# Quantum variational force measurement and the cancellation of nonlinear feedback

S. P. Vyatchanin and A. B. Matsko

*M. V. Lomonosov Moscow State University, 119899 Moscow, Russia*
(Submitted 14 November 1995)
Zh. Éksp. Teor. Fiz. **109**, 1873–1888 (June 1996)

Based on a simplified optical displacement sensor, we consider a small-force quantum detection procedure incorporating cancellation of the back influence of fluctuations. We analyze constraints imposed by nonlinear terms of order $\sim(x/\lambda_0)^2$ ($x$ is the position of the test mass and $\lambda_0$ is the laser wavelength), and we suggest a nonlinear scheme that yields improved sensitivity.
 [S1063-7761(96)00106-0]

## 1. INTRODUCTION

Quantum noise in the mechanical displacement sensor is a key problem in laser interferometer gravitational-wave detectors (as in the LIGO project) and in a number of other fundamental experiments. With continuous position tracking, the back effects of noise restrict measurement sensitivity to that imposed by the standard quantum limit.[1–4] In the simple optical sensor of Fig. 1, the back influence of noise derives from fluctuations in the ponderomotive force due to light pressure, with the result that amplitude fluctuations in the incident wave $E_1$ are transformed into phase fluctuations in the reflected wave $E_2$, with the transformation coefficient increasing with the power in $E_1$. (We assume throughout that the sensor has no intrinsic mechanical noise.)

It is normally assumed that the standard quantum limit determines the magnitude of the minimum detectable force. Thus, for a force that takes the form

$$F_S=F\,\sin(\omega_F t),\qquad 0\leqslant t\leqslant 2\pi/\omega_F, \tag{1}$$

and acts on a free mass $m$ for a time $t_F=2\pi/\omega_F$, the standard quantum limit is $F_{\mathrm{SQL}}\simeq\sqrt{m\hbar\omega_F^2/t_F}$. (All subsequent discussion concerns forces of known form.)

In tracking displacement, it is theoretically possible to measure the force with an error less than $F_{\mathrm{SQL}}$ if the noise generated by the measurement device is correlated in a special way.[2,4] To exceed the standard quantum limit, one can also use a modulated signal source,[5] or a source in a frequency-anticorrelated state[6] or a squeezed state.[7] In the latter case, the squeezing must be frequency-independent, and it is unclear how to bring this about experimentally.

It has been shown[8] that one can exceed the standard quantum limit even with an unmodulated coherent source, without the use of a squeezed, frequency-anticorrelated, or other nonclassical state. This requires that one measure not the phase (which is strongly perturbed by the back influence of noise), but a specially selected quadrature component $B(\theta)$ of the reflected wave (Fig. 2). It is then precisely the ponderomotive nonlinearity (which is in fact the ponderomotive counterpart of the nonlinearity susceptibility $\chi^{(3)}$) responsible for the back influence of noise that results in squeezing of the reflected light. Measurement of $B(\theta)$ makes it possible to "see past" the noise effects.

Significantly, the squeezing in the reflected wave special feature: the angle $\theta$ depends on the frequency, $\theta=\theta(\Omega)$. To measure the frequency-dependent squeezing over a wide bandwidth (i.e., with a short integration time $T\simeq t_F$, which is usually necessary in gravitational radiation detectors), it has been suggested that the phase of the local oscillator be specially modulated during the integration time in a balanced homodyne scheme. The force-measurement error is then determined solely by the initial phase uncertainty in $E_1$, which decreases with increasing optical power.

Radiative damping, however, limits the amount of squeezing actually attainable, and sets the magnitude of the minimum detectable force:

$$F_{\min}=\xi F_{\mathrm{SQL}}\sqrt{\omega_F/\omega_0}. \tag{2}$$

Here $\omega_0$ is the optical pump (laser) frequency and $\xi$ is a numerical factor of order unity. This sensitivity can be achieved at the optimal pump power $W_{\mathrm{opt}}\simeq mc^2/t_F$ ($c$ is the speed of light). An estimate shows that this is too high for experimental purposes: $W_{\mathrm{opt}}\simeq 1\times10^{19}\ W$ at $m=10^3$g and $t_F=10^{-2}$. For a power $W_{\mathrm{SQL}}\ll W\ll W_{\mathrm{opt}}$ ($W_{\mathrm{SQL}}$ is the power required to reach the standard quantum limit), we have $F\simeq F_{\mathrm{SQL}}\sqrt{W_{\mathrm{SQL}}/W}$.[8]

Radiative damping derives from the Doppler effect: if the mirror is moving in the same direction as the incident wave, then the flux falling on it decreases as the path length of the wave increases. As a result, the radiation pressure depends on the velocity of the mirror.

One more factor besides the back effect of noise limits measurement accuracy—the initial position and momentum uncertainties of the mechanical oscillator. To measure a force of order $F_{\min}$, it is necessary, in addition to achieving pump power $W_{\mathrm{opt}}$, that the measured quantity not contain information on the initial conditions. Below, we summarize the additional requirements for this to be the case.

We stress that this procedure does not constitute a quantum nondemolition measurement, as it is not necessary here to identify an unperturbed variable of the mechanical oscillator. Indeed, little information about oscillator position, momentum, or some combination of the two resides in the reflected wave, so the instrument perturbs both strongly. One measures only variations in position resulting from the laser signal pressure. These could thus well be called quantum variational measurements. This further suggests that detecting some signal effect and making a quantum nondemolition measurement are different problems. Each calls for a specific strategy, and the latter generally differ from one another.

 1063-7761/96/061007-08$10.00  

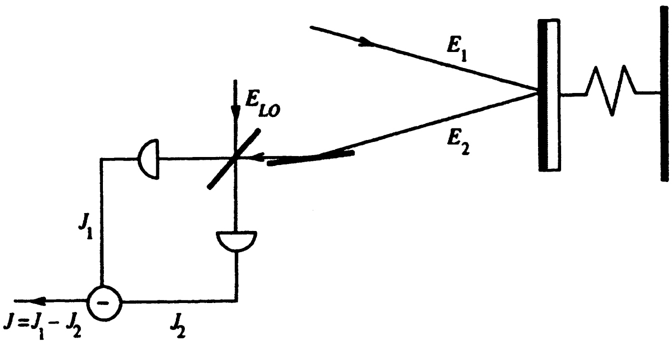


FIG. 1. Simplified optical displacement sensor that detects a change in position of a movable mirror induced by radiation pressure. The electromagnetic plane wave $E_1$ is normally incident upon the surface of a lossless mirror, the position of which varies according to the applied signal force $F_S$. A change in mirror position produces a phase shift in the reflected wave $E_2$ that can be detected using a balanced homodyne layout. The difference photocurrent $J$ is proportional to the quadrature amplitude $B(\theta(t))$ of the signal wave ($\theta(t)$ is determined by the phase of the local oscillator $E_{LO}$). For fluctuations in $E_{LO}$ to be negligible, we must have $|E_{LO}|\gg k|E_1|$, where $k>1$ is the squeezing factor of the signal wave. We assume that the local oscillator produces a field that is in a coherent state. For an optimal choice of $\theta(t)$, the limiting detectable force acting on the movable mirror can be less than $F_{SQL}$.

All of the foregoing ways of achieving sensitivity beyond the standard quantum limit[2–8] have been obtained in the linear approximation in $\omega_0 x/c$ and $e_1/E_0$ (where $x$ is the position operator of the mirror, and $E_0$ and $e_1$ are the mean amplitude and fluctuations of the incident wave). It is only in this approximation that fluctuations of the reflected wave can be described by an orthodox ellipse in the phase diagram

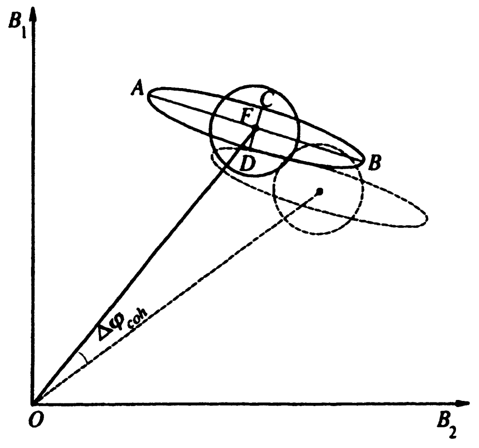


FIG. 2. Phase diagram of reflected-wave amplitude in the linear approximation. If the incident wave is in a coherent state, its fluctuations can be described by a circle in the $B_1B_2$ phase plane that revolves about the origin $O$ at a distance $\sqrt{n}$ (the uncertainties $\langle\Delta B_1^2\rangle$ and $\langle\Delta B_2^2\rangle$ in the quadrature amplitudes are equal). The ponderomotive nonlinearity due to light pressure results in phase/amplitude correlation in the reflected wave. The circle describing fluctuations in the incident wave is transformed upon reflection into an (orthodox) ellipse in the linear approximation. The minimum detectable force will alter the phase of the reflected wave by $\Delta\varphi$, such that the ellipse corresponding to the reflected wave fails to intersect that of the incident wave ($\Delta\varphi=\Delta\varphi_{coh}$). Clearly, rather than the phase, one must then measure the well-defined quadrature component $B_{opt}$. It is then possible to improve upon the standard quantum limit of sensitivity.

(Fig. 2). On the other hand, allowance for higher-order terms ($(\omega_0 x/c)^2$ and $(e_1/E_0)^2$, etc.) induces curvature in the noise ellipse, with a consequent increase in the uncertainty of the quadrature amplitude (line segment $CD$ in Fig. 3 lengthens with increasing curvature). Estimates show that this curvature cannot be neglected at powers much lower than $W_{opt}$.

When allowance is made for nonlinear terms, the uncertainty in the initial conditions gives rise to an additional contribution to the measurement errors. The reason is that the uncertainty enters into terms that contain $x^2$, $x^3$, etc. It thus becomes necessary to first prepare the mechanical system in such a way that positional uncertainties that derive from the initial conditions are negligible in comparison with the perturbation due to the back influence of fluctuations. As we show below, this can be accomplished under far more relaxed conditions than those prescribed by the quantum theory of measurement.

It is well worth noting here that constraints due to curvature of the fluctuation ellipse are not fundamental in nature—they relate exclusively to the linear measurement procedure. In the present paper, we discuss the limitations of the linear scheme, and propose an alternative nonlinear scheme that makes it possible to discriminate among "bent ellipses." The basic idea is that the reflected wave must be transformed in such a way that the bent ellipse describes the state of the resulting amplitude- (photon number-) squeezed field.[1)] Such squeezing can be measured relatively easily with a photodetector. To do so, one adds a reference wave (segment $OO'$ in Fig. 3) to the reflected wave using a beamsplitter (Fig. 4), thereby altering the mean amplitude and phase of the combined wave as required (line segment $OF$ is transformed into $O'F$). The transmission coefficient $T_{sp}$ of the splitter must be small enough that fluctuations in the reference wave can be neglected.

Not only is the squeezing of the reflected wave "curved," it is frequency-dependent as well (which derives from the "memory" mechanism of the mechanical oscillator). Geometrically, this means that fluctuations at different spectral frequencies are described by different "ellipses." We will show that for such squeezing to be observed, the reference wave must be specially modulated during the course of the measurement.

We note immediately that this measurement procedure is not entirely optimal. The curved major axis of the ellipse does not lie exactly on a circle centered at $O'$, due to the presence of third- and higher-order noise terms. Nevertheless, for certain values of the mirror mass $m$ and frequency $\omega_F$, these are minor restrictions, and the sensitivity (2) can be achieved with the nonlinear measurement setup.

To completely negate the back influence of noise, we require some nonlinear transformation of the reflected wave that makes it revert to a state like that of the incident wave while preserving the phase shift of the laser signal. Theoretically, we could pass the reflected wave through a nonlinear medium which, apart from a given constant overall $\chi^{(3)}$, has a frequency-dependent component $\chi^{(3)}(\Omega)$, or we could re-

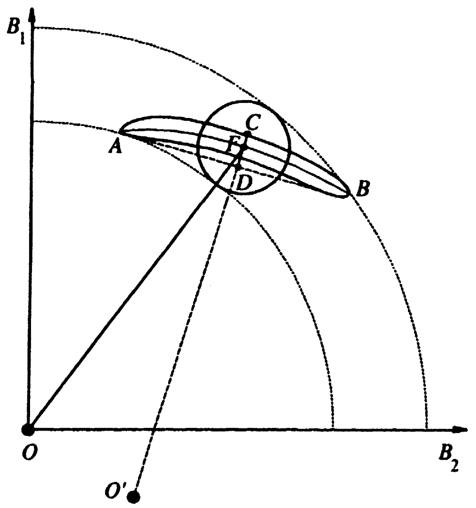


FIG. 3. Phase diagram for the spectral amplitude of the reflected wave. Allowance for nonlinear corrections leads to "curvature" of the fluctuation ellipse, which increase with increasing pump power. This curvature lies behind the additional measurement error in the quadrature component. The error results from the linear projection of all points on the "ellipse" in a selected direction: the line segment $CD$ is longer than in the linear approximation (Fig. 2). The sensitivity of the linear detection scheme is therefore limited. To identify "curved" ellipses, we propose counting photons in the shifted wave (a nonlinear measurement). Due to the addition of a low-amplitude fluctuating shifted field to $E_2$, the origin $O$ of the phase plane is translated to the center of curvature of the "ellipse" (i.e., of the curve $AB$)—to the point $O'$, and the field amplitude is measured relative to the new coordinate system (line segment $O'F$). This procedure makes it possible to significantly reduce the influence of quadratic fluctuations on measurement accuracy. Sensitivity is then limited by the fact that the curve $AB$ is not a proper circle.

flect $E_2$ from a mirror with negative mass. Experimentally, these are probably not feasible

But to enhance sensitivity, the back influence of noise needn't be canceled completely. We can cancel only the nonlinear part due to curvature of the ellipse (for example, by adding nonlinearity to the sensor). The homodyne system could then be modified to actually detect the force, yielding the same measurement accuracy that one would get by completely suppressing the back influence of the noise. We do not address this possibility in the present paper, examining in detail only the approach described above.

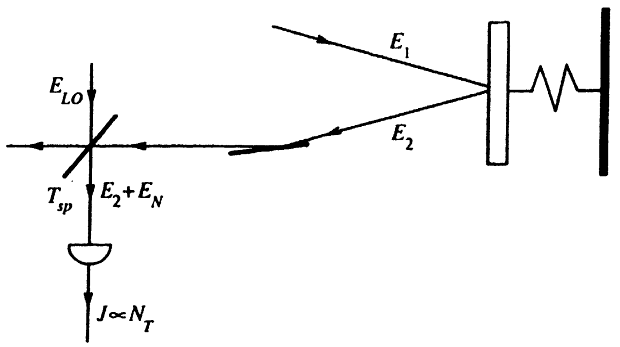


FIG. 4. Photon-counting measurement setup for the shifted wave. The beamsplitter is used to add the shifted field $E_N=\sqrt{T_{\text{sp}}}E_{\text{LO}}$ to $E_2$ ($T_{\text{sp}}$ is the transmission coefficient of the beamsplitter). Let $\sqrt{T_{\text{sp}}}\ll 1/k$ ($k>1$ is the squeezing coefficient of noise in the reflected wave); then fluctuations in the shifted field can be neglected. The detected photocurrent $J$ is then proportional to the number of photons in the wave $E_2+E_N$.

We assume that the incident wave $E_1$ can be in a phase- (or amplitude-) squeezed state (where we have in mind conventional frequency-independent squeezing). This is of interest because when the back influence of noise is completely cancelled (in the linear approximation), the sensitivity is higher for the phase-squeezed state, although the "curvature of the ellipse" is also greater, and there ought to exist an optimum value at which the sensitivity peaks.

## 2. BASIC EQUATIONS

Writing the incident and reflected fields as $E_1\exp(-i\omega_0 t)+E_1^+\exp(i\omega_0 t)$ and $E_2\exp(-i\omega_0 t)+E_2^+\exp(i\omega_0 t)$, respectively, representing $E_1=E_0+e_1$ as a sum of a mean amplitude $E_0$ (assuming $E_0=E_0^*$) and a fluctuating component $e_1$, and likewise taking $E_2=-(E_0+e_2)$, the boundary conditions yield a relationship between $e_1$ and $e_2$:

$$e_2=E_0\left[-1-\frac{2\dot{x}}{c}+\exp\left(i\frac{2\omega_0 x}{c}\right)\right]+e_1\exp\left(i\frac{2\omega_0 x}{c}\right), \tag{3}$$

$$e_1=\int_{-\omega_0}^{\infty}\sqrt{\frac{\hbar\omega_0}{Sc}}\,\lambda_+ a(\omega_0+\Omega)\exp(-i\Omega t)d\Omega, \tag{4}$$

where $\lambda_\pm=\sqrt{1\pm\Omega/\omega_0}$, and $a(\omega)$ is the photon annihilation operator (the commutator $[a(\omega)a^+(\omega')]=\delta(\omega-\omega')$). Fluctuations in the incident wave are squeezed according to

$$a(\omega_0+\Omega)=\sqrt{1+\mu^2}c(\omega_0+\Omega)+\mu c^+(\omega_0-\Omega),$$

where $\mu$ is the (real) squeezing coefficient, and $c$ and $c^+$ are vacuum operators, for which $\langle c(\omega)c^+(\omega')\rangle=\delta(\omega-\omega')$.

These equations must be supplemented by the equation of motion of the mechanical oscillator, which describes the motion of the mirror:

$$\ddot{x}+2\delta_R\dot{x}+\omega_M^2 x=\frac{S}{\pi m}E_0(e_1+e_1^+)+\frac{F_S}{m}. \tag{5}$$

Here $\delta_R=2W/mc^2$ is the radiative damping coefficient, $W=ScE_0^2/2\pi$ is the mean incident power, $S$ is the area of the mirror, and $F_S$ is the force induced by the laser signal. We assume that the system is mass-controlled, with $\omega_M\ll\omega_F$. A numerical demonstration of the smallness of the omitted terms is given in Appendix B.

## 3. LINEAR SCHEME: MEASUREMENT OF QUADRATURE COMPONENT

A modified homodyne design makes it possible to completely suppress the effects of both the back influence of fluctuations and noise due to initial position and momentum uncertainties of the mirror on the measured value of the linear part of the noise. It is well known that the homodyne detector (Fig. 1) can be used to measure the quadrature component of an electromagnetic wave. The difference photocurrent is $J\simeq E_{\text{LO}}(t)E_2^+(t)+\text{H. c.}$ ($E_{\text{LO}}(t)$ and $E_2(t)$ are the complex electric field amplitudes of the reference—the local oscillator—and signal, respectively). For fluctuations in $E_{LO}(t)$ not to make a palpable contribution to the measurements, the LO power needs to be high: $|E_{LO}| \gg k|E_2|$ ($k>1$ is the squeezing coefficient of the signal field $E_2$, and it is assumed that $E_{LO}$ is in a coherent state). The photocurrent $J(t)$ can then be assumed proportional to the quadrature component $B(\theta,t)$:

$$B(\theta,t)=E_2(t)\exp(-i\theta)+E_2^{+}(t)\exp(i\theta), \tag{6}$$

where $\theta$ is governed by the phase of $E_{LO}$. Over some finite integration time $T$, rather than the quadrature component, we detect some mean value

$$B_T=\int_0^T \tilde{\Phi}(t)B(\theta,t)dt. \tag{7}$$

The substance of the modification to the homodyne scheme consists of requiring that the local oscillator be phase modulated. The averaging function $\tilde{\Phi}(t)$ and the LO phase-modulation function $\theta(t)$ can be selected that such that $B_T$ preserves no information about the linear part of the noise derived from the back influence of fluctuations. In essence, measurement of $B_T$ amounts to a linear approximation to the optimal signal detection algorithm described above.

Let $g_S(t)=\tilde{\Phi}(t)\sin\theta(t)$ and $g_C(t)=\tilde{\Phi}(t)\cos\theta(t)$. Noise due to the back influence of fluctuations and noise due to initial uncertainties in the mirror position and momentum will be suppressed when (see Appendix A)

$$g_C(t)+\int_0^t g_S(t)K(t-\tau)d\tau=0, \tag{8}$$

$$g_C(T)=0, \qquad \dot{g}_C(T)=0. \tag{9}$$

Here

$$K(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} K(\Omega)\exp(-i\Omega t)d\Omega$$

is the kernel of the differential equation (5), $K(\Omega)=4\omega_0\delta_R/Z(\Omega)$, and $Z(\Omega)=\omega_M^2-\Omega^2-2i\delta_R\Omega$.

The function $g_S(t)$ can be determined using the methods of optimal filtering theory. We seek to maximize the signal-to-noise ratio under the constraints given by (8) and (9). Recall that if (9) need not be satsified, the optimum filter can be written in the form $g_S(t)=hx_S(t)$, where $h$ is an arbitrary constant and $x_S(t)$ is the response of the mirror position to the incoming signal. In the present case, the filter function can be cast in the form $g_S(t)=h(x_S(t)+\alpha_1 t+\alpha_2)$, where the constants $\alpha_1$ and $\alpha_2$ come from (9). To detect a force of the form (1) acting upon a free mass,

$$\alpha_1=-\frac{2\pi^2+6}{\pi T}\frac{F}{m\omega_F^2}, \qquad \alpha_2=\frac{3}{\pi}\frac{F}{m\omega_F^2}.$$

In the linear approximation, the signal-to-noise ratio can then be put in the form $S/N\simeq\nu(F/F_{SQL})|K(\omega_F)|^{1/2}$; from here on, $\nu$ is a constant of order 1. Due to radiative damping, the magnitude of $|K(\omega_F)|$ is bounded by the constant $\omega_0/\omega_F$. $|K(\omega_F)|$ has it maximum at the peak of the measurement sensitivity (2).

The equations (9) can significantly limit sensitivity. If we formally ignore them, we can obtain a signal-to-noise ratio for the force (1) approximately nine times as high. With a force of a different form acting over the time interval $0\leqslant t\leqslant 2\pi/\omega_F$, for example $F_S=F\cos(\omega_F t)$, this number is somewhat lower, $\sim\sqrt{3}$.

We emphasize that (8) and (9) imply suppression of the back influence of fluctuations only in the linear approximation: Eq. (3) is expanded in a power series, and only terms $\sim x\omega_0/c$ are retained. We can write the signal-to-noise ratio with higher-order terms $\sim(x\omega_0/c)^2$ in the form (the terms in square brackets comes from the nonlinear terms—i.e., the "curvature of the ellipse")

$$\left(\frac{S}{N}\right)^2\simeq\nu_1\frac{F^2}{F_{SQL}^2}|K(\omega_F)|\{(\sqrt{1+\mu^2}-\mu)^2+4\tilde{k}[(1+\sqrt{\pi}|K(\omega_F)|^2)^2|K(\omega_F)|(\sqrt{1+\mu^2}+\mu)^4+|K(\omega_F)|^3]\}^{-1}, \tag{10}$$

where

$$\tilde{k}=\frac{\Delta x_{SQL}^2}{\lambda_0^2}=\frac{\hbar}{m\omega_F}\left(\frac{\omega_0}{c}\right)^2\ll 1,$$

$\Delta x_{SQL}$ is the standard quantum limit on the position measurement of a free mass in time $t\simeq 2\pi/\omega_F$, and $\lambda_0=c/\omega_0$ is the laser pump wavelength.

For an incident wave in a coherent state, the sensitivity peaks when $|K(\omega_F)|\simeq(16\pi\tilde{k})^{-1/5}$, and the minimum detectable force is

$$F_{\text{lin}}^{\text{coh}}\simeq 4\tilde{k}^{1/10}F_{SQL}. \tag{11}$$

The incident wave can be in a squeezed state, whereupon $|K(\omega_F)|\simeq 1$ and $2\mu_{\text{lin}}\simeq(8\tilde{k})^{-1/6}\gg 1$ yield a minimum detectable force of less than

$$F_{\text{lin}}^{\text{squ}}\simeq 2\tilde{k}^{1/6}F_{SQL}. \tag{12}$$

This leads us to conclude that upon measurement of the quadrature component, an initial phase shift in the incident wave yields an improvement. Here the pump power is $W_0=W_{SQL}|K(\omega_F)|$. Note that both $F_{\text{lin}}^{\text{coh}}$ and $F_{\text{lin}}^{\text{squ}}$ are much less than $F_{\min}$, which points up the suboptimality of this detection method.

Noting that $g_S(t)$ plays the role of a filter function and that its form is governed by the form of the force $F_S(t)$, we see that Eqs. (8) and (9) and $g_C(t)$ and $g_S(t)$ yield expressions for the averaging function and the phase-modulation function:

$$\tilde{\Phi}(t)=\sqrt{g_S^2(t)+g_C^2(t)}, \tag{13}$$

$$\tan\theta=-g_S(t)/g_C(t). \tag{14}$$

Figure 5 shows the form taken by $\theta$ and $\tilde{\Phi}$ given a force of the form (1).

When (9) holds in the linear approximation, the uncertainty $\Delta x_{\text{ini}}$ in the initial conditions does not affect sensitivity. The initial conditions can be neglected when there are nonlinear corrections if

$$\Delta x_{\text{ini}}\ll\Delta x_{\text{dist}}, \tag{15}$$

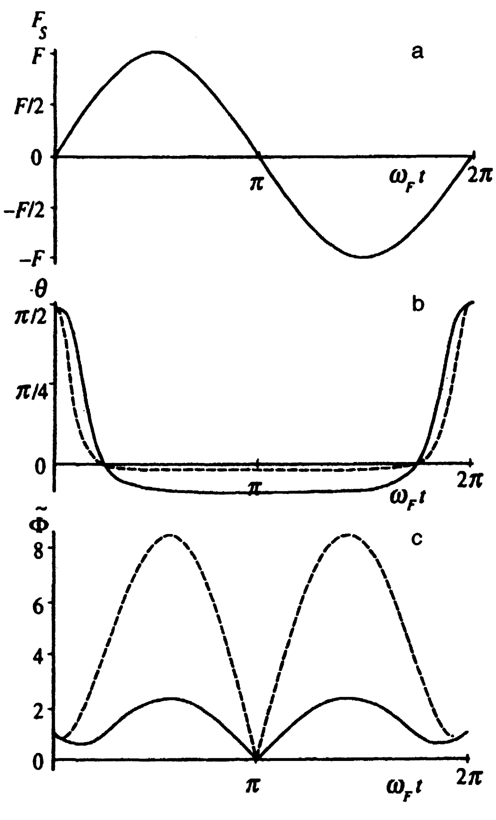


FIG. 5. For a force $F_S$ with the form shown in (a) and an integration time $T=2\pi/\omega_F$, we plot the phase $\theta(t)$ (b) and averaging function $\tilde{\Phi}(t)$ (c) for $4\omega_0\delta_R/\omega_F^2=5$ (solid curves) and $4\omega_0\delta_R/\omega_F^2=20$ (dashed curves).

where $\Delta x_{\text{dist}}$ is the position perturbation due to fluctuations in the light pressure. For an integration time $T\simeq t_F$, we have

$$\Delta x_{\text{dist}}\simeq\Delta x_{\text{SQL}}\sqrt{|K(\omega_F)|}(\sqrt{1+\mu^2}+\mu).$$

We can write $\Delta x_{\text{ini}}$ in terms of the initial uncertainties in position ($\Delta x_0$) and momentum ($\Delta p_0$):

$$\Delta x_{\text{ini}}\simeq\sqrt{\Delta x_0^2+\left(\frac{\Delta p_0 T}{m}\right)^2}\simeq\Delta x_{\text{SQL}}\sqrt{\frac{\epsilon_0}{\hbar\omega_F}}.$$

Here $\epsilon_0$ characterizes the initial uncertainty in the energy of the mirror, which is governed by noise in the device used to prepare the system for measurement. The minimum value, $\epsilon_0\simeq\hbar\omega_F$, corresponds to the standard quantum limit for position measurement on a free mass in time $T$. We see by comparing $\Delta x_{\min}$ and $\Delta x_{\text{dist}}$ that (15) can be satisfied even when $\epsilon_0\gg\hbar\omega_F$.

Without going so far as to specify an optimized setup procedure, we now examine one that is more easily implemented experimentally, namely using the same measurement wave, and not requiring that it be either amplitude- or phase-modulated. Assume that the incident wave is in a coherent state for a long time prior to the measurement, and that its power is kept at approximately $W_0$; then the mirror position will be substantially perturbed, and (15) will not be satisfied. We now show that this perturbation can be measured, and thus compensated. By continuously tracking the mirror position via the phase of the reflected wave over an integration time $\tau=T/\sqrt{|K(\omega_F)|}$, the position can be measured to within $\Delta x_0\simeq\Delta x_{\text{SQL}}|K(\omega_F)|^{-1/4}$, and the momentum to within $\Delta p_0\simeq\Delta x_{\text{SQL}}|K(\omega_F)|^{1/4}/mT$ (for example, by twice measuring the position in time $\tau$). These errors correspond to the standard quantum limit attainable with integration time $\tau$ and power $W_0$. Hence, it can easily be shown that $\Delta x_{\text{ini}}\simeq\Delta x_{\text{SQL}}|K(\omega_F)|^{1/4}$, and (15) holds. This procedure can be invoked as needed to obtain sequentially sampled measurements of the force. Information on initial conditions can be obtained by recording the photocurrent for a time $\tau$ at the end of the preceding measurement (it is clear from Fig. 5 that it is precisely the phase of the reflected wave that is recorded at the end of the time interval $[0,T]$).

## 4. NONLINEAR SCHEME: PHOTON COUNTING

In the nonlinear scheme of Fig. 4, one measures

$$N_T=\int_0^T\Phi_1(t)[E_2^+(t)+E_N^+(t)][E_2(t)+E_N(t)]dt, \quad (16)$$

where $\Phi_1(t)$ is the averaging function, $E_{\text{LO}}$ is the reference field, and $E_N=\sqrt{T_{\text{sp}}}E_{\text{LO}}$ is the field to be added to the reflected field $E_2$. We assume that the transmission coefficient of the beamsplitter is small, $T_{\text{sp}}\ll 1$, so that we can neglect fluctuations in the field $E_N$. We further assume that the wave $E_N(t)$ can be amplitude- and phase-modulated. Finally, we adopt the notation $E_0+E_N=\Phi_2(t)\exp[i\theta(t)]$, $\Phi_1(t)\Phi_2(t)=\Phi(t)$, $G_S(t)=\Phi(t)\sin\theta(t)$, and $G_C(t)=\Phi(t)\cos\theta(t)$. Then when

$$G_C(t)+\int_0^t G_S(t)K(t-\tau)d\tau=0,$$
$$G_C(T)=0,\qquad \dot{G}_C(T)=0, \quad (17)$$

$$G_C(t)+E_0\Phi(t)=f(t) \quad (18)$$

the noise due to the back influence of fluctuations will be compensated up to terms $\sim\omega_0 x/c$ (17) and $\sim(\omega_0 x/c)^2$ (18), and measurement sensitivity will be dominated by third-order noise $\sim(\omega_0 x/c)^3$ and the initial uncertainty in the phase of incident wave (see Appendix B). Here $f(t)$ is an arbitrary small function, smooth over the interval $[0,T]$, that satisfies $\sup|f(t)|\ll(\omega_0/c)\sup|x(t)|$. $G_S(t)$ is obtained according to the dictates of optimal filtering theory (in the same way as $g_S(t)$). Obviously (17) is consistent with (8) and (9). The signal-to-noise ratio can be written in the form

$$\left(\frac{S}{N}\right)^2\simeq\nu_2\frac{F^2}{F_{\text{SQL}}^2}|K(\omega_F)|\left\{(\sqrt{1+\mu^2}-\mu)^2+\frac{16\pi}{3}\tilde{k}^2\left[|K(\omega_F)|^4(\sqrt{1+\mu^2}+\mu)^6\right]\right\}^{-1}. \quad (19)$$

Here the term in square brackets accounts for third-order fed-back noise. If $E_1$ is in a coherent state ($\mu=0$), then (19) will be at a maximum when $|K(\omega_F)|\simeq(4\tilde{k})^{-1/2}$. The least detectable force will then be

$$F_{\text{nonlin}}^{\text{coh}}\simeq 2\tilde{k}^{1/4}F_{\text{SQL}}. \quad (20)$$

If on the other hand the incident wave is in a squeezed state, the sensitivity will be unaltered. The signal-to-noise ratio in (19) will peak when $|K(\omega_F)|(\sqrt{1+\mu^2}+\mu)^2 \simeq (4\tilde{k})^{-1/2}$, and

$$F_{\rm nonlin}^{\rm squ}=F_{\rm nonlin}^{\rm coh}\simeq 2\tilde{k}^{1/4}F_{\rm SQL}. \tag{21}$$

Clearly, as the phase-squeezing rises ($\mu>0$), the required pump power decreases ($W_0\simeq W_{\rm SQL}|K(\omega_F)|$). The nonlinear scheme is more sensitive than the linear, but it does not completely cancel the back influence of fluctuations (as we noted in the Introduction). Even with imperfect cancellation, however, it is possible to achieve the optimal sensitivity (2) if the optimum value of $|K(\omega_F)|$, for which (19) is at a maximum, is at least equal to $|K(\omega_F)|_{\max}\simeq\omega_0/\omega_F$, which value is a consequence of radiative damping.

In finding the optimum sensitivity, we have omitted initial conditions. Inasmuch as the measurement-induced position disturbance $\Delta x_{\rm dist}$ in the nonlinear filtering scheme can be greater than in the linear scheme, constraints on the value of $\epsilon_0$ at which this is a legitimate approximation will be less stringent than for linear filtering.

We now examine the feasbility of the measurement scheme. We know that $\theta(t)$ is given by (14). Thus,

$$\Phi(t)=E_0^{-1}[f(t)-G_C(t)], \tag{22}$$

$$\Phi_1(t)=E_0\frac{\sqrt{G_S^2(t)+G_C^2(t)}}{f(t)-G_C(t)}. \tag{23}$$

There is a certain amount of leeway in choosing $f(t)$. If we choose $f(t)=0$ (essentially perfect cancellation of quadratic noise in the fed-back effects of fluctuations), then measurement of a force of the form (1) requires that $\Phi_1(t)\to\infty$ at the initial time ($t\to 0$). But this then means that the reference power must be infinite at the initial time. To avoid this, we can, for example, choose $f(t)$ to be $\beta G_S(t)$ ($\beta$ is a constant with $\beta\ll\Delta x/\lambda_0$). The quadratic part of the fed-back noise will then be small, and it can be neglected at finite values of $\Phi_1(t)$.

## 5. DISCUSSION

Let us now assess the potential improvement provided by the foregoing methods, and the conditions necessary to achieve that sensitivity. We adopt the following parameters for the displacement sensor: $\omega_0=10^{15}\text{s}^{-1}$, $\omega_F=10^3\text{s}^{-1}$, $\omega_M=10^3\text{s}^{-1}$, and $m=10^3$g. Plugging these into Eq. (11), we obtain $F_{\rm lin}^{\rm coh}\simeq 10^{-2}F_{\rm SQL}$. The required pump power is $W_{\rm lin}^{\rm coh}\simeq 10^{-8}W_{\rm opt}$ ($W_{\rm opt}$ was defined in the Introduction). If the incident wave is in a squeezed state, then (12) yields $F_{\rm lin}^{\rm squ}\simeq 10^{-4}F_{\rm SQL}$, requiring a pump power of $W_{\rm lin}^{\rm squ}\simeq 5\times 10^{-13}W_{\rm opt}$ and squeezing of $\mu_{\rm lin}\simeq 10^4$. Note that both $F_{\rm lin}^{\rm coh}$ and $F_{\rm lin}^{\rm squ}$ are much greater than $F_{\min}$.

For the nonlinear scheme, Eq. (20) tells us that $F_{\rm nonlin}^{\rm coh}\simeq 10^{-6}F_{\rm SQL}$ at a pump power $W_{\rm nonlin}^{\rm coh}\simeq W_{\rm opt}$. For the given system parameters, we then achieve the optimal sensitivity (2): $F_{\rm nonlin}^{\rm coh}\simeq F_{\min}$. If phase-squeezing is initially present ($\mu>0$), we can get essentially the same sensitivity (Eq. (21)) at lower pump power: $W_{\rm nonlin}^{\rm squ}\simeq W_{\rm lin}^{\rm squ}(\sqrt{1+\mu_{\rm nonlin}^2}+\mu_{\rm nonlin})^{-2}$.

The required power is too high, so actual experiments (LIGO, for example) must use an interferometric sensor (Fabry–Perot resonator), in which an additional high-reflectance mirror ($T_{\rm FP}\ll 1$) precedes the scanning mirror. This sort of sensor, with a coherent pump working at optical resonance ($\exp(iL_0\omega_0/c)=1$, $L_0$ is the mirror separation in the loaded resonator, $\omega_0$ is the pump laser frequency), was previously discussed[8–10] in the linear approximation, the result being that it required a factor $16/[T_{\rm FP}^2+\phi^2(\omega_F)]$ less pump power than the conventional displacement sensor discussed above. The parameter $\phi(\omega_F)=4L_0\omega_F/c\ll 1$ comes from dispersion in the resonator.

Analysis of an interferometric sensor with a resonant pump—but one that takes second- and third-order corrections into account—shows that along with improved power consumption comes a loss of peak measurement sensitivity as measured via the methods described above. The underlying reason for this degradation is the increase in system nonlinearity that results from the enhanced reaction of the sensor to changes in mirror position. In fact, whereas a displacement of magnitude $x$ in a conventional sensor leads to a phase shift $\simeq 2x/\lambda_0$ in the reflected wave, the corresponding shift in an interferometric sensor is $\simeq 2x/(\lambda_0 T_{\rm FP})$. Thus, nonlinear terms involving the back influence of fluctuations grow, resulting in a loss of sensitivity.

It must also be mentioned that dispersion will alter the conditions required to cancel the initial uncertainties in mirror position and momentum. Dispersion can be considered small when $\phi(\omega_F)\leqslant T_{\rm FP}$, and all that is required to obtain expressions for the least detectable force is to replace $\tilde{k}\simeq\Delta x_{\rm SQL}^2/\lambda_0^2$ with $\tilde{k}'\simeq\Delta x_{\rm SQL}^2/(\lambda_0^2 T_{\rm FP}^2)$ in the equations derived above. The consequences of reversing the inequality require more detailed study.

The sensitivity of an interferometric sensor must be enhanced if its nonlinearity is to be reduced. This can probably be achieved by adding frequency-dependent mechanical nonlinearity to the measurement system.

We close with sincere thanks to V. B. Braginskiĭ, D. N. Klyshko, and F. Ya. Khalili for detailed discussion of these results. This work was supported by the National Science Foundation of the United States (grant PHY-9503642).

## APPENDIX A

Substituting (3) into (6) and expanding the resulting expression in $x/\lambda_0$ and $e_1/E_0$, to second order in small quantities we obtain

$$B_T=\int_0^T[B_S(t)+B_{\rm I}(t)+B_{\rm II}(t)]dt, \tag{A1}$$

where

$$B_S(t)=4E_0\frac{x_S(t)}{\lambda_0}g_S(t),$$

$$B_{\rm I}(t)=4E_0\left[g_S(t)\frac{x}{\lambda_0}-g_C(t)\frac{\dot{x}}{c}\right]+(e_1+e_1^+)g_C(t)$$
$$-i(e_1-e_1^+)g_S(t),$$

$$B_{\rm II}(t)=2\frac{x}{\lambda_0}\left[-2E_0\frac{x}{\lambda_0}g_C(t)+i(e_1-e_1^+)g_C(t)\right.$$
$$\left.+(e_1+e_1^+)g_S(t)\right].$$

Here $x_S(t)$ is the mirror displacement induced by the incident signal beam.

To analyze the feasibility of suppressing the linear component of fed-back fluctuations, let $g_C(t)$ satisfy (8). Then in the linear approximation (dropping $B_{\rm II}$)

$$B_T=\frac{\pi m}{SE_0}[g_C(t)\dot{x}(t)-\dot{g}_C(t)x(t)-2\delta_R g_C(t)x(t)]|_0^T$$
$$+\int_0^T\left[-i(e_1-e_1^+)+4E_0\frac{x_S}{\lambda_0}\right]g_S dt, \quad \text{(A2)}$$

This means that if $g_S(t)$ is chosen so that $g_C(T)=0$ and $\dot{g}_C(T)=0$, $B_T$ in the linear approximation will not depend on either the initial position and momentum uncertainties of the mirror or on the back influence of fluctuations.

To assess the limitations stemming from inclusion of the nonlinear term $B_{\rm II}$, we assume that the sensor is a mass-controlled oscillator. We are then able to treat the noise attributable to $B_{\rm II}$ as being white over the frequency range of interest, and independent of the noise due to $B_{\rm I}$. Making use of optimal filtering theory and (8), we easily derive the signal-to-noise ratio (10).

## APPENDIX B

We represent $N_T$ in a form analogous to (A1):

$$N_T=\int_0^T(N_S+N_{\rm I}+N_{\rm II}+N_{\rm III})dt. \quad \text{(B1)}$$

Given (17) and (18), we have

$$N_S=B_S, \qquad N_{\rm I}=B_{\rm I},$$

$$N_{\rm II}=2\frac{x}{\lambda_0}\left[-2E_0 f(t)\frac{x}{\lambda_0}+i(e_1-e_1^+)f(t)\right.$$
$$\left.+(e_1+e_1^+)G_S(t)\right], \quad \text{(B2)}$$

$$N_{\rm III}=2\left(\frac{x}{\lambda_0}\right)^2\left[E_0G_S(t)\frac{4x}{3\lambda_0}+i(e_1-e_1^+)G_S(t)\right.$$
$$\left.+(e_1+e_1^+)f(t)\right].$$

When (17) holds, measurement sensitivity will clearly be dominated by the initial state of the pump field ($N_{\rm I}$) and third-order nonlinear noise in the fed-back fluctuations ($N_{\rm III}$). As in Appendix A, we treat this noise as being white over the frequency range of interest, and hence immediately obtain (19).

## APPENDIX C

A number of terms were omitted from Eqs. (3) and (5). Here we justify the approximations entailed.

The exact boundary condition for the incident and reflected waves can be written in the form (relative to a coordinate system attached to the mirror)

$$\tilde{E}_2(t)=-\tilde{E}_1(t). \quad \text{(C1)}$$

In going from (C1) to (3), we neglect terms

$$\simeq\frac{\dot{x}}{c}e_1(t), \qquad \simeq\frac{\dot{x}}{c}\frac{x}{\lambda_0}E_0,\dots . \quad \text{(C2)}$$

The exact equation for a mechanical oscillator takes the form

$$\ddot{x}+\omega_M^2x=\frac{S}{2\pi m}(E_1^+E_1+E_2^+E_2-\langle E_1^+E_1\rangle$$
$$-\langle E_2^+E_2\rangle)+\frac{F_S}{m}. \quad \text{(C3)}$$

Here $\langle\dots\rangle$ denotes averaging over states. In going to Eq. (5), we neglect terms

$$\simeq\dot{x}\delta_R\frac{e_1+e_1^+}{E_0}, \qquad \simeq\frac{S}{\pi m}e_1^+e_1,\dots \quad \text{(C4)}$$

on the right-hand side of (C3).

It is not immediately apparent which of the terms in (C2) and (C4) can be neglected, as they are subsequently multiplied by the considerably different functions $g_S(t)$ and $g_C(t)$.

Substitute (C2) and (C4) into (6), and then into (6). We then find that the term $B_{\rm II}$ in (A1) must also incorporate the terms

$$\simeq g_C(t)E_0\frac{\dot{x}}{c}\frac{e_1+e_1^+}{E_0}, \qquad \simeq g_S(t)E_0\frac{\dot{x}}{c}\frac{x}{\lambda_0},$$
$$\simeq g_S(t)E_0\frac{x}{\lambda_0}\frac{\delta_R}{\omega_F}\frac{e_1+e_1^+}{E_0}, \qquad \simeq g_C(t)\frac{e_1^+e_1}{E_0}.$$

To cancel the linear part of the fed-back fluctuations, we must then have $g_S(t)E_0x/\lambda_0\simeq g_C(t)(e_1+e_1^+)$. Furthermore, given that

$$\frac{x}{\lambda_0}\simeq\frac{\Delta x_{\rm dist}}{\lambda_0}, \qquad \frac{\dot{x}}{c}\simeq\frac{\omega_F}{\omega_0}\frac{x}{\lambda_0},$$
$$\frac{e_1+e_1^+}{E_0}\simeq(\sqrt{1+\mu^2}+\mu)\sqrt{\frac{\hbar\omega_0\omega_F}{mc^2\delta_R}},$$

we find that with the previously adopted parameters, $x/\lambda_0>(e_1+e_1^+)/E_0,\dot{x}/c$. We now adopt the system parameters quoted in Sec. 5, taking the numerical values for the free parameters (pump power, squeezing). We can then easily show that all omitted small quantities are less than the dominant term $B_{\rm II}$ ($\simeq 4E_0g_C(t)(x/\lambda_0)^2$) for linear detection, and are less than the dominant term $B_{\rm III}$ ($\simeq g_S(t)E_0(x/\lambda_0)^3$) for nonlinear detection. Hence, these can legitimately be neglected in the measurement schemes considered above.

[1)]F. Ya. Khalili, Private communication.


[1]V. B. Braginskiĭ, Zh. Éksp. Teor. Fiz. **53**, 1436 (1967) [sic].
[2]Yu. I. Vorontsov, *Theory and Methods of Macroscopic Measurements* [in Russian], Nauka, Moscow (1989).
[3]Yu. I. Vorontsov, Usp. Fiz. Nauk **164**, No. 1, 89 (1994) [Phys. Usp. **36**, 81 (1994)].
[4]V. B. Braginsky and F. Ya. Khalili, in *Quantum Measurement*, K. S. Thorne (ed.), Cambridge University Press, New York (1992), p. 106.
[5]A. V. Syrtsev and F. Ya. Khalili, Zh. Éksp. Teor. Fiz. **106**, 744 (1994) [JETP **79**, 409 (1994)].
[6]V. B. Braginskiĭ and F. Ya. Khalili, Zh. Éksp. Teor. Fiz. **94**, 151 (1988) [Sov. Phys. JETP **67**, 84 (1988)].
[7]M. T. Jaekel and S. Reynaud, Europhys. Lett. **13**, No. 4, 301 (1990).
[8]S. P. Vyatchanin and A. B. Matsko, Zh. Éksp. Teor. Fiz. **104**, 2668 (1993) [JETP **77**, 218 (1993)]; S. P. Vyatchanin, E. A. Zubova, and A. B. Matsko, Opt. Comm. **109**, 492 (1994); S. P. Vyatchanin and E. A. Zubova, Opt. Comm. **111**, 303 (1994).



Translated by Marc Damashek