. Projecting the pressure forces onto the vertical axis and integrating along the surface of the body [near which the flow can be described by the equations of ideal hydrodynamics (see Ref. 1)], we find, as in Ref. 13 (§ 125) that the lift coefficient is given by

$$C_y = \frac{4\alpha}{\sqrt{M^2-1}},$$

where $\alpha \lesssim \delta$ is the angle of attack.

Thus, despite the fundamental differences between the flow structure at large distances from the body the nonlinearity and dispersion have no effect on the macroscopic flow characteristics—the drag and lift. Of course, this is true only insofar as the KdV approximation holds.

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