Evolution of the quasiprobability distribution function of the radiation from a nonlinear optical amplifier

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A study is carried out of the evolution of the classical phase portrait of the density operator of nonlinearly amplified light. A single-mode optical amplifier is considered, with quantum-statistical fluctuations of the two-level active medium and the light being amplified. The Heisenberg–Langevin equations for the field operators are obtained by adiabatic elimination of the atomic variables in the van der Pol approximation. The Fokker–Planck equation for the P-function, corresponding to the Heisenberg–Langevin equations, was solved numerically using the polynomial-expansion method. The initial quasiprobability distribution function corresponded to a pure coherent state. The Wigner quasiprobability distribution was calculated with the help of an integral transform of the P-function. Such quantum-statistical parameters of the amplifier radiation as the mean photon number and its variance, the Mandel parameter, and the mean field and the variance of the phase–amplitude quadratures of the field were calculated. © *1996 American Institute of Physics.* [S1063-7761(96)00405-2]

1. INTRODUCTION

The search for light sources possessing nonclassical properties, i.e., obeying sub-Poisson statistics of the photon number or squeezed in amplitude or phase, is presently one of the most active problems of quantum optics. The mathematical apparatus of the quantum theory of open systems developed in the last 30 years, which is to say, of systems possessing fluctuations and dissipation, allows us to quantitatively investigate the quantum-statistical properties of radiation arising from such well-known nonlinear optical processes as laser generation, parametric scattering, optical bistability, four-wave mixing, etc.

An analysis of the quantum-statistical properties of laser radiation based on the P-representation of the Glauber–Sudarshan density operator was carried out in Refs. 1–3. The authors of Ref. 1, who used the polynomial-expansion method to solve the Fokker–Planck equation for the field variables, found steady-state distribution functions for the quasiprobability of the field amplitude for the two-level model in the case of exact resonance, and also developed a scheme for calculating the dynamics of the P-function under these conditions. The Fokker–Planck equation for the P distribution function of the quasiprobability of the field amplitude for population inversion in the two-level model was solved with the help of the polynomial-expansion method in Ref. 2 for the case of exact resonance. An analysis of the influence of resonance detuning on the quantum-statistical properties of the radiation of a two-level laser under steady-state conditions near lasing threshold was carried out in Ref. 3. However, so far there have been no detailed studies of the evolution of the quasiprobability distribution function of the radiation from a laser or nonlinear optical amplifier—at least not any that are known to us—although changes in the statistics of a quantum field accompanying nonlinear amplification are an urgent problem of quantum optics, which has as its proper object of study, in addition to the aforementioned nonclassical properties of light, the quantitative characteristics of the coherence properties and quantum-noise parameters of optical sources.

It is well known that for a number of states of light the interpretation of the Glauber P-representation of the radiation density operator as a distribution function is possible only under certain conditions, since the P-function is not in general positive-definite. In addition, the Fokker–Planck equation for it often does not have solutions in the class of integrable functions, or the solution is a singular function expressible in terms of the δ-function or its derivatives.[4] The use of other representations of the density operator in classical phase space, e.g., the Wigner representation, allows us in some cases to find distribution functions possessing the necessary properties and suitable for calculating the observable mean quantities.

The field observables corresponding to any one of the different ways of representing the density operator, where this mode of representation corresponds to a choice of how to order the creation and annihilation operators $a^+(a)$ of the field, are calculated with the help of the quasiprobability distribution function $\mathcal{F}(\alpha,\alpha^*,t,s)$ by means of c-number integration in the α phase plane

$$\langle\{(a^+)^n a^m\}^{(s)}\rangle=\int d^2\alpha(\alpha^*)^n\alpha^m\mathcal{F}(\alpha,\alpha^*,t,s), \qquad (1.1)$$

where the notation $\{\dots\}^{(s)}$ means that the s-type ordering procedure has been applied to the operator enclosed in braces. Among the various quasiprobability distributions in wide use at the present time, we may cite the Glauber–Sudarshan function[5,6] for normal ordered operators, the Q-function for antinormal ordering of the operators,[7,8] and the Wigner function[8,9] for symmetrized ordering of the operators. The two-dimensional Fourier transform of the following generalized characteristic function can serve as a formal definition for all three types of quasiprobability distributions:

$$\chi(\xi,\xi^*,t,s)=\left\langle \exp\left(\xi a^+ - \xi a + \frac{s}{2}|\xi|^2 \right) \right\rangle, \tag{1.2}$$

where $s=1$ corresponds to the Glauber–Sudarshan quasiprobability, $s=0$ to the Wigner function, and $s=-1$ to the Q-function.

If the photon detector relies on photon absorption, then the operators corresponding to the measured quantities should be normal ordered, since such a detector should not give any readings in the absence of photons. The Glauber–Sudarshan representation of the density operator in the classical phase plane corresponding to this situation can also be obtained with the help of a theorem about the trace, according to which the trace of the product of two arbitrary operators, one of which is in normal ordered form, and the other in antinormal ordered form, is equal to the integral over the phase plane of the product of the corresponding classical functions obtained by replacing the operators with the corresponding c-numbers: $a^+ \to \alpha^*$, $a^- \to \alpha$. In this case, according to Eq. (1.1), we obtain for the Glauber–Sudarshan quasiprobability distribution P (Ref. 10)

$$\rho(a^+,a,t)=\pi \mathscr{A} P^{(a)}(\alpha,\alpha^*,t), \tag{1.3}$$

$$\langle \mathscr{O}^{(n)}(t) \rangle = \int d^2\alpha \mathscr{O}^{(n)}(\alpha^*,\alpha) P(\alpha,\alpha^*,t), \tag{1.4}$$

where $\mathscr{O}^n$ is an arbitrary normal ordered operator. Equality (1.3) means that the quasiprobability distribution function can be obtained from the density operator written in the antinormal ordered form by replacing the creation and annihilation operators with c-number variables, with the proportionality coefficient needed to maintain normalization.

We use the above definition of the quasiprobability distribution function (1.3), (1.4) in the present work to derive the Fokker–Planck equation for the P-function of the field in a nonlinear optical amplifier, starting with the Heisenberg–Langevin equations. The procedure used here to make the transition from the operator equations to the equations for the c-number variables was realized within the framework of normal ordering of the operators entering into the quantum equations, which allows us to obtain the classical Fokker–Planck equation for the commuting variables, thereby preserving all of the quantum-mechanical properties of light. By means of a numerical solution of the generalized Fokker–Planck equation obtained by expanding the solution over the full orthonormal basis of Laguerre polynomials and a Fourier series expansion, we have found the quasiprobability functions of light propagating in an inverted active medium with saturation.

2. FOKKER–PLANCK EQUATION FOR THE QUASIPROBABILITY DISTRIBUTION FUNCTION AND THE HEISENBERG–LANGEVIN EQUATIONS

In the present paper we use a method in which the Glauber–Sudarshan quasiprobability function is calculated by solving the Fokker–Planck equation, whose form in turn is obtained from the Heisenberg–Langevin equations for the field operators.

Let the Heisenberg–Langevin equation of motion for the creation and annihilation operators in the presence of fluctuations and dissipation have the general form

$$\frac{d}{dt} a = A_a^{(n)}(a^+,a) + f_a(t),$$

$$\frac{d}{dt} a^+ = A_{a^+}^{(n)}(a^+,a) + f_{a^+}(t), \tag{2.1}$$

where $A_i^{(n)}$ is the shift operator and f_i is the operator of random sources, due to the interaction with the reservoir, $i=a,\ a^+$.

In the Markov approximation for the Gaussian random sources $f_a(t)$ and $f_{a^+}(t)$ we take them to be δ-correlated:

$$\langle f_M(t) f_{M'}(t') \rangle = 2D_{MM'}(t)\delta(t-t'), \quad M,M'=a,a^+. \tag{2.2}$$

In the case under consideration of normal ordered operators of the drift vectors of the Heisenberg–Langevin equations, corresponding to the definition of the Glauber–Sudarshan quasiprobability, we find the equation of motion for the mean of the normal ordered operator:

$$\mathscr{O}(a^+(t),a(t)) = \mathscr{O}^{(n)}(a^+(t),a(t)) = e^{\xi a^+} e^{\eta a}. \tag{2.3}$$

In this case the normal ordered characteristic function generating the Glauber–Sudarshan quasiprobability function, according to Eq. (1.2), can be expressed in terms of the operator (2.3) as follows:

$$\chi(\xi,\eta,t) = \langle \mathscr{O}(a^+(t),a(t)) \rangle$$
$$= \langle e^{\xi a^+} e^{\eta a} \rangle = \langle \mathscr{O}^{(n)}(a^+(t),a(t)) \rangle. \tag{2.4}$$

The complex c-function associated with the operator $\mathscr{O}$ can be obtained with the help of the operation of normal ordering according to

$$\mathscr{O}(a^+(t),a(t)) = \mathscr{O}^{(n)}(a^+(t),a(t))$$
$$= :\mathscr{O}^{(n)}(\alpha^*(t),\alpha(t)): = :e^{\xi\alpha^* + \eta\alpha}: \tag{2.5}$$

Here $:\ldots:$ means that the operator $\mathscr{O}$ can be obtained by replacing the numerical variables in the normal ordered c-function by the creation and annihilation operators: $\alpha^* \to a^+$, $\alpha \to a$.

Applying the method presented in Ref. 10, pp. 221–225, and assuming that the random processes are Markovian and the random sources f_{a^+} and f_a, containing contributions from fluctuations from a large number of independent particles of the active medium and the reservoir, are Gaussian, we arrive, directly from the Heisenberg–Langevin equations containing the normally ordered drift vectors A_{a^+} and A_a and diffusion coefficients $D_{MM'}$ (2.2), at the following equation of motion for $\langle \mathscr{O} \rangle$, i.e., for the characteristic function we have

$$\frac{d\chi}{dt} = -\langle :L\mathscr{O}^{(n)}(\alpha^*,\alpha): \rangle, \tag{2.6}$$

where the differential operator on the right-hand side is

$$L=A_\alpha \frac{\partial}{\partial\alpha}+A_{\alpha*} \frac{\partial}{\partial\alpha^*}+D_{\alpha\alpha} \frac{\partial^2}{\partial\alpha\partial\alpha}+2D_{\alpha*\alpha} \frac{\partial^2}{\partial\alpha^*\partial\alpha} +D_{\alpha*\alpha*} \frac{\partial^2}{\partial\alpha^*\partial\alpha^*}. \tag{2.7}$$

The functions A_α, $A_{\alpha*}$, and $D_{\mu\mu'}$ here were obtained from the corresponding operators A_a, A_{a^+}, and $D_{MM'}$ by the substitution $a^+\to\alpha^*$, $a^-\to\alpha$. Equation (2.7) takes into account that as a consequence of the aforementioned normal ordering of the operators in the diffusion coefficients we have

$$D_{a^+a}(a^+,a,t)=:D^{(n)}_{\alpha*\alpha}(\alpha^*,\alpha,t):=:D^{(n)}_{\alpha\alpha*}(\alpha^*,\alpha,t):. \tag{2.8}$$

Equality (2.8) means that in the case under consideration, only the normal ordered diffusion coefficients are present in the equations. According to the definition of the quasiprobability function [Eq. (1.3)], in the case of a normal ordered characteristic function we have

$$\chi(\xi,\eta,t)=\mathrm{Tr}(\mathscr{O},\rho)=\int d^2\alpha\mathscr{O}^{(n)}(\alpha^*,\alpha)P(\alpha,\alpha^*,t) =\langle\mathscr{O}^{(n)}(t)\rangle. \tag{2.9}$$

Thus, the expression for the quantum-statistical mean (2.9) contains only c-number expressions. From Eq. (2.4), taking into account Eq. (2.6), we find

$$\frac{d\chi}{dt}=-\mathrm{Tr}(\rho(a,a^+,t):L\mathscr{O}^{(n)}(\alpha^*,\alpha):) =-\int d^2\alpha P(\alpha,\alpha^*,t)L\mathscr{O}^{(n)}(\alpha^*,\alpha). \tag{2.10}$$

Substituting the expression for the operator (2.7) in Eq. (2.10) and integrating by parts, assuming in so doing that $P(\alpha,\alpha^*,t)$ decays rapidly with increasing $|\alpha|$, i.e., that it is small outside the limits of integration, we immediately find

$$\frac{d\chi}{dt}=\int d^2\alpha\mathscr{O}^{(n)}(\alpha^*,\alpha)\frac{\partial P}{\partial t} =-\int d^2\alpha\mathscr{O}^{(n)}(\alpha^*,\alpha)L^+P. \tag{2.11}$$

From the last equality in Eq. (2.11) it follows that

$$\frac{\partial P}{\partial t}=-L^+P, \tag{2.12}$$

$$L^+=-\frac{\partial}{\partial\alpha}A_\alpha-\frac{\partial}{\partial\alpha^*}A_{\alpha*}+\frac{\partial^2}{\partial\alpha\partial\alpha}D_{\alpha\alpha} +\frac{\partial^2}{\partial\alpha^*\partial\alpha}2D_{\alpha*\alpha}+\frac{\partial^2}{\partial\alpha^*\partial\alpha^*}D_{\alpha*\alpha*}. \tag{2.13}$$

Equation (2.12) is the Fokker–Planck equation for the Glauber–Sudarshan quasiprobability function, since everywhere above we assumed that A_ν and $D_{\mu\mu'}$ are in normal ordered form. The use in the above procedure of the characteristic function $\chi(\xi,\eta,t)$ in antinormal form (in the field operators) leads to the corresponding generalized Fokker–Planck equation for the Q-function, and using symmetric ordering we arrive at the equation for the Wigner function. Here, as was shown in Ref. 11, the Wigner function is related to the Glauber–Sudarshan function by the following equation:

$$W(\alpha,\alpha^*,t)=\frac{2}{\pi}\int d^2\beta e^{-2|\alpha-\beta|^2}P(\beta,\beta^*,t). \tag{2.14}$$

3. FOKKER–PLANCK EQUATION FOR AN OPTICAL AMPLIFIER

We have considered the propagation of an electromagnetic plane wave in an active medium containing two-level atoms fixed in space and inverted by an external incoherent pump. The Hamiltonian of the system of N_A atoms with transition frequency ω_A, interacting with a monochromatic wave of frequency ω_c and with the corresponding reservoirs residing in a stationary state of thermodynamic equilibrium and being sources of fluctuations and dissipation, has the form

$$H=\hbar\omega_c a^+a+\sum_{i=1}^{N_A}\frac{\hbar\omega_A}{2}\sigma_i^Z+\hbar g\sum_{i=1}^{N_A}(a^+\sigma_i^- \times\exp(-ikZ_i)+\sigma_i^+a\exp(ikZ_i))+\sum_{i=1}^{N_A}(\Gamma_i^+\sigma_i^- +\Gamma_i\sigma_i^+\Gamma_{PH}\sigma_i^Z)+\Gamma_f^+a+\Gamma_f a^++H_R. \tag{3.1}$$

The first two terms of the Hamiltonian (3.1) characterize the free energy of the normal mode of the field and the system of atoms, where $a^+(a)$ are the creation (annihilation) operators of the normal mode of the field, and σ_i^Z is the diagonal Pauli operator for the ith spin-1/2 atom:

$$\sigma_i^Z=(|2\rangle\langle2|-|1\rangle\langle1|)_i. \tag{3.2}$$

The third term of the Hamiltonian describes the interaction of the field with the atom, with coupling constant g, in the dipole approximation, where this coupling constant is proportional to the projection of the matrix element of the transition between the levels of the two-level atom onto the polarization direction of the field mode, and inversely proportional to the square root of the quantization volume of the normal mode of the field oscillator, assumed in what follows to be infinitely large. The off-diagonal Pauli operators $\sigma_i^-(\sigma_i^+)$ represent the polarization of the ith atom

$$\sigma_i^-=(|1\rangle\langle2|)_i,\quad \sigma_i^+=(|2\rangle\langle1|)_i, \tag{3.3}$$

and at all times satisfy

$$\sigma_i^\pm\sigma_i^\mp=\frac{1}{2}(1\pm\sigma_i^Z), \tag{3.4}$$

$$\sigma_i^Z\sigma_i^Z=1, \tag{3.5}$$

where Z_i is the position of the ith atom and $k=\omega_c/c$.

The next two terms of the Hamiltonian describe the interaction of atoms with their reservoirs; the first term characterizes fluctuations of the medium and thus describes spontaneous emission. The operators Γ_i^+ and Γ_i here correspond to the reservoir of the ith atom and possess the properties of a Markov random process. The next term, proportional to

σ_i^Z, corresponds to interaction of the atoms with the reservoir, leading to phase jumps of the dipole moment due to elastic collisions with the particles of the reservoir. The next two terms, containing the field operators, are that part of the Hamiltonian that describes the zero-point fluctuations of the electromagnetic vacuum acting on the field mode under consideration. The term H_R is the free energy operator of the reservoirs. In the Heisenberg picture, the equation of motion for any operator of the system $M(t)$ has the form

$$i\hbar \frac{d}{dt} M(t)=[M(t),H(t)]. \tag{3.6}$$

Within the framework of the Markov approximation used to describe the reservoirs of the atomic and field subsystems, it is possible to obtain from the exact equation of motion (3.6) approximate effective equations of motion for the system operators, effectively taking into account the influence of the unobservable reservoirs with the help of the dissipative and fluctuational terms arising as a result of eliminating the reservoir operators[10] and including the fluctuation sources in the so-obtained equations of motion. The approximations employed here require that the correlation times of the random sources be much smaller than the characteristic dissipation times of the subsystems. Using the results of Ref. 10, for the above-described Heisenberg–Langevin equations it is possible to obtain

$$\frac{d}{dt} M(t)=\frac{1}{i\hbar}[M(t),H_s(t)]+D_M(t)+F_M(t)$$
$$\equiv A_M(t)+F_M(t), \tag{3.7}$$

where $H_s(t)$ is that part of the Hamiltonian (3.1) corresponding to the observable system of atoms and field, $D_M(t)$ is the dissipative term of the equation of motion, and $F_M(t)$ are Gaussian random operators, δ-correlated when taking the mean over the reservoir variables:

$$\langle F_M(t)F_{M'}(t')\rangle_R=\langle 2D_{MM'}(t)\rangle_R\delta(t-t'). \tag{3.8}$$

Writing the equation of motion for correlation functions of the form (3.8), it is not hard to obtain the corresponding relation for the diffusion coefficients $\langle 2D_{MM'}(t)\rangle$:

$$\langle 2D_{MM'}(t)\rangle_R=\frac{d}{dt}\langle M(t)M'(t)\rangle_R-\langle M(t)A_{M'}(t)\rangle_R$$
$$-\langle A_M(t)M'(t)\rangle_R. \tag{3.9}$$

For the collective operators of the model, defined as

$$P(t)=\sum_{i=1}^{N_A}\sigma_i^-\exp(-ikZ_i), \tag{3.10}$$

$$D(t)=\frac{1}{2}\sum_{i=1}^{N_A}\sigma_i^Z, \tag{3.11}$$

we find the following Heisenberg–Langevin equations of motion from Eq. (3.7):

$$\frac{d}{dt}a(t)=-\frac{ck}{2}a(t)-i\omega_c a(t)-igP(t)+F_a(t),$$

$$\frac{d}{dt}P(t)=-(\gamma_\perp+i\omega_A)P(t)+2igD(t)a(t)+F_P(t),$$

$$\frac{d}{dt}D(t)=-\gamma_\parallel\left[D(t)-\frac{1}{2}\frac{w_{12}-w_{21}}{w_{12}+w_{21}}N_A\right]+ig[a^+(t)P(t)$$
$$-P^+(t)a(t)]+F_D(t), \tag{3.12}$$

where ck is the dissipation constant of the field, c is the speed of light, $\gamma_\parallel=w_{12}+w_{21}$, $\gamma_\perp=\gamma_{PH}+\gamma_\parallel/2$. Here w_{12} and w_{21} are the pump and relaxation rate constants of the two-level atom. We next introduce the slow system operators

$$a(t)=\tilde{a}(t)\exp(-i\Omega t) \tag{3.13}$$

for the amplitude of the light field, and for the polarization of the medium

$$P(t)=\tilde{P}(t)\exp(-i\Omega t), \tag{3.14}$$

thus assuming that under steady-state conditions the frequency of the oscillations of the light field is close to Ω and in general differs from both the frequency of the quantum mode of the quasi-monochromatic traveling plane wave and the resonant frequency of the atomic transition. Substituting Eqs. (3.13) and (3.14) into Eq. (3.12), we obtain

$$\frac{d}{dt}\tilde{a}(t)=-\left[i(\omega_c-\Omega)+\frac{kc}{2}\right]\tilde{a}(t)-ig\tilde{P}(t)+\tilde{F}_a(t),$$

$$\frac{d}{dt}\tilde{P}(t)=-[i(\omega_A-\Omega)+\gamma_\perp]\tilde{P}(t)+2igD(t)\tilde{a}(t)+\tilde{F}_P(t),$$

$$\frac{d}{dt}D(t)=-\gamma_\parallel[D(t)-D_0]+ig[\tilde{a}^+(t)\tilde{P}(t)$$
$$-P^+(t)\tilde{a}(t)]+\tilde{F}_D(t),$$

$$D_0=\frac{1}{2}(w_{12}-w_{21})/(w_{12}+w_{21}). \tag{3.15}$$

Restricting ourselves to the case $\gamma_\parallel,\gamma_\perp\gg kc/2\equiv\tilde{k}$, we adiabatically eliminate the variables $P(t)$ and $D(t)$ in the operator equations (3.15) (we drop the tilde), setting dP/dt and dD/dt equal to zero. We arrive at the following equations for the field operators:

$$\dot{a}=-k(1-i\delta)a+\frac{2g^2D_0}{\gamma_\perp(1+i\delta)}\times\left[1+\frac{4g^2f}{\gamma_\perp^2(1+\delta^2)}a^+a\right]^{-1}a+G_a(t),$$

$$\dot{a}^+=-k(1+i\delta)a^++\frac{2g^2D_0}{\gamma_\perp(1-i\delta)}a^+\times\left[1+\frac{4g^2f}{\gamma_\perp^2(1+\delta^2)}a^+a\right]^{-1}+G_{a^+}(t), \tag{3.16}$$

where $f=\gamma_\perp/\gamma_\parallel$ and

$$\delta=\frac{\omega_A-\Omega}{\gamma_\perp}=-\frac{\omega_c-\Omega}{k}. \tag{3.17}$$

Assuming that

$$\frac{4g^2f}{\gamma_\perp^2(1+\delta^2)}\langle a^+a\rangle_R \ll 1, \tag{3.18}$$

and noting that $\langle G(t)\rangle_R=0$, we find for the field operator $\langle a\rangle_R$ averaged over the reservoir

$$\langle \dot a\rangle_R \approx k(1-i\delta)\left\{\left[\frac{2D_0g^2}{k\gamma_\perp(1+\delta^2)}-1\right] - \frac{4D_0g^4f}{k\gamma_\perp^3(1+\delta^2)^2}\langle a^+a\rangle_R\right\}\langle a\rangle_R,$$

$$\langle \dot a^+\rangle_R \approx k(1+i\delta)\langle a^+\rangle_R\left\{\left[\frac{2D_0g^2}{k\gamma_\perp(1+\delta^2)}-1\right] - \frac{4D_0g^4f}{k\gamma_\perp^3(1+\delta^2)^2}\langle a^+a\rangle_R\right\}. \tag{3.19}$$

In the case when condition (3.18) is fulfilled, we write the equations of motion for the mean field operators in normal ordered form.

Reduction of the equations of motion for field operators of the form (2.1), (2.2) corresponding to Eqs. (3.19) to normal ordered form and application of the normal ordering procedure to the diffusion operators allows us, in the calculations that are to follow, to go from non-commutative field operators to c-number variables $a^+\to\alpha^*$, $a^-\to\alpha$ and to c-number stochastic equations of motion for α^* and α, treating these variables as formally independent.

We introduce the following dimensionless variables:

$$\alpha=\xi\beta,\quad I=|\beta|^2,\quad Z=T\zeta,\quad Z=ct. \tag{3.20}$$

Removing the average over the reservoir in Eqs. (3.19) and including the c-number Langevin sources in the resulting stochastic equations, we arrive, with the help of the variables (3.20), at equations of the form

$$\frac{d\beta}{d\zeta}=(1-i\delta)(p-I)\beta+F_\beta(\zeta),$$

$$\frac{d\beta^*}{d\zeta}=(1+i\delta)\beta^*(p-I)+F_{\beta^*}(\zeta), \tag{3.21}$$

$$p=[2D_0g^2-k\gamma_\perp(1+\delta^2)](2D_{\alpha^*\alpha}D_0g^4f\gamma_\perp)^{-1/2}\gamma_\perp,$$

$$\xi=\left[\frac{D_{\alpha^*\alpha}\gamma_\perp^3(1+\delta^2)^2}{8D_0g^4f}\right]^{1/4},$$

$$T=\left[\frac{c^2\gamma_\perp^3(1+\delta^2)^2}{2D_{\alpha^*\alpha}D_0g^4f}\right]^{1/2},$$

$$\langle F_{\beta^*}(\zeta)F_\beta(\zeta')\rangle=4\delta(\zeta-\zeta'). \tag{3.22}$$

Assuming that the diffusion coefficients depend weakly on ζ, according to Refs. 12–14 we may set

$$2D_{\alpha^*\alpha}\approx\frac{4g^2N_Aw_{12}}{\gamma_\parallel\gamma_\perp(1+\delta^2)},\quad 2D_{\alpha^*\alpha^*}\approx 2D_{\alpha\alpha}\approx 0. \tag{3.23}$$

In order to find the Fokker–Planck equation corresponding to the resulting stochastic process, we use the procedure presented in Sec. 2 of this paper, and the correspondence between the quantum and classical quantities for the drift vectors and second moments (diffusion coefficients):

$$A_a^{(n)}(a^+,a)=:A_\alpha^{(n)}(\alpha^*,\alpha):,\quad A_{a^+}^{(n)}(a^+,a)=:A_{\alpha^*}^{(n)}(\alpha^*,\alpha):,$$

$$D_{aa}^{(n)}=:D_{\alpha\alpha}^{(n)}:,\quad D_{a^+a^+}^{(n)}=:D_{\alpha^*\alpha^*}^{(n)}:,$$

$$D_{a^+a}^{(n)}=:D_{\alpha^*\alpha}^{(n)}:=:D_{\alpha\alpha^*}^{(n)}:$$

From Eqs. (2.14) and (2.15) with the help of Eqs. (3.21) and (3.22) we find the following Fokker–Planck equation for the quasiprobability distribution $P(\beta,\beta^*,\zeta)$:

$$P(\beta,\beta^*,\zeta)=2P(I,\phi,\zeta), \tag{3.24}$$

$$\frac{\partial P(I,\phi,\zeta)}{\partial\zeta}=\left[2\frac{\partial}{\partial I}(I-p)I+4\frac{\partial}{\partial I}I\frac{\partial}{\partial I}+\frac{1}{I}\frac{\partial^2}{\partial\phi^2}+(I-p)\delta\frac{\partial}{\partial\phi}\right]P(I,\phi,\zeta). \tag{3.25}$$

The mean values determined by the statistics of the field of the nonlinear amplifier, found with the help of the Glauber–Sudarshan quasiprobability distribution, are given according to Eq. (1.1) by

$$\langle(a^+)^na^m\rangle=\xi^{n+m}\langle(b^+)^nb^m\rangle =\xi^{n+m}\int d^2\beta P(\beta,\beta^*,\zeta)(\beta^*)^n\beta^m =\frac{\xi^{n+m}}{2}\int_0^{2\pi}d\phi\int_0^\infty dIP(I,\phi,\zeta)I^{(n+m)/2}\times\exp\{i\phi(m-n)\}. \tag{3.26}$$

4. SOLUTION OF THE FOKKER–PLANCK EQUATION

We calculated the quasiprobability distribution function $P(\beta,\beta^*,\zeta)$ numerically by expanding the solution of the Fokker–Planck equation over a complete basis of orthogonal functions. To determine the dependence of the quasiprobability distribution function on the dimensionless length of the amplifier ζ, we expanded the solution in generalized Laguerre polynomials, which give the dependence on the intensity I, and in a Fourier series in the phase of the field, ϕ:

$$P(I,\phi,\zeta)=\sum_{n=0}^\infty\sum_{m=-\infty}^\infty C_n^m(\zeta)e^{-I/\alpha}\left(\frac{I}{\alpha}\right)^{|m|/2}e^{im\phi}L_n^{|m|}\left(\frac{I}{\alpha}\right), \tag{4.1}$$

$$L_n^{|m|}(x)=\sum_{k=0}^n(-1)^k\binom{n+|m|}{n-k}\frac{x^k}{k!}, \tag{4.2}$$

where α is an arbitrary constant.

Upon substituting the expansion (4.1), (4.2) into the Fokker–Planck equation (3.25) and invoking completeness and orthogonality of the Laguerre polynomials and Fourier harmonics, and then applying some well-known relations between the generalized Laguerre polynomials,[1,15] we obtain the following system of differential equations for the in-general complex expansion coefficients of the quasiprobability function:

$$C_n^m(\zeta)=(n+|m|+1)\alpha(2n+|m|)C_{n+1}^m(\zeta) +\{-\alpha[2n(3n+3|m|+1)+|m|(|m|+1)]+p(2n +|m|)\}C_n^m(\zeta)+n\left[\alpha(6n+3|m|-2)-2p -\frac{4}{\alpha}\right]C_{n-1}^m(\zeta)-2\alpha(n-1)nC_{n-2}^m(\zeta)+im\delta[\alpha(2n +|m|+1)-p]C_n^m(\zeta)-im\delta\alpha nC_{n-1}^m-im\delta\alpha(n +|m|+1)C_{n+1}^m(\zeta). \tag{4.3}$$

Invoking the orthogonality properties of the Laguerre polynomials, we arrive at the following relations for the mean values characterizing the field of the amplifier:

$$\langle b(\zeta)\rangle=\langle E(\zeta)\rangle=2\pi\alpha^{3/2}C_0^{-1}(\zeta), \tag{4.4}$$

$$\langle b^+(\zeta)\rangle=\langle E^*(\zeta)\rangle=2\pi\alpha^{3/2}C_0^{+1}(\zeta), \tag{4.5}$$

$$\langle(b^+(\zeta)b(\zeta))^n\rangle=\langle:|E(\zeta)|^2:\rangle=\langle n(\zeta)\rangle=\langle I(\zeta)\rangle =2\pi\alpha^2[C_0^0(\zeta)-C_0^1(\zeta)], \tag{4.6}$$

$$\langle b(\zeta)b(\zeta)\rangle=\langle E^2(\zeta)\rangle=4\pi\alpha^2C_0^{-2}(\zeta), \tag{4.7}$$

$$\langle b^+(\zeta)b^+(\zeta)\rangle=\langle(E^*(\zeta))^2\rangle=4\pi\alpha^2C_0^{+2}(\zeta), \tag{4.8}$$

$$\langle(b^+(\zeta)b(\zeta)b^+(\zeta)b(\zeta))^{(n)}\rangle =\langle:I^2(\zeta):\rangle=4\pi\alpha^3[C_0^0(\zeta)-2C_1^0(\zeta)+C_2^0(\zeta)], \tag{4.9}$$

$$\langle:I^2(\zeta):\rangle=\langle n^2(\zeta)\rangle-\langle n\rangle. \tag{4.10}$$

Here we have used the boundary condition $\phi(0)=0$.

We assume that the radiation entering the optical amplifier is in a pure coherent state

$$\rho(\zeta_0)=|\beta_0\rangle\langle\beta_0|, \tag{4.11}$$

and that the corresponding quasiprobability distribution function in the Glauber representation has the form

$$P(\beta,\beta^*,\zeta_0)=\delta^{(2)}(\beta-\beta_0) =\delta(\mathrm{Re}\,\beta-\mathrm{Re}\,\beta_0)\delta(\mathrm{Im}\,\beta-\mathrm{Im}\,\beta_0). \tag{4.12}$$

In intensity–phase variables we then obtain

$$P(I,\phi,\zeta_0)=2\delta(I-I_0)\delta(\phi-\phi_0). \tag{4.13}$$

Using relation (4.13) in expansion (4.1), we directly find the following initial conditions for the solution of system (4.3), setting $\phi(0)=0$:

$$C_n^m(0)=\frac{1}{\pi\alpha}\frac{n!}{(n+|m|)!}x_0^{|m|/2}L_n^{|m|}(x_0), \qquad \alpha x_0=I_0. \tag{4.14}$$

As was shown in Refs. 1 and 16, varying the scaling parameter of the intensity $\alpha>0$ allows us in the course of the numerical calculations, after we have found the optimal value of α, to improve the convergence of series (4.1) and reduce the number of terms of the expansion $n_{\max}$, $m_{\max}$ needed to reach the required accuracy.

As we have seen above, in the case of an initial pure coherent state of the radiation entering the amplifier, the initial quasiprobability function is a singular function of the field variables, and consequently, strictly speaking, all of the expansion terms (4.1) should be present in a calculation of the exact solution of the Fokker–Planck equation, at least in the immediate vicinity of $\zeta=0$. This kind of practical complication in the numerical solution is absent in the case of antinormal or symmetrized ordering of the field operators generating the Q quasiprobability function and the Wigner function and the generalized Fokker–Planck equations corresponding to them. In these cases the initial values of the quasiprobability function are smooth functions.

5. RESULTS OF NUMERICAL CALCULATION

We solved the system of equations for the expansion coefficients of the Glauber–Sudarshan quasiprobability (4.1) numerically, using the Runge–Kutta method. For the initial state of the radiation entering the amplifier, we used the pure coherent state (4.11) with a singular quasiprobability function of the form (4.12). With the help of the well-known Glauber–Sudarshan integral transform (2.10), we found the Wigner quasiprobability function. Invoking relations (4.4)–(4.10), we calculated such quantum-statistical characteristics of the amplified radiation as the Mandel parameter

$$Q(\zeta)=\frac{\langle(\Delta n(\zeta))^2\rangle-\langle n(\zeta)\rangle}{\langle n(\zeta)\rangle}=\frac{\langle:(\Delta I(\zeta))^2:\rangle}{\langle I(\zeta)\rangle}. \tag{5.1}$$

For radiation in a pure coherent state or in a mixed coherent state with indeterminate phase, the Mandel parameter is equal to unity. In a nonclassical sub-Poisson state, the Mandel parameter becomes less than zero: $-1<Q(\zeta)<0$, i.e., $\langle:(\Delta I(\zeta))^2:\rangle<0$.

Another statistical characteristic of the amplifier is the variance (of which there are two) of the amplitude–phase quadratures of the field

$$X_+(\zeta)=\frac{b(\zeta)+b^+(\zeta)}{2},\quad X_-(\zeta)=\frac{b(\zeta)-b^+(\zeta)}{2i}, \tag{5.2}$$

$$\langle(\Delta X_+(\zeta))^2\rangle=\frac{1}{4}+\frac{\langle b^2(\zeta)+(b^+(\zeta))^2\rangle}{4}+\frac{\langle n(\zeta)\rangle}{2}-\frac{\langle b(\zeta)+b^+(\zeta)\rangle^2}{4},$$

$$\langle(\Delta X_-(\zeta))^2\rangle=\frac{1}{4}-\frac{\langle b^2(\zeta)+(b^+(\zeta))^2\rangle}{4}+\frac{\langle n(\zeta)\rangle}{2}+\frac{\langle b(\zeta)-b^+(\zeta)\rangle^2}{4}. \tag{5.3}$$

The initial field is in a minimum-uncertainty coherent state, for which the two quadratures (5.2) are equal to 1/4. A field in a nonclassical (quadrature-squeezed) state has one of the quadratures less than 1/4.

In the calculations of the quantum-statistical properties of the nonlinear amplifier we took the mean photon number and the phase at the amplifier input to be $\langle n(0)\rangle=1$ and $\langle\phi(0)\rangle=0$. Figures 1–4 present contour plots of the

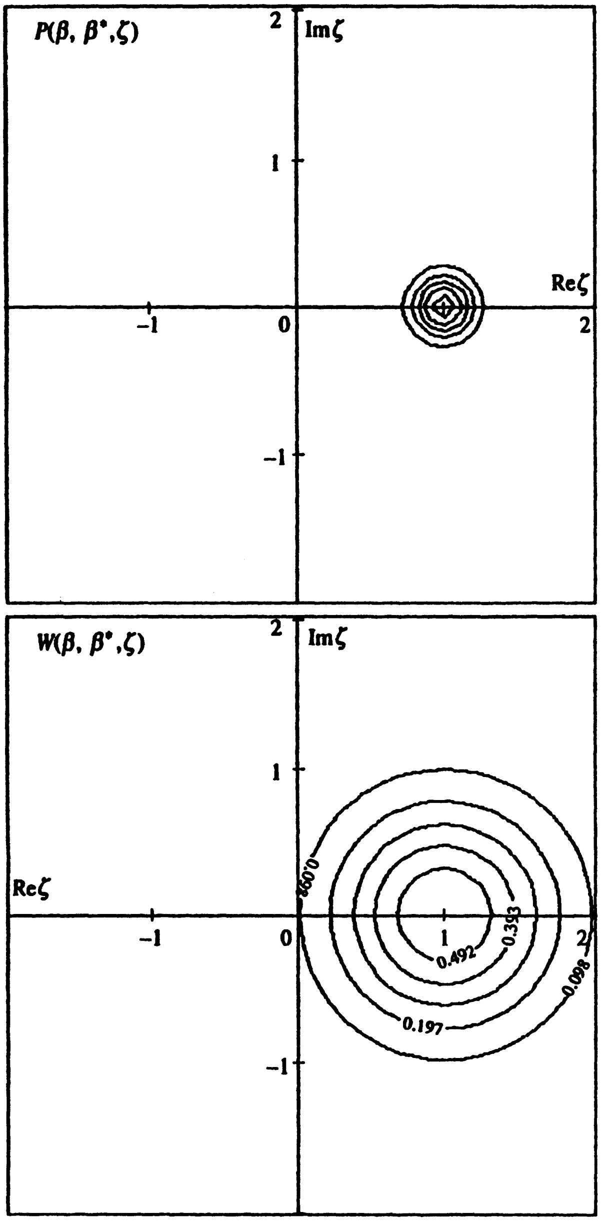

FIG. 1. Isolines of the Glauber–Sudarshan quasiprobability distribution function $P(\beta,\beta^*,\zeta)$ and the Wigner function $W(\beta,\beta^*,\zeta)$ of an optical amplifier for various values of the dimensionless length of the amplifier ζ for detuning $\delta=0$ and gain coefficient $p=1$. Field intensity at the input: $\langle I_0\rangle=1$, phase of the field at the input: $\langle\phi_0\rangle=0$; $\zeta=0.01$, $\langle n\rangle=1.04$, $Q=0.07$, $\langle(\Delta X_+)^2\rangle=0.27$, $\langle(\Delta X_-)^2\rangle=0.27$.

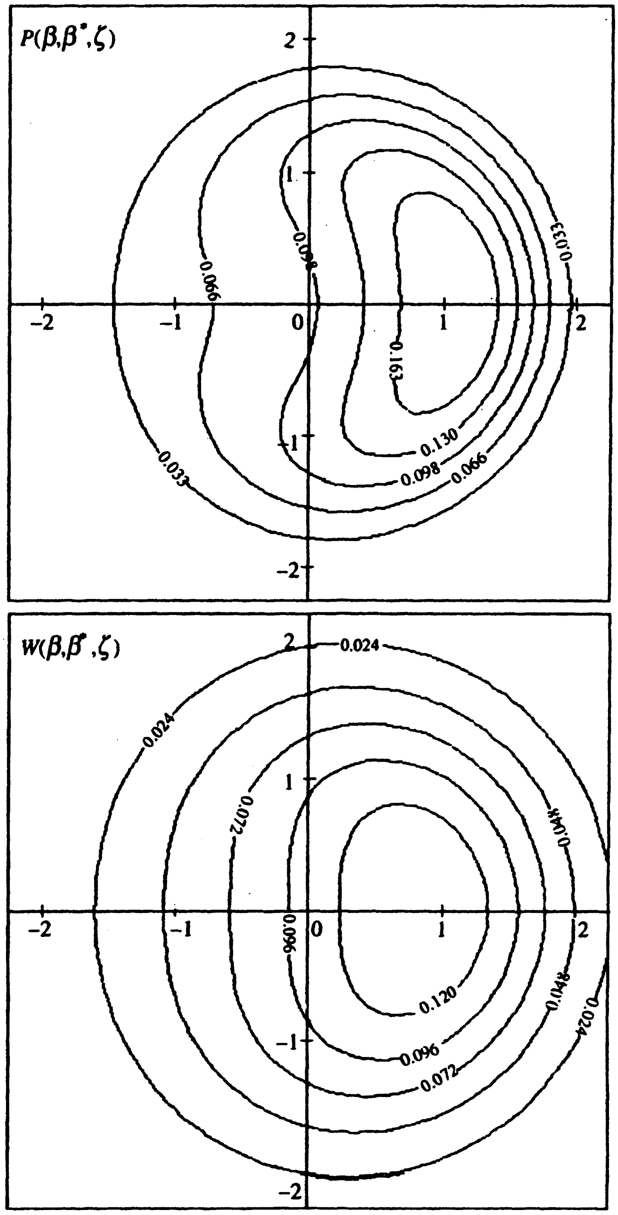

FIG. 2. Same as in Fig. 1, but for $\zeta=1.0$, $\langle n\rangle=1.58$, $Q=0.69$, $\langle(\Delta X_+)^2\rangle=0.93$, $\langle(\Delta X_-)^2\rangle=1.0$.

Glauber–Sudarshan quasiprobability functions $P(\beta,\beta^*,\zeta)$ and the Wigner function $W(\beta,\beta^*,\zeta)$ in the complex β=plane. The initial stage of the evolution of the quasiprobability function in the case of exact resonance is shown in Fig. 1 for the small value $\zeta=0.01$. Since the initial Wigner quasiprobability function, in contrast to the initial Glauber–Sudarshan function, is not singular at $\zeta=0$, but a smooth function for purely coherent light, for small ζ the magnitude of the field already possesses a large spread in the β-density, whereas the Glauber–Sudarshan function remains localized near I_0. With increasing ζ the difference between the Wigner and Glauber–Sudarshan functions diminishes; however, for $\zeta>1.5$ the shape of these functions begins to differ fundamentally: the Wigner function, as a result of diffusion in the β-plane, spreads into a larger region of the β-plane, all the while preserving its Gaussian shape, while the P-function, with increasing ζ, acquires a more complicated shape and a local minimum appears in the region Re $\beta<0$.

In the first stage of the evolution of the quasiprobability function the main process determining its shape is diffusion of the field intensity in the β-plane. In the second stage, diffusion of the phase brings the quasiprobability function into a phase-isotropic form. With increasing ζ, the mean value of the field $\langle\beta(\zeta)\rangle$ tends to zero and the position of the minimum observed in the P-function tends to $\beta=0$. The final form of the P- and W-functions at $\zeta=3.5$ is depicted in

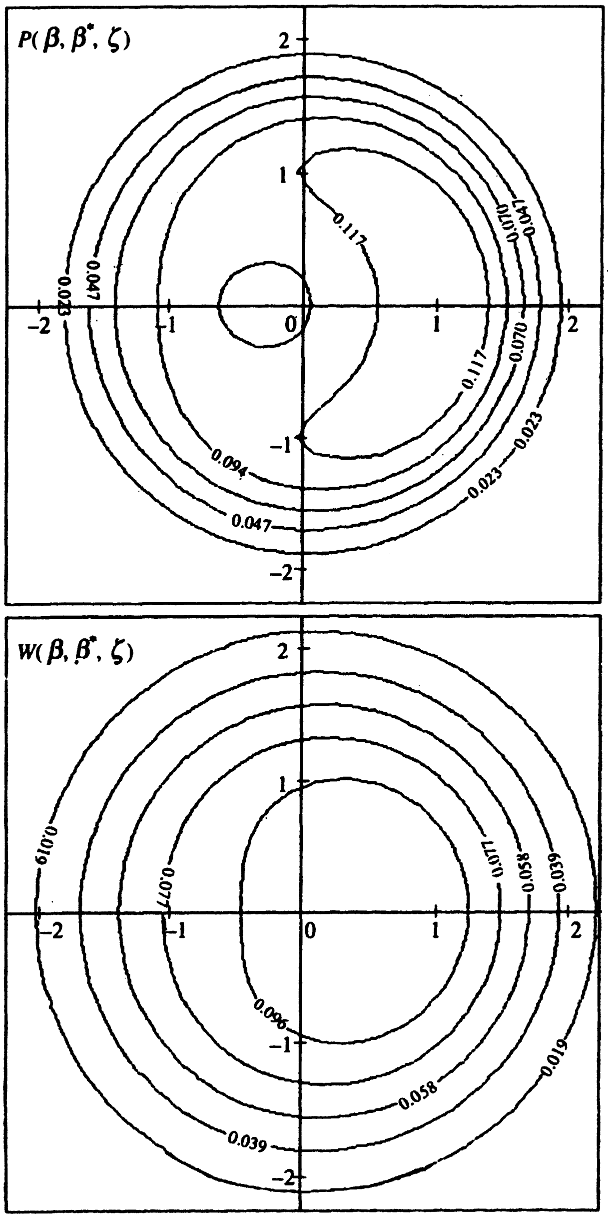

FIG. 3. Same as in Fig. 1, but for $\zeta=2.0$, $\langle n\rangle=1.58$, $Q=0.69$, $\langle(\Delta X_+)^2\rangle=1.03$, $\langle(\Delta X_-)^2\rangle=1.04$.

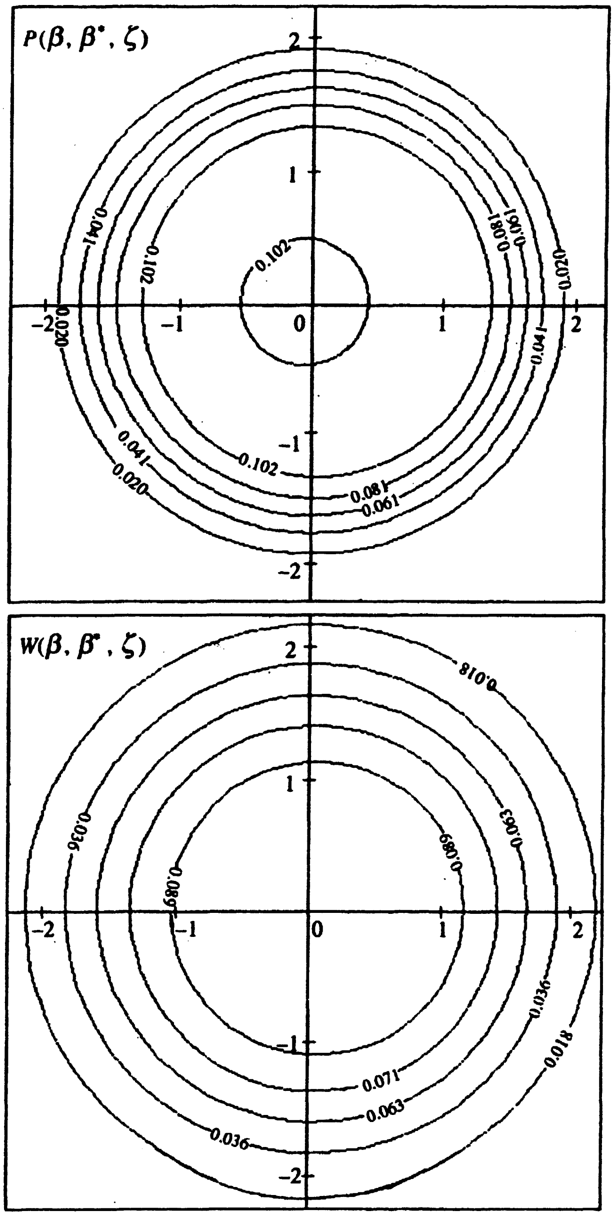

FIG. 4. Same as in Fig. 1, but for $\zeta=3.5$, $\langle n\rangle=1.58$, $Q=0.69$, $\langle(\Delta X_+)^2\rangle=1.04$, $\langle(\Delta X_-)^2\rangle=1.04$.

Fig. 4. From the figure it is clear that the shape of the P-function differs fundamentally from that of the W-function in the steady state; the P-function is a "pillow" centered at $\beta=0$, i.e., it is a sum of the spontaneous noise, or coherent vacuum with mean $\langle n\rangle=0$, and the superposition of coherent states with $\langle n\rangle\neq 0$ and $\langle E\rangle=\sqrt{\langle n\rangle}e^{i\phi}$ for all values of ϕ. But the Wigner function in this low-gain case gain has the Gaussian shape characteristic of noise, i.e., the coherent vacuum.

Figures 5–8 depict the P and W quasiprobability functions in the case of high gain $p=10$ and exact resonance. Comparison with Figs. 1–4 shows that the evolution of the quasiprobability functions in the low-gain case differs qualitatively from the high-gain case. For small values $\zeta>0.1$, the uncertainty region of the radiation is already prolate along the Im β axis, i.e., $\langle(\Delta X_-(\zeta))^2\rangle>\langle(\Delta X_+(\zeta))^2\rangle>1/4$, and the Mandel parameter $Q(\zeta)$ for the high-gain case turns out to be significantly smaller than the corresponding value for the case. Since the mean intensity (photon number) of the radiation is large in the high-gain case, the intensity fluctuations are reduced due to saturation, and by the early stages of evolution the variance of the photon number becomes significantly smaller. Diffusion of the modulus of the field in the β-plane is characteristic of low gain, and is comparatively small in the high-gain case. On the other hand, diffusion of the phase is the dominant process determining the evolution of the quasiprobability at high gains, while at low gains, diffusion of the modulus of the field is the dominant process. It is clear from Fig. 8 that for a high gain in the steady state the P- and W-functions coincide, and as a result of the evolution a state of the field is formed which has super-Poisson statistics $Q(\zeta)>1$ and arbitrary phase, with

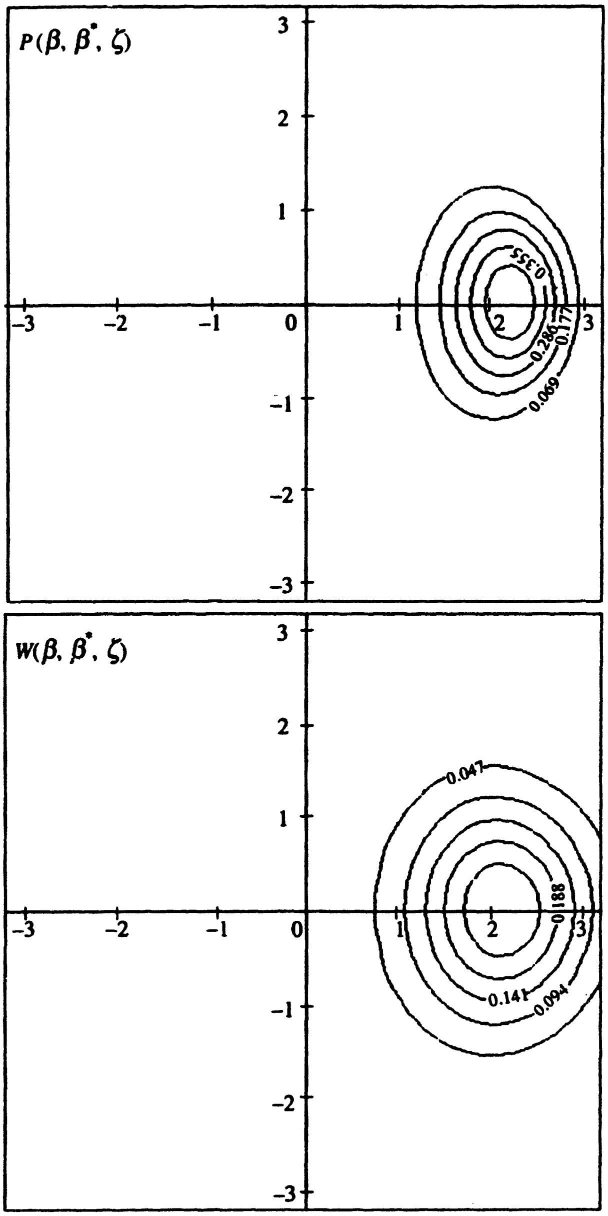

FIG. 5. Same as in Fig. 1, but at a gain $p=10$ for $\zeta=0.1$, $\langle n\rangle=4.78$, $Q=0.73$, $\langle(\Delta X_+)^2\rangle=0.48$, $\langle(\Delta X_-)^2\rangle=0.66$.

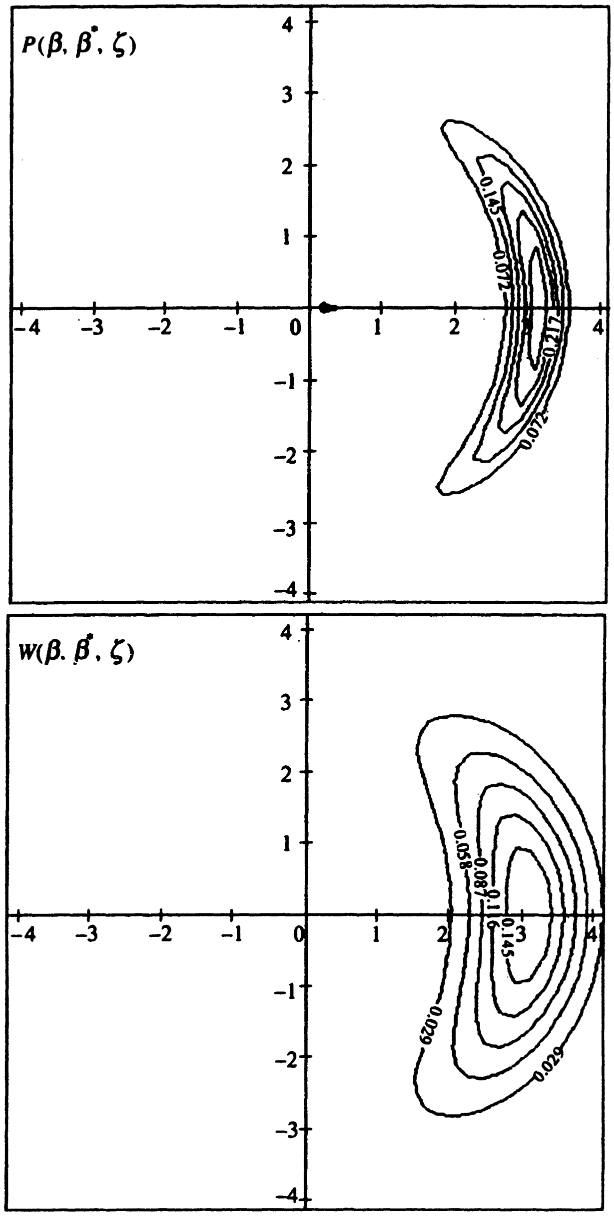

FIG. 6. Same as in Fig. 1, but at a gain $p=10$ for $\zeta=0.8$, $\langle n\rangle=10.0$, $Q=0.2$, $\langle(\Delta X_+)^2\rangle=0.60$, $\langle(\Delta X_-)^2\rangle=2.31$.

amplitude–phase quadratures $\langle(\Delta X_\pm(\zeta))^2\rangle\approx1/4+\langle n(\zeta)\rangle/2$.

At both low and high gain, the main step in the formation of the quasiprobability distribution function takes place after reaching complete saturation of the gain, i.e., at an unvarying mean photon number.

Figures 9–12 present the results of calculations in the case of finite detuning of the frequency of the radiation from the frequency of the atomic transition of the active medium. As can be seen from the plots, the detuning injects a fundamentally new character into the evolution of the quasiprobability function. In the first stage of linear amplification for $\zeta<0.02$, detuning is manifested by a rotation of the maximum of the quasiprobability function about the origin without any change in the symmetry of the distribution. At large ζ the quasiprobability function acquires a spiral shape, as a result of the fact that diffusion of the phase takes place faster for fields of lower intensity than for fields of higher intensity. On the other hand, such a shape of the quasiprobability distribution is indicative of the phase sensitivity of the amplification in the presence of detuning. Nonlinear amplification with saturation leads to the result that a signal with larger phase experiences less amplification, which leads to the formation of the spiral shape of the quasiprobability distribution (see Fig. 11). The quasiprobability functions in this case are a superposition of coherent states characterized by the entire spectrum of the parameters $\langle n\rangle\neq0$, $\langle\phi\rangle\neq0$, and $\langle(\Delta n)^2\rangle\neq0$. In this case, comparison shows that the evolution of the Wigner function takes place substantially faster than the evolution of the P-function so that by $\zeta\simeq0.07$, the W-function reaches a state close to steady-state.

The calculations show that the direction of rotation of

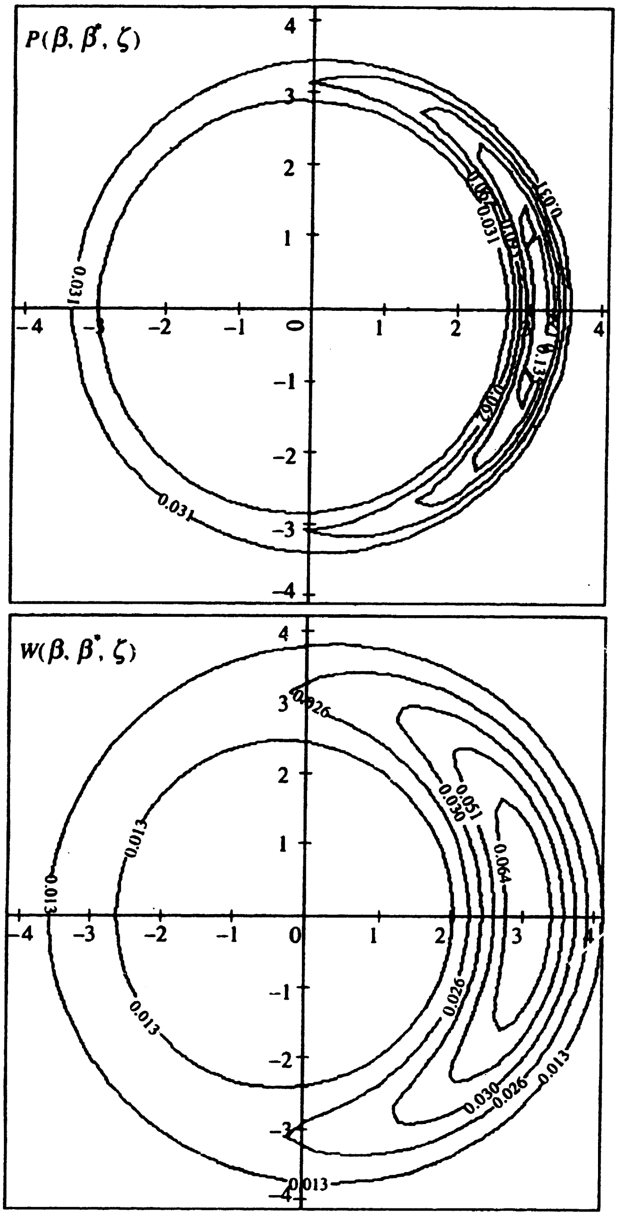

FIG. 7. Same as in Fig. 1, but at a gain $p=10$ for $\zeta=2.5$, $\langle n\rangle=10.0$, $Q=0.2$, $\langle(\Delta X_+)^2\rangle=1.21$, $\langle(\Delta X_-)^2\rangle=4.27$.

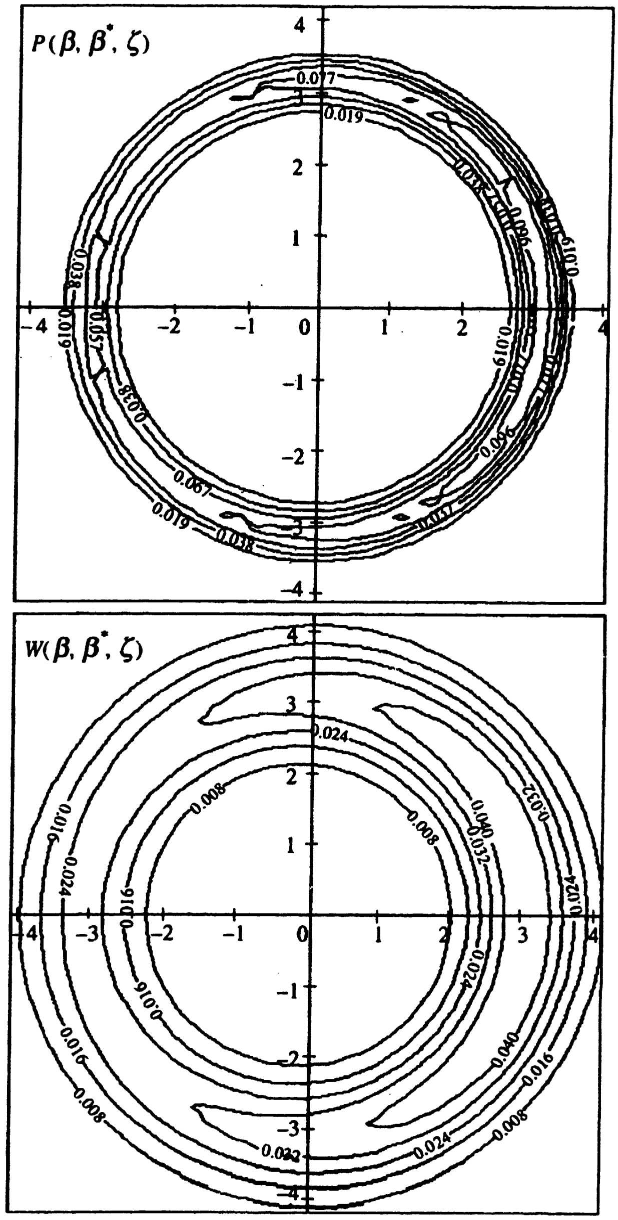

FIG. 8. Same as in Fig. 1, but at a gain $p=10$ for $\zeta=12.0$, $\langle n\rangle=10.0$, $Q=0.2$, $\langle(\Delta X_+)^2\rangle=4.51$, $\langle(\Delta X_-)^2\rangle=5.22$.

the quasiprobability distribution function in the case of finite positive detuning depends on the magnitude of the detuning and on the gain. Thus, for $p=5$, $\delta=5$ the rotation is counterclockwise: $d\langle\phi\rangle/d\zeta>0$, and for $p=1$, $\delta=10$ it is clockwise: $d\langle\phi\rangle/d\zeta<0$.

In all of the calculations at small $\zeta<0.01$, we observed negative values of the Glauber–Sudarshan quasiprobability distribution. Such a nonclassical state of light in the earliest stages of amplification is due, in our opinion, to the idealized initial conditions of the calculation, since it is assumed that the field entering the amplifier is in a pure coherent state with singular P-function. However, the calculations show that in these cases, there are no values of the amplifier parameters for which sub-Poisson or quadrature-squeezed states of the field are realized. The field statistics even at high gains—for which the van der Pol oscillator approximation is applicable—turn out to be substantially super-Poisson.

In all of the calculations for $\zeta>0.1$, either the Glauber–Sudarshan function is positive, or the absolute value of the negative values is small in comparison with the maximum positive values.

6. CONCLUSION

We have investigated the evolution of the Glauber–Sudarshan quasiprobability distribution function (P-function) and the Wigner W-function for a nonlinear optical amplifier. The Fokker–Planck equation for the Glauber–Sudarshan P-function has been solved numerically with the help of the polynomial-expansion method. The Wigner function was found with the help of the well-known

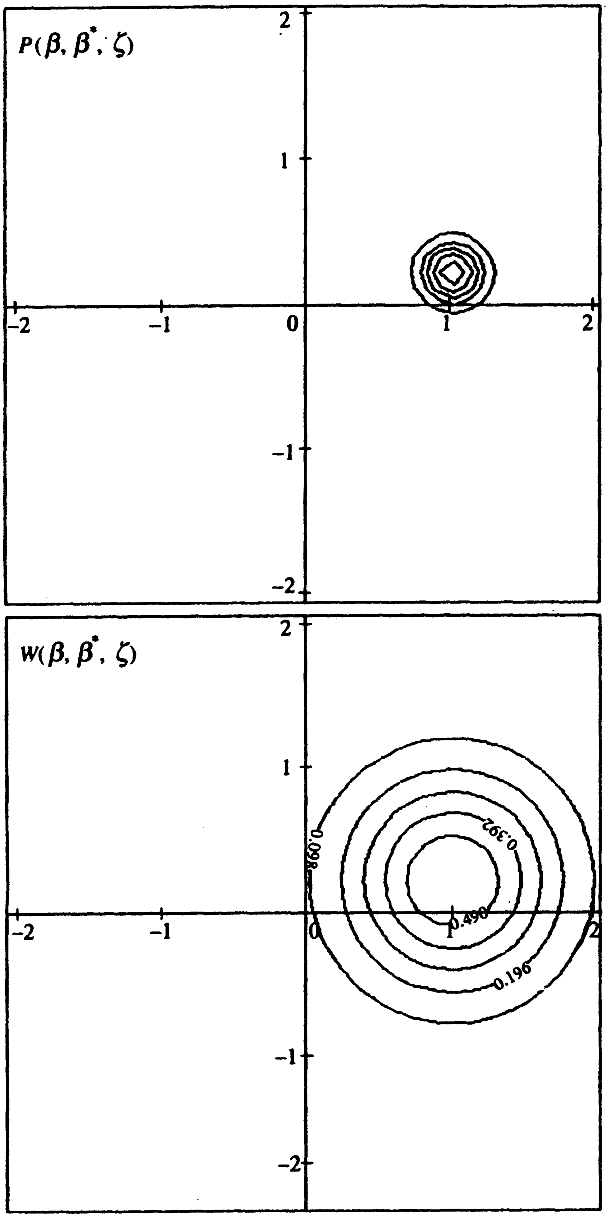

FIG. 9. Same as in Fig. 1, but for a detuning $\delta=5$ and gain $p=5$ for $\zeta=0.01$, $\langle n\rangle=1.12$, $Q=0.08$, $\langle(\Delta X_+)^2\rangle=0.27$, $\langle(\Delta X_-)^2\rangle=0.27$.

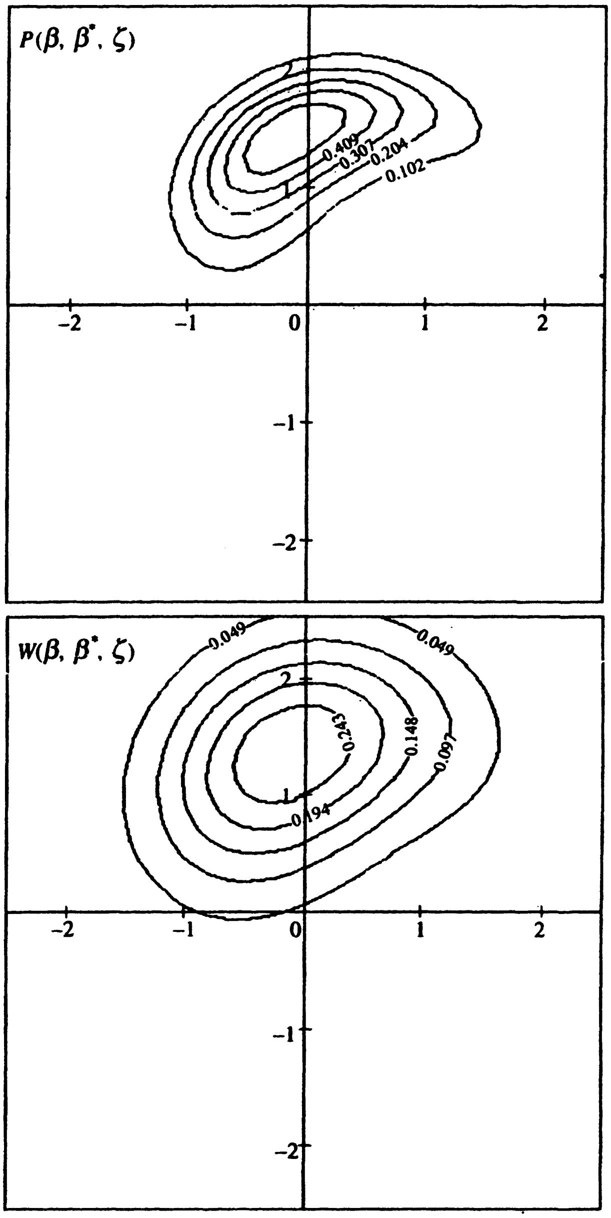

FIG. 10. Same as in Fig. 1, but for a detuning $\delta=5$ and gain $p=5$ for $\zeta=0.1$, $\langle n\rangle=2.35$, $Q=0.59$, $\langle(\Delta X_+)^2\rangle=0.7$, $\langle(\Delta X_-)^2\rangle=0.46$.

Glauber–Sudarshan integral transform. We calculated quantum-statistical means characterizing the amplifier radiation, such as the mean photon number, the mean field amplitude, the variance of the photon number, the Mandel parameter, and the variances of the phase–amplitude quadratures of the field.

A study of the quantum-statistical properties of the field of an amplifier with saturation has been carried out for different values of the gain and detuning of the field from the frequency of the transition of the two-level atom. We have uncovered a qualitative difference in the evolution of the quasiprobability function at exact resonance for low and high gains. At low gain, the dominant process governing the shape of the quasiprobability function is diffusion of the modulus of the field in the phase plane. At high gain, fluctuations of the modulus of the field, i.e., of the intensity, are significantly weaker and the dominant process governing the shape of the quasiprobability function is phase diffusion. In the first stage of evolution at low gain, the field intensity fluctuations grow, and in the second stage after stabilization of the intensity the phase fluctuations begin to grow. The variance of the intensity and the Mandel parameter grow monotonically with increasing amplifier length. At high gains, phase fluctuations dominate in the first stage; the variance of the phase grows much faster than that of the intensity. At later times, for long amplifiers the variance of the intensity is reduced.

Detuning has a large effect on the evolution of the quasiprobability. A nonlinear amplifier with detuning exhibits phase sensitivity; as in the case of the Kerr nonlinearity,[17] the degree of amplitude gain depends on the magnitude of the phase of the field, but in contrast to the Kerr nonlinearity,

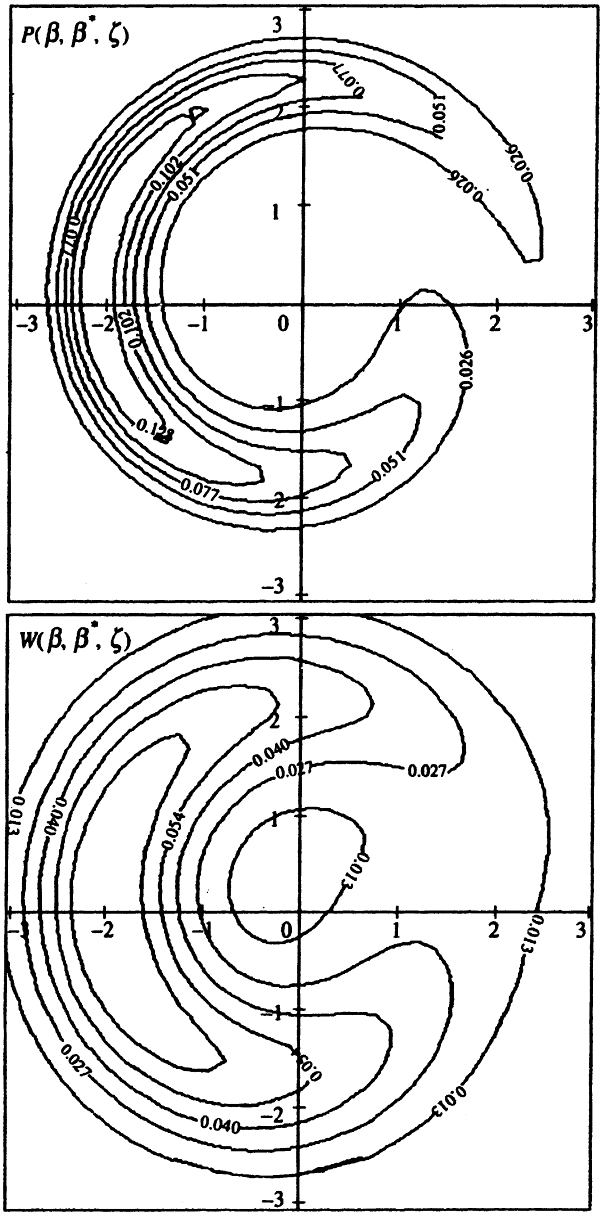

FIG. 11. Same as in Fig. 1, but for a detuning $\delta=5$ and gain $p=5$ for $\zeta=0.3$, $\langle n\rangle=4.22$, $Q=0.63$, $\langle(\Delta X_+)^2\rangle=2.03$, $\langle(\Delta X_-)^2\rangle=2.34$.

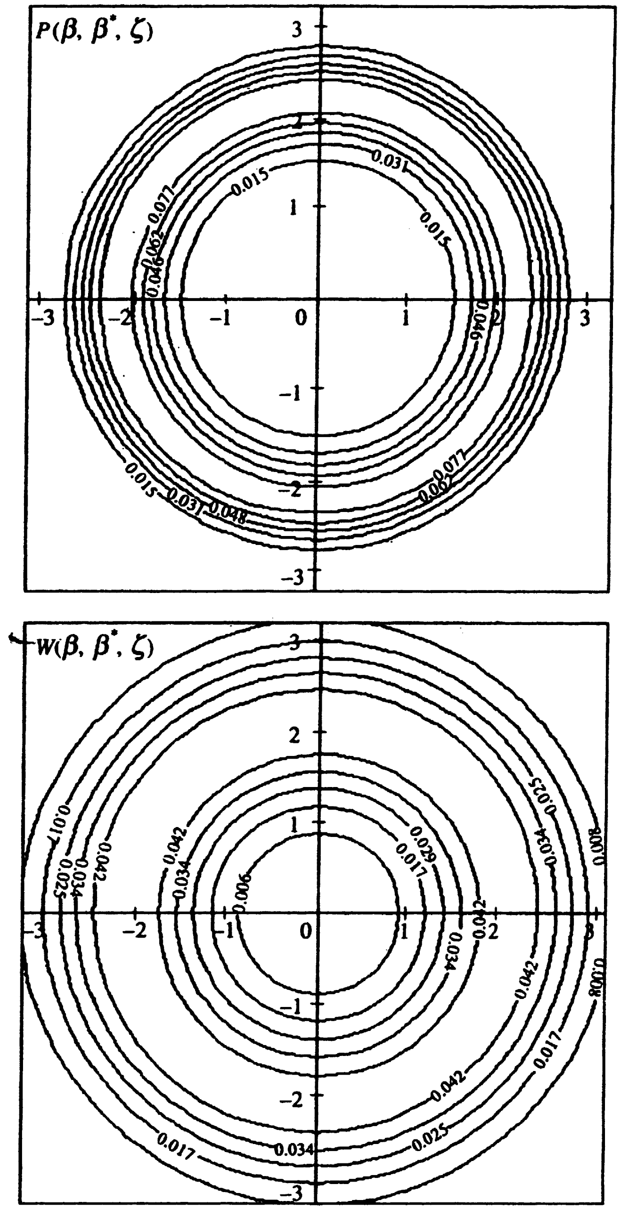

FIG. 12. Same as in Fig. 1, but for a detuning $\delta=5$ and gain $p=5$ for $\zeta=0.9$, $\langle n\rangle=4.99$, $Q=0.4$, $\langle(\Delta X_+)^2\rangle=2.74$, $\langle(\Delta X_-)^2\rangle=2.74$.

amplification with saturation results in large amplitudes undergoing less rotation about the origin. Phase diffusion takes place at lower intensities faster than at higher intensities, and the uncertainty in the field amplitude is greater for large values of the phase.

Our investigations have shown that the nonlinear amplifier considered here, even for a short amplifier, adds significant noise to a coherent input signal. We have shown that nonlinear amplification introduces major limitations in the use of such an amplifier in systems of coherent information transmission. These include rotation and diffusion of the phase of the signal in the presence of detuning, and the injection into the transmitted signal of a high level of additional intensity noise, which plays a substantial role even at low gain.

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