Energy–momentum tensor of particles created by an external field

A. I. Nikishov

P. N. Lebedev Physics Institute, Russian Academy of Sciences, 117924 Moscow, Russia
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Normal ordering has been used to determine the energy–momentum tensor of particles created in a Minkowski space after an external field has been turned on. The method used does not require regularization, as it yields a finite result directly. The expectation value $\langle 0^{\rm in}|:T_{\mu\nu}:|0^{\rm in}\rangle$ is treated on an equal footing with other observables, such as the current density and the mean number of particles created, i.e., it makes physical sense only after the quantity in question has actually come into being. As it applies to particles created by a mirror that is accelerated over a finite interval of time in $(1+1)$ spacetime, $\langle 0^{\rm in}|:T_{\mu\nu}:|0^{\rm in}\rangle$ is not the same as the regularized quantity $\langle 0^{\rm in}|:T_{\mu\nu}:|0^{\rm in}\rangle_{\rm reg}$, although when integrated over all space, the 00-components of these tensors yield the same total energy for the particles created. [S1063-7761(96)00206-5]

1. INTRODUCTION

The energy–momentum tensor is a source of the gravitational field, so there is ample justification for studying it closely (see, e.g., Refs. 1–3). One by-product of a plethora of such research has been the elevation of the energy–momentum tensor to the rank of an especially important quantity, with a considerably wider range of applicability than, say, the particle concept (see Ref. 4 for an extreme view).

Nevertheless, in the present paper, we wish to revert to the original setting of the problem in order to discuss the feasibility of a more cautious approach: perhaps, despite its (probably only seeming) locality, the energy–momentum tensor in fact depends on global conditions, and therefore requires some finite domain in which to come into being, as is the case, for instance, in particle creation. Indeed, the operator for the energy–momentum tensor is defined in terms of a field operator Ψ, which depends on a complete set of solutions of the wave equation and particle creation and annihilation operators. The sense of the latter operators depends on the sign of the frequency of the solutions, and can only be determined in a region in which the particle creation process (or in any event, the creation process for those particles with the same quantum numbers as the solution) has already come to an end.[5,6]

Thus, there is reason to believe that the energy–momentum tensor can only be defined when certain conditions are satisfied. It is useful, in any case, to compare the vacuum mean of the normal-ordered energy–momentum tensor after turning off the field with the vacuum mean of the regularized energy–momentum tensor. This we shall do in Sec. 3 for the situation of greatest interest (due to its connection with the Hawking evaporation of black holes), that of radiation by an accelerated mirror. But first, to develop the intuition required for the subsequent discussion, we consider in Sec. 2 a more clearcut case, in which an electromagnetic field acting over some finite time interval t engenders a nonvanishing expectation value $\langle 0^{\rm in}|:T_{\mu\nu}:|0^{\rm in}\rangle$ and particle density, which we examine as $t\to\infty$. Even in this simple situation there are *in* and *out* particle creation and annihilation operators, and exactly which operators are to be normal-ordered is a problem whose solution is suggested either by examination of the structure of the mathematical machinery (see Ref. 7 and further references therein) or by trial and error. Here, in a word, we take the latter approach.

2. ENERGY–MOMENTUM TENSOR OF PARTICLES CREATED BY AN ELECTROMAGNETIC FIELD

In general, the total energy–momentum tensor of the field consists of more than just the energy–momentum tensor of created particles. Below we shall see that after some necessary averaging over time and/or space, the energy–momentum tensors of the field and created particles coincide.

For simplicity, let the complete sets of *in* and *out* solutions of the wave equation, $\psi_k^{\rm in}$ and $\psi_k^{\rm out}$ be characterized by the single set of quantum numbers k, i.e.,

$$\psi_k^{(+)\rm in}=\alpha_k\psi_k^{(+)\rm out}+\beta_k\psi_k^{(-)\rm out}. \tag{1}$$

Superscript $+$ and $-$ here denote the sign of the frequency of the solution. Then the *in* and *out* creation and annihilation operators for particles, a_k^+ and a_k, and antiparticles, b_k^+ and b_k, are related by[6]

$$a_k^{\rm out}=\alpha_k a_k^{\rm in}\mp\beta_k^* b_k^{+\rm in},\qquad a_k^{+\rm out}=\alpha_k^* a_k^{+\rm in}\mp\beta_k b_k^{\rm in},$$
$$b_k^{+\rm out}=\beta_k a_k^{\rm in}+\alpha_k^* b_k^{+\rm in},\qquad b_k^{\rm out}=\beta_k^* a_k^{+\rm in}+\alpha_k b_k^{\rm in}. \tag{2}$$

The upper sign corresponds to spin-1/2 particles, and the lower, to spin 0.

In the field operator

$$\Psi=\sum_k\,[a_k^{\rm in}\psi_k^{(+)\rm in}+b_k^{+\rm in}\psi_k^{(-)\rm in}] \tag{3}$$

$a_k^{\rm in}$ and $b_k^{+\rm in}$ are the particle annihilation and antiparticle creation operators as $t\to-\infty$, at which time the external field has yet to turn on. As $t\to\infty$, the external field has already turned off, and (3) naturally takes the form

$$\Psi=\sum_k\,[a_k^{\rm out}\psi_k^{(+)\rm out}+b_k^{(+)\rm out}\psi_k^{(-)\rm out}]. \tag{4}$$

To represent Ψ at intermediate times, we need to know $\psi_k^{(\pm)}$ in the range when particles are being created. But in the particle formation range (a range that depends on the set of quantum numbers k), there is no sensible way to distinguish solutions at $(+)$ and $(-)$ frequencies.[5] We therefore refrain from consideration of the intermediate time range, and take Ψ in the form (4) as $t\to\infty$.

We shall concern ourselves below with the consequences of this course of action. A different, widely adopted approach is reviewed in Ref. 7.

Note that means over the *out* vacuum of the normal-ordered operators for the energy–momentum tensor or current tensor, expressed in terms of the creation and annihilation *out* operators, vanish as expected. We shall be working in the Heisenberg picture, and will be interested in means over the *in* vacuum.

2.1. Spin-1/2 particles

The energy–momentum tensor operator takes the form[1,2]

$$T_{\mu\nu}=\frac{i}{2}[\bar{\Psi}\gamma_{(\nu}\Psi_{,\mu)}-\bar{\Psi}_{,(\mu}\gamma_{\nu)}\Psi,$$

$$\Psi_{,\mu}=(\partial_\mu+ieA_\mu)\Psi. \tag{5}$$

The symbol (ν,μ) denotes symmetrization over μ and ν:

$$a_{(\mu\nu)}=\frac{1}{2}(a_{\mu\nu}+a_{\nu\mu}).$$

For simplicity, we assume that both the field and the potential vanish as $t\to\pm\infty$. Noting that

$$\langle 0^{\rm in}|a_k^{+\rm in}=\langle 0^{\rm in}|b_k^{+\rm in}=a_k^{\rm in}|0^{\rm in}\rangle=b_k^{\rm in}|0^{\rm in}\rangle=0, \tag{6}$$

and making use of (2) and (4), we obtain the mean over the *in* vacuum of one of the terms in (5):

$$\begin{aligned}\langle 0^{\rm in}|:\bar{\Psi}\gamma_\nu\Psi_{,\mu}:|0^{\rm in}\rangle=\sum_k\{&|\beta_k|^2[\bar{\psi}_k^{(+)\rm out}\gamma_\nu\psi_{k,\mu}^{(+)\rm out}\\&-\bar{\psi}_k^{(-)\rm out}\gamma_\nu\psi_{k,u}^{(-)\rm out}]\\&-\alpha_k\beta_k^*\bar{\psi}_k^{(-)\rm out}\gamma_\nu\psi_{k,\mu}^{(+)\rm out}\\&-\alpha_k^*\beta_k\bar{\psi}_k^{(+)\rm out}\gamma_\nu\psi_{k,\mu}^{(-)\rm out}\}.\end{aligned} \tag{7}$$

As usual, a pair of colons denotes normal ordering, and we denote the simple derivative $\partial_\mu\Psi$ by $\Psi_{,\mu}$. Some simple index arithmetic then yields all of the remaining terms in the expression for

$$\langle 0^{\rm in}|:T_{\mu\nu}:|0^{\rm in}\rangle. \tag{8}$$

We refer to terms proportional to $\alpha_k\beta_k^*$ and $\alpha_k^*\beta_k$ as interference terms. These vanish when averaged over an interval longer than the Compton time $\hbar/mc^2$. Measurements cannot be made over a time interval $\sim\hbar/mc^2$ due to attendant pair creation.

Apart from the interference terms, one can say that (8) comprises contributions from each of the modes k. The contribution of positive-frequency modes can be obtained from (5) by making the substitution $\Psi\to\psi^{(+)\rm out}$. It should be clear from (7) that the weight of this mode is given by the probability $|\beta_k|^2$ of its creation. The contribution of antiparticle modes equals that of particle modes: the minus sign of the second term in square brackets in (7) is balanced by the fact that we take the derivative of $\psi_k^{(-)}$ with respect to x^μ, while in the first term we take the derivative of $\psi_k^{(+)}$ [see Eq. (9)].

In the case at hand, it can be assumed that

$$\psi_k^{(\pm)\rm out}\propto e^{\pm ipx},\quad \psi_{k,\mu}^{(\pm)\rm out}=\pm ip_\mu\psi_k^{(\pm)\rm out},\quad k=(\mathbf{p},s), \tag{9}$$

and s is the two-valued spin index.

Integrating (8) over all space for $\nu=\mu=0$ yields the total energy

$$\sum_k|\beta_k|^2 2p_0,\qquad p_0=\sqrt{m^2+\mathbf{p}^2}, \tag{10}$$

where $2p_0$ is the energy of state k. This should not be surprising, as the amplitude for pair creation from the vacuum with quantum numbers k is β_k.

It is useful to compare the expression (8) for the energy–momentum tensor with the corresponding expression for the current, which can be obtained from (7) by premultiplying by i and omitting the differentiation. Although the resulting expression vanishes (because the first and second terms in square brackets in (7) now cancel), the number density of particles alone (or pairs) is well-defined, and is given by the analog of the first of the terms in square brackets in (7) when $\nu=0$. The total number of particles is determined by integrating the density over all space, i.e., by taking the sum in (10) without the factor $2p_0$.

If one can reasonably speak of a number density of particles only after the field has turned off, then the same applies to the energy–momentum tensor.

2.2 Spin-0 particles

Instead of (5), we now have for the energy–momentum tensor operator[1–3]

$$T_{\mu\nu}=\Psi^*_{,\mu}\Psi_{,\nu}+\Psi^*_{,\nu}\Psi_{,\mu}-g_{\mu\nu}\mathscr{L},$$

$$\mathscr{L}=\Psi^*_{,\mu}\Psi^{,\mu}-m^2\Psi^*\Psi. \tag{11}$$

Instead of (7), we have

$$\begin{aligned}\langle 0^{\rm in}|:\Psi^*_{,\mu}\Psi_{,\nu}:|0^{\rm in}\rangle=\sum_k\{&|\beta_k|^2[\psi_{k,\mu}^{*(+)\rm out}\psi_{k,\nu}^{(+)\rm out}\\&+\psi_{k,\mu}^{*(-)\rm out}\psi_{k,\nu}^{(-)\rm out}]\\&+\alpha_k\beta_k^*\psi_{k,\mu}^{*(-)\rm out}\psi_{k,\nu}^{(+)\rm out}\\&+\alpha_k^*\beta_k\psi_{k,\mu}^{*(+)\rm out}\psi_{k,\nu}^{(-)\rm out}\}.\end{aligned} \tag{12}$$

The remaining terms can be obtained from this expression via straightforward index manipulations. In particular, for the term containing m^2 in (11), indices μ and ν must be omitted. Disregarding the interference terms, we can state as before that the expression for (8) consists of a sum of terms in which each mode is weighted by $|\beta_k|^2$. The contribution of mode k is obtained from (11) by letting $\Psi\to\psi_k^{(+)}$ for particles and $\Psi\to\psi_k^{(-)}$ for antiparticles. As expected, these contributions are equal.

The current operator is of the form

$$j_\mu = i[\Psi^* \Psi_{,\mu} - \Psi^*_{,\mu} \Psi]. \tag{13}$$

The vacuum mean can once again be obtained from Eq. (12) by omitting one of the differentiation indices. The results are analogous to the spinor case.

3. MOVING MIRRORS

Attempts to understand the evaporation of black holes—or even just to reconcile it with prevailing physical notions—unleashed a torrent of work on modeling the various aspects of the problem. References to early work can be found in Ref. 2; later studies are listed, for example, in Refs. 9 and 10. The simplest of these models is that of the emission of massless scalar particles by an accelerated mirror in $(1+1)$-spacetime.

One remarkable achievement of this model is the simple expression that it gives for the regularized energy–momentum tensor. In particular, if $v=f(u)$ is the mirror's equation of motion in coordinates $v \equiv t+x$, $u \equiv t-x$, then the components of the regularized energy–momentum tensor are[2]

$$\langle 0^{\rm in}|T_{\mu\nu}|0^{\rm in}\rangle_{\rm reg} = t_{\mu\nu},$$

$$t^{00}=t^{11}=t^{10}=t^{01}=\frac{1}{24\pi}\left[\frac{3}{2}\left(\frac{f''}{f'}\right)^2 - \frac{f'''}{f'}\right], \tag{14}$$

i.e., for any motion $f(u)$, they are determined by just the first three derivatives of $f(u)$.

In many ways, pair creation by a field is analogous to particle creation by a mirror. In the latter case, however, there is one complicating factor: no matter how large the value of the time t, the *out* wave[2]

$$\psi_\omega^{(+)\rm out} = \frac{1}{\sqrt{2|\omega|L}}[\exp(-i\omega g(v)) - \exp(-i\omega u)] \tag{15}$$

contains an incoming wave $\exp[-i\omega g(v)]$ along with the outgoing wave $\exp(-i\omega u)$. It is precisely the presence of this wave that ensures satisfaction of the boundary condition—in the present case, the vanishing of the wave at the mirror. Here $g(v)$ is the inverse function of $f(u)$, i.e., $g(f(u))=u$. In our previous example, $\psi^{(+)\rm out}$ consisted of just a single wave [see (9)].

If we now write the *in* wave as

$$\psi_\omega^{(+)\rm in} = \frac{1}{\sqrt{2|\omega|L}}[\exp(-i\omega v) - \exp(-i\omega f(u))], \tag{16}$$

we find that the amplitude for the creation of an *out* wave is given by a relation analogous to (1):

$$\psi_\omega^{(+)\rm in} = \sum_{\omega'} [\alpha_{\omega'\omega}\psi_{\omega'}^{(+)\rm out} + \beta_{\omega'\omega}\psi_{\omega'}^{(-)\rm out}]. \tag{17}$$

The fact that $\alpha_{\omega'\omega}$ and $\beta_{\omega'\omega}$ have off-diagonal elements is merely a minor technical complication. It is somewhat unpleasant, however, that the interpretation of (17) must be modified slightly. In contrast to (1), we can no longer claim that $\beta_{\omega'\omega}$ is the creation amplitude for the wave $\psi^{(-)\rm out}$ as $t \to +\infty$, given that there were initially no particles in the state ω. Instead, $\beta_{\omega'\omega}$ is the creation amplitude for a negative-frequency outgoing wave, $-\exp(i\omega u)/\sqrt{2|\omega|L}$ (see (15)), where L is the size of the system under consideration. Of course, if we look upon the solution (15) as the limiting case of waves described by packets, then everything is the same as before: for large enough t, the incoming wave is completely transformed into an outgoing wave.

With this new slant on a theory with emission from a mirror in mind, we now present for the first time an intuitive derivation of the expression for the energy–momentum tensor. Due to the simple structure of the waves (15) and (16), we can adopt the more compact notation

$$\psi_\omega^{(+)} = \psi_\omega, \qquad \psi_\omega^{(-)} = \psi_\omega^*. \tag{18}$$

Thus, if the vacuum was in the state ω at $t \to -\infty$, the amplitude of the negative-frequency wave is

$$\phi_\omega^{(-)}(u) = -\sum_{\omega'} \beta_{\omega'\omega}\frac{\exp(i\omega' u)}{\sqrt{2|\omega'|L}}, \qquad u=t-x. \tag{19}$$

The contribution of this wave to component T_{00} of the energy–momentum tensor is[11]

$$\left|\frac{\partial\phi_\omega^{(-)}}{\partial t}\right|^2 + \left|\frac{\partial\phi_\omega^{(-)}}{\partial x}\right|^2 = 2\left|\frac{\partial\phi_\omega^{(-)}}{\partial t}\right|^2 = 2\left|\sum_{\omega'} \beta_{\omega'\omega}\sqrt{\frac{\omega'}{2L}}\exp(i\omega' u)\right|^2. \tag{20}$$

As expected from physical considerations, Eq. (20) depends solely on u: the emitted energy departs from the mirror. This constitutes an argument supporting Eq. (19).

If we now sum (20) over ω, we obtain the component of interest:

$$T_{00} = 2\sum_\omega \left|\frac{\partial\phi_\omega^{(-)}}{\partial t}\right|^2 = 2\sum_\omega \left|\sum_{\omega'} \beta_{\omega'\omega}\sqrt{\frac{\omega'}{2L}}\exp(i\omega' u)\right|^2. \tag{21}$$

Notice that even if Eq. (19) diverges in the infrared when we sum over ω', the derivative in (20) still exists.

By definition, $\beta_{\omega'\omega} \propto L^{-1}$, where L is the size of the given system. For the sake of compactness, we write the sums over states as integrals:

$$\sum_\omega \to \int \frac{d\omega L}{2\pi}. \tag{22}$$

Since

$$T^{00} = T^{11} = T^{01} = T^{10}(u), \qquad u=t-x \tag{23}$$

and all components of $T_{\mu\nu}$ depend only on u, we have the conservation law

$$\partial_0 T^{00} + \partial_1 T^{01} = 0. \tag{24}$$

Integrating T_{00} over all x in (21), we obtain

$$2\int dx \sum_{\omega,\omega',\omega''} \frac{\sqrt{\omega''\omega'}}{2L}\beta_{\omega'\omega}\beta^*_{\omega''\omega}\exp(i(\omega'-\omega'')u) = \sum_{\omega,\omega'} \omega'|\beta_{\omega',\omega}|^2. \tag{25}$$

This is the total energy of created particles. Since we are modeling a strictly neutral field, antiparticles make no contribution to (25).

Equations (23) and (24) also hold for the regularized energy–momentum tensor (14). It can be shown[13] that Eq. (14) can be written as a sum of two terms, the first being associated with the energy of the emitted particles, and the other with a "shot noise" energy:

$$t_{00}^{\rm rad}=\frac{1}{48\pi}\left(\frac{f''}{f'}\right)^2, \qquad t_{00}^{S}=\frac{1}{24\pi}\left[\left(\frac{f''}{f'}\right)^2-\frac{f'''}{f'}\right]$$
$$=-\frac{1}{24\pi}\left(\frac{f''}{f'}\right)'. \tag{26}$$

It might be supposed that T_{00} in (21) is the same as $t_{00}^{\rm rad}$ Examination of so-called quasihyperbolic motion of a mirror, which is substantially different from hyperbolic motion only at very large times $|t|$, shows that (21) differs both from (14) and $t_{00}^{\rm rad}$ in (26) (see Appendix). Thus, the feasibility of an instantaneous determination of $T_{00}(u)$ at any u assumed in (14) is at variance with (21), even at $t\to+\infty$.

We now show that the intuitively derived Eq. (21) for the energy–momentum tensor also follows, neglecting interference terms, from the procedure detailed in Sec. 2, if by the negative-frequency out wave we mean only the outgoing wave $-\exp(i\omega u)/\sqrt{2|\omega|}L$ (see (15)).

We begin by considering the operator of the strictly neutral field[11]

$$\Psi=\Psi^{+}=\sum_{\omega}\,[a_\omega^{\rm out}\psi_\omega^{\rm out}+a_\omega^{+\rm out}\psi_\omega^{*\rm out}]. \tag{27}$$

Here we use the notation of (18). The energy–momentum tensor operator is[12]

$$T_{\mu\nu}=\frac{1}{2}\begin{bmatrix}\dot\Psi^2+\Psi'^2 & \Psi'\dot\Psi+\dot\Psi\Psi' \\ \Psi'\dot\Psi+\dot\Psi\Psi' & \dot\Psi^2+\Psi'^2\end{bmatrix}, \qquad \mu,\nu=0,1, \tag{28}$$

a dot denotes differentiation with respect to t, and a prime denotes differentiation with respect to x. In place of (2), we have

$$a_{\omega'}^{\rm out}=\sum_{\omega}\,[\alpha_{\omega'\omega}a_\omega^{\rm in}+\beta_{\omega'\omega}^{*}a_\omega^{+\rm in}],$$

$$a_{\omega'}^{+\rm out}=\sum_{\omega}\,[\alpha_{\omega'\omega}^{*}a_\omega^{+\rm in}+\beta_{\omega'\omega}a_\omega^{\rm in}]. \tag{29}$$

Substituting (27) into (28) and calculating (8), we obtain

$$\langle 0^{\rm in}|:T_{00}:|0^{\rm in}\rangle=2\sum_{\omega\omega'\bar\omega}\left\{\beta_{\omega'\bar\omega}\beta_{\omega\bar\omega}^{*}\dot\psi_{\omega'}^{*\rm out}\dot\psi_{\omega}^{\rm out}+\frac{1}{2}[\alpha_{\omega'\bar\omega}\beta_{\omega\bar\omega}^{*}\dot\psi_\omega^{\rm out}\dot\psi_{\omega'}^{\rm out}+{\rm c.c.}]\right\}. \tag{30}$$

Dropping interference terms and letting $\psi^{\rm out}$ represent only the outgoing wave, we obtain (21). The sum over $\bar\omega$ in (30) is a sum over primordial vacuum states with quantum numbers $\bar\omega$. Sums over ω and ω' yield the contributions of the energy–momentum tensors constructed out of secondary nonquantized states $\psi_{\omega'}$ and ψ_ω:

$$t_{00}(\psi_{\omega'}\psi_\omega)=\dot\psi_{\omega'}^{*}\dot\psi_\omega+\psi_{\omega'}'^{*}\psi_\omega'=2\dot\psi_{\omega'}^{*}\dot\psi_\omega, \tag{31}$$

(see (28), where Ψ is a quantized hermitian operator, and there is thus[11] an extra factor of 1/2). The factor $\beta_{\omega'\omega}\beta_{\omega\bar\omega}^{*}$ incorporates the probability of occurrence of the quantity (31).

We can now write the analog of the particle flux. For a strictly neutral field, the vector current density operator takes the form[14]

$$d_\mu=i[\Psi^{(-)}\Psi_{,\mu}^{(+)}-\Psi_{,\mu}^{(-)}\Psi^{(+)}], \tag{32}$$

where, according to (27),

$$\Psi^{(+)}=\sum_{\omega}\,a_\omega^{\rm out}\psi_\omega^{\rm out}, \qquad \Psi^{(-)}=\sum_{\omega}\,a_\omega^{+\rm out}\psi_\omega^{*\rm out}. \tag{33}$$

Simple calculations involving (32) and (6) then yield

$$\langle 0^{\rm in}|:d_\mu:|0^{\rm in}\rangle=\sum_{\omega\omega'\bar\omega}\,\beta_{\omega'\bar\omega}\beta_{\omega\bar\omega}^{*}d_\mu(\psi_{\omega'},\psi_\omega), \tag{34}$$

$$d_\mu(\psi_{\omega'},\psi_\omega)=i\psi_{\omega'}^{*}\overleftrightarrow{\partial}_\mu\psi_\omega=i[\psi_{\omega'}^{*}\partial_\mu\psi_\omega-(\partial_\mu\psi_{\omega'}^{*})\psi_\omega]. \tag{35}$$

The total number of particles is given by the integral of the 0-component of (34) with respect to x, which then yields the right-hand side of (25) without the factor ω' under the summation sign.

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APPENDIX

In view of the importance of the lack of identity between the regularized and normal-ordered energy–momentum tensors, we wish to compare their asymptotic behavior over the *in* vacuum at $u=t-x\to\pm\infty$ for the special ("quasihyperbolic") motion of the mirror

$$x(t)=\beta[B-\sqrt{B^2+t^2}], \qquad \beta<1. \tag{A1}$$

In u,v coordinates, the motion is given by

$$v(u)\equiv f(u)=(1-\beta^2)^{-1}\{(1+\beta^2)u+2\beta B-2\beta\sqrt{u^2+2\beta Bu+B^2}\}, \tag{A2}$$

or

$$u(v)\equiv g(v)=(1-\beta^2)^{-1}\{(1+\beta^2)v-2\beta B+2\beta\sqrt{v^2-2\beta Bv+B^2}\}. \tag{A3}$$

From (A2) we find

$$f'(u)=\frac{1+\beta^2}{1-\beta^2}-\frac{2\beta}{1-\beta^2}\,\frac{u+\beta B}{\sqrt{u^2+2\beta Bu+B^2}},$$

$$f''(u)=-\frac{2\beta B^2}{(u^2+2\beta Bu+B^2)^{3/2}},$$

$$f'''(u)=\frac{6\beta B^2(u+\beta B)}{(u^2+2\beta Bu+B^2)^{5/2}}. \tag{A4}$$

As $u\to\pm\infty$, the derivative $f'(u)$ tends to a constant limit that depends on the sign of u:

$$f'(u)|_{u\to+\infty}\to\frac{1-\beta}{1+\beta},\qquad f'(u)|_{u\to-\infty}\to\frac{1+\beta}{1-\beta}. \tag{A5}$$

From (26), (A4), and (A5), we then have the following behavior as $u\to\pm\infty$:

$$t_{00}^{\rm rad}\Big|_{u\to\pm\infty}\to\frac{1}{12\pi}\left[\frac{\beta B^2}{u^3 f'(\pm\infty)}\right]^2,$$

$$t^S_{u\to\pm\infty}\to-\frac{\beta B^2 u}{4\pi|u|^5 f'(\pm\infty)}. \tag{A6}$$

We show below that the mean of the energy–momentum tensor (21) over *in* states as $u\to\pm\infty$ is independent of the sign of u, and it decrease more gradually with $|u|$ (as u^{-2}).

From (A17) we have

$$\beta_{\omega'\omega}\equiv\beta_{z'z}=\frac{1}{L}\sqrt{\frac{\omega'}{\omega}}\int_{-\infty}^{\infty}du\ \exp(-i\omega' u-i\omega f(u))$$

$$=B\sqrt{\frac{\omega'}{\omega}}\int_{-\infty}^{\infty}d\zeta\ \exp\left[-iz'\zeta-iz\frac{f(u)}{B}\right],$$

$$u=B\zeta,\qquad z'=B\omega',\qquad z=B\omega. \tag{A7}$$

Inserting the expression for $f(u)$ from (A2), we obtain

$$\int_0^\infty dx[\exp(bx)+\exp(-bx)]\exp(-c\sqrt{x^2+a^2})$$

$$=\frac{2ac}{\sqrt{c^2-b^2}}K_1(a\sqrt{c^2-b^2}), \tag{A8}$$

where the integral can be evaluated by differentiating expression (2.3.17.13) in Ref. 15. The result is

$$\beta_{\omega'\omega}=-i\frac{4\beta B\sqrt{zz'}}{L}\frac{K_1(y)}{y}\exp[i(z'-z)\beta],$$

$$y=[4zz'+(z-z')^2(1-\beta^2)]^{1/2}. \tag{A9}$$

Here $K_1(y)$ is the modified Bessel function of the second kind. Making use of (19) and (A9), we have

$$\frac{\partial\phi_\omega^{(-)}(u)}{\partial t}=\frac{iL}{2\pi\sqrt{2L}B^{3/2}}\int_0^\infty dx\sqrt{x}\ \exp\left(i\frac{x}{B}u\right)\beta_{xz}$$

$$=\frac{\beta}{\pi}\sqrt{\frac{2z}{LB}}\exp(-i\beta z)\int_0^\infty dx\ x$$

$$\times\exp(ix\eta)y^{-1}K_1(y), \tag{A10}$$

where the variable z' has been called x, and $u/B+\beta=\eta$. The squared modulus of this expression, which enters into T_{00} (see (21)), is independent of the sign of η; in other words, at large $\zeta=u/B$, it does not depend on the sign of u.

We now find the asymptotic behavior of (21) as $u\to\pm\infty$, i.e., as $\eta\to\infty$. In this case, $x_{\rm eff}\sim 1/\eta\ll 1$, and as will shortly be apparent, $z_{\rm eff}\ll 1$. Then

$$\int_0^\infty dx\ xe^{i\eta x}y^{-1}K_1(y)\approx\int_0^\infty dx\ xy^{-2}e^{i\eta x}$$

$$=\frac{1}{1-\beta^2}\left\{\frac{x_2}{x_2-x_1}E_1(i\eta x_2)\right.$$

$$\left.-\frac{x_1}{x_2-x_1}E_1(i\eta x_1),\quad \eta\gg 1.\right. \tag{A11}$$

Here

$$y^2=(1-\beta^2)(x-x_1)(x-x_2),\quad x_1=-z\frac{1+\beta}{1-\beta},$$

$$x_2=-z\frac{1-\beta}{1+\beta},\quad E_1(z)=\int_z^\infty e^{-t}\frac{dt}{t}. \tag{A12}$$

To continue, it is necessary to square the absolute value of (A10) and integrate with respect to z from 0 to ∞; see (21) and (A10). With this and (A11) in mind, we make use of the following indefinite integral $\mu=\beta-\alpha$:

$$\int dz\ z[e^{-\mu z}E_1(\alpha z)E_1(-\beta z)+e^{\mu z}E_1(-\alpha z)E_1(\beta z)]$$

$$=-e^{-\mu z}\left(\frac{z}{\mu}+\frac{1}{\mu^2}\right)E_1(\alpha z)E_1(-\beta z)+e^{\mu z}\left(\frac{z}{\mu}-\frac{1}{\mu^2}\right)E_1$$

$$\times(-\alpha z)E_1(\beta z)+\frac{1}{\mu\beta}[e^{-\beta z}E_1(-\beta z)+e^{\beta z}E_1(\beta z)]$$

$$-\frac{1}{\mu\alpha}\left[e^{\alpha z}E_1(\alpha z)+e^{-\alpha z}E_1(-\alpha z)-\frac{2\ln z}{\alpha\beta}\right.$$

$$\left.+\mu^{-2}[E_1(\beta z)E_1(-\beta z)+E_1(-\alpha z)E_1(\alpha z)].\right. \tag{A13}$$

This expression can be checked by differentiation. The other required integral can be obtained from this one by taking the limit $u\to 0$, i.e., $\alpha\to\beta$. The final result is

$$T_{00}\Big|_{\eta\to\infty}=\frac{1}{(2\pi)^3(\eta B)^2}\left\{\frac{1+\beta^2}{\beta}\ln\frac{1+\beta}{1-\beta}\right.$$

$$\left.-\frac{(1-\beta^2)^2}{4\beta^2}\ln^2\frac{1+\beta}{1-\beta}-1\right\}. \tag{A14}$$

As expected, this expression tends to zero as $\beta\to 0$, i.e., for a stationary mirror.

In the integral with respect to z, the dominant values are located near $z\propto\eta^{-1}$, so that $y_{\rm eff}\ll 1$ and the approximation (A11) is justified at large enough η. Thus, $T_{00}\propto u^{-2}$ in (A14), and it differs from both expressions in (A6).

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Note added in proof (26 March 1996): It can easily be shown that the vacuum means of the normal-ordered energy–momentum tensors obtained in Sec. 2 are exactly the same as the corresponding quantities obtained by means of regularization. Furthermore, due to the fact that ψ^{in} at $t \to -\infty$ and ψ^{out} at $t \to +\infty$ are the same, no expansions are necessary to separate the divergent part, as we are only interested in values of the energy–momentum tensor at $t \to +\infty$.

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