Bosonization of ZF algebras: direction toward a deformed Virasoro algebra

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We review bosonization of conformal field theory (CFT) and show how it can be applied to the study of representations of Zamolodchikov–Faddeev (ZF) algebras. In the bosonic construction we obtain an explicit realization of the chiral vertex operators interpolating between irreducible representations of the deformed Virasoro algebra. The commutation relations of these operators are determined by the elliptic matrix of IRF type and their matrix elements are given as contour integrals of meromorphic functions. © *1996 American Institute of Physics.* [S1063-7761(96)00306-X]

1. INTRODUCTION

The development of CFT was initiated in the fundamental work[1] by Belavin, Polyakov and Zamolodchikov (BPZ) where the system of axioms describing CFT was proposed. The main idea of BPZ is that fields in CFT are classified by irreducible representations of the Virasoro algebra. From a mathematical point of view, studying of the CFT is equivalent to the description of representations of the Virasoro algebra and deriving the matrix elements of vertex operators interpolating between different irreducible representations of the Virasoro algebra $\mathscr{L}_\lambda$ specified by the highest weight Δ_λ. Among vertex operators there is a set of basic ones, called primary operators:

$$\Phi^{\lambda\mu}_{\Delta}(\zeta):\mathscr{L}_\mu\to\mathscr{L}_\lambda\otimes\mathbf{C}(\zeta)\zeta^{\Delta_\lambda-\Delta_\mu}.$$

Matrix elements of these operators ("conformal blocks") are multivalued analytical functions. Knowing conformal blocks one can reconstruct physical correlation functions which satisfy the requirement of locality. The analytical properties of conformal blocks[2] are dictated by the commutation relations in the algebra of the chiral vertex operators $\Phi^{\lambda\mu}_{\Delta}(\zeta)$:

$$\Phi^{\lambda_3\lambda_4}_{\Delta_1}(\zeta_1)\Phi^{\lambda_4\lambda_1}_{\Delta_2}(\zeta_2)|_{\mathscr{L}_{\lambda_1}}=\sum_{\lambda_2}\mathbf{W}_{\Delta_1\Delta_2}\begin{bmatrix}\lambda_3 & \lambda_2\\ \lambda_4 & \lambda_1\end{bmatrix}\times\Phi^{\lambda_3\lambda_2}_{\Delta_2}(\zeta_2)\Phi^{\lambda_2\lambda_1}_{\Delta_1}(\zeta_1)|_{\mathscr{L}_{\lambda_1}}. \quad (1.1)$$

This quadratic algebra is found to be associative, which follows from the fact that the matrix $\mathbf{W}_{\Delta_1\Delta_2}$ satisfies the Yang–Baxter equation (YBE) in the IRF form.[3]

It is important that CFT can be considered alternatively as the representation theory of this algebra[4,5] which will be referred below as a Zamolodchikov–Faddeev (ZF) algebra of IRF type. In this way, Virasoro algebra can be considered as an algebra of transformations preserving the commutation relations. Under the appropriate choice of the set of irreducible representations of the Virasoro algebra $\mathscr{L}_\lambda$ the representations of the chiral vertex algebra (1.1) possess the realization in the direct sum $\oplus_\lambda\mathscr{L}_\lambda$. The ZF algebra of IRF type is deeply connected with another associative quadratic algebra

$$Z_a(\zeta_1)Z_b(\zeta_2)=\mathbf{R}^{cd}_{ab}Z_d(\zeta_2)Z_c(\zeta_1), \quad (1.2)$$

where the matrix $\mathbf{R}^{ab}_{cd}$ is a solution of the Yang–Baxter equation. We call (1.2) a Zamolodchikov–Faddeev algebra[6,7] of vertex type. It can be realized in the extension of the space $\oplus_\lambda\mathscr{L}_\lambda$ by taking irreducible representations of Virasoro algebra with proper multiplicities:

$$\pi_Z=\oplus_\lambda\mathscr{L}_\lambda\otimes\mathscr{V}_\lambda, \quad (1.3)$$

where $\mathscr{V}_\lambda$ are some finite-dimensional vector spaces.[8]

Thus, CFT can be described in terms of two algebras which have different forms and require, at first glance, different approaches of investigation. The first one is infinite-dimensional Virasoro algebra, while the second is the associative quadratic algebra which is determined by some finite-dimensional matrices. However, both algebras are deeply interconnected and can be represented in the same space. The initial success of CFT was based on the well developed representation theory of the Virasoro algebra.[9] At the same time, the ZF algebra approach seems to be more general, since algebraic structures like (1.1), (1.2) with matrices **R** and **W** depending on $\zeta_1\zeta_2^{-1}$, play a crucial role in the two-dimensional integrable models of both statistical mechanics[10–14] and quantum field theory (see e.g. Ref. 15). In the hierarchy of the solutions of YBE the constant solutions corresponding to the algebras (1.1), (1.2) are the simplest ones. They might be obtained from trigonometric and elliptic **R** and **W** matrices as a result of the well-known limiting procedure. Conversely, one can regard more complicated solutions of YBE as parametric deformations of the constant ones. It is reasonable to expect that ZF algebras corresponding to the trigonometric and elliptic matrices **R**, **W** are deformations of the algebras of conformal vertex operators. In this way we run into the following questions:

1. Is it possible to describe the representations of elliptic and trigonometric ZF algebras using the methods of CFT or, more explicitly, can ZF algebras be realized in the direct sum of irreducible representations of some infinite-dimensional algebras generalizing the Virasoro algebra?

2. What is the exact form of the commutation relations of these deformed Virasoro algebras and their geometrical and physical meaning?

The present work is mainly devoted to studying the first problem. We construct the representation of the elliptic deformation of the conformal vertex operator algebra (1.1). Our

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main idea is to deform in an appropriate manner the bosonization procedure developed in Refs. 2, 9, 16 for CFT. Let us recall that the central objects in the bosonization are screening operators. Indeed, to realize the irreducible representations of the Virasoro algebra in Fock space, one needs to know only the explicit bosonic realization of intertwining (screening) operators. It is remarkable that the explicit form of the commutation relations of the Virasoro algebra is not really used. We describe a generalization of the conformal bosonization based on deformed screening operators, proposed in Ref. 17. It allows one to get the explicit bosonic realization of the chiral vertex operators which satisfy the commutation relations of the form (1.1) with elliptic **W** matrices. Quadratic associative algebras obtained in this approach seem to be deeply connected to algebras of vertex operators found in Ref. 18. In particular, we expect that the deformed Virasoro algebra is some reduction of the elliptic $sl(2)$ algebra, just as ordinary Virasoro algebra is a result of quantum Hamiltonian reduction of the affine Kac-Moody algebra $\hat{s}l(2)$.[19]

The largest part of this work is devoted to bosonization of CFT. In the simplest cases we re-examine why and when bosonization is still working. The material from the first and the third sections seems to be known to experts and it is contained (but sometimes in obscure form) in the works.[2,9,16] Our aim here is just to emphasize those subtle points which appear to be crucial in the generalization of bosonization.[17] When we understand and correctly formulate the conformal case, we will be able to construct the representation of the elliptic ZF algebra. In particular, we obtain an integral representation for matrix elements of vertex operators generalizing the Dotsenko–Fateev formulas for conformal blocks. For instance, in the simplest case of four-point function it is equivalent to the following integral representation of q-hypergeometric functions:

$$\int_C \frac{dz}{2\pi i} z^{c-1} \frac{(q^{(1+a)/2}z^{-1};q)_\infty}{(q^{(1-a)/2}z^{-1};q)_\infty} \frac{(q^{(1+b)/2}\zeta z;q)_\infty}{(q^{(1-b)/2}\zeta z;q)_\infty} = q^{c(1-a)/2} \frac{\Gamma_q(c+a)}{\Gamma_q(c+1)\Gamma_q(a)} F_q(a+c,b,c+1;q^{1-(a+b)/2}\zeta). \quad (1.4)$$

Note, that matrix elements of deformed vertex operators are written in the form of ordinary contour integrals rather than Jackson's integrals in the bosonization scheme of Ref. 20. We also consider the trigonometric limit of elliptic construction, which corresponds to the sin-Gordon model.[17,21,22] In this limit we show how to reconstruct the ZF algebra of vertex type from the IRF ZF algebra.[23]

Let us agree on some notational conventions.

i) Throughout this work we will denote objects which have similar meaning by the same letter, distinguishing it by "prime" symbol. It should not be confused with the derivation symbol ∂.

ii) For technical reasons it is convenient for us to carry out the ordering procedure in exponential operators like $e^{i\phi}$ in the final step. There is no difference between this prescription and the ordinary one in the conformal case, and it is not fundamental but is useful in the deformed case. It will be explained in Sec. 4.

2. PRELIMINARIES

2.1. Let $R_{ab}^{cd}(t)$ be a numerical matrix depending on the complex parameter t. The indices a,b,c,d take values in the set of integers. One can consider this matrix as an operator acting on the tensor product $\mathscr{V}\otimes\mathscr{V}$, where $\mathscr{V}$ is a finite dimensional linear space, in which vectors are specified by indices a. We will call the following algebraic equation a Yang–Baxter equation of vertex type:[3]

$$R_{a_1a_2}^{c_1c_2}(t_1t_2^{-1})R_{c_1a_3}^{b_1c_3}(t_1t_3^{-1})R_{c_2c_3}^{b_2b_3}(t_2t_3^{-1}) = R_{a_1c_3}^{c_1b_3}(t_1t_3^{-1})R_{a_2a_3}^{c_2c_3}(t_2t_3^{-1})R_{c_1c_2}^{b_1b_2}(t_1t_2^{-1}). \quad (2.1)$$

This equation plays a fundamental role in 2D Integrable Models of both quantum field theory and statistical mechanics. At present many of its nontrivial solutions have been found (see e.g., Ref. 24). In this work we will be interested in the simplest one when the space $\mathscr{V}$ has dimension 2. We will label the basis vectors of $\mathscr{V}$ by $\pm$. In 1972 Baxter found the following remarkable solution of (2.1):[25]

$$\begin{aligned} R_{--}^{--}(t,p,q) &= \frac{\Theta_q(p^2q^{1/2})\Theta_q(tq^{1/2})}{\Theta_q(q^{1/2})\Theta_q(p^2tq^{1/2})}, \\ R_{+-}^{+-}(t,p,q) &= R_{-+}^{-+}(t,p,q) = p\,\frac{\Theta_q(p^2q^{1/2})\Theta_q(t)}{\Theta_q(q^{1/2})\Theta_q(p^2t)}, \\ R_{+-}^{-+}(t,p,q) &= R_{-+}^{+-}(t,p,q) = t^{1/2}\frac{\Theta_q(p^2)\Theta_q(tq^{1/2})}{\Theta_q(q^{1/2})\Theta_q(p^2t)}, \\ R_{++}^{--}(t,p,q) &= R_{++}^{--}(t,p,q) \\ &= p^{-1}t^{-1/2}q^{1/4}\frac{\Theta_q(p^2)\Theta_q(t)}{\Theta_q(q^{1/2})\Theta_q(p^2tq^{1/2})}, \end{aligned} \quad (2.2)$$

where

$$\Theta_q(s) = (q;q)_\infty(s;q)_\infty(qs^{-1};q)_\infty,$$

is the Jacobi elliptic function and we use the standard notation:

$$(z;q_1,\ldots,q_k)_\infty = \prod_{n_1,\ldots n_k=0}^{\infty}(1-zq_1^{n_1}\ldots q_k^{n_k}).$$

The R-matrix depends on two additional parameters p, q. In this work we assume that the parameter q is real and $0\leq q<1$. At the same time p will be a complex number such that $|p|^2=1$. In this case the matrix elements are real numbers if $|t|^2=1$. Together with the Yang–Baxter equation the matrix R_{ab}^{cd} also satisfies the so-called unitarity condition:

$$R_{a_1a_2}^{b_1b_2}(t)R_{b_1b_2}^{c_1c_2}(t^{-1}) = \delta_{a_1}^{c_1}\delta_{a_2}^{c_2}. \quad (2.3)$$

As was pointed out by A. B. Zamolodchikov,[6] Eqs. (2.1), (2.3) can be treated as self-consistency and associativity conditions respectively in the formal algebra

$$Z_{a_1}(t_1)Z_{a_2}(t_2) = S_{a_1a_2}^{b_1b_2}(t_1t_2^{-1})Z_{b_2}(t_2)Z_{b_1}(t_1). \quad (2.4)$$

This algebra was also considered in the work of L. D. Faddeev,[7] and we will call it a ZF algebra.

In the limit $q \to 0$ the matrix simplifies greatly since $\Theta_q(t) \to 1-t$, and the nontrivial elements of $R_{ab}^{cd}(t,p)$ read:

$$\begin{aligned} &R_{++}^{++}(t,p)=R_{--}^{--}(t,p)=1, \\ &R_{+-}^{+-}(t,p)=R_{-+}^{-+}(t,p)=p\,\frac{1-t}{1-p^2t}, \\ &R_{+-}^{-+}(t,p)=R_{-+}^{+-}(t,p)=t^{1/2}\,\frac{1-p^2}{1-p^2t}. \end{aligned} \tag{2.5}$$

We can also get a solution of the Yang–Baxter equation which does not depend on the spectral parameter t by the following limiting procedure:

$$R_{ab}^{cd}(\sigma,p)=p^{\sigma/2}\lim_{L\to+\infty} e^{(a-c)\sigma L/4}R_{ab}^{cd}(e^{\sigma L},p), \tag{2.6}$$

where $\sigma=\pm$. The nonzero elements of the matrix $R_{ab}^{cd}(+,p)$ are defined by the relations

$$\begin{aligned} &R_{++}^{++}(+,p)=R_{--}^{--}(+,p)=p^{1/2}, \\ &R_{+-}^{+-}(+,p)=R_{-+}^{-+}(+,p)=p^{-1/2}, \\ &R_{+-}^{-+}(+,p)=p^{-1/2}(p-p^{-1}). \end{aligned} \tag{2.7}$$

Note that the matrices $R(+,p)$ and $R(-,p)$ are connected by the relation

$$R_{ab}^{cd}(-,p)=[R^{-1}]_{ba}^{dc}(+,p), \tag{2.8}$$

which can be regarded as an analogue of the unitarity condition for the constant solution of the Yang–Baxter equation. The matrix (2.7) coincides with the fundamental **R** matrix for quantum algebra $U_p(sl(2))$.[26] We will need some facts concerning this algebra, so let us recall them here.

2.2. The quantum universal enveloping algebra $U_p(sl(2))$ is an algebra on generators $X^{\pm},T$ subject to the following relations:

$$\begin{aligned} &X^+X^- - p^2X^-X^+ = \frac{1-T^2}{1-p^{-2}}, \\ &TX^{\pm}=p^{\mp 2}X^{\pm}T. \end{aligned} \tag{2.9}$$

It is well known that (2.9) has Hopf algebra structure. In particular, the comultiplication Δ is defined by

$$\Delta(X^{\pm})=T\otimes X^{\pm}+X^{\pm}\otimes 1, \quad \Delta(T)=T\otimes T. \tag{2.10}$$

The parameter p is a complex number. The most relevant cases for us will be those with $p=e^{i\pi(\xi+1)/\xi}$ or $e^{i\pi\xi/(\xi+1)}$, where ξ is a real irrational number greater then 1. In these cases the algebra $U_p(sl(2))$ admits irreducible representations in the finite dimensional spaces $\mathscr{V}_l$ with the dim $\mathscr{V}_l=l$. Construction of such representations is quite similar to the construction of irreducible representations of $sl(2)$ Lie algebra with spin $j=(l-1)/2$.[27,28] The basis vectors $\mathbf{e}_l^m \in \mathscr{V}_l$ of the representation are specified by the index $m=-j,-j+1,\dots,j$ and the conditions

$$T\mathbf{e}_l^m=p^{-2m}\mathbf{e}_l^m, \quad X^-\mathbf{e}_l^m=\mathbf{e}_l^{m-1}. \tag{2.11}$$

The existence of Hopf structure means that the tensor product $\mathscr{V}_{l_1}\otimes\mathscr{V}_{l_2}$ of two representations of $U_p(sl(2))$ would also carry the structure of a representation of this algebra. Moreover, if ξ is an irrational number, then this representation turns to be completely reducible. Since the tensor product of two finite-dimensional irreducible representations can be represented as a direct sum of irreducible representations, one can write down a Clebsch–Gordan decomposition of the form:

$$\mathbf{e}_{2j_1+1}^{m_1}\otimes\mathbf{e}_{2j_2+1}^{m_2}=\sum_l \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m_1+m_2 \end{pmatrix}_p \mathbf{e}_{2j+1}^{m}. \tag{2.12}$$

In present work we will need the explicit form of the following Clebsch–Gordan coefficients:

$$\begin{aligned} &\begin{pmatrix} 1/2 & j & j+1/2 \\ 1/2 & m & m+1/2 \end{pmatrix}_p = p^{-2j}\frac{[j+m+1]_p}{[2j+1]_p}, \\ &\begin{pmatrix} 1/2 & j & j+1/2 \\ -1/2 & m & m-1/2 \end{pmatrix}_p = \frac{p^{-j+m}}{[2j+1]_p}, \\ &\begin{pmatrix} 1/2 & j & j-1/2 \\ 1/2 & m & m+1/2 \end{pmatrix}_p = (-1)^{2j}\frac{[j-m]_p}{[2j+1]_p}, \\ &\begin{pmatrix} 1/2 & j & j-1/2 \\ -1/2 & m & m-1/2 \end{pmatrix}_p = (-1)^{2j+1}\frac{p^{-j+m-1}}{[2j+1]_p}, \end{aligned} \tag{2.13}$$

here we use the notation $[x]_p=(p^x-p^{-x})/(p-p^{-1})$.

3. FREE-FERMION ZF ALGEBRA AND $c=-2$ VIRASORO ALGEBRA

In this section we will discuss the ZF algebra generated by the operators $Z_a(\zeta)(a=\pm,\zeta\in C)$, satisfying the simple commutation relations

$$Z_a(\zeta_1)Z_b(\zeta_2)=-Z_b(\zeta_2)Z_a(\zeta_1), \quad \zeta_1\neq\zeta_2. \tag{3.1}$$

We consider its irreducible representations characterized by operator product expansions:

$$\begin{aligned} &Z_{\pm}(\zeta_2)Z_{\mp}(\zeta_1)=\pm\frac{2\zeta_1\zeta_2}{(\zeta_2-\zeta_1)^2}+O(1), \\ &Z_{\pm}(\zeta_2)Z_{\pm}(\zeta_1)=O(1), \quad \zeta_1\to\zeta_2. \end{aligned} \tag{3.2}$$

This algebra is well known in the physical literature as the so-called $b-c$ system $(Z_+=b, Z_-=\partial c)$. Its representations admit decompositions into a direct sum of irreducible representations of the Virasoro algebra. Let us note now that the commutation relations (3.1) and operator product expansions (3.2) are invariant with respect to a linear transformation

$$Z_a[r]\to\mathbf{G}^{ab}Z_b[r],$$

if the determinant of **G** is equal to unity. We will concentrate on the case when this symmetry is not broken and the representation space of the ZF algebra is classified by the $SL(2)$ symmetry together with the Virasoro symmetry. In this case the Virasoro algebra has central charge equal to -2. We will try to analyze these well-known results in a form which leaves room for generalization.

3.1. Irreducible representations of the free-fermion algebra

The algebra (3.1) possesses two different types of representation denoted by π_Z^R (Ramond) and π_Z^{NS} (Neveu–Schwartz), according to the type of boundary conditions imposed on the generators $Z_a(\zeta)$:

$$\begin{aligned} &\text{R:}\quad Z_a(e^{2\pi i}\zeta)=Z_a(\zeta),\\ &\text{NS:}\quad Z_a(e^{2\pi i}\zeta)=-Z_a(\zeta). \end{aligned} \tag{3.3}$$

For this reason, the Z_a have the following decomposition in Laurent series:

$$Z_a(\zeta)=\sum_{r\neq 0} Z_a[r]\zeta^{-r}, \tag{3.4}$$

where $r\in\mathbf{Z}$ for the R sector and $r\in\mathbf{Z}+1/2$ for the NS sector. It follows from (3.1), (3.2), that the modes $Z_a[r]$ obey the anticommutation relations

$$\{Z_a[r],Z_b[m]\}_+=2ra\,\delta_{a+b,0}\delta_{r+m,0}. \tag{3.5}$$

The spaces π_Z^R and π_Z^{NS} are defined as Fock modules of the fermionic algebra (3.5) created by the operators $Z_a[r]$, $r<0$ under the action on the corresponding R and NS vacuum states. The vacuum states are specified by the condition that they are annihilated by any operator $Z_a[r]$ with $r>0$. We assume that R and NS vacuum states are scalars with respect to the $SL(2)$ transformations. Then $SL(2)$ structure of the spaces $\pi_Z^{R,NS}$ is uniquely determined by the condition that ZF operators are arranged in the fundamental $SL(2)$ doublet.

The next fact which we will need in what follows is that the dual space $\pi_Z^{R,NS\star}$ also admits the structure of a representation of the algebra (3.5). This is shown as follows. Given a linear space π_Z, its dual $\pi_Z^\star$ is the set of linear maps $\pi_Z^\star:\pi_Z\to\mathbf{C}$ determined by the action

$$\mathbf{u}\overset{\mathbf{u}^\star}{\longrightarrow}\langle\mathbf{u}^\star,\mathbf{u}\rangle.$$

Choose the dual basis by the canonical pairing $\langle\mathbf{u}_j^\star,\mathbf{u}_k\rangle=\delta_{jk}$. Since the algebra (3.5) admits the anti-involution

$$\mathscr{A}\{Z_a[r]\}=Z_{-a}[-r], \tag{3.6}$$

the spaces $\pi_Z^\star=\pi_Z^{R,NS\star}$ can be endowed with the structure of the representation of the ZF algebra through the formula

$$\langle\mathbf{u}_1^\star Z_a^\star[n],\mathbf{u}_2\rangle=\langle\mathbf{u}_1^\star,\mathscr{A}\{Z_a[n]\}\mathbf{u}_2\rangle. \tag{3.7}$$

3.2. Bosonization of fermionic ZF algebra

Now we want to show how the representations of the ZF algebra (3.1), (3.2) can be realized in direct sums of boson Fock modules. We start with the space π_Z^R. Let $\{b_n,P,Q|n\in\mathbf{Z}/\{0\}\}$ be a set of operators satisfying the commutation relations:

$$\begin{aligned} &[b_m,b_n]=m\,\delta_{m+n,0},\\ &[P,Q]=-i,\quad [P,b_n]=[Q,b_n]=0. \end{aligned} \tag{3.8}$$

The bosonic Fock module $\mathscr{F}_p$ is generated by the action of creation operators b_{-n}, $n>0$ on the highest vector $\mathbf{f}_p:b_n\mathbf{f}_p=0$, $n>0$; $P\mathbf{f}_p=p\mathbf{f}_p$. In the direct sum of the Fock modules $\oplus_{k\in Z}\mathscr{F}_{k-1/2}$ the action of the following operators is well-defined:

$$\begin{aligned} X&=\oint\frac{dz}{2\pi iz}e^{-2i\phi(z)},\\ X'&=\oint\frac{dz}{2\pi iz}e^{i\phi(z)}, \end{aligned} \tag{3.9}$$

where the integration contours are around the origin and

$$\phi(z)=Q-i\,P\ln z+\sum_{\substack{m\in Z\\ m\neq 0}}\frac{b_m}{i\,m}z^{-m}.$$

It is easy to see that the operator X' is nilpotent, $X'^2=0$. Let us define the following spaces:

$$\begin{aligned} \mathscr{F}^+_{k-1/2}&=\mathrm{Ker}_{\mathscr{F}_{k-1/2}}[X'],\\ \mathscr{F}^-_{k-1/2}&=\mathscr{F}_{k-1/2}/\mathscr{F}^+_{k-1/2}. \end{aligned} \tag{3.10}$$

Since the operators X, X', P obey the commutation relations

$$[X,X']=0,\quad [P,X]=-2X,\quad [P,X']=X', \tag{3.11}$$

they act as

$$\begin{aligned} &\mathscr{F}^\pm_{k+1/2}\overset{X}{\longrightarrow}\mathscr{F}^\pm_{k-3/2},\\ &\mathscr{F}^-_{k-1/2}\overset{X'}{\longrightarrow}\mathscr{F}^+_{k+1/2}\overset{X'}{\longrightarrow}0. \end{aligned} \tag{3.12}$$

In the simple case we are considering now, it is not hard to guess the expressions of $Z_a(\zeta)$ through the field ϕ. Finally, the bosonization of π_Z^R is described by proposition 3.1.

Proposition 3.1. The map $\pi_Z^R\to\oplus_{k\in Z}^\infty\mathscr{F}^+_{k-1/2}$ given by the identification of the R-vacuum state of π_Z^R with the vector $\mathbf{f}_{1/2}$ and the bosonization rules

$$\begin{aligned} Z_+(\zeta)&=e^{i\phi(\zeta)},\\ Z_-(\zeta)&=[X,Z_+(\zeta)]=2\zeta\partial_\zeta e^{-i\phi(\zeta)} \end{aligned} \tag{3.13}$$

is an isomorphism of modules.

Note that it follows from formulae (3.11) that the operators X and $H=P-1/2$ can be regarded as generators of the Borel subalgebra of the $sl(2)$ algebra acting in the space π_Z^R.

By proposition 3.1, one can treat π_Z^R as a direct sum of Virasoro algebra modules. The crucial observation here is that the subspaces $\mathrm{Ker}_{\mathscr{F}^+_{l-1/2}}[X^l]$, $l>0$ can be endowed with the structure of an irreducible representation of the Virasoro algebra. To explain this important property let us introduce the $SL(2)$ scalar field $T(\zeta)$ by the formula

$$-4\zeta^2T(\zeta)=Z_+(\zeta)Z_-(\zeta)-Z_-(\zeta)Z_+(\zeta). \tag{3.14}$$

In terms of the bosonic field ϕ it reads

$$2T(\zeta)=f^2-\partial_\zeta f, \tag{3.15}$$

where $f=i\partial_\zeta\phi(\zeta)-1/2\zeta$. It is easy to check that the $L_n:T(\zeta)=\sum_{n\in Z}L_n\zeta^{-n-2}$ generate the Virasoro algebra Vir_c

$$[L_n,L_m]=(n-m)L_{n+m}+\frac{c}{12}(n^3-n) \tag{3.16}$$

with $c=-2$. Let us denote by $\mathscr{L}_{l-1/2}$, $l>0$ the irreducible Verma module with highest weight

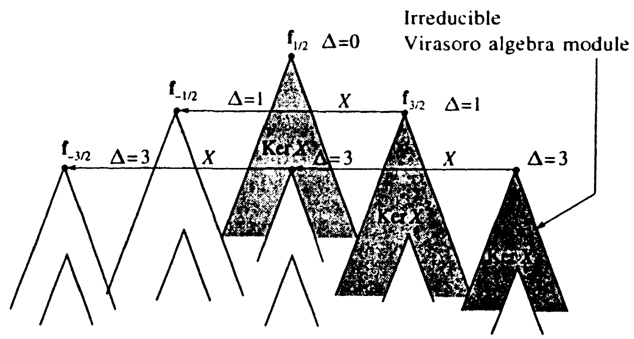

FIG. 1. The Virasoro structure of the space $\oplus_{k\in Z}\mathscr{F}^{+}_{k-1/2}$.

$$\Delta_{l-1/2}=\frac{l(l-1)}{2} \tag{3.17}$$

built on the highest vector $\mathbf{v}_{l-1/2}$. If generators of the Virasoro algebra are realized as (3.15) then any highest weight vector $\mathbf{f}_{l-1/2}$ of Fock space turns to be the highest-weight vector of the Verma module of the Virasoro algebra with highest weight (3.17). Moreover, consider non-zero vectors in bosonic space π^{R}_{Z} which can be obtained from $\mathbf{f}_{l-1/2}$ by the action of operator X^k, $k=0, 1,\ldots$. We claim that the generators L_n with $n>0$ annihilate any such vector while L_0 acts by multiplication on constant $\Delta_{l-1/2}$. This follows from the fact that generators (3.15) commute with the operator X as can be easily checked.

Let us consider the formal tensor product of the irreducible Virasoro algebra module $\mathscr{L}_{l-1/2}$ and l-dimensional $sl(2)$ irreducible representation $\mathscr{V}^{+}_{l}$ with basis vectors $\mathbf{e}^{m+}_{l}$, where $m=-j,\ldots,j$ and $l=2j+1$, $j=0, 1/2, 1,\ldots$. We will denote it as $\mathscr{L}_{l-1/2}\otimes\mathscr{V}^{+}_{l}$. The meaning of the additional index $\langle\langle+\rangle\rangle$ will be clarified later. Standard arguments [9,29] suffice to prove the following proposition describing the Virasoro structure of the space π^{R}_{Z} (Fig. 1):

Proposition 3.2. The map $\pi^{\mathrm{R}}_{Z}\rightarrow\oplus^{\infty}_{l=1}\mathscr{L}_{l-1/2}\otimes\mathscr{V}^{+}_{l}$ given by the correspondence $X^{j-m}\mathbf{f}_{l-1/2}\rightarrow\mathbf{v}_{l-1/2}\otimes\mathbf{e}^{m+}_{l}$ $(l=2j+1)$, and the bosonizaton rule (3.15) is an isomorphism of modules of the Virasoro algebra with
i)

$$\mathrm{Ker}_{\mathscr{F}^{+}_{l-1/2}}[X^l]\simeq\mathscr{L}_{l-1/2},\quad l>0; \tag{3.18}$$

ii)

$$\mathscr{F}^{+}_{2m+1/2}\simeq\oplus^{\infty}_{k=0}\mathscr{L}_{2|m|+2k+1/2}\otimes\mathbf{e}^{m+}_{2|m|+2k+1},\quad m\in\frac{1}{2}\mathbf{Z}. \tag{3.19}$$

Notice, that such as operators L_n and X' commute, then the space $\mathscr{F}^{-}_{k-1/2}$ has the similar Virasoro structure as $\mathscr{F}^{+}_{k+1/2}$:

$$\mathscr{F}^{-}_{2m-1/2}\simeq\oplus_{k=0}\mathscr{L}_{2|m|+2k+1/2}\otimes\mathbf{e}^{m-}_{2|m|+2k+1},\quad m\in\frac{1}{2}\mathbf{Z}, \tag{3.20}$$

where we have denoted by $\mathbf{e}^{m-}_{l}$ the basis vectors in the l-dimensional space $\mathscr{V}^{-}_{l}$. Note that the vector $\mathbf{v}_{l-1/2}\otimes\mathbf{e}^{-m,-}_{l}$, where $\mathbf{v}_{l-1/2}$ is the highest vectors in the Virasoro module, is identified with the state from $\mathscr{F}^{-}_{-2m-1/2}$ which is the preimage of $X^{j+m}\mathbf{f}_{-l+1/2}$ with respect to the action of the operator X'. In particular, $\mathbf{v}_{l-1/2}\otimes\mathbf{e}^{-j,-}_{l}\equiv\mathbf{f}_{-l+1/2}$ mod $\mathrm{Ker}\,X'$. The difference which appears in this case is in the fact that irreducible Virasoro algebra modules are identified with factor spaces rather than subspaces of Fock space.

3.3. Scalar product in Fock space

Now we turn to the bosonization of the dual representation $\pi^{\mathrm{R}\star}_{Z}$ of the ZF algebra. This problem is closely related to the proper choice of the scalar product in the direct sum of Fock spaces $\oplus_{k\in Z}\mathscr{F}_{k-1/2}$. The obvious guess is

$$\begin{aligned}(\mathbf{u}_1 b_n,\mathbf{u}_2)_0&=-(\mathbf{u}_1,b_{-n}\mathbf{u}_2)_0,\\(\mathbf{u}_1 P,\mathbf{u}_2)_0&=-(\mathbf{u}_1,P\mathbf{u}_2)_0.\end{aligned} \tag{3.21}$$

However, it does not conform to the conjugation condition (3.7) for the operators Z_a. In thinking about this problem, it is rather natural to introduce an independent bosonic representation for $\pi^{\mathrm{R}\star}_{Z}$ and then try to identify it with some subspaces in the Fock space $\oplus_{k\in Z}\mathscr{F}_{k-1/2}$. To do this, we define a new set of generators $\{b^{\star}_{n},P^{\star},Q^{\star}|n=\pm1,\pm2,\ldots\}$ with the same commutation relations as in (3.8). Let the analogs of the integral operators (3.9) be

$$\begin{aligned}X'^{\star}&=\oint\frac{dz}{2\pi iz}\,e^{i\phi^{\star}(z)},\\X^{\star}&=\oint\frac{dz}{2\pi iz}\,e^{-2i\phi^{\star}(z)}.\end{aligned} \tag{.3.22}$$

Repeating the above analysis step by step, one can describe the bosonization of the representation $\pi^{\mathrm{R}\star}_{Z}$ and introduce the Virasoro algebra structure in it. The formulae for this case are rather evident. In particular, $\pi^{\mathrm{R}\star}_{Z}\simeq\oplus_{k\in Z}\mathscr{F}^{\star+}_{k-1/2}$, where we denote by $\mathscr{F}^{\star+}_{k-1/2}$ the kernel of the operator $X'^{\star}$: $\mathscr{F}^{\star}_{k-1/2}\rightarrow\mathscr{F}^{\star}_{k+1/2}$. And any space $\mathscr{F}^{\star+}_{k+1/2}$ turns to be isomorphic to a direct sum of irreducible Virasoro algebra modules:

$$\mathscr{F}^{\star+}_{2m+1/2}\simeq\oplus^{\infty}_{k=0}\mathscr{L}^{\star}_{2|m|+2k+1/2}\otimes\mathbf{e}^{\star m}_{2|m|+2k+1}. \tag{3.23}$$

Here the states $X^{\star j-m}\mathbf{f}^{\star}_{l-1/2}$ $(l=2j+1)$ correspond to the states $\mathbf{v}^{\star}_{l-1/2}\otimes\mathbf{e}^{\star m}_{l}$. Notice, that the symbol $\star$ in $\mathscr{L}^{\star}_{l-1/2}$ is used in order to emphasize that the generators of Virasoro algebra are built from fields $\phi^{\star}$ rather than from ϕ's. To proceed further, let us note, that the decompositions (3.20), (3.23) of the spaces $\pi^{\mathrm{R}\star}_{Z}$ and $\oplus_{k\in\mathbf{Z}}F^{-}_{k-1/2}$ include identical sets of irreducible representations of the Virasoro algebra. Indeed, any two representations $\mathscr{L}_{l-1/2}$ and $\mathscr{L}^{\star}_{l-1/2}$ are isomorphic as $c=-2$ Virasoro algebra representations, since they have the same highest weight $\Delta_{l-1/2}$. In the bosonic realization this means that one can identify the Virasoro algebra generators.[1)] To extend the isomorphism of Virasoro algebra modules to an isomorphism of $\pi^{\mathrm{R}\star}_{Z}$ and $\oplus_{k\in\mathbf{Z}}\mathscr{F}^{-}_{k-1/2}$ one needs to establish the correspondence between vectors in $\mathscr{V}^{\star}_{l}$ and $\mathscr{V}^{-}_{l}$. Let us identify the vectors $\mathbf{e}^{m\star}_{l}\in\mathscr{V}^{\star}_{l}$ and $c^{j}_{m}\mathbf{e}^{-m,-}_{l}\in\mathscr{V}^{-}_{l}$. The constants c^{j}_{m} can be fixed by the condition that the operators X, $X^{\star}$ and $H=-P-1/2$ acting in the space $\oplus_{k\in Z}\mathscr{F}^{-}_{k-1/2}\equiv\oplus_{k\in Z}\mathscr{F}^{\star+}_{k-1/2}$ generate the $sl(2)$ Lie algebra:

$$[X,X^{\star}]=H,\quad[H,X]=2X,\quad[H,X^{\star}]=-2X^{\star}. \tag{3.24}$$

Then we will have $c^j_m=(j-m)!(2j)!/(j+m)!$. Hence we conclude that the representation $\pi_Z^{R\star}$ can be realized as follows:

$$\pi_Z^{R\star}=\oplus_{k\in Z}\mathcal{F}^-_{k-1/2}. \tag{3.25}$$

Note that by virtue of this identification, the action of the operator $X^\star$ in the space $\oplus_{k\in Z}\mathcal{F}_{k-1/2}$ can be uniquely specified by the condition that the following diagram is commutative:

$$\begin{array}{ccccc} & \mathcal{F}^-_{k-3/2} & \overset{X'}{\longrightarrow} & \mathcal{F}^+_{k-1/2} & \\ X^\star & \downarrow & X' & \downarrow & X^\star. \\ & \mathcal{F}^-_{k+1/2} & \longrightarrow & \mathcal{F}^+_{k+3/2} & \end{array} \tag{3.26}$$

Now we are able to describe the total symmetry algebra Symm which acts in the space $\pi_Z^R\oplus\pi_Z^{R\star}$. First, it includes an infinite-dimensional Virasoro algebra with central charge $c=-2$. Second, it contains an $sl(2)$ subalgebra. Finally, the operators X' and $X'^\star$ generate the Clifford subalgebra

$$\{X',X'^\star\}_+=1,\quad X'^2=[X'^\star]^2=0. \tag{3.27}$$

Note that the operators $X',X'^\star$ commute with Virasoro generators and X, $X^\star$ but do not commute with H. The decomposition of the space $\pi_Z^R\oplus\pi_Z^{R\star}$ into a direct sum of irreducible representations of the algebra Symm has the form

$$\pi_Z^R\oplus\pi_Z^{\star R}=\oplus_{l=0}^\infty\mathcal{L}_{l-1/2}\otimes\mathcal{V}_l\otimes\mathcal{V}_2', \tag{3.28}$$

where we have denoted the l-dimensional irreducible representation of the algebra $su(2)$ as $\mathcal{V}_l$, and $\mathcal{V}_2'$ is a two-dimensional representation of the Clifford algebra (3.27). The decomposition (3.28) makes clear the symbolic notations introduced earlier. Namely, the vectors $\mathbf{e}_l^{m\pm}$ are basis vectors in the irreducible l-dimensional representations of $sl(2)$ algebra with spin $j=(l-1)/2$ and momentum projection m. Any pair of vectors $\mathbf{e}_l^{m-}$ and $\mathbf{e}_l^{m+}$ is arranged in a doublet of the Clifford algebra.

Let us discuss now the Hilbert structure of the space $\pi_Z^R\oplus\pi_Z^{R\star}$. This space has a canonical scalar product $(\mathbf{u}_1,\mathbf{u}_2)$ induced by a condition $(\mathbf{u}_1,\mathbf{u}_2)=(\mathbf{u}_1^\star,\mathbf{u}_2^\star)=0$, $(\mathbf{u}_1^\star,\mathbf{u}_2)=(\mathbf{u}_1,\mathbf{u}_2^\star)^*=\langle\mathbf{u}_1^\star,\mathbf{u}_2\rangle$ for any $\mathbf{u}_{1,2}\in\pi_Z^R$, $\mathbf{u}_{1,2}^\star\in\pi_Z^{R\star}$. From the other side, such a scalar product is equivalently described as follows:

$$(\mathbf{e}_{2j'+1}^{n\,b},\mathbf{e}_{2j+1}^{m\,a})=(-1)^{j+m}\frac{2^{2j}(2j)!}{2j+1}\,\delta_{j,j'}\,\delta_{m+n,0}\,\delta_{a+b,0} \tag{3.29}$$

and the scalar product in the irreducible representations of the Virasoro algebra is defined by the conjugation conditions

$$(\mathbf{v}_1L_n,\mathbf{v}_2)=(\mathbf{v}_1,L_{-n}\mathbf{v}_2). \tag{3.30}$$

We will also assume that the scalar product of the highest Virasoro vectors has the form

$$(\mathbf{v}^\star_{l_1-1/2},\mathbf{v}_{l_2-1/2})=l_1\delta_{l_1,l_2}. \tag{3.31}$$

Notice that the scalar product (3.29), (3.30) is drastically different from (3.21).

3.4. NS sector

Now let us shortly describe the bosonization of the NS sector of the representation of the ZF algebra. The consideration here is quite similar to what has been done for R sector and we present only the results. The space π_Z^{NS} can be realized as a following direct sum of Fock modules:

$$\pi_Z^{NS}=\oplus_{k\in Z}\mathcal{F}_k, \tag{3.32}$$

where the NS-vacuum is identified with $\mathbf{f}_0$. The expression for the operators $Z_a(\zeta)$ in terms of the bosonic field ϕ is given by formula (3.13) again. The main difference here in comparison with the Ramond sector case is that the operator X' does not act in the NS sector. The reason for this is that the integration contour in the definition of X' is not closed.

The states in the Fock module $\mathcal{F}_k$ can be classified with respect to the action of a direct product of the Virasoro and $sl(2)$ algebras:

$$\mathcal{F}_{2m}\simeq\oplus_{k=0}^\infty\mathcal{L}_{2|m|+2k}\otimes\mathbf{e}^m_{2|m|+2k+1},\quad m\in\frac{1}{2}\mathbf{Z}, \tag{3.33}$$

where $\mathcal{L}_l$, $l>0$, denotes the irreducible modules of the Virasoro algebra built on the highest vector $\mathbf{v}_l$ with highest weight

$$\Delta_{l-1}=\frac{(l-1)^2}{2}-\frac{1}{8}. \tag{3.34}$$

The vectors $\mathbf{e}_l^m$ are basis vectors in the l-dimensional irreducible representation of the $sl(2)$-algebra. Again, the correspondence between representations is given by the maps of the vectors $X^{j-m}\mathbf{f}_l\rightarrow\mathbf{v}_l\otimes\mathbf{e}_l^m$ and the bosonization rules for generators of the Virasoro algebra and the $sl(2)$ algebra. The scalar product in the space π_Z^{NS} can be introduced by analogy with that for the R sector. Notice that in this case the space π_Z^{NS} can be considered as a self-dual representation of the ZF algebra:

$$\pi_Z^{NS}=\pi_Z^{NS\star}.$$

3.5. Vertex operators for Vir_{-2} algebra

Now, we wish to find the exact meaning of the operators $Z_\pm$ in terms of the Symm algebra. Before going on, let us explain the subject on which we will focus. In the case under consideration the algebra Symm turns to be a direct product of infinite-dimensional Virasoro algebra and a finite algebra. We expect, that this statement is general, in the sense that it is possible to associate with any ZF algebra another algebra which would be a direct product of two parts, an infinite-dimensional one (like the Virasoro algebra) and finite ones. We will demonstrate by means of the simple example (3.1), (3.2) that these operators can be identified with vertex operators for the total symmetry algebra Symm. If we keep in mind the decompositions (3.28), it is rather natural to study the action of the operators $Z_\pm$ on the Virasoro states separately from the action on the finite-dimensional part of the representation space. The operators $Z_\pm$ have spins 1/2 and momentum projections $\pm1/2$ with respect to the $sl(2)$ algebra. Then, if the $\mathbf{e}_l^m$ are basis vectors in the irreducible representations $\mathcal{V}_l$ and $\mathbf{v}\in\mathcal{L}_p$, where $p=l-1/2$, $p=l-1$ respectively for the R and NS sectors, the operators $Z_a(\zeta)$ in the representations π_Z^{R} and π_Z^{NS} can be represented in the form

$$Z_a(\zeta)\mathbf{v}\otimes\mathbf{e}_l^m=\sum_{b=\pm1}\begin{pmatrix}1/2 & j & j+b/2\\ a/2 & m & m+a/2\end{pmatrix}\Phi_{21}^{p+b\ p}(\zeta)\mathbf{v}\otimes\mathbf{e}_{l+b}^{m+a/2}, \tag{3.35}$$

where $l=2j+1$. The numerical coefficients in this formula coincide with the Clebsch–Gordan coefficients for the $sl(2)$ algebra. At the same time the operators Φ_{21} act as follows:

$$\Phi_{21}^{p\pm1\ p}(\zeta):\quad \mathscr{L}_{\Delta_p}\xrightarrow{\Phi_{21}}\mathscr{L}_{\Delta_{p\pm1}}\otimes\mathbf{C}[\zeta]\zeta^{\Delta_{p\pm1}-\Delta_p}, \tag{3.36}$$

where $\mathbf{C}[\zeta]$ denotes the Laurent series in ζ. Using the operators $Z^{\star}\in$ and $(\pi_Z^{\mathrm{R,NS}\star}$ one can define the operators $\Phi_{21}^{\star}$ by similar expressions. It follows from the formula (3.7) that the operators Φ_{21} and $\Phi_{21}^{\star}$ in (3.36) satisfy the relation

$$(\mathbf{v}_1^{\star}\Phi_{21}^{\star\ p\ p\pm1}(\zeta),\mathbf{v}_2)=(\mathbf{v}_1^{\star},\Phi_{21}^{p\ p\pm1}(\zeta)\mathbf{v}_2), \tag{3.37}$$

where $\mathbf{v}_1^{\star}\in\mathscr{L}_p^{\star}$ and $\mathbf{v}_2\in\mathscr{L}_{p\pm1}$.

We claim that the operators (3.36) are vertex operators for the Virasoro algebra. Indeed, using the bosonization (3.13) and conjugation condition (3.37) one can show that they satisfy the following commutation relations with the Virasoro algebra generators L_n:

$$[L_n,\Phi_{21}^{\pm}(\zeta)]=\zeta^{n+1}\partial_\zeta\Phi_{21}^{\pm}(\zeta)+\Delta_{21}\ n\ \zeta^n\ \Phi_{21}^{\pm}(\zeta), \tag{3.38}$$

where $\Delta_{21}=1$. Here and in what follows we will use the short notation $\Phi_{21}^{\pm}$ for the operators $\Phi_{21}^{p\pm1\ p}$.

Thus, the ZF algebra appears to be naturally connected with the chiral primary operators of the Virasoro algebra. The last objects are of great importance for our next constructions. It is well known[1,2] that chiral primary operators are uniquely determined by the commutation relations with L_n, the intertwining property and conjugation condition. From now on in this section let us assume that the symbol $\mathscr{L}_p$ denotes irreducible representation of the $c=-2$ Virasoro algebra with highest weight

$$\Delta_p=\frac{p^2}{2}-\frac{1}{8},$$

where $p\in\mathbf{R}$ rather than just being integer or half-integer. This generalization is very useful, since, considering the general case we will be able separate the general features of the bosonic construction from the specific properties of the $SL(2)$-invariant fermion model (3.1), (3.2).

Usually, special attention is paid to studying the properties of the two chiral primary operators.[1,2] The first of these operators is given by Φ_{21} with the properties (3.36)–(3.38), while the second can be introduced by the following conditions:

1. Intertwining properties,

$$\Phi_{12}^{p\mp p/2}(\zeta):\quad \mathscr{L}_p\xrightarrow{\Phi_{12}}\mathscr{L}_{p\mp1/2}\otimes\mathbf{C}[\zeta]\zeta^{\Delta_{p\mp1/2}-\Delta_p}. \tag{3.39}$$

2. Conjugation conditions,

$$(\mathbf{v}_1^{\star}\Phi_{12}^{\star\ p\ p\mp1/2}(\zeta),\mathbf{v}_2)=(\mathbf{v}_1^{\star},\Phi_{12}^{p\ p\mp1/2}(\zeta),\mathbf{v}_2). \tag{3.40}$$

3. Commutation relations with Virasoro algebra generators,

$$[L_n,\Phi_{12}^{\pm}(\zeta)]=\zeta^{n+1}\partial_\zeta\Phi_{12}^{\pm}(\zeta)+\Delta_{12}\ n\ \zeta^n\Phi_{12}^{\pm}(\zeta), \tag{3.41}$$

with $\Delta_{12}=-1/8$.

Again, the abbreviated notation $\Phi_{12}^{\pm}$ stands for $\Phi_{12}^{p\mp p/2}$ respectively. The properties (3.36)–(3.38) and (3.39)–(3.41) describe the chiral primary operators Φ_{21} and Φ_{12} uniquely up to a constant multiplier. This means that by using only these formulae one can reconstruct the matrix elements of any combination of such operators. Indeed, as was shown by Belavin, Polyakov and Zamolodchikov,[1] by virtue of the formulae (3.38)–(3.41) the matrix elements of operators built from Φ_{21} and Φ_{12} turn to be solutions of certain linear differential equations. For instance, the functions

$$\begin{aligned}G_p^{\pm}(\zeta_1\zeta_2^{-1})&=(\mathbf{v}_p^{\star},\Phi_{21}^{\pm}(\zeta_2)\Phi_{21}^{\mp}(\zeta_1)\mathbf{v}_p),\\ G_p'^{\pm}(\zeta_1\zeta_2^{-1})&=(\mathbf{v}_p^{\star},\Phi_{12}^{\pm}(\zeta_2)\Phi_{12}^{\mp}(\zeta_1)\mathbf{v}_p),\end{aligned} \tag{3.42}$$

where $\mathbf{v}_p$ is the highest vector in the irreducible Virasoro module $\mathscr{L}_p$, satisfy second-order ordinary differential equations:

$$L_{\Delta_{21}}G_p^{\pm}(\zeta)=0,\quad L_{\Delta_{12}}G_p'^{\pm}(\zeta)=0. \tag{3.43}$$

The explicit form of the linear differential operator L_Δ is given by:

$$L_\Delta=\left\{\frac{3}{2(2\Delta+1)}\partial_\zeta^2+\left(\frac{1}{\zeta}+\frac{1}{\zeta-1}\right)\partial_\zeta-\frac{\Delta_p}{\zeta^2}-\frac{\Delta}{(1-\zeta)^2}+\frac{2\Delta}{\zeta(\zeta-1)}\right\}\zeta^{-\Delta}. \tag{3.44}$$

According to this formula the functions (3.42) have the following asymptotics under $\zeta\to0$:

$$G_p^{\pm}(\zeta)=O(\zeta^{1/2\mp p}),\quad G_p'^{\pm}(\zeta)=O(\zeta^{1/8\pm p/2}).$$

To within normalization, the solutions of the equations (3.43) with the above boundary conditions can easily be expressed in terms of hypergeometric functions $F(a,b,c;\zeta)$ as

$$\begin{aligned}G_p^{\pm}(\zeta)&=\zeta^{1/2\mp p}\frac{1\mp2p+\zeta(1\pm2p)}{(1-\zeta)^2},\\ G_p'^{\pm}(\zeta)&=\frac{\Gamma(\pm p+1/2)}{\pi^{1/2}\Gamma(\pm p+1)}\zeta^{1/8\pm p/2}(1-\zeta)^{1/4}\\ &\times F\left(\pm p+\frac{1}{2},\frac{1}{2},\pm p+1;\zeta\right).\end{aligned} \tag{3.45}$$

Let us now choose indices $p\neq0$, 1/2, 1,... . Knowing the explicit formulae (3.45) one can find the commutation relations of the algebra of chiral primary operators. This can be done by using the relation

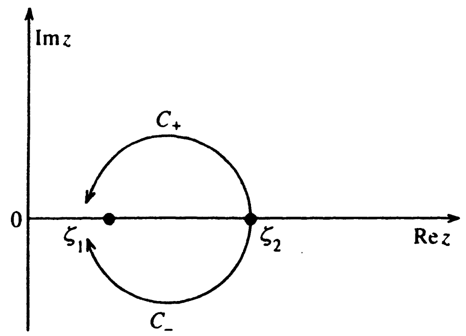

FIG. 2. Two contours $C_\pm$ for analytical continuations of the functions $G_p'^{\pm}(\zeta)$.

$$F(a,b,c;z)=\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}F(a,a-c+1,a-b+1;z^{-1})+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}\times(-z)^{-b}F(b,b-c+1,b-a+1;z^{-1}). \tag{3.46}$$

Straightforward calculation shows that the operators $\Phi_{1,2}$ generate a ZF algebra of IRF type:

$$\Phi_{12}^a(\zeta_1)\Phi_{12}^b(\zeta_2)|_{\mathscr{L}_p}=\sum_{c+d=a+b}\mathbf{W}\left[\begin{matrix}p+(a+b)/2 & p+c/2\\ p+b/2 & p\end{matrix}\middle|\sigma_{12}\right]\times\Phi_{12}^d(\zeta_2)\Phi_{12}^c(\zeta_1)|_{\mathscr{L}_p}. \tag{3.47}$$

The precise form of the nontrivial elements of the matrix $\mathbf{W}$ is the following:

$$\mathbf{W}\left[\begin{matrix}p\pm1 & p\pm1/2\\ p\pm1/2 & p\end{matrix}\middle|\sigma\right]=e^{i\pi\sigma/4},$$

$$\mathbf{W}\left[\begin{matrix}p & p\pm1/2\\ p\pm1/2 & p\end{matrix}\middle|\sigma\right]=\mp\frac{\exp[-i\pi\sigma(1\pm4p)/4]}{\sin\pi p}, \tag{3.48}$$

$$\mathbf{W}\left[\begin{matrix}p & p\mp1/2\\ p\pm1/2 & p\end{matrix}\middle|z\right]=\pm e^{-\pi i\sigma/4}\cot\pi p.$$

The commutation relations (3.47) should be considered as the rule for analytical continuation of functions from the region $|\zeta_1|>|\zeta_2|$ to $|\zeta_1|<|\zeta_2|$ along the paths $C_{\sigma_{12}},\sigma_{12}=\pm$ (see Fig. 2). Thus, the dependence of the matrix $\mathbf{W}$ on z appears only in the choice of the contour used in analytical continuation. The commutation relations of the operators Φ_{21} are simpler. It holds for any value of p:

$$\Phi_{21}^a(\zeta_1)\Phi_{21}^b(\zeta_2)=-\Phi_{21}^b(\zeta_2)\Phi_{21}^a(\zeta_1). \tag{3.49}$$

If the index p is an integer or half-integer, then the commutation relations of the chiral primaries Φ_{12}^a become undetermined and the algebra of chiral primaries would be meaningless. The technical reason for this inconsistency is that for half-integer values of p the hypergeometric function in $G_p'^{\pm}(\zeta)$ degenerates. This phenomena is well-known. It appears when one studies degenerate modules of Virasoro algebra with rational central charge.[2] To avoid additional complications we omit such cases, considering either special values of central charge c and modules in generic position (further in this section and in Sec. 4), or degenerate modules but for general central charge (as in Sec. 5 below).

In spite of its failure with the algebra of chiral primaries, the model with $c=-2$ and $p=0$, 1/2, 1,... elaborated above has several attractive features. First, the algebra of the operators Φ_{21} is still the same as for general p. The fermionic representation of these operators drastically simplifies the analysis of the construction. Second, in this model we have an explicit bosonic realization of a finite-dimensional subalgebra in the symmetry algebra which is given by operators X, X', P. And finally, such model gives us example how to find the bosonization of chiral primary operators. One expects that the form of the bosonic realization of chiral operators should be the same, regardless of whether the action of the intertwining operators X, X' is defined or not.

3.6. Bosonization of vertex operators

The definitions of the chiral primaries do not assume a bosonization procedure. However, we will find a bosonic realization of such operators because it admits direct generalization for less trivial cases. Since the chiral primary operators do not depend on the realization of $\mathscr{L}_p$, its matrix elements are the same for any irreducible Virasoro module with given conformal dimension Δ_p. Let us consider at first the cases when p is positive integer or half-integer. It is convenient to realize the irreducible Virasoro modules $\mathscr{L}_p$, ($\mathscr{L}_p^\star$) as subspaces (factorspaces) $\mathrm{Ker}_{\mathscr{F}_p}[X^l]$, $\mathscr{F}_{-p}/\mathrm{Im}\,[X^l]$ in the Fock modules, where $l=p+1/2$ for the R sector and $l=p+1$ for the NS sector. Then we have:

Proposition 3.3. The following vertex operators of the representation π_Z can be realized as bosonic operators in the Fock submodules

i)

$$\Phi_{21}^+(\zeta)\mathbf{v}=e^{i\phi(\zeta)}\mathbf{v},$$
$$\Phi_{21}^{\star+}(\zeta)\mathbf{v}^\star=\int_C\frac{dz}{2\pi iz}e^{-2i\phi(z)}e^{i\phi(\zeta)}\mathbf{v}^\star, \tag{3.50}$$

ii)

$$\Phi_{12}^{\star-}(\zeta)\mathbf{v}^\star=e^{-i\phi(\zeta)/2}\mathbf{v}^\star,$$
$$\Phi_{12}^-(\zeta)\mathbf{v}=\int_C\frac{dz}{2\pi iz}e^{i\phi(z)}e^{-i\phi(\zeta)/2}\mathbf{v}. \tag{3.51}$$

where $\mathbf{v}\in\mathscr{L}_p$ and $\mathbf{v}^\star\in\mathscr{L}_p^\star$.

The counterclockwise integration contour C is chosen to begin and end at the origin of the complex z-plane. It encloses all singularities whose positions are determined by the vector $\mathbf{v}^\star(\mathbf{v})$. Note, that because of this prescription the integrals are well defined.

This proposition is rather evident since part i) is a direct consequence of Eq. (3.13) while the proof of part ii) is based on the following important property of the operator $e^{-i\phi(\zeta)/2}$:

$$X|_{\pi^{\rm NS}}\, e^{-i\phi(\zeta)/2} = -e^{-i\phi(\zeta)/2} X|_{\pi^{\rm R}}. \tag{3.52}$$

Now we want to check our guess that Eqs. (3.50), (3.51) describe the bosonization of the chiral primaries for any $p>0$. Note, that the irreducible Virasoro algebra modules $\mathscr{L}_p(\mathscr{L}_{p\star})$ in the case of general p are isomorphic to the Fock modules $\mathscr{F}_p(\mathscr{F}_{-p})$. Indeed, the action of the Virasoro algebra in $\mathscr{F}_p$ can be determined through the formulae (3.15), but there are no intertwining operators between Fock modules.[3)] Therefore the operators (3.50), (3.51) act from an irreducible Virasoro algebra module into an irreducible one. Hence one needs to check only the commutation relations of these operators with the Virasoro algebra generators. But it is rather evident such as the integrals in (3.50), (3.51) are well defined in the general case too. Now straightforward computation proves that proposition 3.3 is still true for the chiral primary operators in the case of general p.

The bosonization prescription (3.50), (3.51) and the conjugation condition (3.37), (3.40) allow us to work out the integral representation for any matrix element of the operators $\Phi_{12}^{\pm}$ and $\Phi_{21}^{\pm}$. For instance, let us write down alternative derivation of the function $G_p'^{\pm}(\zeta)$. It would seem at first glance that calculation of matrix elements using the scalar product (3.29), (3.30) is a very nontrivial problem. The essence of the bosonization method is in the fundamental fact that the scalar product (3.21) restricted to the submodules of the Fock modules i) $\mathrm{Ker}_{\mathscr{F}_p}[X^l]$, $\mathscr{F}_{-p}/\mathrm{Im}[X^l]$, $l=p+1/2$ for R sector, $l=p+1$ for NS sector; p is a positive integer or half-integer, ii) $\mathscr{F}_p$, $\mathscr{F}_{-p}$, for general p, which are isomorphic to irreducible modules of the Virasoro algebra, coincides with (3.30). Using this fact and (3.50), (3.51) we get the formula

$$(\mathbf{v}_p^\star, \Phi_{12}^+(\zeta_2)\Phi_{12}^-(\zeta_1)\mathbf{v}_p) = \int_C \frac{dz}{2\pi i z}\,(\mathbf{f}_{-p}, e^{-i\phi(\zeta_2)/2} e^{i\phi(z)} e^{-i\phi(\zeta_1)/2}\mathbf{f}_p)_0, \tag{3.53}$$

where the integration contour C is determined by the same prescription as in (3.51). The formula (3.53) leads to an integral representation for the functions $G_p'^{+}(\zeta)$ if we use the well-known rules for averaging exponential operators in the bosonic Fock space:

$$(\mathbf{f}_{-p}, e^{i\ell_n\phi(\zeta_n)}\dots e^{i\ell_1\phi(\zeta_1)}\mathbf{f}_p)_0 = \prod_{k>m}(\zeta_k-\zeta_m)^{\ell_k\ell_m}\prod_k \zeta_k^{l_k^2/2-l_kp}\,\delta_{\ell_1+\dots+\ell_n,0}. \tag{3.54}$$

Then the function (3.53) can be rewritten in term of hypergeometric functions via the integral representation:

$$\int_C \frac{dz}{2\pi i}\, z^{c-1}(1-z^{-1})^{-a}(1-\zeta z)^{-b} = \frac{\Gamma(c+a)}{\Gamma(c+1)\Gamma(a)}\,F(a+c,b,c+1;\zeta), \quad \mathrm{Re}[c]>0 \tag{3.55}$$

(see Fig. 3). To find the function $G_p'^{-}$ we can use the relation

$$[G_p'^{+}(\zeta^{-1})]^\star = G_{-p}'^{-}(\zeta), \quad |\zeta|=1, \tag{3.56}$$

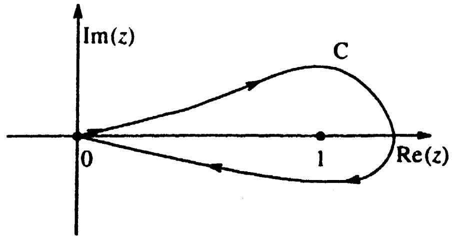

FIG. 3. Integration contour for the hypergeometric function.

which follows from the conjugation conditions (3.40). In this way we reproduce Eq. (3.45).

4. DEFORMED VIRASORO ALGEBRA WITH $c=-2$

In the previous section we have developed a bosonization procedure to study the representations of the ZF algebra. The important step there was the introduction of integral operators X, X' acting in the bosonic space $\oplus_{k\in\mathbb{Z}}\mathscr{F}_{k/2}$. In this space we introduced the action of the $c=-2$ Virasoro algebra by defining generators $T(z)=\Sigma L_n z^{-n-2}$ as composite operators constructed from Heisenberg generators b_n. The property that Virasoro algebra generators commute with X and X' allowed us to classify the representation space of ZF algebra by the action of the Symm algebra. Moreover, we pointed out that the operators $Z_\pm$ themselves can be constructed as linear combinations of the vertex operators Φ_{21} of the Virasoro algebra. We also determined another elementary chiral primary Φ_{12}, the commutation relations of which are defined by the constant **R**-matrix of IRF type.

Of course, representations of the algebra (3.1), (3.2) could be investigated directly in the fermionic language. However, the bosonization procedure can be applied in constructions where a fermionic description is lacking. One of the lessons we have learned above is that the ZF algebra, Virasoro algebra, algebras of chiral primary operators and algebras of screening operators are mutually related in the bosonic picture. We would like to examine further these interrelations, considering less trivial situations.

It will be convenient to start with bosonic space and the deformed integral operators X and X' rather than with ZF algebra. Indeed, the operators X and X' could be treated as completely specifying the construction. For instance, the universal enveloping algebra of the Virasoro algebra could be considered as a subalgebra of operators from the Heisenberg algebra commuting with X and X', while the operators Φ_{21}. Φ_{12} were specified with respect to the Virasoro algebra, etc. From this point of view, generalizations of the proposed construction are determined by appropriate deformations of the screening operators.

The well-known generalization of the proposed bosonic construction originates in the deformation of the operators X and X' in a way which leads to the Virasoro algebra with central charge $-2<c<1$. This direction will be discussed later.

There exists, however, another class of deformations[17] related to generalizations of the Virasoro algebra. These deformations will be called hereafter x-deformations. Again,

x-deformation is determined by the appropriate continuous deformation of the screening operators. In contrast with the Virasoro algebra cases, x-deformation leads to more general associative algebras of the deformed vertex operators Φ_{21}, Φ_{12}.

In the present section we want to elaborate the simplest example of x-deformation corresponding to $c=-2$ Virasoro algebra. In this particular instance the deformation has a remarkably simple form. In working out this toy model we try to extract information on the essential properties of the general x-deformation. Namely, we argue that the whole Fock space can be decomposed under the action of operators X and X' into direct sum of irreducible representations of the deformed Virasoro algebra proposed in Ref. 31. This algebra is determined as the subalgebra of operators from the universal enveloping algebra of the Heisenberg algebra which commute with X and X'. It should be clear that such commutants always generate an associative algebra. Indeed, the linear space of operators commuting with X and X' is closed under the operation of multiplication, while the associativity condition follows from the fact that $T(\zeta)$ are constructed from generators of the Heisenberg algebra. Hence, in the general case one can also expect that Fock space will be divided into direct a sum of representations of some infinite-dimensional algebra generalizing the Virasoro algebra.

Another purpose of this example is to demonstrate how one can find the deformation of the chiral primary operators Φ_{12} and Φ_{21}, or more explicitly, how to obtain the bosonic realization for such operators. We will show that the commutation relations of operators Φ_{12} are determined by an elliptic **R** matrix of IRF type. This justifies the proposed deformation.

In what follows we will use the same notations as before, pointing out only those definitions and formulae which contain significant distinctions. For instance, the symbol $\mathscr{L}$ will mean the irreducible representation of the deformed $c=-2$ Virasoro algebra, Φ_{12} will be the deformation of the correspondent chiral primary etc.

4.1. Deformation of the Virasoro algebra with $c=-2$

Let us start by considering of the bosonic space $\pi_Z = \oplus_{k\in Z}\mathscr{F}_{k-1/2}$ which occurred in Sec. 3. The x-deformation of the construction given in the previous section is determined by the following redefinition of the operator X:

$$X=\oint \frac{dz}{2\pi i z}\exp\{-i(\phi(zx)+\phi(zx^{-1}))\}, \qquad (4.1)$$

where the deformation parameter is a real number satisfying $0<x<1$. If the second operator X' remains unchanged, then it is not hard to see that formulae (3.10)–(3.12) are valid. Let us introduce new operators, the exact meaning of which will become transparent later:

$$Z_+(\zeta)=e^{i\phi(\zeta)}, \quad Z_-(\zeta)=[X,Z_+(\zeta)]. \qquad (4.2)$$

The integral in the definition of Z_- can be computed explicitly:

$$Z_-(\zeta)=\frac{e^{-i\phi(\zeta x^2)}-e^{-i\phi(\zeta x^{-2})}}{x-x^{-1}}. \qquad (4.3)$$

In accordance with the scheme of the previous section let us first describe the structure of the subspaces $\mathrm{Ker}_{\mathscr{F}^+_{l-1/2}}[X^l]$ in Fock space. To do this one must find operators which generalize (3.14). It can easily be verified that the action of the following operators in the Fock space π_Z commute with X and X':

$$-\zeta^2T^\nu(\zeta) =\frac{Z_+(\zeta x^{-2\nu-1})Z_-(\zeta x^{2\nu+1})-Z_-(\zeta x^{-2\nu-1})Z_+(\zeta x^{2\nu+1})}{2(x+x^{-1})}, \qquad (4.4)$$

where $\nu\in\mathbf{Z}$. Note that

$$T^\nu(\zeta)\equiv T^{-\nu-1}(\zeta).$$

In the limit $x\to1$ the operators $T^\nu(\zeta)$ become the Virasoro algebra generators given by formula (3.15). Straightforward calculation confirms that modes of the Laurent expansion $T^\nu(\zeta)=\Sigma_{n\in Z}L^\nu_n\zeta^{-n-2}$ generate the so-called deformed Virasoro algebra.[31]

$$\begin{aligned}2[L^\nu_n,L^\mu_m]=&[(\mu+1)n-(\nu+1)m]_{x^2}\,L^{\nu+\mu+1}_{n+m}+[(\mu+1)\\ &\times n+\nu m]_{x^2}\,L^{\mu-\nu}_{m+n}-[\mu n+(\nu+1)m]_{x^2}\\ &\times L^{\nu-\mu}_{n+m}-[\mu n-\nu m]_{x^2}\,L^{\nu+\mu}_{n+m}\\ &-\frac{\delta_{m+n,0}}{4(x^2-x^{-2})^2}\,(C^{\nu,\mu}_n+C^{\nu,-1-\mu}_n\\ &+C^{-1-\nu,\mu}_n+C^{-1-\nu,-1-\mu}_n),\end{aligned} \qquad (4.5)$$

where

$$C^{\nu,\mu}_n=\frac{[n(\mu+\nu)]_{x^2}}{[\mu+\nu]_{x^2}}-\frac{[n(\nu-\mu)]_{x^2}}{[\nu-\mu]_{x^2}}.$$

We denote this algebra by $\mathrm{Vir}_{-2,x}$. Let us adjoin to the algebra (4.5) a derivation D with property $[D,L^\nu_n]=nL^\nu_n$ for any $\nu\in\mathbf{Z}$. In bosonic Fock spaces this operator can be realized as $D=\Sigma_{m>0}b_{-m}b_m+P^2/2-1/8$. It provides the universal enveloping algebra of $\mathrm{Vir}_{-2,x}$ with structure of a Z-graded algebra. Then the triangular decomposition of $\mathrm{Vir}_{-2,x}$ is a decomposition into elements of positive $(L^\nu_n,n<0)$, zero (D,L^ν_0) and negative $(L^\nu_n,n<0)$ degrees for any $\nu\in\mathbf{Z}$. By definition, the Verma module of the deformed Virasoro algebra is a Z-graded module generated by the operators $L^\nu_n,n<0$ acting on the unique highest weight vector $\mathbf{v}$. The Verma module inherits the structure of the Z-graded space from the grading of the universal enveloping algebra. The vector $\mathbf{v}$ is the highest weight vector of the Verma module with weight Δ if it is annihilated by any operator $L^\nu_n,n>0$, and when elements with zero grading act on it, it yields $L^\nu_0 v=\Delta^\nu\mathbf{v}$; $D\mathbf{v}=\Delta\mathbf{v}$. In the general situation the Verma module is degenerate, i.e., it contains invariant subspaces created by the action of the operators L^ν_n, $n<0$, on null vectors. Null vectors, by definition, obey the equation $L^\nu_n\mathbf{v}^0=0$, $n>0$; $D\mathbf{v}^0=(\Delta+N)\mathbf{v}^0$ for some integer N (the degeneration level).

In order to obtain an irreducible representation, we have to set all null-vectors together with the whole subspace generated by it equal to zero; that is we have to factorize the Verma module over all invariant submodules.

Although the operator X and its commutants in Fock space have been deformed, the description of the structure of the Fock space $\oplus_k \mathscr{F}^+_{k-1/2}$ in terms of irreducible $\mathrm{Vir}_{-2,x}$ modules is almost the same as before. Indeed, let the generators L^ν_n be realized through (4.4). Then one can demonstrate that the highest vectors $\mathbf{f}_{l-1/2}$ of the Fock module are the highest vectors of the Verma modules of $\mathrm{Vir}_{-2,x}$. It is possible to show that when generators L^ν_n act on the highest vectors of Fock space $\mathbf{f}_{l-1/2}\in\mathscr{F}^+_{l-1/2}$ $l=1, 2,\ldots$, they create subspaces $\mathrm{Ker}_{\mathscr{F}^+_{l-1/2}}[X^l]$ which are isomorphic to the irreducible $\mathrm{Vir}_{-2,x}$ modules $\mathscr{L}_{l-1/2}$, respectively. Note, that the character of the irreducible module of $\mathrm{Vir}_{-2,x}$ coincides with that of the $c=-2$ Virasoro algebra:

$$\mathrm{Tr}_{\mathscr{L}_{l-1/2}}[q^{D-c/24}]=q^{1/12}\frac{q^{l(l-1)/2}-q^{l(l+1)/2}}{\Pi^\infty_{n=1}(1-q^n)}. \quad (4.6)$$

The numbers of the states on a given level will be conserved in a general x-deformation too. We recall that a similar situation occurs in the representation theory of quantum groups,[26–28] where the characters of deformed irreducible modules remain the same as the undeformed ones if the deformation parameter is not a primitive root of unity.

The spaces $\mathscr{F}^+_{l-1/2}$ and $\mathscr{F}^\star_{l+1/2}$ can be decomposed into a direct sum of $\mathrm{Vir}_{-2,x}$ modules analogously to (3.23) and (3.20), where, we should, of course, bear in mind that the symbol $\mathscr{L}_{l-1/2}$ now means the irreducible module of the deformed Virasoro algebra.

Now the whole algebra Symm acting on the space $\pi^{\mathrm{R}}_Z\oplus\pi^{\mathrm{R}\star}_Z$ is the tensor product of the infinite-dimensional deformed Virasoro algebra with $c=-2$, $0<x<1$ and a finite-dimensional part generated by the operators X, $X^\star$, X', $X'^\star$ and P with commutation relations (3.24)–(3.27). For this reason, the decomposition of the space $\pi^{\mathrm{R}}_Z\oplus\pi^{\mathrm{R}\star}_Z$ into a direct sum of irreducible representation of the algebra Symm has the form (3.28). The scalar product in the space $\oplus_{k\in\mathbf{Z}}\mathscr{F}_{k-1/2}$ is given by (3.29) and the conjugation conditions are $(\mathbf{v}_1 L^\nu_n,\mathbf{v}_2)=(\mathbf{v}_1 L^\nu_{-n},\mathbf{v}_2)$. Moreover, in calculations one can use the scalar product (3.21). The arguments here are practically the same as in Sec. 3.6.

In the NS sector the finite-dimensional part of the Symm algebra is still given by the $sl(2)$ algebra and as in the undeformed situation we have the isomorphism $\mathrm{Ker}_{\mathscr{F}_l}X^l\cong\mathscr{L}_l$. In addition, the decomposition of Fock spaces into a direct sum of irreducible modules of the deformed Virasoro algebra is described by the Eq. (3.33) again. The scalar product in the space π^{NS}_Z is the same as (3.29), (3.30), and the space π^{NS}_Z can be treated as selfdual, $\pi^{\mathrm{NS}}_Z=\pi^{\mathrm{NS}\star}_Z$.

4.2. Deformation of vertex operators

Now we want to establish the x-deformation of chiral primary operators $\Phi^\pm_{21}$ and $\Phi^\pm_{12}$. The construction of the first operator $\Phi^\pm_{21}$ is rather evident. Indeed, it can be carried out in the Fock spaces in the same manner as (3.50):

$$\Phi^+_{21}(\zeta)\mathbf{v}=e^{i\phi(\zeta)}\mathbf{v},\quad \mathbf{v}\in\mathscr{L}_p,$$
$$\Phi^{\star+}_{21}(\zeta x^2)\mathbf{v}^\star=\int_C\frac{dz}{2\pi iz}\exp\{-i[\phi(zx)+\phi(zx^{-1})]\}\exp[i\phi(\zeta)]\mathbf{v}^\star,\quad \mathbf{v}^\star\in\mathscr{L}^\star_p, \quad (4.7)$$

where we explicitly show the states on which such operators are well-defined. As before, we take contours in the integral $\Phi^{\star+}_{21}(\zeta)$ beginning and ending at the origin, and enclose all singularities whose positions are determined by the vector $\mathbf{v}^\star$. Defining the conjugation properties of $\Phi^\pm_{21}(\zeta)$ for $|\zeta|=1$ as

$$(\mathbf{u}^\star\Phi^{\star\pm}_{21}(\zeta x^2),\mathbf{v})=(\mathbf{u}^\star,\Phi^\mp_{21}(\zeta)\mathbf{v}), \quad (4.8)$$

where $\mathbf{u}^\star\in\mathscr{L}^\star_p$ and $\mathbf{v}\in\mathscr{L}_{p\pm1}$, one can easily obtain any matrix element of such operators. The bosonic realization (4.7) possesses to find that the commutation relations of these vertex operators with the deformed Virasoro algebra are given by the formula

$$2\zeta^{-n}[L^\nu_n,\Phi^\pm_{21}]=\frac{x^{2\nu+3}\Phi^\pm_{21}(\zeta x^{4\nu+4})-x^{-2\nu-3}\Phi^\pm_{21}(\zeta x^{-4\nu-4})}{x^2-x^{-2}}-\frac{x^{2\nu-1}\Phi^\pm_{21}(\zeta x^{4\nu})-x^{-2\nu+1}\Phi^\pm_{21}(\zeta x^{-4\nu})}{x^2-x^{-2}}. \quad (4.9)$$

To define the deformed chiral primary operators Φ_{12} (3.39), we recall that the crucial property in proving proposition 3.3 was the equation (3.52). We require that this basic relation be preserved in the deformed theory in the following sense. The second chiral primary has to be constructed in terms of a new field ϕ' such that $\lim_{x\to1}\phi'(z)=-(1/2)\phi(z)$ and

$$X_{|\pi_{\mathrm{NS}}}e^{i\phi'(\zeta)}=-e^{i\phi'(\zeta)}X_{|\pi_{\mathrm{R}}}. \quad (4.10)$$

Let ϕ' be constructed from generators P, Q, b'_n as:

$$\phi'(\zeta)=-\frac{1}{2}(Q-i\,P\ln\zeta)-\sum_{\substack{m\in\mathbf{Z}\\ m\neq0}}\frac{b'_m}{i\,m}\zeta^{-m}, \quad (4.11)$$

then condition (4.10) is obviously satisfied if the new creation-annihilation operators are

$$b'_n=(x^n+x^{-n})^{-1}\,b_n. \quad (4.12)$$

Notice, that, in consequence of this definition the formula expressing X' in terms of the field $\phi'(\zeta)$ has the form

$$X'=\oint\frac{dz}{2\pi iz}\exp\{-i[\phi'(zx)+\phi'(zx^{-1})]\}. \quad (4.13)$$

Comparing this with Eq. (4.1) one sees that the integral operators X, X' are constructed from ϕ and ϕ' respectively in a remarkably symmetrical form. We will see that this property allows direct generalization for an arbitrary x-deformation with $0<x<1$. Now the chiral primary operators intertwining the R and NS sectors can be realized as

$$\Phi_{12}^{*-}(x^2\zeta)\mathbf{v}^* = e^{i\phi'(\zeta)}\mathbf{v}^*, \quad \mathbf{v}^* \in \mathscr{L}_p^*,$$

$$\Phi_{12}^-(\zeta)\mathbf{v} = \int_C \frac{dz}{2\pi i z} \exp\{-i[\phi'(zx) + \phi'(zx^{-1})]\}\exp[i\phi'(\zeta)]\mathbf{v}, \quad \mathbf{v} \in \mathscr{L}_p. \tag{4.14}$$

The integral in $\Phi_{12}^-(\zeta)$ is well defined if the contour is chosen as in (3.51). These bosonic prescriptions together with the conjugation condition

$$(\mathbf{u}^*\Phi_{12}^{*\pm}(\zeta x^2), \mathbf{v}) = (\mathbf{u}^*, \Phi_{12}^{\mp}(\zeta)\mathbf{v}), \tag{4.15}$$

where $|\zeta|=1$, completely specify the action of the operators $\Phi_{12}^{\pm}(\zeta)$ in Fock space. The matrix elements of products of such operators can be derived by the standard bosonization technique. We leave the explicit calculations till Sec. 6, where it will be worked out as particular case $\xi=1$ in the context of the general x-deformation. So let us just note, that by knowing the matrix elements one can obtain the commutation relations of these operators in the usual way. The essential difference between this case and undeformed one is that these commutation relations are determined by an elliptic **R** matrix of IRF type rather than constant **R**-matrix:

$$\Phi_{12}^a(\zeta_1)\Phi_{12}^b(\zeta_2)|\mathscr{L}_p = \sum_{c+d=a+b} \mathbf{W}'\begin{bmatrix} p+(a+b)/2 & p+c/2 \\ p+b/2 & p \end{bmatrix}\left.\frac{\zeta_1}{\zeta_2}\right] \times \Phi_{12}^d(\zeta_2)\Phi_{12}^c(\zeta_1)|\mathscr{L}_p. \tag{4.16}$$

The nontrivial elements of the matrix $\mathbf{W}'$ are:

$$\mathbf{W}'\begin{bmatrix} p\mp 1 & p\mp 1/2 \\ p\mp 1/2 & p \end{bmatrix}\left.\zeta\right] = r(\zeta),$$

$$\mathbf{W}'\begin{bmatrix} p & p\mp 1/2 \\ p\mp 1/2 & p \end{bmatrix}\left.\zeta\right] = r(\zeta)\zeta^{\pm p - 1/2} \times \frac{\Theta_{x^4}(x^2)\Theta_{x^4}(x^{\pm 4p}\zeta)}{\Theta_{x^4}(x^2\zeta)\Theta_{x^4}(x^{\pm 4p})}, \tag{4.17}$$

$$\mathbf{W}'\begin{bmatrix} p & p\pm 1/2 \\ p\mp 1/2 & p \end{bmatrix}\left.\zeta\right] = -r(\zeta)x^{\pm 2p}\zeta^{-1/2} \times \frac{\Theta_{x^4}(x^{2(\pm 2p+1)})\Theta_{x^4}(\zeta)}{\Theta_{x^4}(x^{\pm 4p})\Theta_{x^4}(x^2\zeta)},$$

where

$$r(\zeta) = \zeta^{1/4}\exp\left[\sum_{m=1}^{+\infty} \frac{\zeta^m - \zeta^{-m}}{m(x^m + x^{-m})^2}\right].$$

The associativity of the algebra (4.16) follows from the fact that the matrix $\mathbf{W}'$ is a solution of the Yang–Baxter equation. This justifies the proposed deformation of the chiral primary operators Φ_{12}.

4.3. The x-deformation of fermions ZF algebra

Thus far we have been interested in the vertex operators of the $\mathrm{Vir}_{-2,x}$ algebra. To construct the vertex operators of Symm algebra, we must take into account the remaining finite-dimensional part of the symmetry algebra.

Let $\mathbf{e}_l^m$ be basis vectors of an irreducible spin $j=(l-1)/2$ representation $\mathscr{V}_l$ of the algebra $sl(2)$, with $\mathbf{v}\in\mathscr{L}_l$. Define the operators $Z_\pm(\zeta)$ acting on the spaces π_Z^{R} and π_Z^{NS} by Eq. (3.35). Straightforward computation shows that the operators $Z_\pm(\zeta)$ are generators of a ZF algebra given by the commutation relations

$$Z_a(\zeta_1)Z_b(\zeta_2) = -Z_b(\zeta_2)Z_a(\zeta_1), \quad \zeta_1 \neq \zeta_2 \tag{4.18}$$

and the operator product expansion

$$Z_\pm(\zeta_2)Z_\mp(\zeta_1) = \pm\frac{(x+x^{-1})\zeta_1\zeta_2}{(\zeta_2-\zeta_1 x^2)(\zeta_2-\zeta_1 x^{-2})} + O(1),$$
$$Z_\pm(\zeta_2)Z_\pm(\zeta_1) = O(1). \tag{4.19}$$

It is quite evident now that the operators (4.2) introduced in the beginning of this section are just the generators of ZF algebra (4.18), (4.19).

5. NONDEFORMED VIRASORO ALGEBRA WITH $c<1$

In the present section we recall how to construct representations of a ZF algebra (more explicitly, a pair of ZF algebras) with constant **R** matrix corresponding to the quantum group $U_q(sl(2))$. As before, the representation space of the ZF algebra can be realized as a direct sum of irreducible representations of the Virasoro algebra with central charge $c<1$. The operators of the ZF algebras will be expressed in terms of vertex operators of the Virasoro algebra Vir_c. Thus, the main objects of our investigation will be the irreducible representations of the Virasoro algebra and vertex operator algebra. To describe these objects we will use the bosonization method, examples of which were presented in the previous sections.

In the case under consideration the explicit realization of the Virasoro algebra generators L_n in the bosonic Fock space $\mathscr{F}_p$ is well known.[2] Then the universal enveloping algebra $U(\mathrm{Vir}_c)$ turn to be subalgebra in the universal enveloping algebra of the Heisenberg algebra. Moreover, any highest-weight vector in the Fock module will be the highest-weight vector of Verma module of Vir_c. The original motivation for bosonization of the Virasoro algebra[9] is that in Fock space one can explicitly construct the intertwining operators between Verma modules. In the physical literature these operators have been historically called screening operators.[2] According to the definition, the intertwining operators commute with any element of $U(\mathrm{Vir}_c)$, hence it has to map singular vectors of a Verma module into singular ones. So the analysis of the structure of the reducible Verma modules becomes very simple. Indeed, to determine the singular vectors one needs to find such vectors in the Fock space which are mapped by the action of screening operators into the highest weight vectors, or such vectors which can be obtained from the highest weight vectors by the action of screening operators. For this reason, knowledge of all possible intertwining operators is equivalent to the knowledge of the invariant subspaces in the Verma modules. This subject has been discussed extensively in the literature. It is well known that the structure of the Verma modules of Vir_c depends critically on the arithmetical properties of the real number c. Having in the mind x-deformation, we will omit the complicated cases of completely reducible Verma modules, just considering the general case, i.e., the case 2 in the Feigin–Fuks classification.[9] Our task is to recall the essential features which occur in the bosonization of representations of the Virasoro algebra and ZF algebra. (The analysis of more complicated cases can be found in Refs. 16 and 32.) In particular, we would like to clear up the idea that knowledge of the intertwining operators between the Verma modules of Vir_c is sufficient to describe the irreducible representations of the Virasoro algebra and the algebra of the vertex operators without appealing to the commutation relations of the generators L_n. Another very important point we recall and constantly use in the construction is the existence of the remarkable discrete symmetry, which in our notations is just a replacement $\xi\leftrightarrow -1-\xi$ ($\alpha_-\leftrightarrow\alpha_+$ in that of the Dotsenko–Fateev work[2]). This symmetry, the origin of which is still hardly understood, seems to be essential not only in CFT[30] but also in the general theory of integrable models.[17,18]

Our present treatment is based on a slightly different point of view on bosonization (see also Secs. 3 and 4). Namely, we consider the screening operators as the basic objects which completely determine the whole construction. In particular, the Virasoro algebra can be treated as an algebra of generators in the Fock space which commute with the corresponding screening operators. In this approach we can investigate the irreducible representations of the Virasoro algebra independently on what basis in $U(\mathrm{Vir}_c)$ or what commutation relations are really involved. A similar idea was applied in the development of the theory of W-algebra.[33] Hence, the most important step is the introduction of correctly defined screening operators. These operators have a remarkably simple form. They are given by powers of the operators X and X' which are deformations of (3.9). We explicitly list the subspaces in the Fock space where these integral operators are well defined. As soon as we determine the screening operators, we are able to describe the irreducible representations of the Virasoro algebra as submodules (or factor modules) of the Fock modules. Our next step is the definition of the chiral vertex operators of the Virasoro algebra. To do this, we must specify a basis L_n in $U(\mathrm{Vir}_c)$ since, in general, knowledge of the intertwining operators is not enough to uniquely determine vertex operators. Our task is to extract the necessary properties of these operators which are determined by screening operators rather than by the choice of the basis L_n. Defining the bosonization of the chiral primaries, we demonstrate that matrix elements of these operators can easily be computed using the Wick theorem. Study of the analytical properties of four-point functions shows that the commutation relations of the chiral primary operators Φ_{12}, Φ_{21} are determined by two different constant **W** matrices, which are solutions of the Yang–Baxter equation of IRF type. To construct the ZF algebras of "vertex type"[8] one needs take into account the multiplicities of the irreducible representations of the Virasoro algebra. We argue that there are two ZF algebras associated with the algebras of the chiral primaries Φ_{12} and Φ_{21} respectively. The **R** matrices of these ZF algebras correspond to the **R** matrices of the quantum groups $U_p(sl(2))$ with different deformation parameters p.[30] These **R** matrices are connected by the transformation $\xi\leftrightarrow -1-\xi$ which arises in the symmetry between Φ_{12} and Φ_{21} (or $\phi'\leftrightarrow\phi$). The irreducible representation of a pair of ZF algebras coincides with the direct sum of Fock modules, so it admits classification by the representations of the Virasoro algebra.

Bosonization is just a useful method to study the representations of the Virasoro and ZF algebras. Of course, the results do not depend on it. In particular, without any bosonization the irreducible representations of a pair of $[\xi\leftrightarrow -1-\xi]$-symmetrical ZF algebras are isomorphic to a direct sum of irreducible representations of the Virasoro algebra.

This construction can be regarded as a case $x=1$ of the general two-parameter deformation with parameters x and ξ. In the following section we will generalize the main statements for a $x\neq 1$ deformation.

3.1. Let us introduce the free bosonic field

$$\phi(z)=\sqrt{\frac{\xi+1}{2\xi}}\,(Q-i\,P\,\ln z)+\sum_{\substack{m\in\mathbf{Z}\\ m\neq 0}}\frac{b_m}{i\,m}\,z^{-m},\qquad(5.1)$$

where the commutation relation of the null modes P, Q is defined by $[Q,P]=i$, while the determining relation for the modes b_n is

$$[b_m,b_n]=\frac{\xi+1}{2\xi}\,m\,\delta_{m+n,0}.\qquad(5.2)$$

Next, we will consider cases with deformation parameter $\xi>1$. Moreover, in order to avoid additional complications we assume that this number is irrational. Note that the case $\xi=1$ corresponds to the example discussed in Sec. 3. Together with the field ϕ, it is convenient to define another one:

$$\phi'(z)=-\sqrt{\frac{\xi}{2(\xi+1)}}\,(Q-i\,P\,\ln z)-\sum_{\substack{m\in\mathbf{Z}\\ m\neq 0}}\frac{b'_m}{i\,m}\,z^{-m},\qquad(5.3)$$

where

$$\xi\,b_m=(\xi+1)b'_m.$$

As we will see below, there is a remarkable symmetry with respect to the transformation $\xi\leftrightarrow -1-\xi$. This is the reason why we introduce independent notation for field $\phi'(z)$. In the case $x=1$ the fields $\phi'(z)$ and $\phi(z)$ are just proportional. However, in the x-deformed construction their connection turns to be more complicated.

In the set of Fock modules

$$\mathscr{F}_{k,k'}\equiv\mathscr{F}_{[(\xi+1)k-\xi k']/\sqrt{2\xi(\xi+1)}},\qquad k,k'\in\mathbf{Z}\qquad(5.4)$$

one might introduce the following formal operators:

$$X^{l}=\int_{C_1}\cdots\int_{C_l}\frac{dz_1}{2\pi i z_1}\cdots\frac{dz_l}{2\pi i z_l}\,e^{-2i\phi(z_1)}\cdots e^{-2i\phi(z_l)},$$

$$X'^{l'}=\int_{C_1}\cdots\int_{C_{l'}}\frac{dz_1}{2\pi i z_1}\cdots\frac{dz'_l}{2\pi i z_{l'}}\,e^{-2i\phi'(z_1)}\cdots e^{-2i\phi'(z'_l)}.\qquad(5.5)$$

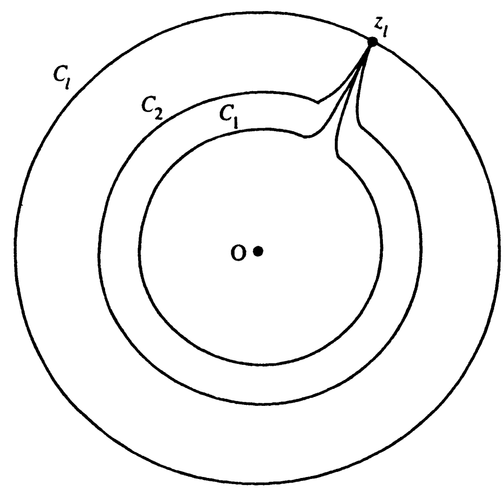

FIG. 4. The integration contours.

Here we have used the Felder prescription[16] for the integration contours (see Fig. 4). Namely, any contour C_i, $1\leqslant i\leqslant l-1$, has begins and ends at a point z_l chosen on the unit circle. It encloses the origin and all singularities depending on the variables $z_k (k<i)$. The last integration over the variable z_l is performed along the circle C_l. In essential contrast with the case $c=-2$, the action of these operators is ill defined in the whole set of Fock modules, since the last integration contour $C_l(C_{l'})$ in the definitions (5.5) is not closed. Hence, first of all, we must specify where the operators (5.5) act.

Proposition 5.1. The action of the operator $X^l(X'^{l'})$ is defined only on the Fock modules $\mathscr{F}_{l,k}$ $(\mathscr{F}_{k,l'})$, $k\in\mathbf{Z}$ and l, $l'>0$. Then

$$\begin{aligned}
&\mathrm{Ker}_{\mathscr{F}_{l,l'}}[X^l]=\mathrm{Ker}_{\mathscr{F}_{l,l'}}[X'^{l'}],\\
&\mathrm{Ker}_{\mathscr{F}_{l,-l'}}[X^l]=\mathrm{Ker}_{\mathscr{F}_{l,-l'}}[X'^{l'}]=0,\\
&\mathrm{Im}_{\mathscr{F}_{l,l'}}[X^l]=\mathscr{F}_{-l,l'},\\
&\mathrm{Im}_{\mathscr{F}_{l,l'}}[X'^{l'}]=\mathscr{F}_{l,-l'},\\
&\mathrm{Im}_{\mathscr{F}_{l,-l'}}[X^l]=\mathrm{Im}_{\mathscr{F}_{-l,l'}}[X'^{l'}]\subset\mathscr{F}_{-l,-l'}.
\end{aligned}\tag{5.6}$$

We will call the integral operators (5.5) defined on the corresponding spaces screening operators. Considering the screening operators as basic objects, one can define Virasoro algebra as follows:

Definition. Let the operators P, Q, b_n, $n\in\mathbf{Z}$ satisfy the commutation relations (5.2) and screening operators be given by (5.5). Virasoro algebra is a subalgebra in the universal enveloping algebra of the Heisenberg algebra of operators b_n and P. An element from the universal enveloping of Heisenberg algebra belongs to Virasoro algebra if it commutes with the action of the screening operators.

In the present case one can simply write down explicit expressions for the basic generators of the space of invariants in the universal enveloping of the Heisenberg algebra:

$$\begin{aligned}
&L_n=\sum_{k+m=n} b_m b'_k-(\sqrt{2\xi(\xi+1)}P+n)(b_n-b'_n), \quad n\neq 0,\\
&L_0=\sum_{m>0}(b_{-m}b'_m+b'_{-m}b_m)+\frac{P^2}{2}-\frac{1}{4\xi(\xi+1)}.
\end{aligned}\tag{5.7}$$

The L_n obey Virasoro commutation relations with the central charge $c=1-6/\xi(\xi+1)$. So, the Fock space is given the structure of the Virasoro module. The highest Fock state $\mathbf{f}_{k,k'}\equiv\mathbf{f}_{|(\xi+1)k-\xi k'|/\sqrt{2\xi(\xi+1)}}$ will also be the highest weight vector of the Verma module of Vir with the conformal dimension

$$\Delta_{k,k'}=\frac{((\xi+1)k-\xi k')^2-1}{4\xi(\xi+1)}.\tag{5.8}$$

The most important property of the operators (5.5) is that they commute with generators of the form (5.7); i.e., they are intertwining operators between Verma modules of the Virasoro algebra. Knowing intertwining operators one might study the structure of the Verma modules $\mathscr{M}_{k,k'}$ of Vir_c and construct the irreducible representations $\mathscr{L}_{k,k'}$ as subspaces or factor spaces of Fock spaces. Thus, introduce the following notations

$$\mathscr{L}_{k,k'}=\begin{cases}\mathrm{Ker}_{\mathscr{F}_{k,k'}}[X^k], & \text{if } k,k'\geqslant 1;\\ \mathscr{F}_{k,k'}/\mathrm{Im}_{\mathscr{F}_{-k,k'}}[X^{-k}], & \text{if } k,k'\leqslant -1;\\ \mathscr{F}_{k,k'}, & \text{otherwise.}\end{cases}\tag{5.9}$$

We claim that the following proposition holds:

Proposition 5.2. Let the generators of the Virasoro algebra are given by Eqs. (5.7) and the parameter $\xi>1$ be an irrational number. Then 1. The space $\mathscr{L}_{k,k'}$ for any integer numbers k, k' is an irreducible representation of the Virasoro algebra with central charge $c=1-6/\xi(\xi+1)$. The highest weight vector of $\mathscr{L}_{k,k'}$ coincides with the vector $\mathbf{f}_{k,k'}$ and has the conformal dimensions (5.8). 2. For $kk'\leqslant 0$ or k, $k'<0$, the Verma module $\mathscr{M}_{k,k'}$ constructed from the highest weight vector $\mathbf{f}_{k,k'}$ coincides with the total Fock module $\mathscr{F}_{k,k'}$.

We would like now to comment propositions 5.1, 5.2.[32] As we have noted, any highest weight vector $\mathbf{f}_{k,k'}$ of the Fock module $\mathscr{F}_{k,k'}$ is found to be the highest weight vector of the Verma module $\mathscr{M}_{k,k'}$ of the Virasoro algebra. Generically speaking, Fock space $\mathscr{F}_{k,k'}$ does not coincide neither with the Verma module $\mathscr{M}_{k,k'}$, nor with the irreducible module $\mathscr{L}_{k,k'}$ of Vir_c. It might contain some subspaces which are invariant with respect to the action of generators (5.7). Since the screening operators (5.5) commute with any generator from the universal enveloping of Vir_c then they have to map an invariant subspace of the Vir into invariant one. In the case under consideration the structure of the embedding of the Verma modules given by the action of screening operators is rather simple:

(i) Consider first the Fock module $\mathscr{F}_{l,l'}$, l, $l'>0$. According to proposition 5.1 this module contains a vector $\mathbf{f}^0_{l,l'}$ such that $X^l\mathbf{f}^0_{l,l'}=\mathbf{f}_{-l,l'}(X'^{l'}\mathbf{f}^0_{l,l'}=\mathbf{f}_{l,-l'})$. This vector cannot be obtained by the action of the Virasoro algebra generators on the highest weight vector $\mathbf{f}_{l,l'}$ and the Fock submodule constructed from the state $\mathbf{f}^0_{l,l'}$ turns to be invari-

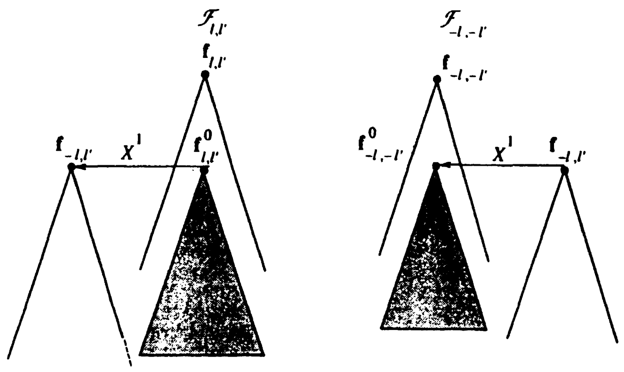

FIG. 5. The structure of spaces $\mathcal{F}_{l,l'}$ and $\mathcal{F}_{-l,-l'}$.

ant space under the action of the generators of the Virasoro algebra. Indeed, if $\mathbf{f}^0_{l,l'}$ is produced by the action of any element from the universal enveloping of Vir_c on the vector $\mathbf{f}_{l,l'}$, then the operator $X^l(X'^{l'})$ would map it into zero rather than into $\mathbf{f}_{-l,l'}(\mathbf{f}_{l,-l'})$, since $X^l(X'^{l'})$ commutes with any generator of Vir_c and $X^l\mathbf{f}_{l,l'}=X'^{l'}\ \mathbf{f}_{l,l'}=0$. Proposition 5.2 means that the subspace $\mathrm{Ker}_{\mathcal{F}_{l,l'}}[X^l] = \mathrm{Ker}_{\mathcal{F}_{l,l'}}[X'^{l'}]$ is an irreducible representation $\mathcal{L}_{l,l'}$ of the Virasoro algebra. The modules $\mathcal{F}_{-l,l'} = \mathrm{Im}_{\mathcal{F}_{l,l'}} X^l$ and $\mathcal{F}_{l,-l'} = \mathrm{Im}_{\mathcal{F}_{l,l'}} X'^{l'}$ coincide with the Verma modules Vir_c and do not have any invariant subspaces. Hence it can be identified with $\mathcal{L}_{l,-l'}$ and $\mathcal{L}_{-l,l'}$ correspondingly.

(ii) The Fock space $\mathcal{F}_{-l,-l'}$ is a reducible Verma module of Vir_c with unique null-vector $\mathbf{f}^0_{-l,-l'}$ which is the image of the highest weight vector $\mathbf{f}_{l,-l'}(\mathbf{f}_{l,-l'})$ under the action of the operator $X^l(X'^{l'})$:

$$\mathbf{f}^0_{-l,-l'}=X^l\mathbf{f}_{l,-l'}=X'^{l'}\mathbf{f}_{-l,l'}\in\mathcal{F}_{-l,-l'}.$$

The irreducible representation $\mathcal{L}_{-l,-l'}$ of Vir turn to be isomorphic to factor space $\mathcal{F}_{-l,-l'}/\mathrm{Im}_{\mathcal{F}_{l,-l'}}[X^l] = \mathcal{F}_{-l,-l'}/\mathrm{Im}_{\mathcal{F}_{-l,l'}}[X'^{l'}]$ of Fock space (see Fig. 5).

The most important objects for us are the sets of irreducible representations $\{\mathcal{L}_{l,l'}|l,l'>0\}$ and $\{\mathcal{L}^*_{l,l'}|l,l'>0\}$ where, by definition, $\mathcal{L}^*_{l,l'} = \mathcal{L}_{-l,-l'}$. These spaces can be endowed by the scalar product through the procedure which was explained in the section 3 on the example of Virasoro algebra with central charge $c=-2$. Namely, we introduce dual field ϕ^* and repeat analysis above for dual representations. Then we identify dual modules of Virasoro algebra with subspaces (or factor spaces) of Fock space by demanding the condition $T^*(\zeta)=T(\zeta)$. Again, choosing the field ϕ as basic field and determining dual field via Riccati type equation, we destroy the symmetry between ϕ and ϕ^*. For instance, Fock submodule $\mathcal{L}^*_{l,l'} = \mathrm{Ker}_{\mathcal{F}^*_{l,l'}}[X^{*l}]$ will be identified with factor module $\mathcal{L}_{-l,-l'} = \mathcal{F}_{-l,-l'}/\mathrm{Im}_{\mathcal{F}_{l,-l'}}[X^l]$ etc. Thus, we have:

Proposition 5.3. Let irreducible representation $\mathcal{L}_{l,l'}$ and dual Verma module $\mathcal{M}^*_{l,l'}$ of Virasoro algebra are realized as bosonic modules $\mathrm{Ker}_{\mathcal{F}_{l,l'}}[X^l]$ and $\mathcal{F}_{-l,-l'}$ correspondingly, where $l=1, 2,...$ Then the scalar product $(,)_0$ (3.21) restricted on the vectors from $\mathcal{L}_{l,l'}$ and $\mathcal{M}^*_{l,l'}$ coincides with the following:

$$(\mathbf{u}L_n,\mathbf{v})=(\mathbf{u},L_{-n}\mathbf{v}). \tag{5.10}$$

Let us explain the meaning of this statement. Consider at first vector $\mathbf{u}\in \mathrm{Ker}_{\mathcal{F}_{l,l'}}[X^l]\cong\mathcal{L}_{l,l'}$. Then due to both scalar products $(,)_0$ and (5.10) $\mathbf{u}$ has to be orthogonal to any vector $\mathbf{v}\notin\mathcal{F}_{-l,-l'}$. Let now $\mathbf{v}\in\mathcal{F}_{-l,-l'}$. We know from the proposition 5.2 that $\mathcal{F}_{-l,-l'}\cong\mathcal{M}^*_{l,l'}$. It is clear that taking $\mathbf{v}\in\mathrm{Im}_{\mathcal{F}_{l,-l'}}[X^l]$ we would find that this vector is orthogonal to any vector from $\mathrm{Ker}_{\mathcal{F}_{l,l'}}[X^l]\cong\mathcal{L}_{l,l'}$. The reason for this is that subspace $\mathrm{Im}_{\mathcal{F}_{l,-l'}}[X^l]$ is generated by null vector in the $\mathcal{M}^*_{l,l'}$ which is orthogonal to $\mathcal{L}_{l,l'}$. So, the only vectors which have non-trivial scalar product are $\mathbf{u}\in\mathrm{Ker}_{\mathcal{F}_{l,l'}}[X^l]$ and $\mathbf{v}\in\mathcal{F}_{-l,-l'}/\mathrm{Im}\,\mathcal{F}_{l,-l'}[X^l]\cong\mathcal{L}^*_{l,l'}$. But scalar products (3.21) and (5.10) are the same in these spaces. It should be clear from explicit bosonic realization of generators L_n (5.7).

Now let us turn to consideration of the vertex operator algebra acting on the set of irreducible representations of Vir_c. It is easy to see that this algebra is generated by the following operators:

$$\Phi^{\pm}_{21}(\zeta)\ :\ \mathcal{L}_{l,l'}\xrightarrow{\Phi^{\pm}_{21}}\mathcal{L}_{l\pm1,l'}\otimes\mathbf{C}[\zeta]\zeta^{\Delta_{l\pm1,l'}-\Delta_{l,l'}}, \tag{5.11}$$

$$\Phi^{\pm}_{12}(\zeta)\ :\ \mathcal{L}_{l,l'}\xrightarrow{\Phi^{\pm}_{12}}\mathcal{L}_{l,l'\pm1}\otimes\mathbf{C}[\zeta]\zeta^{\Delta_{l,l'\pm1}-\Delta_{l,l'}}. \tag{5.12}$$

The commutation relations of these operators with the Virasoro generators are given by Eqs. (3.38), (3.41), where $\Delta_{1,2}$ and $\Delta_{2,1}$ are defined as in (5.8). In addition, the following conjugation conditions hold:

$$(\mathbf{u}^\star\Phi^{\star\pm}_{21}(\zeta),\mathbf{v})=(\mathbf{u}^\star,\Phi^{\mp}_{21}(\zeta)\mathbf{v}),$$
$$(\mathbf{u}^\star\Phi^{\star\pm}_{12}(\zeta),\mathbf{v})=(\mathbf{u}^*,\Phi^{\mp}_{12}(\zeta)\mathbf{v}), \tag{5.13}$$

where $\mathbf{u}^\star\in\mathcal{L}^*_{l,l'}$ and $\mathbf{v}\in\mathcal{L}_{l\pm1,l'}(\mathcal{L}_{l,l'\pm1})$.

Now we wish to use the example of the operators Φ_{21} to explain how one can find the bosonic realization of such vertex operators. We introduce the formal operator

$$V_-(\zeta)=\int_C\frac{dz}{2\pi iz}\,e^{-2i\phi(z)}e^{i\phi(\zeta)}, \tag{5.14}$$

where the integration contour C is chosen as in (3.50). The action of this operator is defined only in the Fock modules $\mathcal{F}_{k,k'}$ with $(\xi+1)k-\xi k'<0$ and

$$V_-(\zeta)\ :\ \mathcal{F}_{k,k'}\xrightarrow{V_-}\mathcal{F}_{k-1,k'}.$$

The chiral primary operator which increases value of k by unity can obviously be constructed as $V_+(\zeta)=e^{i\phi(\zeta)}$:

$$V_+(\zeta)\ :\ \mathcal{F}_{k,k'}\xrightarrow{V_+}\mathcal{F}_{k+1,k'}. \tag{5.15}$$

It is easy to see that the operators (5.14), (5.15), satisfy the same commutation relations with the Virasoro algebra as Φ_{21}. Moreover, the following proposition holds:

Proposition 5.4. Let l, l' be positive integers such that $(\xi+1)l-\xi\, l'<0$. Then the following diagrams are commutative:

i)

$$\begin{array}{ccc} \mathscr{F}_{l,l'} & \xrightarrow{V_\pm} & \mathscr{F}_{l\pm1,l'} \\ X'^{l'} \downarrow & & \downarrow\ (-1)^{l'}X'^{l'}, \\ \mathscr{F}_{l,-l'} & \xrightarrow{V_\pm} & \mathscr{F}_{l\pm1,-l'} \end{array} \tag{5.16}$$

ii)

$$\begin{array}{ccc} \mathscr{F}_{-l,-l'} & \xrightarrow{V_\pm} & \mathscr{F}_{-l\pm1,-l'} \\ X'^{l'} \uparrow & & \uparrow\ (-1)^{l'}X'^{l'} \\ \mathscr{F}_{-l,l'} & \xrightarrow{V_\pm} & \mathscr{F}_{-l\pm1,l'}. \end{array} \tag{5.17}$$

Propositions 5.1–5.4 ensure that $V_\pm$ maps one highest-weight representation of Vir_c into another and provide all properties of the vertex operators. However, we can to identify such bosonic operators with the vertex operators Φ_{21} only on the states, where $V_\pm$ are defined, not in the whole Fock space. Let $\mathbf{v}$ and $\mathbf{v}^\star$ be vectors from $\mathscr{L}_{l,l'}$ and $\mathscr{L}^\star_{l,l'}$ respectively, then

$$\begin{aligned} &V_+(\zeta)\mathbf{v}=\Phi^+_{21}(\zeta)\mathbf{v}\ \ (l>0,l'>0,(\xi+1)l-\xi\, l'>0),\\ &V_+(\zeta)\mathbf{v}^\star=\Phi^{\star -}_{21}(\zeta)\mathbf{v}^\star\ \ (l>1,l'>0,(\xi+1)l-\xi\, l'<0),\\ &V_-(\zeta)\mathbf{v}=\Phi^-_{21}(\zeta)\mathbf{v}\ \ (l>1,l'>0,(\xi+1)l-\xi\, l'<0),\\ &V_-(\zeta)\mathbf{v}^\star=\Phi^{\star +}_{21}(\zeta)\mathbf{v}^\star\ \ (l>0,l'>0,(\xi+1)l-\xi\, l'>0), \end{aligned} \tag{5.18}$$

where we explicitly write down in the parentheses the restrictions on the labels l,l' of the spaces $\mathscr{L}_{l,l'}$ where the bosonic realization is defined correctly. The bosonization for vertex operators of another type (Φ_{12} and $\Phi^\star_{12}$) can be carried out in similar fashion. The result is:

$$\begin{aligned} &V'_+(\zeta)\mathbf{v}=\Phi^+_{12}(\zeta)\mathbf{v}\ \ (l>0,l'>0,(\xi+1)l-\xi\, l'<0),\\ &V'_+(\zeta)\mathbf{v}^\star=\Phi^{\star -}_{12}(\zeta)\mathbf{v}^\star\ \ (l>0,l'>1,(\xi+1)l-\xi\, l'>0),\\ &V'_-(\zeta)\mathbf{v}=\Phi^-_{12}(\zeta)\mathbf{v}\ \ (l>0,l'>1,(\xi+1)l-\xi\, l'>0),\\ &V'_-(\zeta)\mathbf{v}^\star=\Phi^{\star +}_{12}(\zeta)\mathbf{v}^\star\ \ (l>0,l'>0,(\xi+1)l-\xi\, l'<0), \end{aligned} \tag{5.19}$$

where $\mathbf{v}\in\mathscr{L}_{l,l'}$ and $\mathbf{v}^\star\in\mathscr{L}^\star_{l,l'}$. The operators $V'_\pm(\zeta)$ are

$$\begin{aligned} &V'_+(\zeta)=e^{i\phi'(z)},\\ &V'_-(\zeta)=\int_C \frac{dz}{2\pi i z}\, e^{-2i\phi'(z)}e^{i\phi'(\zeta)}, \end{aligned} \tag{5.20}$$

and its properties are quite similar.

The bosonization prescriptions (5.18), (5.19) together with the conjugation conditions (5.13) allow us to calculate any matrix element of the vertex operators Φ_{12}, Φ_{21}. Using the technique described in the previous sections, one can calculate the functions $G^\pm_p$, $G'^\pm_p$ (3.42) for any number ξ and then find the commutation relations in the algebra of the vertex operators. We arrive at the following result:[20,34,35]

$$\begin{aligned} &\Phi^a_{21}(\zeta_1)\Phi^b_{21}(\zeta_2)|_{\mathscr{L}_{l,l'}}\\ &\quad=\sum_{c+d=a+b}\mathbf{W}\left[\begin{matrix} l+a+b & l+c\\ l+b & l\end{matrix}\middle|\sigma_{12},p\right]\\ &\quad\times\Phi^d_{21}(\zeta_2)\Phi^c_{21}(\zeta_1)|_{\mathscr{L}_{l,l'}},\\ &\Phi^a_{12}(\zeta_1)\Phi^b_{12}(\zeta_2)|_{\mathscr{L}_{l,l'}}\\ &\quad=\sum_{c+d=a+b}\mathbf{W}\left[\begin{matrix} l'+a+b & l'+c\\ l'+b & l'\end{matrix}\middle|\sigma_{12},p'\right]\\ &\quad\times\Phi^d_{12}(\zeta_2)\Phi^c_{12}(\zeta_1)|_{\mathscr{L}_{l,l'}}, \end{aligned} \tag{5.21}$$

where $p=e^{i\pi(\xi+1)/\xi}$ and $p'=e^{i\pi\xi/(\xi+1)}$. As before, these formulae define the rule for analytical continuation of the matrix elements of the vertex operators from the region $|\zeta_1|>|\zeta_2|$ to $|\zeta_2|>|\zeta_1|$ along the contours $C_{\sigma_{12}}$, $\sigma_{12}=\pm$ depicted in Fig. 2. The nontrivial elements of $\mathbf{W}$ read explicitly:

$$\begin{aligned} &\mathbf{W}\left[\begin{matrix} l\pm2 & l\pm1\\ l\pm1 & l\end{matrix}\middle|\sigma\right]=p^{\sigma/2},\\ &\mathbf{W}\left[\begin{matrix} l & l\pm1\\ l\pm1 & l\end{matrix}\middle|\sigma\right]=\mp\frac{p^{-(1/2\pm l)\sigma}}{[l]_p},\\ &\mathbf{W}\left[\begin{matrix} l & l\pm1\\ l\mp1 & l\end{matrix}\middle|\sigma\right]=-p^{-\sigma/2}\frac{[l\pm1]_p}{[l]_p}, \end{aligned} \tag{5.22}$$

where $[l]_p=(p^l-p^{-l})/(p-p^{-1})$. It is easy also to find the commutation relation

$$\Phi^a_{21}(\zeta_1)\Phi^b_{12}(\zeta_2)=a\ b\ d(\sigma_{12})\Phi^b_{12}(\zeta_2)\Phi^a_{21}(\zeta_1), \tag{5.23}$$

with

$$d(\sigma)=e^{-i\pi\sigma/2}. \tag{5.24}$$

Let us discuss the structure of the commutation relations. The algebra (5.21), (5.23) is self-consistent, since we have $d(\sigma_{12})d(\sigma_{21})=1$ ($\sigma_{12}=-\sigma_{21}$) and the matrices $\mathbf{W}$, $\mathbf{W}'$ satisfy the so-called unitarity condition:

$$\sum_l \mathbf{W}\left[\begin{matrix} l_4 & l\\ l_1 & l_2\end{matrix}\middle|\sigma_{21}\right]\mathbf{W}\left[\begin{matrix} l_4 & l_3\\ l & l_2\end{matrix}\middle|\sigma_{12}\right]=\delta_{l_1,l_3}. \tag{5.25}$$

The associativity condition for this algebra is provided by the Yang–Baxter equation in IRF form.[34,35]

$$\begin{aligned} &\sum_l \mathbf{W}\left[\begin{matrix} l_4 & l_3\\ l & l_2\end{matrix}\middle|\sigma_{23}\right]\mathbf{W}\left[\begin{matrix} l_5 & l_4\\ l_6 & l\end{matrix}\middle|\sigma_{13}\right]\mathbf{W}\left[\begin{matrix} l_6 & l\\ l_1 & l_2\end{matrix}\middle|\sigma_{12}\right]\\ &\quad=\sum_l \mathbf{W}\left[\begin{matrix} l_5 & l_4\\ l & l_3\end{matrix}\middle|\sigma_{12}\right]\mathbf{W}\left[\begin{matrix} l & l_3\\ l_1 & l_2\end{matrix}\middle|\sigma_{13}\right]\mathbf{W}\left[\begin{matrix} l_5 & l\\ l_6 & l_1\end{matrix}\middle|\sigma_{23}\right], \end{aligned} \tag{5.26}$$

and analogously for $\mathbf{W}'$.

3.3. We wish now to construct the representations of ZF algebras associated with algebras of the chiral primaries Φ_{21}.[8] It was explained in connection with the simple example $\xi=1$ (Sec. 3), that to describe the representations of ZF algebras one must take into consideration a finite-dimensional algebra U together with Virasoro algebra. Then the irreducible representations of two ZF algebras are decomposed into a direct sum of irreducible representations of the whole symmetry algebra

$$\text{Symm}=\text{Vir}_c\otimes U. \tag{5.27}$$

It will be shown that this procedure leads to ZF algebras in the general situation ($\xi\neq 1$) too. The finite-dimensional subalgebra U of the whole symmetry algebra for the $\xi\neq 1$ case was considered in Ref. 30. It is found to be a direct product of two quantum algebras of the form (2.9). We refer to the concept of quantum algebras and their representations in the Sec. 2.2.

Let us consider the direct product $U_p(sl(2))\otimes U_{p'}(sl(2))$ of two quantum algebras with generators $(X^{\pm},T)$ and $(X'^{\pm},T')$ respectively. The commutation relations of the two algebras are determined by (2.9), where the deformation parameters p, p' are given by

$$p=e^{i\pi(\xi+1)/\xi},\quad p'=e^{i\pi\xi/(\xi+1)}. \tag{5.28}$$

We will concentrate on the situation when ξ is a real irrational number greater then 1. In this case, as was noted in section 2, the representations of the quantum algebra (2.9) are similar to representations of an ordinary $sl(2)$ algebra.[26,27] For this reason, the algebra $U_p(sl(2))\otimes U_{p'}(sl(2))$ admits a set of finite-dimensional irreducible representations parametrized by a pair of non-negative integers l, l'. We denote these representations as $\mathscr{V}_{l,l'}$. The basis vectors of the space $\mathscr{V}_{l,l'}$ are given by $\mathbf{e}^{m,m'}_{l,l'}=\mathbf{e}^m_l\otimes\mathbf{e}^{m'}_{l'}$, where the $\mathbf{e}^m_l$ were described in (2.11).

Let us consider now the whole symmetry algebra $\text{Symm}=\text{Vir}_c\otimes U_p(sl(2))\otimes U_{p'}(sl(2))$. The irreducible representation of this algebra is given by the tensor product

$$\pi_Z=\oplus_{l,l'}\mathscr{L}_{l,l'}\otimes\mathscr{V}_{l,l'}. \tag{5.29}$$

The scalar product in this space is induced by the scalar product in $\mathscr{L}_{l,l'}$ and in $\mathscr{V}_{l,l'}$. We claim that π_Z has the structure of an irreducible representation of two ZF algebras. Indeed, let us define the action of the generators of ZF algebras on the space (5.29) by the formulae[8]:

$$Z_a(\zeta)\mathbf{v}\otimes\mathbf{e}^{m,m'}_{l,l'}=(-1)^{j'-m'}\sum_{b=\pm1}\begin{pmatrix}1/2 & j & j+b/2\\ a/2 & m & m+a/2\end{pmatrix}_p \times\Phi^b_{21}(\zeta)\mathbf{v}\otimes\mathbf{e}^{m+a/2,m'}_{l+b,l'}, \tag{5.30}$$

$$Z'_a(\zeta)\mathbf{v}\otimes\mathbf{e}^{m,m'}_{l,l'}=(-1)^{j-m}\sum_{b=\pm1}\begin{pmatrix}1/2 & j' & j'+b/2\\ a/2 & m' & m'+a/2\end{pmatrix}_{p'} \times\Phi^b_{12}(\zeta)\mathbf{v}\otimes\mathbf{e}'^{m,m'+a/2}_{l,l'+b}. \tag{5.31}$$

Here, as usual, $l=2j+1$, $l'=2j'+1$. Using the precise expressions (2.13) and (5.18), (5.19) one can find that for $\zeta_1\neq\zeta_2$ the operators (5.30), (5.31) satisfy the commutation relations:

$$Z_a(\zeta_1)Z_b(\zeta_2)=R^{cd}_{ab}(\sigma_{12},p)Z_d(\zeta_2)Z_c(\zeta_1),$$
$$Z'_a(\zeta_1)Z'_b(\zeta_2)=R^{cd}_{ab}(\sigma_{12},p')Z'_d(\zeta_2)Z'_c(\zeta_1), \tag{5.32}$$
$$Z_a(\zeta_1)Z'_b(\zeta_2)=a\ b\ d(\sigma_{12})Z'_b(\zeta_2)Z_a(\zeta_1).$$

The meaning of the parameter $\sigma_{12}=\pm$ is the same as in (3.49). The explicit form of the matrix $R^{cd}_{ab}(\sigma_{12},p)$ is defined by the relations (2.7), (2.8).

6. DEFORMED VIRASORO ALGEBRA AND ELLIPTIC ZF ALGEBRAS

Now we want to consider the general case of the deformed Virasoro algebra $\text{Vir}_{c,x}$ with $0<x<1$ and the central charge

$$c=1-\frac{6}{\xi(\xi+1)}, \tag{6.1}$$

where $\xi>1$ is again an irrational number. Our method, various aspects of which were demonstrated above, is based on a very simple idea. The structure of representations of the deformed Virasoro algebra is determined by the deformed intertwining operators. The problem of finding of proper deformation of screening operators is not quite evident. Here we will use the deformation suggested in Ref. 17. Having the deformed screening operators depending on two continuous parameters x and ξ, we claim that there is a corresponding two parameter algebra $\text{Vir}_{c,x}$ which coincides with the Virasoro algebra with central charge (6.1) in the limit $x\to1$. Indeed, for given parameters ξ, x any operator constructed from annihilation-creation operators of Fock space belongs, by definition, to the universal enveloping algebra of the deformed Virasoro algebra if it commutes with the screening operators. Unfortunately, at the present time, we do not know explicit expressions for a proper basis of the generators of the deformed Virasoro algebra. Therefore the generalization of $\Phi^{\pm}_{21}$, $\Phi^{\pm}_{12}$ will be based only on properties of the deformed screening operators. In this way, we obtain the bosonic representation of the elliptic ZF algebra of the deformed vertex operators. We will show using examples how to carry out real computations of matrix elements of the vertex operators. The main point we wish to emphasize in this section is that the ideas and technique developed in CFT can be generalized for other integrable models.

6.1. Screening operators for the $\text{Vir}_{c,x}$ algebra

Let the bosonic fields $\phi(z)$ and $\phi'(z)$ be given by (5.1), (5.3). We assume that the modes b_m satisfy the deformed commutation relations[17]

$$[b_m,b_n]=m\frac{[m]_x[(\xi+1)m]_x}{[2m]_x[\xi m]_x}\delta_{m+n,0}, \tag{6.2}$$

and

$$b'_m[m(\xi+1)]_x=b_m[m\xi]_x. \tag{6.3}$$

Here we use the notation $[a]_x=(x^a-x^{-a})/(x-x^{-1})$. One can treat these commutation relations as a two-parameter deformation of the Heisenberg algebra (3.8) with real parameters $\xi>1$ and $0<x<1$. The meaning of a deformation with parameter ξ was discussed in Sec. 5, while the simple example of x-deformation was presented in Sec. 4. The Fock module $\mathscr{F}_p$ for the algebra (6.2) can be constructed like an ordinary one. As a vector space it is isomorphic to the undeformed Fock space considered in the previous sections, because $\mathscr{F}_p$ is still covered by vectors $\otimes_{k=1}^{\infty}\mathbf{C}b_{-n_1}\dots b_{-n_k}\mathbf{f}_p$ with $n_i>0$. The module $\mathscr{F}_p$ can be endowed with the structure of the Z-graded module if we introduce the grading operator by

$$D=\frac{1}{2}\sum_{m>0}\frac{[2m]_x}{[m]_x}(b_{-m}b'_m+b'_{-m}b_m)+\frac{P^2}{2}-\frac{1}{4\xi(\xi+1)}. \tag{6.4}$$

The x-deformation (6.2) does not change the number of states on every level, and the character of the Fock module will be the same as before. We note here that in choosing deformation parameter x as a real positive number, we are considering x-deformation at a generic point. As before, the main object of our consideration will be the set of the Fock space $\{\mathscr{F}_{k,k'}\equiv\mathscr{F}_{|(\xi+1)k-\xi k'|/\sqrt{2\xi(\xi+1)}}|k,k'\in\mathbf{Z}\}$. Introduce the two-parameters family of formal operators

$$\begin{aligned}X^l&=\int_{C_1}\dots\int_{C_l}\frac{dz_1}{2\pi i z_1}\dots\frac{dz_l}{2\pi i z_l}\exp\{-i[\phi(z_1x)\\&\quad+\phi(z_1x^{-1})]\}\dots\exp\{-i[\phi(z_lx)+\phi(z_lx^{-1})]\},\\X'^{l'}&=\int_{C_1}\dots\int_{C_{l'}}\frac{dz_1}{2\pi i z_1}\dots\frac{dz_{l'}}{2\pi i z_{l'}}\exp\{-i[\phi'(z_1x)\\&\quad+\phi'(z_1x^{-1})]\}\dots\exp\{-i[\phi'(z_{l'}x)\\&\quad+\phi'(z_{l'}x^{-1})]\}.\end{aligned} \tag{6.5}$$

Evidently, these integral operators are deformations of the operators (5.5). We have seen an example of such deformation (4.1) in section 4. Note that X^l and $X'^{l'}$ are related by the transformation $\xi\leftrightarrow-1-\xi$ (or equivalently, $\phi\leftrightarrow\phi'$, $l\leftrightarrow l'$,). The important fact is that, as in the case $x=1$, the operators X^l and $X'^{l'}$ obey:

Proposition 6.1. The action of operator X^l ($X'^{l'}$) is defined only on the Fock modules $\mathscr{F}_{l,k}(\mathscr{F}_{k,l'})$, $k\in\mathbf{Z}$ and l, $l'>0$. Then

$$\begin{aligned}&\mathrm{Ker}_{\mathscr{F}_{l,l'}}[X^l]=\mathrm{Ker}_{\mathscr{F}_{l,l'}}[X'^{l'}],\\&\mathrm{Ker}_{\mathscr{F}_{l,-l'}}[X^l]=\mathrm{Ker}_{\mathscr{F}_{l,-l'}}[X'^{l'}]=0,\\&\mathrm{Im}_{\mathscr{F}_{l,l'}}[X^l]=\mathscr{F}_{-l,l'},\\&\mathrm{Im}_{\mathscr{F}_{l,l'}}[X'^{l'}]=\mathscr{F}_{l,-l'},\\&\mathrm{Im}_{\mathscr{F}_{l,-l'}}[X^l]=\mathrm{Im}_{\mathscr{F}_{-l,l'}}[X'^{l'}]\subset\mathscr{F}_{-l,-l'}.\end{aligned} \tag{6.6}$$

Now we would like to give the:

Definition. Let the operators P, Q, b_n, $n\neq0$ satisfy the commutation relations (6.2) and the screening operators be given by (6.5). The x-deformed Virasoro algebra $\mathrm{Vir}_{c,x}$ is a subalgebra in the universal enveloping algebra of the deformed Heisenberg algebra of operators b_n and P. An element from the universal enveloping of the Heisenberg algebra belongs to the deformed Virasoro algebra if it commutes with the screening operators.

An example of the deformed Virasoro algebra was given in Sec. 4. This algebra inherits $\mathbf{Z}$ grading from the algebra (6.2) of operators b_n. Due to the definition, the screening operators are intertwining operators for representations of the deformed Virasoro algebra. Using proposition 6.1 one can investigate the irreducible representations of $\mathrm{Vir}_{c,x}$. Let the spaces $\mathscr{L}_{k,k'}$ be defined as in (5.9). We assume that at the general point of x-deformation the analogue of proposition 5.2 is also correct. It can be rewritten now as:

Conjecture 6.1. Let the parameter ξ be irrational number, $\xi>1$. Then 1. The space $\mathscr{L}_{k,k'}$ for any integer number k, k' is an irreducible representation of the deformed Virasoro algebra with central charge defined by (3.16). The highest weight vector of $\mathscr{L}_{k,k'}$ coincides with the vector $\mathbf{f}_{l,l'}$ and its weight is given by (5.8). 2. For $kk'\geqslant0$ or k, $k'<0$, the Verma modules $\mathscr{M}_{k,k'}$ constructed from highest vector $\mathbf{f}_{k,k'}$ coincide with Fock modules $\mathscr{F}_{k,k'}$. Now we must make one more assumption concerning the conjugation condition for $\mathrm{Vir}_{c,x}$. Such as in the Fock space there exists a natural scalar product then we expect that the following statement is true:

Conjecture 6.2. Let irreducible representation $\mathscr{L}_{l,l'}$ and dual Verma module $\mathscr{M}^{\star}_{l,l'}$, of the deformed Virasoro algebra $\mathrm{Vir}_{c,x}$ be realized as the bosonic modules $\mathrm{Ker}_{\mathscr{F}_{l,l'}}[X^l]$ and $\mathscr{F}_{-l,-l'}$ respectively, where $l=1,2,\dots$. Then the scalar product $(,)_0$ given by (3.21), restricted to the vectors from $\mathscr{L}_{l,l'}$ and $\mathscr{M}^{\star}_{l,l'}$, conforms with inner conjugation in the deformed Virasoro algebra.

Unfortunately, at present we can not construct a proper basis of generators in the deformed Virasoro algebra and prove the above assumptions. Nevertheless, let us try to work out the consequences of these statements. We will see below that such a conjugation condition leads to the proper anti-involution in the algebras of chiral primaries.

Consider now the generalization of the operators $V_{\pm}(\zeta)$ and $V'_{\pm}(\zeta)$ introduced in Sec. 5. In the x-deformed case we will define it as follows:

$$\begin{aligned}V_+(\zeta)&=e^{i\phi(\zeta)},\\V_-(\zeta)&=\eta^{-1}\int_C\frac{dz}{2\pi i z}\exp\{-i[\phi(zx)\\&\qquad+\phi(zx^{-1})]\}\exp[i\phi(\zeta)],\\V'_+(\zeta)&=e^{i\phi'(\zeta)},\\V'_-(\zeta)&=\eta'^{-1}\int_C\frac{dz}{2\pi i z}\exp\{-i[\phi'(zx)\\&\qquad+\phi'(zx^{-1})]\}\exp[i\phi'(\zeta)],\end{aligned} \tag{6.7}$$

where the prescription for the integration contours is the same as before. The constant η, η' will be specified later to provide a convenient normalization of the operators (6.7). Note that the action of the operator $V_-(V'_-)$ is well defined only on the Fock modules $\mathscr{F}_{k,k'}$ with $(\xi+1)k-\xi k'<0$ ($(\xi+1)k-\xi k'>0$). One can prove:

Proposition 6.2. Let $(\xi+1)l-\xi l'<0$ and l, $l'>0$. Then the action of the operators $V_{\pm}(\zeta)$ is described by the commutative diagrams (5.16), (5.17); the action of the operators $V'_{\pm}(\zeta)$ is defined by the following commutative diagrams

i)

$$\begin{array}{ccc} \mathscr{F}_{l,l'} & \xrightarrow{V'_{\pm}} & \mathscr{F}_{l,l'\pm1} \\ X^l \downarrow & & \downarrow (-1)^l X^l \\ \mathscr{F}_{-l,l'} & \xrightarrow{V'_{\pm}} & \mathscr{F}_{-l,l'\pm1}, \end{array} \tag{6.8}$$

ii)

$$\begin{array}{ccc} \mathscr{F}_{-l,-l'} & \xrightarrow{V'_{\pm}} & \mathscr{F}_{-l,-l'\pm1} \\ X^l \uparrow & & \uparrow (-1)^l X^l \\ \mathscr{F}_{l,-l'} & \xrightarrow{V_{\pm}} & \mathscr{F}_{l,-l'\pm1}. \end{array} \tag{6.9}$$

Let us illustrate this proposition for a simple example.

6.2. Example of calculations

From proposition 6.1 follows that if the Fock module $F_{-1,-1}$ is regarded as a $\mathrm{Vir}_{c,x}$ module, then it contains a singular vector $X'\mathbf{f}_{-1,1}$. It is easy to show that this vector is proportional to the state $\partial_t\{t\exp\{-i[\phi'(tx)+\phi'(tx^{-1})]\}\}|_{t=0}\mathbf{f}_{-1,1}\sim b'_{-1}\mathbf{f}_{-1,-1}$. According to proposition 6.2, the operator $V_-(\zeta_1)$ acting on this state maps it into the null vector in the Verma module $\mathscr{M}_{-2,-1}\simeq\mathscr{F}_{-2,-1}$. Due to our conjecture on the scalar product, such a null vector has to be orthogonal to all states from the irreducible module $\mathscr{L}_{2,1}$ and, in particular, to the state $V_+(\zeta_2)\mathbf{f}_{1,1}$:

$$\zeta_1^{-1}f(\zeta_1\zeta_2^{-1})=(\mathbf{f}_{1,1}V_+(\zeta_2),\quad V_-(\zeta_1)X'\mathbf{f}_{-1,1})=0. \tag{6.10}$$

Let us check this formula. As a consequence of the bosonic representation it can be represented in the form:

$$\zeta_1^{-1}f(\zeta_1\zeta_2^{-1})=\partial_t\left\{t\int_C\frac{dz}{2\pi iz}(\mathbf{f}_{1,1},e^{i\phi(\zeta_2)}e^{-i\bar{\phi}(z)}\times e^{i\phi(\zeta_1)}e^{-i\bar{\phi}'(t)}\mathbf{f}_{-1,1})_0\right\}\Bigg|_{t=0}, \tag{6.11}$$

where abbreviated notation $\bar{\phi}(z)$ and $\bar{\phi}'(z)$ is introduced for $\phi(zx)+\phi(zx^{-1})$ and $\phi'(tx)+\phi'(tx^{-1})$ respectively.

The technique for calculating of similar vacuum expectation values in a bosonic space was developed by Dotsenko and Fateev.[2] Let us recall how it works in the case of x-deformed operators. First of all, it is convenient to extract the contribution coming from the zero modes of operators in (6.11). Using the commutation relation for P, Q it is easy to find that ordering these operators results in the product of the form $\zeta_1^{(\xi-1)/4\xi}\zeta_2^{-(\xi+3)/4\xi}z^{1/\xi}t^{-1}$. The ordering of the oscillator modes b_n, as usual, means that annihilation operators have to stand on the right, while creation operators should be on the left. It is convenient to distinguish two cases here. In the first case one must carry out ordering of oscillators from different exponentials, while in the second from the same exponent.

First we want to remind the reader the procedure in the first case. For instance, let us specify the ordering of the expression $e^{i\phi(\zeta_2)}e^{i\phi(\zeta_1)}$. Using Campbell–Baker–Hausdorff formula $e^Ae^B=e^{A+B}e^{|A.B|/2}$ one can find that contributions appearing from coupling of these exponentials are equal to

$$g(\zeta_1\zeta_2^{-1})=\exp\{-[\phi_+(\zeta_2),\phi_-(\zeta_1)]\}, \tag{6.12}$$

where we denote by $\phi_{\pm}$ the positive and negative frequency parts of the field ϕ respectively. Straightforward computation using the commutation relations (6.2) leads to the following representation of the function $g(\zeta_1\zeta_2^{-1})$:

$$g(z)=\exp\left[-\sum_{m=1}^{+\infty}\frac{[m]_x[m(\xi+1)]_x}{m[2m]_x[m\xi]_x}z^m\right]. \tag{6.13}$$

The sum in this expression converges only for $|z|<1$. Its analytical continuation over the whole complex plane is given by the following infinite product:

$$g(z)=\frac{(z;x^{2\xi})_\infty(x^4z;x^{2\xi},x^4)_\infty(x^{4+2\xi}z;x^{2\xi},x^4)_\infty}{(x^2z;x^{2\xi})_\infty(x^6z;x^{2\xi},x^4)_\infty(x^{2+2\xi}z;x^{2\xi},x^4)_\infty}. \tag{6.14}$$

The contributions coming from averaging other pairs of exponents can be obtained in the same fashion. Slightly different procedure is required to order the oscillators in the same exponent. Proceeding as above, one finds formally that the ordering of $e^{i\phi(\zeta_1)}$ gives constant ρ such that $\rho^2=g(1)$. However, as is seen from (6.14), $g(z)$ has a simple zero at $z=1$. We adopt the following conventions:

$$\rho^2=\lim_{z\to1}\frac{1-x^2}{1-z}g(z). \tag{6.15}$$

Ordinary, one use exponential operators in the normal ordered form from the very beginning. Then there is no need to order oscillators belonging to the same exponential. It is more convenient for us to carry out this ordering in the final step. Indeed, in the deformed case this step results in nontrivial constants depending on the deformation parameter, unlike the conformal case where renormalization has the similar form for every exponential. We introduce constants like ρ since they provide the natural normalization of operators. Now we present the final expression for the averaging of the integrand in (6.11):

$$\begin{aligned}(\mathbf{f}_{1,1},e^{i\phi(\zeta_2)}e^{-i\bar{\phi}(z)}e^{i\phi(\zeta_1)}e^{-i\bar{\phi}'(t)}\mathbf{f}_{-1,1})_0 &=\rho^2\bar{\rho}\bar{\rho}'\zeta_1^{(\xi-1)/4\xi}\zeta_2^{-(\xi+3)/4\xi}z^{1/\xi}t^{-1}g(\zeta_1\zeta_2^{-1})\\ &\times w(\zeta_1z^{-1})w(z\zeta_2^{-1})u(t\zeta_1^{-1})u(t\zeta_2^{-1})\bar{h}(tz^{-1}).\end{aligned} \tag{6.16}$$

Here $g(z)$ is defined by (6.14), while the other functions yield

$$w(z)=\frac{(x^{1+2\xi}z;x^{2\xi})_\infty}{(x^{-1}z;x^{2\xi})_\infty},\qquad u(z)=1-z,\qquad \bar{h}(z)=\frac{1}{(1-zx)(1-zx^{-1})}. \tag{6.17}$$

As noted above, the constants ρ, $\bar{\rho}$, $\bar{\rho}'$ are nontrivial functions of the deformation parameter x. They are given by

$$\rho^2=(1-x^2)\,\frac{(x^{2\xi};x^{2\xi})_\infty}{(x^{2+2\xi};x^{2\xi})_\infty}\,g^{-1}(x^2),$$

$$\bar\rho^2=(1-x^2)\,\frac{(x^{-2};x^{2\xi})_\infty}{(x^{2+2\xi};x^{2\xi})_\infty}, \tag{6.18}$$

$$\bar\rho'^2=(1-x^2)\,\frac{(x^2;x^{2+2\xi})_\infty}{(x^{2\xi};x^{2+2\xi})_\infty}.$$

The integral from expression (6.16) can be expressed in terms of q-special functions. Let us introduce the necessary notations. We will need definitions of the q-gamma function $\Gamma_q(a)$ and q-hypergeometric function $F_q(a,b,c;z)$. They are usually defined by

$$\Gamma_q(a)=(1-q)^{1-a}\,\frac{(q;q)_\infty}{(q^a;q)_\infty}, \tag{6.19}$$

$$F_q(a,b,c;z)=1+\sum_{n=1}^{\infty}\frac{(q^a;q)_n(q^b;q)_n}{(q^c;q)_n(q;q)_n}\,z^n, \tag{6.20}$$

where $(q^a;q)_n=\Pi_{p=0}^{n-1}(1-q^{a+p})$. There exists the following integral representation for q-hypergeometric function, generalizing formula (3.55):

$$\int_C\frac{dz}{2\pi i}\,z^{c-1}\,\frac{(q^{(1+a)/2}z^{-1};q)_\infty}{(q^{(1-a)/2}z^{-1};q)_\infty}\,\frac{(q^{(1+b)/2}\zeta z;q)_\infty}{(q^{(1-b)/2}\zeta z;q)_\infty}$$
$$=q^{c(1-a)/2}\,\frac{\Gamma_q(c+a)}{\Gamma_q(c+1)\Gamma_q(a)}\,F_q(a+c,b,c+1;q^{1-(a+b)/2}\zeta). \tag{6.21}$$

Notice, that in this remarkable formula we have an ordinary contour integral rather than Jackson's one. When this integral representation is used the calculation becomes trivial and leads to the formula:

$$f(\zeta)=\text{const}\,g(\zeta)[F_{x^{2\xi}}(2\xi^{-1},1+\xi^{-1},\xi^{-1};\zeta x^{-2})$$
$$-(1+\zeta)F_{x^{2\xi}}(1+2\xi^{-1},1+\xi^{-1},1+\xi^{-1};\zeta x^{-2})]. \tag{6.22}$$

In the special case under consideration we need the following particular expressions for hypergeometric functions:

$$F_q(a,b,b;z)=\frac{(q^az;q)_\infty}{(z;q)_\infty},$$
$$F_q(2b-2,b,b-1;z)=(1+q^{b-1}z)\,\frac{(q^{2b-1}z;q)_\infty}{(z;q)_\infty}. \tag{6.23}$$

Substituting these expressions into (6.22), one finds that $f(\zeta)$ is identically zero. That proves the orthogonality of the vectors $V_-(\zeta_1)X'\mathbf{f}_{-1,1}$ and $V_+(\zeta_2)\mathbf{f}_{1,1}$. Analogously, one can prove that the following matrix element is zero:

$$(\mathbf{f}_{1,1}V'_+(\zeta_2),\quad V'_-(\zeta_1)X\mathbf{f}_{1,-1})=0. \tag{6.24}$$

The procedure here is similar to those carried out before. The only difference is that averaging the integrand leads to the expression

$$(\mathbf{f}_{1,1},e^{i\phi'(\zeta_2)}e^{-i\bar\phi'(z)}e^{i\phi'(\zeta_1)}e^{-i\bar\phi(t)}\mathbf{f}_{1,-1})_0$$
$$=\rho'^2\bar\rho'\bar\rho\zeta_1^{(\xi+2)/4(\xi+1)}\zeta_2^{-(\xi-2)/4(\xi+1)}z^{-1/(\xi+1)}$$
$$\times t^{-1}g'(\zeta_1\zeta_2^{-1})w'(\zeta_1z^{-1})w'(z\zeta_2^{-1})u(t\zeta_1^{-1})$$
$$\times u(t\zeta_2^{-1})\bar h(tz^{-1}). \tag{6.25}$$

We emphasize here that the function $u(\zeta) = \exp[\phi'_+(\zeta_2),\bar\phi_-(\zeta_1)]$ in this expression is given by the same formula as before, although it arises from the ordering of other exponentials. This is consequence of the very special form of the proposed x-deformation. The functions $\bar h$, $\bar\rho$, $\bar\rho'^2$ have been written in (6.17) and (6.18), while the others have the form

$$g'(z)=\frac{(x^2z;x^{2+2\xi},x^4)_\infty(x^{4+2\xi}z;x^{2+2\xi},x^4)_\infty}{(x^4z;x^{2+2\xi},x^4)_\infty(x^{2+2\xi}z;x^{2+2\xi},x^4)_\infty},$$

$$w'(z)=\frac{(x^{1+2\xi}z;x^{2+2\xi})_\infty}{(xz,x^{2+2\xi})_\infty}, \tag{6.26}$$

$$\rho'^2=\frac{(x^2;x^{2+2\xi})_\infty}{(x^{2\xi+2};x^{2+2\xi})_\infty}\,g'^{-1}(x^2).$$

6.3. Vertex operators for the deformed Virasoro algebra

Proposition 6.2 means that the operators (6.7) are vertex operators which interpolate between irreducible representations of the deformed Virasoro algebra. We assume that the bosonic operators $V_\pm$ can be identified with "quantum" analogues of the operators $\Phi_{21}^{\pm}$ by the formulae

$$V_+(\zeta)\mathbf{v}=\Phi_{21}^{+}(\zeta)\mathbf{v}\quad(l>0,l'>0,\ (\xi+1)l-\xi\,l'>0),$$
$$V_+(\zeta)\mathbf{v}^\star=\Phi_{21}^{\star-}(\zeta x^2)\mathbf{v}^\star\quad(l>1,l'>0,\ (\xi+1)l-\xi\,l'<0),$$
$$V_-(\zeta)\mathbf{v}=\Phi_{21}^{-}(\zeta)\mathbf{v}\quad(l>1,l'>0,\ (\xi+1)l-\xi\,l'<0), \tag{6.27}$$
$$V_-(\zeta)\mathbf{v}^\star=\Phi_{21}^{\star+}(\zeta x^2)\mathbf{v}^\star\quad(l>0,l'>0,\ (\xi+1)l-\xi\,l'>0).$$

Bosonic realization of operator $\Phi_{12}^{\pm}$ is determined by similar expressions:

$$V'_+(\zeta)\mathbf{v}=\Phi_{12}^{+}(\zeta)\mathbf{v}\quad(l>0,l'>0,\ (\xi+1)l-\xi\,l'<0),$$
$$V'_+(\zeta)\mathbf{v}^\star=\Phi_{12}^{\star-}(\zeta x^2)\mathbf{v}^\star\quad(l>0,l'>1,\ (\xi+1)l-\xi\,l'>0),$$
$$V'_-(\zeta)\mathbf{v}=\Phi_{12}^{-}(\zeta)\mathbf{v}\quad(l>0,l'>1,(\xi+1)l-\xi\,l'>0), \tag{6.28}$$
$$V'_-(\zeta)\mathbf{v}^\star=\Phi_{12}^{\star+}(\zeta x^2)\mathbf{v}^\star\quad(l>0,l'>0,\ (\xi+1)l-\xi\,l'<0),$$

If we adjoin the conjugation conditions

$$(\mathbf{v}_1^\star\Phi_{21}^{\star\pm}(\zeta x^2),\mathbf{v}_2)=(\mathbf{v}_1^\star,\Phi_{21}^{\mp}(\zeta)\mathbf{v}_2),$$
$$(\mathbf{v}_1^\star\Phi_{12}^{\star\pm}(\zeta x^2),\mathbf{v}_2)=(\mathbf{v}_1^\star,\Phi_{12}^{\mp}(\zeta)\mathbf{v}_2), \tag{6.29}$$

where $\mathbf{v}_1^\star\in\mathscr{L}_{l,l'}^\star$, $\mathbf{v}_2\in\mathscr{L}_{l,l'}$ and $|\zeta|=1$, then formulae (6.27), (6.28) uniquely specify the action of x-deformed vertex operators on irreducible representation of $\mathrm{Vir}_{c,x}$. From the bosonic representation one can get all information on the vertex operators. In principle, our construction is very similar to those known in CFT. The difference appears only in the explicit form of the vacuum averaging of exponentials. For instance, let us write down the matrix elements of the product of two x-deformed vertex operators. It can be obtained by using the formula (6.21) and knowing the coupling of corresponding exponentials. If we specify the constants η and η' in the definition (6.7) by

$$\eta=\eta_{x^{2\xi}}\left(\frac{\xi+1}{\xi}\right), \quad \eta'=\eta_{x^{2+2\xi}}\left(\frac{\xi}{\xi+1}\right), \tag{6.30}$$

where

$$\eta_q^2(a)=(1-x^2)\,\frac{q^{a(a-1)/2}(1-q)}{\Gamma_q(a)\Gamma_q(1-a)},$$

then we easily find that x-deformation of the functions $G_p^{\pm}$ (3.42) has the form

$$(\mathbf{v}^{\star}_{l,l'},\Phi^{\pm}_{2,1}(\zeta_2)\Phi^{\mp}_{2,1}(\zeta_1)\mathbf{v}_{l,l'})$$
$$=C\,\frac{g(\zeta_1\zeta_2^{-1})}{g(x^2)}\,G_{x^{2\xi}}\left(\mp\frac{\xi+1}{\xi}\,l\pm l',\ \frac{\xi+1}{\xi};\zeta_1\zeta_2^{-1}\right),$$

$$(\mathbf{v}^{\star}_{l,l'},\Phi^{\pm}_{1,2}(\zeta_2)\Phi^{\mp}_{1,2}(\zeta_1)\mathbf{v}_{l,l'})$$
$$=C'\,\frac{g'(\zeta_1\zeta_2^{-1})}{g'(x^2)}\,G_{x^{2+2\xi}}\left(\mp\frac{\xi}{\xi+1}\,l'\pm l,\ \frac{\xi}{\xi+1};\zeta_1\zeta_2^{-1}\right). \tag{6.31}$$

Both formulae here are described by the same function $G_q(c,a;z)$ taken with different parameters. It is given by the expression

$$G_q(c,a;z)=q^{(1-a)(2c+a)/4}(1-q)^{2a-2}\,\frac{\Gamma_q(a+c)}{\Gamma_q(c+1)}\,z^{a/4+c/2}$$
$$\times F_q(a+c,a,c+1;q^{1-a}z). \tag{6.32}$$

If we choose the constants η, η' as in (6.30), then the constants C, C' will have the form

$$C=(1-x^2)\Gamma_{x^{2\xi}}\left(\frac{\xi+1}{\xi}\right), \quad C'=\frac{1-x^{2+2\xi}}{\Gamma_{x^{2+2\xi}}1/(\xi+1)}.$$

This normalization of the vertex operators is convenient, since it provides the following normalization of the functions (6.33):

$$(\mathbf{v}^{\star}_{1,1},\Phi^{-}_{21}(\zeta_2)\Phi^{+}_{21}(\zeta_1)\mathbf{v}_{1,1})=\frac{1-x^2}{1-x^{-2}\zeta_1\zeta_2^{-1}}+\dots,$$
$$\zeta_1\to x^2\zeta_2,$$
$$(\mathbf{v}^{\star}_{1,1},\Phi^{-}_{12}(x^2\zeta)\Phi^{+}_{12}(\zeta)\mathbf{v}_{1,1})=1. \tag{6.33}$$

In what follows we see that expressions (6.31) also determine the commutation relations of the deformed operators as it was in the conformal case. For this reason, in order to describe arbitrary matrix element of operators $\Phi^{\pm}_{12}$, $\Phi^{\mp}_{21}$ it is enough to present explicit formulae for the functions

$$\mathcal{T}_{nm}(\zeta_1,\dots\zeta_{2n},\zeta'_1,\dots,\zeta'_{2m})=(\mathbf{v}^{\star}_{11},\Phi^{-}_{12}(\zeta'_{2m})\dots\Phi^{-}_{12}(\zeta'_{m+1})$$
$$\times\Phi^{+}_{12}(\zeta'_m)\dots\Phi^{+}_{1,2}(\zeta'_1)\Phi^{-}_{21}(\zeta_{2n})\dots\Phi^{-}_{21}(\zeta_{n+1})$$
$$\times\Phi^{+}_{21}(\zeta_n)\dots\Phi^{+}_{21}(\zeta_1)\mathbf{v}_{11}). \tag{6.34}$$

To calculate these matrix elements of the vertex operators together with the functions (6.14), (6.17), (6.26) we need know also the explicit form of the following averages:

$$\bar g(\zeta_1\zeta_2^{-1})=\exp\{-[\bar\phi_+(\zeta_2),\bar\phi_-(\zeta_1)]\},$$
$$\bar g'(\zeta_1\zeta_2^{-1})=\exp\{-[\bar\phi'_+(\zeta_2),\bar\phi'_-(\zeta_1)]\}, \tag{6.35}$$
$$h(\zeta_1\zeta_2^{-1})=\exp\{-[\phi_+(\zeta_2),\phi'_-(\zeta_1)]\}.$$

Carrying out the standard procedure which was explained above, it is not hard to obtain the formulae

$$\bar g(z)=(1-z)\,\frac{(x^{-2}z;x^{2\xi})_\infty}{(x^{2+2\xi}z;x^{2\xi})_\infty},$$
$$\bar g'(z)=(1-z)\,\frac{(x^2z;x^{2+2\xi})_\infty}{(x^{2\xi}z;x^{2+2\xi})_\infty}, \tag{6.36}$$
$$h(z)=\frac{(x^3z;x^4)_\infty}{(xz;x^4)_\infty}.$$

The bosonization technique allows one to represent the functions (6.34) in the form of contour integrals from meromorphic functions:[4)]

$$\mathcal{T}_{nm}(\zeta_1,\dots\zeta_{2n},\zeta'_1,\dots,\zeta'_{2m})$$
$$=\rho^{2n}(\bar\rho\eta^{-1})^n\rho'^{2m}(\bar\rho'\eta'^{-1})^m$$
$$\times\int_{C_1}\dots\int_{C_n}\prod_k\frac{dz_k}{2\pi iz_k}\int_{S_1}\dots\int_{S_m}\prod_k\frac{dz'_k}{2\pi iz'_k}$$
$$\times f^n_{(\xi+1)/\xi}(\zeta_1,\dots,\zeta_{2n}|z_1,\dots z_n)f^m_{\xi/(\xi+1)}$$
$$\times(\zeta'_1,\dots,\zeta'_{2m}|z'_1,\dots z'_m)\prod_{i<j}g(\zeta_i\zeta_j^{-1})\prod_{i<j}\bar g(z_iz_j^{-1})$$
$$\times\prod_{i<n+j}w(\zeta_iz_j^{-1})\prod_{n+i\leqslant j}w(z_i\zeta_j^{-1})\prod_{i<j}g'(\zeta'_i\zeta_j'^{-1})$$
$$\times\prod_{i<j}\bar g'(z'_iz_j'^{-1})\prod_{i<m+j}w'(\zeta'_iz_j'^{-1})$$
$$\times\prod_{m+i\leqslant j}w'(z'_i\zeta_j'^{-1})\prod_{i,j}h(\zeta_i\zeta_j'^{-1})\prod_{i,j}\bar h(z_iz_j'^{-1})$$
$$\times\prod_{i,j}u(\zeta_iz_j'^{-1})\prod_{i,j}u(z_i\zeta_j'^{-1}). \tag{6.37}$$

Here we denote by the symbols $f^n_{(\xi+1)/\xi}$, $f^m_{\xi/(\xi+1)}$ the functions representing the contributions of null modes in averaging of corresponding exponentials. Their explicit form is given by the relations:

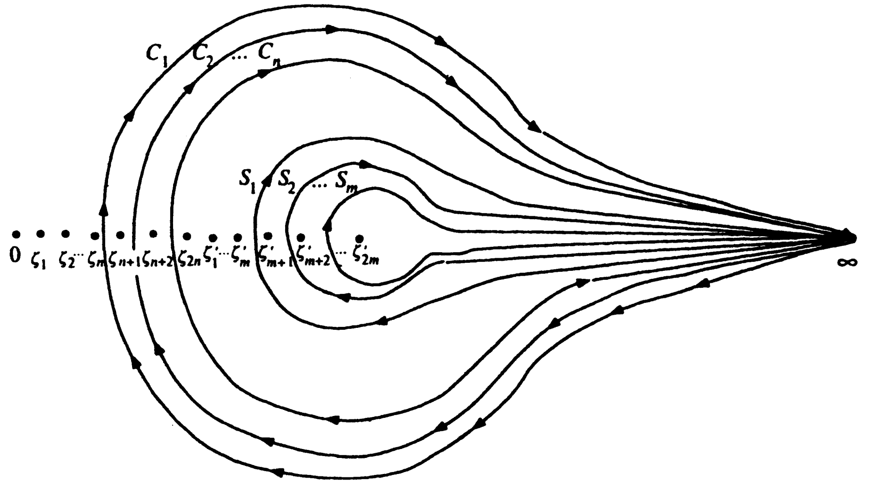

FIG. 6. Integration contours.

$$f_a^n(\zeta_1,\dots\zeta_{2n}|z_1,\dots,z_n)$$

$$=\prod_{k=1}^{n} \zeta_k^{a(1+2k)/4-1/2}\zeta_{n+k}^{a(1+2n-2k)/4-1/2} z_k^{a(k-n-1)+1}. \tag{6.38}$$

The integration contours $\{C_i\}_{i=1}^n$ and $\{S_i\}_{i=1}^m$ are shown in Fig. 6. Note that any contour C_i encloses those singularities of integrand which depend on $\zeta_k(2n\geqslant k\geqslant n+i)$, $z_k(k>i)$, $\zeta'_k(2n\geqslant k\geqslant 1), z'_k(n\geqslant k\geqslant 1)$. At the same time a contour S_i encloses all singularities determined by $\zeta'_k(2m\geqslant k\geqslant m+i)$, $z'_k(k>i)$.

6.4. Elliptic ZF algebra of IRF type

Now let us turn to consideration of the commutation relations for the vertex operators. They can be derived from the rules for analytical continuation of the functions (6.31) by using the following relations for the q-hypergeometric function:[36]

$$F_q(a,b,c;z)=\frac{\Gamma_q(c)\Gamma_q(b-a)}{\Gamma_q(b)\Gamma_q(c-a)}\frac{\Theta_q(q^a z)}{\Theta_q(z)}F_q(a,a-c+1,a-b+1;q^{c+1-a-b}z^{-1})+\frac{\Gamma_q(c)\Gamma_q(a-b)}{\Gamma_q(a)\Gamma_q(c-b)}\frac{\Theta_q(q^b z)}{\Theta_q(z)}F_q(b,b-c+1,b-a+1;q^{c+1-a-b}z^{-1}). \tag{6.39}$$

We have:

Proposition 6.3. The vertex operators Φ_{12}, Φ_{21} obey the following commutation relations:

$$\Phi_{21}^a(\zeta_1)\Phi_{21}^b(\zeta_2)|_{\mathscr{L}_{l,l'}}=\sum_{c+d=a+b}\mathbf{W}\left[\begin{matrix} l+a+b & l+c\\ l+b & l\end{matrix}\middle|\frac{\zeta_1}{\zeta_2}\right]\times\Phi_{21}^d(\zeta_2)\Phi_{21}^c(\zeta_1)|_{\mathscr{L}_{l,l'}},$$

$$\Phi_{12}^a(\zeta_1)\Phi_{12}^b(\zeta_2)|_{\mathscr{L}_{l,l'}}=\sum_{c+d=a+b}\mathbf{W}'\left[\begin{matrix} l'+a+b & l'+c\\ l'+b & l'\end{matrix}\middle|\frac{\zeta_1}{\zeta_2}\right]\Phi_{12}^d(\zeta_2)\times\Phi_{12}^c(\zeta_1)|_{\mathscr{L}_{l,l'}},$$

$$\Phi_{21}^a(\zeta_1)\Phi_{12}^b(\zeta_2)=a\ b\ d\left(\frac{\zeta_1}{\zeta_2}\right)\Phi_{12}^b(\zeta_2)\Phi_{21}^a(\zeta_2). \tag{6.40}$$

The function $d(z)$ appearing at the formula above has the form:

$$d(z)=z^{-1/2}\frac{\Theta_{x^4}(xz)}{\Theta_{x^4}(x^3 z)}, \tag{6.41}$$

while the matrices $\mathbf{W}$, $\mathbf{W}'$ can be represented in the form

$$\mathbf{W}\left[\begin{matrix} l_4 & l_3\\ l_1 & l_2\end{matrix}\middle|\zeta\right]=\zeta^{(\xi+1)/2\xi}\frac{g(\zeta^{-1})}{g(\zeta)}\hat{\mathbf{W}}\left[\begin{matrix} l_4 & l_3\\ l_1 & l_2\end{matrix}\middle|\zeta,\frac{\xi+1}{\xi},x^{2\xi}\right],$$

$$\mathbf{W}'\left[\begin{matrix} l'_4 & l'_3\\ l'_1 & l'_2\end{matrix}\middle|\zeta\right]=\zeta^{\xi/2(\xi+1)}\frac{g'(\zeta^{-1})}{g'(\zeta)}\times\hat{\mathbf{W}}\left[\begin{matrix} l'_4 & l'_3\\ l'_1 & l'_2\end{matrix}\middle|\zeta,\frac{\xi}{\xi+1},x^{2+2\xi}\right]. \tag{6.42}$$

Here $\hat{\mathbf{W}}$ denotes the following matrices:

$$\hat{\mathbf{W}}\left[\begin{matrix} l\pm 2 & l\pm 1\\ l\pm 1 & l\end{matrix}\middle|\zeta,a,q\right]=1,$$

$$\hat{\mathbf{W}}\left[\begin{matrix} l & l\pm 1\\ l\pm 1 & l\end{matrix}\middle|\zeta,a,q\right]=\zeta^{a(\mp l-1)}\frac{\Theta_q(q^a)\Theta_q(q^{\mp al}\zeta)}{\Theta_q(q^a\zeta)\Theta_q(q^{\mp al})}, \tag{6.43}$$

$$\hat{\mathbf{W}}\left[\begin{matrix} l & l\mp 1\\ l\pm 1 & l\end{matrix}\middle|\zeta,a,q\right]=-q^{\mp a^2 l}\zeta^{-a}\frac{\Theta_q(q^{a(\mp l+1)})\Theta_q(\zeta)}{\Theta_q(q^{\mp al})\Theta_q(q^a\zeta)}.$$

The matrices $\mathbf{W}$, $\mathbf{W}'$ satisfy the unitarity condition and the Yang–Baxter equation in IRF form.[3,37] These properties ensure the self-consistency and associativity conditions, respectively, for the algebra of vertex operators (6.40). In addition, the matrices $\mathbf{W}$ and $\mathbf{W}'$ also satisfy the crossing symmetry equation:

$$\mathbf{W}'\left[\begin{matrix} l_4' & l_3' \\ l_2' & l_1' \end{matrix}\middle| x^2 z^{-1}\right] = -\frac{\kappa_{l_3'}}{\kappa_{l_4'}}\mathbf{W}'\left[\begin{matrix} l_2' & l_4' \\ l_1' & l_3' \end{matrix}\middle| z\right], \tag{6.44}$$

where

$$\kappa_{l'} = -x^{\xi(l'-1)(\xi l'-1)/(\xi+1)} \frac{\Theta_{x^{2+2\xi}}(x^{2\xi l'})}{\Theta_{x^{2+2\xi}}(x^{2\xi})}.$$

A similar equation holds for the matrix $\mathbf{W}$. Using the commutation relation (6.40) and the property (6.44) one can show that the quadratic combination $\kappa_{l'}^{-1}(\kappa_{l'-1}\Phi_{12}^{+}(\zeta)\Phi_{12}^{-}(x^2\zeta)+\kappa_{l'+1}\Phi_{12}^{-}(\zeta)\Phi_{12}^{+}(x^2\zeta))$ is a central element in the algebra of vertex operators. In the chosen normalization (6.33) we have:

$$\kappa_{l'-1}\Phi_{12}^{+}(\zeta)\Phi_{12}^{-}(x^2\zeta)+\kappa_{l'+1}\Phi_{12}^{-}(\zeta)\Phi_{12}^{+}(x^2\zeta) = -\kappa_{l'}. \tag{6.45}$$

Note that the conjugation (6.29) conforms with the commutation relations (6.40) and condition (6.45). This justifies the proposed choice of the scalar product in the conjecture 4.2 and the bosonization rules (6.27), (6.28).

6.5. Trigonometric ZF algebra

Now let us briefly discuss the limiting cases of x-deformed construction. The first obvious limit is the conformal one. It can be obtained when the parameter x tends to 1 while the variable $\ln\zeta$ remains finite and nonzero. The bosonization formulae for the matrix elements of the vertex operators in this limit are obviously equivalent to the Dotsenko–Fateev integral representation for conformal blocks in CFT.[2] In particular, one can find that the complicated function $g(z)$, $g'(z)$, $w(z)$, $w'(z)$ in the limit $x\to 1$, $i\ln z\sim 1$ become correspondingly $(1-z)^{(\xi+1)/2\xi}$, $(1-z)^{\xi/2(\xi+1)}$, $(1-z)^{-(\xi+1)/\xi}$, $(1-z)^{-\xi/(\xi+1)}$.

Another limiting case corresponds to the case when the elliptic matrices $\mathbf{W}$ transform into trigonometric ones.[6,38] It can be found as follows. Let us write down the variables x, ζ in the form $z=e^{-i\epsilon\beta}$, $x^2=e^{-\pi\epsilon}$ and then look at the limit $\epsilon\to 0$, assuming that β is finite. Note, that it is convenient to carry out the modular transformation $q=e^{i\pi\tau}\leftrightarrow e^{-i\pi/\tau}$ of the theta functions $\Theta_q(\zeta)$ first. Then the limits can be found straightforwardly. It is not hard to show that functions (6.37), as well as the commutation relations (6.40) are well defined in this limit. These expressions can be treated naturally as vacuum averages of certain operators

$$\Phi_{21}^{\pm}(\beta) = \lim_{\epsilon\to 0}\Phi_{21}^{\pm}(e^{-i\epsilon\beta}),$$
$$\Phi_{12}^{\pm}(\alpha) = \lim_{\epsilon\to 0}\Phi_{12}^{\pm}(e^{-i\epsilon\alpha}), \tag{6.46}$$

which act in the set of spaces $\lim_{\epsilon\to 0}\mathscr{L}_{l,l'}$.[5)] Using the bosonic realization of the ZF algebra of IRF type, one can construct in this limit representations of the ZF algebra of vertex type.[23] Indeed, the finite-dimensional part of the total symmetry algebra Symm will coincide with those considered in the conformal case, i.e., it is given by the direct product of two quantum groups $U_p(sl(2))\otimes U_{p'}(sl(2))$. Let us now introduce space π_Z as

$$\pi_Z = \oplus_{l,l'=1}^{\infty}\mathscr{L}_{l,l'}\otimes\mathscr{V}_{l,l'}, \tag{6.47}$$

where $\mathscr{V}_{l,l'}$ is the ll'-dimensional irreducible representation of the algebra $U_p(sl(2))\otimes U_{p'}(sl(2))$, and define the action of the operators $Z_\pm(\beta)$, $Z'_\pm(\alpha)$ by a formula analogous to (5.30), (5.31). It is convenient to consider the following simple redefinition of these operators

$$Z_\pm(\beta)\to e^{\mp\beta/2\xi}Z_\pm(\beta),$$
$$Z'_\pm(\alpha)\to e^{\mp\alpha/2(\xi+1)}Z'_\pm(\alpha). \tag{6.48}$$

Then one can show that the operators $Z_a(\beta)$, $Z'_a(\alpha)$, generate the ZF algebra of vertex type:

$$Z_a(\beta_1)Z_b(\beta_2) = S_{ab}^{cd}(\beta_1-\beta_2)Z_d(\beta_2)Z_c(\beta_1),$$
$$Z'_a(\alpha_1)Z'_b(\alpha_2) = R_{ab}^{cd}(\alpha_1-\alpha_2)Z'_d(\alpha_2)Z'_c(\alpha_1), \tag{6.49}$$
$$Z_a(\beta)Z'_b(\alpha) = a\ b\ d(\beta-\alpha)Z'_b(\alpha)Z_a(\beta).$$

The function $d(\beta)$ is the limiting value of the function (6.41). Its explicit form is just $d(\beta)=\tan(\pi/4+i\beta/2)$. The matrices $\mathbf{S}_{ab}^{cd}(\beta)$, $\mathbf{R}_{ab}^{cd}(\alpha)$ have the following form:

$$\mathbf{S}_{ab}^{cd}(\beta) = s(\beta)R_{ab}^{cd}(e^{-2\beta/\xi}, e^{i\pi(\xi+1)/\xi}),$$
$$\mathbf{R}_{ab}^{cd}(\alpha) = r(\alpha)R_{ab}^{cd}(e^{-2\alpha/(\xi+1)}, e^{i\pi\xi/(\xi+1)}), \tag{6.50}$$

where the nontrivial elements of the matrix $R_{ab}^{cd}(t,p)$ are given by (2.7). The explicit forms of the functions $r(\alpha)$, $s(\beta)$ is complicated and it can be found in Ref. 17. The ZF algebra (6.49) was introduced in the context of the sine-Gordon model and its physical meaning was discussed in Refs. 17, 21, and 22.

7. OPEN PROBLEMS

Now we run into the problem how to construct the representations of the elliptic ZF algebra of vertex type.[39] Let us give an abstract definition of this object. It is a quadratic algebra of the form (6.49) where the matrices $\mathbf{R}_{ab}^{cd}$, $\mathbf{S}_{ab}^{cd}$ have the following form:

$$\mathbf{S}_{ab}^{cd}(\beta) = \exp\left\{-i\frac{\xi+1}{2\xi}\epsilon\beta\right\}\frac{g(e^{i\epsilon\beta})}{g(e^{-i\epsilon\beta})}$$
$$\times R_{ab}^{cd}(e^{-2\beta/\xi}, e^{i\pi(\xi+1)/\xi}, e^{-4\pi/\epsilon\xi}),$$
$$\mathbf{R}_{ab}^{cd}(\alpha) = \exp\left\{-i\frac{\xi}{2(\xi+1)}\epsilon\alpha\right\}\frac{g'(e^{i\epsilon\alpha})}{g'(e^{-i\epsilon\alpha})} \tag{7.1}$$
$$\times R_{ab}^{cd}(e^{-2\alpha/(\xi+1)}, e^{i\pi\xi/(\xi+1)}, e^{-4\pi/\epsilon(\xi+1)}).$$

The nontrivial elements of the matrix $R_{ab}^{cd}(t,p)$ are defined by (2.2). The function $d(\beta)$ coincides with (6.41) where, z is equal to $e^{-i\epsilon\beta}$. Note that the matrices (7.1) satisfy unitarity, crossing symmetry and the Yang–Baxter equation, and at the limit $\epsilon\to 0$ they transform into (6.50). As we have seen in the example above, to construct the ZF algebra of vertex type from the IRF algebra of vertex operators, one must know the comultiplication in the finite-dimensional subalgebra of the symmetry algebra. This problem might be very nontrivial since this subalgebra seems to be related to the Sklyanin algebra.[40] At the same time, we want to emphasize that the Sklyanin algebra itself is a deformation of $U_p(sl(2))$, while we expect this new algebra to be a deformation of a tensor product of the quantum groups $U_p(sl(2))\otimes U_{p'}(sl(2))$, in order to give the limits which are consistent with our constructions in the trigonometric and conformal cases.

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9. NOTES ADDED IN PROOF

When the work was completed we received a number of preprints devoted to deformed Virasoro algebra and related problems. In Ref. 41 the x-deformation for the classical Virasoro algebra and classical **W**-algebras was proposed. The x-deformed Virasoro algebra with arbitrary central charge $c=1-6/\xi(\xi+1)$ was described in Ref. 42. Further generalization of the x-deformation for the case of **W**-algebras was provided in Refs. 43 and 44.[6)]

10. APPENDIX

In this Appendix we collect the explicit expressions for the functions and constants which are necessary to compute the matrix elements of the vertex operators in formula (6.37). The functions g, w, $\bar g$, g', w', $\bar g'$, h, u, $\bar h$ are defined as

$$\begin{aligned}
&g(\zeta_1\zeta_2^{-1})=\exp\{-[\phi_+(\zeta_2),\phi_-(\zeta_1)]\},\\
&w(\zeta_1\zeta_2^{-1})=\exp\{[\phi_+(\zeta_2),\bar\phi_-(\zeta_1)]\},\\
&\bar g(\zeta_1\zeta_2^{-1})=\exp\{-[\bar\phi_+(\zeta_2),\bar\phi_-(\zeta_1)]\},\\
&g'(\zeta_1\zeta_2^{-1})=\exp\{-[\phi'_+(\zeta_2),\phi'_-(\zeta_1)]\},\\
&w'(\zeta_1\zeta_2^{-1})=\exp\{[\phi'_+(\zeta_2),\bar\phi'_-(\zeta_1)]\},\\
&\bar g'(\zeta_1\zeta_2^{-1})=\exp\{-[\bar\phi'_+(\zeta_2),\bar\phi'_-(\zeta_1)]\},\\
&h(\zeta_1\zeta_2^{-1})=\exp\{-[\phi_+(\zeta_2),\phi'_-(\zeta_1)]\},\\
&u(\zeta_1\zeta_2^{-1})=\exp\{[\phi_+(\zeta_2),\bar\phi'_-(\zeta_1)]\},\\
&\bar h(\zeta_1\zeta_2^{-1})=\exp\{-[\bar\phi_+(\zeta_2),\bar\phi'_-(\zeta_1)]\}.
\end{aligned}\tag{10.1}$$

They are given by:

$$\begin{aligned}
&g(z)=\frac{(z;x^{2\xi})_\infty(x^4z;x^{2\xi},x^4)_\infty(x^{4+2\xi}z;x^{2\xi},x^4)_\infty}{(x^2z;x^{2\xi})_\infty(x^6z;x^{2\xi},x^4)_\infty(x^{2+2\xi}z;x^{2\xi},x^4)_\infty},\\
&w(z)=\frac{(x^{1+2\xi}z;x^{2\xi})_\infty}{(x^{-1}z;x^{2\xi})_\infty},\\
&\bar g(z)=(1-z)\,\frac{(x^{-2}z;x^{2\xi})_\infty}{(x^{2+2\xi}z;x^{2\xi})_\infty},\\
&g'(z)=\frac{(x^2z;x^{2+2\xi},x^4)_\infty(x^{4+2\xi}z;x^{2+2\xi},x^4)_\infty}{(x^4z;x^{2+2\xi},x^4)_\infty(x^{2+2\xi}z;x^{2+2\xi},x^4)_\infty},\\
&w'(z)=\frac{(x^{1+2\xi}z;x^{2+2\xi})_\infty}{(xz,x^{2+2\xi})_\infty},\\
&\bar g'(z)=(1-z)\,\frac{(x^2z;x^{2+2\xi})_\infty}{(x^{2\xi}z;x^{2+2\xi})_\infty},\\
&h(z)=\frac{(x^3z;x^4)_\infty}{(xz;x^4)_\infty},\quad u(z)=1-z,\\
&\bar h(z)=\frac{1}{(1-zx)(1-zx^{-1})}.
\end{aligned}\tag{10.2}$$

The constants ρ, $\bar\rho$, ρ', $\bar\rho'$ defined by

$$\begin{aligned}
&\rho^2=\lim_{z\to1}\frac{1-x^2}{1-z}\,g(z),\quad \bar\rho^2=\lim_{z\to1}\frac{1-x^2}{1-z}\,\bar g(z),\\
&\rho'^2=\lim_{z\to1}g'(z),\quad \bar\rho'^2=\lim_{z\to1}\frac{1-x^2}{1-z}\,\bar g'(z)
\end{aligned}\tag{10.3}$$

have the following values:

$$\begin{aligned}
&\rho^2=(1-x^2)\,\frac{(x^{2\xi};x^{2\xi})_\infty}{(x^{2+2\xi};x^{2\xi})_\infty}\,g^{-1}(x^2),\\
&\bar\rho^2=(1-x^2)\,\frac{(x^{-2};x^{2\xi})_\infty}{(x^{2+2\xi};x^{2\xi})_\infty},\\
&\rho'^2=\frac{(x^2;x^{2+2\xi})_\infty}{(x^{2\xi+2};x^{2+2\xi})_\infty}\,g'^{-1}(x^2),\\
&\bar\rho'^2=(1-x^2)\,\frac{(x^2;x^{2+2\xi})_\infty}{(x^{2\xi};x^{2+2\xi})_\infty}.
\end{aligned}\tag{10.4}$$

1)Because of this identification one can regard the fields $i\partial_\zeta\phi-1/2\zeta$ and $i\partial_\zeta\phi^*-1/2\zeta$ as different solutions of the quantum version of the Riccati equation (3.15).[4]

2)In certain cases (minimal models) it is possible to reduce the space of states so that chiral primaries will be correctly defined and generate an associative algebra.[30]

3)For instance, the action of the integral operators like X, X' is not defined in these modules.

4For convenience, we collect all the necessary averaging of the exponentials in the Appendix.

5)To avoid introducing additional notation, we will denote such spaces by $\mathscr{L}_{l,l'}$ in this subsection.

6)The authors of Refs. 41–44 used the parameters p, q, t which are connected with ξ and x by $p=x^{-2}$, $q=x^{-2(\xi+1)}$, $t=x^{-2\xi}$, Ref. 45.

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