Slow cooling dynamics of the Ising p-spin interaction spin-glass model

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(Submitted 19 June 1995)
Zh. Éksp. Teor. Fiz. **109**, 2094–2114 (June 1996)

We have studied the dynamical behavior of the infinite-range Ising spin-glass model with p-spin interaction above and below the transition into the nonergodic phase. The transition is continuous at sufficiently high external magnetic field. The dynamic critical exponent of the power-law decay of the autocorrelation function at the transition point is shown to decrease smoothly to zero as the field approaches the "tricritical" point from above; in weaker fields the transition is discontinuous. The slow-cooling approach is used to study the nonergodic behavior below the transition at zero external field. It is shown that the anomalous response function $\Delta(t,t')$ contains δ-function as well as regular contributions at any temperature below the phase transition. No evidence of the second phase transition (known to exist within the static replica solution of the same model) is found. At low enough temperatures the slow-cooling solution approaches the one known for the standard Sherrington–Kirkpatrick model.
 [S1063-7761(96)01306-6]

1. INTRODUCTION

The free energy surface of spin glasses in the low-temperature phase has a very complicated structure and consists of an exponentially large number of valleys with infinite barriers between them (at least in the mean-field approximation). This leads to unique dynamical properties. On arbitrarily long (but finite) time scales such systems occupy only one valley and physical quantities depending only on spin variables at a single time differ from their static values derived from the Gibbs distribution. Usually such a behaviour is referred to as "nonergodic" The term "nonergodic" means here that the values of measurable physical quantities (magnetization, susceptibility, etc) cannot be regarded just as functions of the temperature and magnetic field at the measurement point (T,H); rather, they are functionals of the trajectory on the (T,H) plane which lead to the final state one is measuring. A quantitative theoretical approach suitable for the study of such a nonergodic behavior in a classical Sherrington–Kirkpatrick (SK) spin glass[1] and called "slow-cooling theory" was invented in the late 1980's by Ioffe *et al.*[2,3] (see also Refs. 4 and 5) and then developed (in a slightly different version) in Ref. 6.

The simplest and best-known example of nonergodic behavior is the difference between so-called zero-field-cooled (ZFC) and field-cooled (FC) susceptibilities ($\chi_{\rm ZFC}$ and $\chi_{\rm FC}$ below), which is also well described in the static replica-symmetry-breaking approach by Parisi *et al.*[7,8] However, it was shown in Refs. 2 and 5 that the values of the FC susceptibilities obtained in the slow-cooling approach differ from the results of the Parisi theory; moreover, the same applies even to the values of the internal energy (which might be considered a quite robust quantity). Formally one can understand the origin of the difference between the results of the slow-cooling and equilibrium theories as being due to the noncommutativity of two limiting procedures: the thermodynamic limit $N\to\infty$ and stationary-state limit $t_a\to\infty$ (here t_a is the time the system spend in the glassy phase, i.e., the aging time). In the slow-cooling theory it is assumed that the limit $N\to\infty$ is taken first, whereas in the equilibrium theory the limit $t_a\to\infty$ is assumed to be carried out before it. As a result, the transitions between different valleys of the spin-glass phase space which are separated by "infinite" (in the limit $N\to\infty$) barriers are strictly prohibited in the slow-cooling approach, so the contributions of different valleys do not follow the Gibbs distribution in contrast to the case of the equilibrium theory. As a result, there are two sources for the differences between the values of the same physical quantity (e.g., energy) calculated within these two approaches: i) the metastable state (valley) within which the system is typically stuck after slow-cooling is nonoptimal, i.e., has higher free energy than the ground state; ii) in equilibrium there is an additional contribution from the sum over different valleys (a similar quantity is sometimes called configurational entropy or complexity[9]). It is not quite clear to us which of these two sources contributes more; however, there is a clear parallel between the temperature dependence of the configurational entropy of the SK model,[10]

$$S_{\rm con}(T)/N\propto(T_c-T)^6, \qquad (1)$$

and the relative difference between slow-cooling and equilibrium free energies which behaves similarly near T_c (cf. Refs. 2 and 3). Therefore it is natural to expect that either the second cause (item ii)) is the dominant one, or that both of them are of the same importance.

It is seen from (1) that in the vicinity of the spin-glass transition (which is the only analytically tractable region) the disagreement between the results of the two theories for the physical quantities of the SK model is very weak; indeed, the relative difference is found to be of order $(T_c-T)^3$ in $\chi_{\rm FC}$ and of order $(T_c-T)^5$ in the internal energy, and did not receive much attention. The same statement applies to other spin-glass models that are characterized by a continuous Parisi function $q(x)$ which differs only weakly (for T close to T_c) from its maximum value $q(1)$.

There is another family of spin glasses that are characterized by one-step replica symmetry breaking (the Parisi function in these models is a step function.[11–13] Moreover, these models are known to possess dynamic (at $T=T_d$) and static (at $T=T_c$) phase transitions at different temperatures ($T_d>T_c$). Therefore, one expects all effects of history-dependence and nonergodicity to be more dramatic in the models of this second family. Note that the configurational entropy is known to be finite in these models right at the transition point:

$$S_{\mathrm{con}}(T)/N \propto \theta(T_d-T), \tag{2}$$

see Ref. 9 and references therein. Indeed, as will be shown below in the present paper, the slow-cooling solution for the model of that kind differs qualitatively from the results of static-replica theory. Another reason to be interested in this second family is the development of glass models without pre-existing disorder[14,15] whose behavior seems to be similar to the random spin glasses of the second family. Similar dynamical equations were also derived by Leutheusser[16] in connection with structural glass problem.

Recently significant progress was achieved in the investigation of the dynamics of spherical p-spin interaction model,[17,18] which is in some sense exactly solvable: dynamical equations for this model can be written down exactly. In this paper we discuss the more complicated (and more realistic) case of the Ising p-spin interaction spin-glass model, which belongs to the family mentioned above. In the case of infinite p the model is equivalent to the random energy model and exhibits one-step replica symmetry breaking.[13] If p has a finite value, this phase is stable in some vicinity of the transition point only, whereas at lower temperatures a second-phase transition with full continuous replica symmetry breaking takes place.[11] This property distinguishes the Ising version of the p-spin model from the spherical one and it is interesting to examine how it influences the dynamics.

The paper is organized as follows. In Secs. 2 and 3 we introduce the model and derive the mean field generating functional, averaged over disorder, for the correlation and response functions. In Sec. 4 ergodic dynamics is investigated and the transition line $T_d(b)$ (where b is the external magnetic field) is found. It is shown that at sufficiently high magnetic field $b>b_{\mathrm{cr}}$ the phase transition is of continuous nature and is reminiscent of the one known to exist in the SK model at finite field. The dynamic critical exponent ν_1 characterizing the decay of spin-spin autocorrelation function at the dynamic transition line, $C(t) \propto t^{-\nu_1}$, is found as function of b and shown to vanish as $b \to b_{\mathrm{cr}}+0$. In weaker external fields the transition is discontinuous: the nonzero long-time limit of the autocorrelation function $C(t\to\infty)$ appears just at $T=T_d(b)$, at zero field $C(t\to\infty)|_{T_d} = p-2$ at small $p-2$. The "tricritical" field value separating the regions of the first- and second-order spin-glass transitions is identified as $b_{\mathrm{cr}} \propto p-2$. The reason for the existence of the tricritical field b_{cr} is rather simple: in a nonzero external field the spins are already polarized at $T>T_d(b)$, i.e., the local magnetization $\langle\sigma_i\rangle \neq 0$; the interaction between $\tilde{\sigma}=\sigma_i-\langle\sigma_i\rangle$ on different sites i and j will now contain the usual pair-wise random term $\tilde{\sigma}_i\tilde{\sigma}_j\tilde{J}_{ij}$ with relative strength $\propto b$, which tends to produce the usual SK-type transition and competes with the original p-spin interaction. Thus, in strong fields the "induced" interaction wins and the transition is continuous. The phase diagram in $T-b-p$ coordinates is shown schematically in Fig. 1.

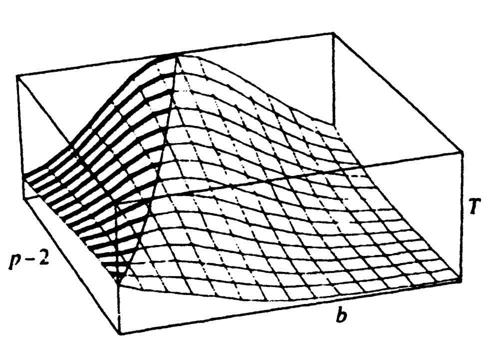

FIG. 1. Phase diagram. Thin lines—continuous transition, thick lines—discontinuous transition.

In Sec. 5 slow cooling of the system in the spin-glass phase is considered. It is shown that the behavior of the nonergodic dynamic response is qualitatively the same in the whole low-temperature phase, i.e., the additional low-T phase transition known from the replica solution[11] is absent. As the temperature goes down, the solution for the anomalous correlation and response functions interpolates smoothly between the one characteristic for spherical $p>2$ spin model[17] and the one for the usual $p=2$ Ising model.[2] We also found a downward jump in the specific heat as temperature decreases through $T=T_d$. Such a behavior is known to exist in real glasses; in our model its origin may be associated with an abrupt drop in the configurational entropy [Eq. (2)] due to the freezing of the system in one of all possible [($\propto \exp(S_{\mathrm{com}})$] metastable states.

Section 6 is devoted to the discussion of the results.

2. THE MODEL

The Ising p-spin interaction spin-glass model is described by the Hamiltonian[11,13]

$$H=-\sum_{1\le i_1<\ldots<i_p\le N} J_{i_1\ldots i_p}\sigma_{i_1}\cdots\sigma_{i_p}-b\sum_{i=1}^{N}\sigma_i, \tag{3}$$

where σ_i is the Ising spin and b is the external field. The spin glass described by this Hamiltonian is a system of N Ising spins interacting via randomly quenched infinite-range interactions $J_{i_1\ldots i_p}$. For simplicity we will take the distribution of constants $J_{i_1\ldots i_p}$ to be Gaussian:

$$P(J_{i_1\ldots i_p})=\sqrt{\frac{N^{p-1}}{\pi J^2 p!}}\exp\left(-\frac{J^2_{i_1\ldots i_p}N^{p-1}}{J^2 p!}\right),$$

$$1\le i_1<\cdots<i_p\le N. \tag{4}$$

We assume Glauber dynamics for σ_i: the probability for σ_i to change its sign during unit time is

$$w\{(\sigma_1,\ldots,\sigma_i,\ldots,\sigma_N)\to(\sigma_1,\ldots,-\sigma_i,\ldots,\sigma_N)\}$$

$$=\frac{1}{2}[1-\sigma_i\tanh(\beta h_i(t))], \tag{5}$$

where the local field is $h_i=\partial H/\partial\sigma_i$ and $\beta=T^{-1}$ is the inverse temperature. In the following, we put $\Gamma=1$ without loss of generality.

The quantities of interest are the average response function

$$G(t,t')=\overline{\frac{\delta\langle\sigma_i(t)\rangle}{\delta\beta b_i(t')}},$$

which vanishes for $t<t'$, and the average spin correlation function

$$C(t,t')=\overline{\langle\sigma_i(t)\sigma_i(t')\rangle}.$$

Here $\langle...\rangle$ means the dynamic average and $\overline{...}$ means an average over disorder.

3. AVERAGE SELF-CONSISTENT GENERATING FUNCTIONAL

In the thermodynamic limit $N\to\infty$, the analysis simplifies and dynamics of the system can be described by a set of self-consistent equations for a single spin. These equations can be derived by introducing a generating functional for Glauber dynamics[19,20] and averaging it over disorder. This functional is defined as

$$Z[\hat\lambda]=\left\langle\exp\left(\sum_{i=1}^{N}\int\hat\lambda_i(t)\sigma_i(t)dt\right)\right\rangle$$

and, as shown in Appendix, can be written as

$$Z=\hat J Z_0\Big|_{h=\beta b}.$$

Here

$$\hat J=\exp\left(\sum_{i_1<i_2<\cdots<i_p}\int dt J_{i_1\cdots i_p}\hat\delta_{i_1}(t)\hat\delta_{i_2}(t)\cdots\hat\delta_{i_p}(t)\right),$$

$$\hat\delta_t=\frac{\delta}{\delta\hat\lambda(t)},\quad \delta_t=\frac{\delta}{\delta h(t)},$$

$Z_0[\hat\lambda]$ is the generating functional for noninteracting spins,

$$Z_0[\hat\lambda]=\exp\left(\sum_{i=1}^{N}\int_{t_0}^{\infty}\hat\lambda_i(t)m_i(t)dt\right),$$

and the m_i obey the equation

$$\partial_t m_i(t)=i\hat\lambda_i(t)[1-m_i^2(t)]-[m_i(t)-\tanh h],$$
$$m_i(t_0)=m_0. \tag{6}$$

It is convenient to use this functional in another form. Note that

$$Z_0[\hat\lambda]=\int\prod_{i=1}^{N}\mathscr{D}\hat\sigma_i\mathscr{D}\sigma_i\mathscr{D}h_i\mathscr{D}\hat h_i\times\exp\left(\sum_{i=1}^{N}\int\hat\lambda_i\sigma_i dt\right)\exp(S),$$

where

$$S=\sum_i\left\{-i\int\hat\sigma_i(\sigma_i-m_i)dt-i\int\hat h_i(h_i-\beta b)dt\right\},$$

$$\partial_t m_i(t)=i\hat\sigma_i(t)[1-m_i^2(t)]-[m_i(t)-\tanh(\beta b)],\quad m_i(t_0)=m_0.$$

Application of the operator $\hat J$ gives

$$Z[\hat\lambda]=\int\prod_{i=1}^{N}\mathscr{D}\hat\sigma_i\mathscr{D}\sigma_i\mathscr{D}h_i\mathscr{D}\hat h_i\exp\left(\sum_{i=1}^{N}\int\hat\lambda_i\sigma_i dt\right)\exp(S), \tag{7}$$

where

$$S=\sum_i\left\{-i\int\hat\sigma_i(\sigma_i-m_i)dt-i\int\hat h_i\left(h_i-\beta\frac{\partial H}{\partial\sigma_i}\right)dt\right\}.$$

This integral is normalized to unity (i.e., $Z[\hat\lambda\equiv0]=1$), so that the average over $J_{i_1\ldots i_p}$ can be performed. The following derivation of the mean-field generating functional is standard and almost coincides with the analogous one derived by Kirkpatrick and Thirumalai[12] for the case of Langevin dynamics. There is no need to repeat this derivation and we write only the final result:

$$Z[\hat\lambda]=\int\mathscr{D}h\mathscr{D}\hat h\mathscr{D}\sigma\mathscr{D}\hat\sigma\exp(S), \tag{8}$$

$$S=-i\int\hat h(h-\beta b)-\int[i\hat\sigma(\sigma-m)-\hat\lambda\sigma]+S_{gl}, \tag{9}$$

$$S_{gl}=-J^2\int\frac{1}{2}\sqrt{\mu(1)\mu(2)}C^{p-1}(1,2)\hat h(1)\hat h(2)+J^2\times\int(p-1)\sqrt{\mu(1)\mu(2)}G(1,2)C^{p-2}(1,2)i\hat h(1)\sigma(2), \tag{10}$$

$$\partial_t m(t)=i\hat\sigma(t)[1-m^2(t)]-[m(t)-\tanh h],$$
$$m(t_0)=m_0 \tag{11}$$

with the obvious simplified notation for the time arguments. Here we use the notation $\mu=p\beta^2J^2/2$. The correlation $C(t,t')$ and response $G(t,t')$ functions and have to be determined self-consistently from (8):

$$C(t,t')=\frac{\delta}{\delta\hat\lambda_t}\frac{\delta}{\delta\hat\lambda_{t'}}Z[\hat\lambda]\Big|_{\hat\lambda\equiv0}$$

and

$$G(t,t')=\frac{\delta}{\delta\hat\lambda_t}\frac{\delta}{\delta\beta b_{t'}}Z[\hat\lambda]\Big|_{\hat\lambda\equiv0}.$$

In the case of $p=2$ the action in (8) coincides with the action of Sommers[19] for the SK model. Below we will be mainly interested in the case $\varepsilon=p-2\ll1$, since we already can consider p as a continuous variable. We will put also $J=1$ to simplify the expressions.

4. DYNAMICS ABOVE THE TRANSITION LINE

In this section we study the dynamical mean-field equation for constant temperature and magnetic field. Under these conditions, the system is expected to be in the paramagnetic phase (see the criterion below), so that the correlation and response functions depend only on the time difference and are related by the fluctuation-dissipation theorem (FDT):

$$G(t)=-\theta(t)\partial_t C(t).$$

Here it is assumed that t_0 in Eq. (6) is equal to $-\infty$, so that m does not depend on m_0. In a nonzero magnetic field, the Edwards–Anderson order parameter $q=\lim_{t\to\infty}C(t)$ is also nonzero at any temperature, so it is convenient to represent the correlator $C(t)$ as $C(t)=\tilde{C}(t)+q$. The part of S_{gl} containing q is

$$\exp\left[-\int \frac{1}{2}\mu q^{p-1}\hat{h}(1)\hat{h}(2)d1d2\right]$$
$$=\left\langle \exp\left(iz\int \hat{h}(t)dt\right)\right\rangle_z,$$

where $\langle...\rangle_z$ means $\int dz/\sqrt{2\pi\mu q^{p-1}}(...)\exp(-z^2/2\mu q^{p-1})$.

In analogy to the previous section one can write $Z[\hat{\lambda}]$ in the following form:

$$Z=\left\langle \hat{J}Z_0\bigg|_{h=\beta b+z}\right\rangle_z,$$

where $\hat{J}$ is

$$\hat{J}=\exp\left[\frac{1}{2}\int \hat{C}(1,2)\frac{\delta}{\delta h(1)}\frac{\delta}{\delta h(2)}+\int \hat{G}(1,2)\frac{\delta}{\delta h(1)}\frac{\delta}{\delta\hat{\lambda}(2)}\right] \tag{12}$$

and

$$\hat{C}(1,2)=\sqrt{\mu(1)\mu(2)}C^{p-1}(1,2)-\sqrt{\mu(1)\mu(2)}q^{p-1},$$
$$\hat{G}(1,2)=(p-1)\sqrt{\mu(1)\mu(2)}C^{p-2}(1,2)G(1,2).$$

The function m in Z_0 obeys Eq. (6).

One can show[19,20] by means of the FDT that the static limits of all correlation functions are independent of the short-time parts of C and G in (12). This leads to the equation

$$q=\langle m^2\rangle, \tag{13}$$

where $m=\tanh(z+\beta b)$ and $\langle...\rangle$ means the average over z. This equation coincides with the replica-symmetric equation found in Refs. 11 and 13.

Moreover, we will see that for $t\gg 1$ the expansion of C and G in βJ from (12) is at the same time an expansion in $\hat{G}(\omega)-G(\omega)|_{\omega=0}$. One can therefore use perturbation theory with respect to $\hat{C}$ and $\hat{G}$.

Following Sommers,[19] we write several terms of this expansion, applying the FDT to each term:

$$G(\omega)=\frac{\langle 1-m^2\rangle}{1-i\omega}+\frac{i\omega\langle(1-m^2)^2\rangle}{(1-i\omega)^2}\left[\int_0^\infty e^{i\omega t}\hat{C}(t)dt+\hat{C}(t)\bigg|_{t=0}\right]-\frac{\omega^2}{(1-i\omega)^3}\langle(1-m^2)^3\rangle$$
$$\times\left[\int_0^\infty e^{i\omega t}\hat{C}(t)dt+\hat{C}(t)\bigg|_{t=0}\right]^2$$
$$+\frac{i\omega}{(1-i\omega)^2}\langle 2m^2(1-m^2)^2\rangle\left[\int_0^\infty e^{i\omega t}(\hat{C}(t))^2dt+(\hat{C}(t)|_{t=0})^2\right]. \tag{14}$$

Let us first consider sufficiently large magnetic fields where the transition is expected to be of second order (the exact criterion will be derived below). The correlator $C(\omega)$ at $\omega\to 0$ diverges at the transition line where

$$\mu_c(p-1)\langle(1-m^2)^2\rangle q_c^{p-2}=1. \tag{15}$$

At $T=T_c$ we have $C(t) = At^{-\nu_1}$ where ν_1 obeys the equation

$$\pi\cot(\pi\nu_1)\langle(1-m^2)^3\rangle\frac{\Gamma(2\nu_1)}{\Gamma^2(\nu_1)}$$
$$=\langle m^2(1-m^2)^2\rangle+\langle(1-m^2)^2\rangle\frac{p-2}{4\mu q^{p-1}(p-1)}. \tag{16}$$

For the case of $p=2$, this equation was derived in Refs. 19 and 21. In contrast to the SK model with $p=2$, ν_1 becomes zero at the tricritical point where

$$\langle(1-m^2)^3\rangle=\langle 2m^2(1-m^2)^2\rangle+\langle(1-m^2)^2\rangle\frac{p-2}{2\mu q^{p-1}(p-1)}. \tag{17}$$

This equation along with Eqs. (13) and (15) determines $T_{\rm tr}$, $q_{\rm tr}$, and $b_{\rm tr}$. Near the tricritical point we can expand the left-hand side of Eq. (16) in ν_1. This expansion contains no first-order term and begins with ν_1^2, so that ν_1 at $b-b_{\rm tr}\ll 1$ is

$$\nu_1\propto\sqrt{b-b_{\rm tr}}. \tag{18}$$

We can also find how $b_{\rm tr}$ tends to zero in the limit $p\to 2$. Expansion of the Eqs. (13), (15), and (17) in m^2 yields

$$b_{\rm tr}\propto\varepsilon=p-2. \tag{19}$$

In fields $b<b_{\rm tr}$ at $T=T_c$, the correlation function $C(t)$ does not vanish in the limit $t\to\infty$ what makes phase transition discontinuous. Consider the case of $b=0$. For $\omega\ll 1$

$$G(\omega)=1+i\omega\int_0^\infty e^{i\omega t}\hat{C}^{p-1}(t)dt+\left[i\omega\int_0^\infty e^{i\omega t}\hat{C}^{p-1}(t)dt\right]^2. \tag{20}$$

All results at $b=0$ will be correct only for $\varepsilon=p-2\ll 1$, since we restrict ourselves to a few terms in the expansion of (12). Assume that $C(t)$ has time-independent part:

$$\lim_{t\to\infty}C(t)=q.$$

Equation (20) yields (to lowest order in q)

$$q-\mu q^{1+\varepsilon}+q^2=0.$$

This equation has the trivial solution $q=0$ and a nontrivial one starting from some $\mu=\mu_c$. This condition determines the transition point:

$$q_c=\varepsilon, \quad \mu_c=1-\varepsilon \ln \varepsilon+\varepsilon. \tag{21}$$

These results will be found below by the adiabatic-cooling method. It should be emphasized that the transition temperature is higher than the one derived in Ref. 11 by means of the replica method.

To determine the critical behavior of the function $C(t)$ at long times, consider its Laplace transform $C(\omega)=\int_0^\infty e^{i\omega t}C(t)dt$. We suppose that for T slightly larger than T_c the function $C(\omega)$ has a pole contribution and a remaining part:

$$C(\omega)=\frac{q}{1/\tau-i\omega}+\tilde{C}(\omega), \tag{22}$$

where the relaxation time τ diverges when $T\to T_c$. At the transition point $\tilde{C}(\omega) \propto \omega^{\nu_2-1}$ (and $\tilde{C}(t) \propto t^{-\nu_2}$). To determine ν_2, consider Eq. (20) at $T=T_c$. The terms of order ω^{ν_2} cancel, in agreement with (21). Comparison of the coefficients in the terms of order $\omega^{2\nu_2}$ gives the equation for ν_2. To leading order in ε

$$\Gamma(1-2\nu_2)=2\Gamma^2(1-\nu_2), \quad \nu_2\approx 0.395. \tag{23}$$

We can also find how the relaxation time τ depends on the temperature T near T_c. Equation (20) at $\omega=0$ yields

$$1=(1+\varepsilon)\mu q^{\varepsilon}-2q.$$

As will be shown below, the characteristic time $\tilde{\tau}$ of the function $\tilde{C}(t)$ is much smaller than τ. In the interval $1/\tau \ll \omega \ll 1/\tilde{\tau}$, to lowest order of ε, we obtain:

$$0=\mu_c-\mu+\frac{2\varepsilon}{\tau}\tilde{C}(\omega)+\frac{i\omega\varepsilon}{\tau}\frac{d}{d\omega}\tilde{C}(\omega)+[i\omega\tilde{C}(\omega)]^2 +\frac{i\omega}{2}\int_0^\infty e^{i\omega t}\tilde{C}^2(t)dt.$$

This equation (for $\varepsilon=1$) was studied by Leutheusser[16] in his description of the dynamics near the liquid-glass transition. Assuming $\tilde{\tau}\propto(\mu_c-\mu)^{-\alpha}$ and $\tau\propto(\mu_c-\mu)^{-\beta}$, one obtains

$$\alpha=\frac{1}{2\nu_2}, \quad \beta=\frac{1+\nu_2}{2\nu_2}. \tag{24}$$

It should be mentioned that for the spherical p-spin interaction model[17] the response function is

$$G(\omega)=\left[1-i\omega\int_0^\infty e^{i\omega t}\hat{C}^{p-1}(t)dt\right]^{-1}. \tag{25}$$

For small ε this expression is identical to (20). Therefore, the asymptotic behavior of the functions $C(t)$ and $G(t)$ is the same as for the spherical model (at least to the lowest order in ε), as one can see from Eqs. (21), (23), and (24).

5. SLOW COOLING

5.1. Adiabatic equations and the transition line

Probably the most direct way to investigate the behavior of the spin glass on a finite time scale is based on slow cooling, starting in the ergodic high-temperature phase. We will assume that the temperature and possibly the magnetic field vary on a time scale of order $t_0 \gg \Gamma^{-1}$. The situation is quite complicated, because now the correlation and response functions depend now on both time arguments and not on the time difference. It is convenient to divide these functions into "fast" and "slow" parts:

$$C(t,t')=\tilde{C}_t(t-t')+q(t,t'),$$

$$G(t,t')=\tilde{G}_t(t-t')+\Delta(t,t').$$

The functions $\tilde{C}_t(t-t')$ and $\tilde{G}_t(t-t')$ decay on the time scale $\bar{t}\ll t_0$ and represent the dynamics in a "pure" state in the system phase space. The relevant time scale of the functions $q(t,t')$ and $\Delta(t,t')$ is t_0. It turns out that a closed system of equations can be obtained for the slow parts of the correlation and response functions. This was first proposed by Ioffe *et al.* in Refs. 2 and 3, where Langevin dynamics was used. An alternative method, which starts from Glauber dynamics, was developed by Horner and Freixa-Pascual in Ref. 6. It can be proved (see Appendix 2) that both methods give identical equations: Here we choose the second one.

Now let us explain briefly the main idea. As was mentioned in the previous section, for $t-t'\gg\bar{t}$ the functions q and Δ are independent of the short-time functions $\tilde{C}_t(t-t')$ and $\tilde{G}_t(t-t')$. Thus, we can replace C and G by q and Δ respectively in the functional (12). Moreover, we can also take $m(t)$ to be $\tanh(\beta b)$. The existence of the terms $\partial_t m(t)$ and $i\hat{\sigma}(t)[1-m^2(t)]$ in Eq. (11) implies that the correlation functions are not equal to the asymptotic values and also have the relaxation parts. For example, the correlator $C(t)=\langle\sigma(t)\sigma(0)\rangle$ of the free spins in the constant field is $C(t)=\tanh^2(\beta b)+[1-\tanh^2(\beta b)]e^{-|t|}$. Without these terms we would obtain $C(t)=\tanh^2(\beta b)$. The actual form of the time-dependent correlation function for the interacting spins is different from simple exponential relaxation of course; however, its particular form is irrelevant for the derivation of the slow-cooling equations for the slow functions $\Delta(t,t')$ and $q(t,t')$.

If we take these remarks into account, the generating functional (12) in the adiabatic limit can be written as

$$Z[\hat{\lambda}]=\hat{J}\exp\left[\int\hat{\lambda}(t)\tanh h(t)dt\right]\Big|_{h=\beta b}, \tag{26}$$

$$\hat{J}=\exp\left[\frac{1}{2}\int\sqrt{\mu(1)\mu(2)}q^{p-1}(1,2)\frac{\delta}{\delta h(1)}\frac{\delta}{\delta h(2)}+(p-1)\int\sqrt{\mu(1)\mu(2)}\Delta(1,2)q^{p-2}(1,2)\frac{\delta}{\delta h(1)}\frac{\delta}{\delta\hat{\lambda}(2)}\right]. \tag{27}$$

As in the previous section, we introduce the functions

$$\hat{q}(t,t')=\sqrt{\mu(t)\mu(t')}q^{p-1}(t,t')-\mu_c q^{p-1}|_{T=T_c+0},$$

$$\hat{\Delta}(t,t')=(p-1)\sqrt{\mu(t)\mu(t')}q^{p-2}(t,t')\Delta(t,t')$$

and rewrite (26) as

$$Z[\hat{\lambda}]=\left\langle \hat{J}\,\exp\left[\int \hat{\lambda}(t)\tanh h(t)dt\right]\Bigg|_{h=\beta b+z}\right\rangle, \tag{28}$$

$$\hat{J}=\exp\left[\frac{1}{2}\int \hat{q}(1,2)\,\frac{\delta}{\delta h(1)}\,\frac{\delta}{\delta h(2)} + \int \hat{\Delta}(1,2)\,\frac{\delta}{\delta h(1)}\,\frac{\delta}{\delta \hat{\lambda}(2)}\right], \tag{29}$$

$$\langle z^2\rangle=\mu_c q^{p-1}|_{T=T_c+0}.$$

The self-consistency conditions are

$$q(t,t')=\hat{\delta}_t\hat{\delta}'_t Z[\hat{\lambda}],\quad \Delta(t,t')=\hat{\delta}_t\delta'_t Z[\hat{\lambda}],$$

where

$$\hat{\delta}_t=\frac{\delta}{\delta\hat{\lambda}(t)},\quad \delta_t=\frac{\delta}{\delta\beta b(t)}.$$

To derive these equations we have to expand $\hat{J}$ given in (29) and retain several terms of this expansion. This method is correct near the transition point for $b=0$ and $\varepsilon\ll 1$ and also for b near or greater than b_{tr} and arbitrary ε. When these conditions hold, the values of $q_{t,t'}$ and $\int\Delta_{t,t'}dt'$ are small. After some algebra the expansion of $\hat{J}$ to the second order in q and Δ results in Eqs. (30) and (31). The essential remark should be done before we write this equation. After the operator $\int\hat{\Delta}(1,2)\delta_1\hat{\delta}_2$ acts on $\exp[\int\hat{\lambda}(t)\tanh\beta h(t)dt]$ the integral $\int\Delta(t,t)[1-\tanh^2(\beta b)]dt$ appears containing the undefined quantity $\Delta(t,t)$. Let us consider what such terms correspond to in the exact (not adiabatic) correlators. To derive the corresponding terms, the operator $\delta/\delta\hat{\lambda}$ should be applied only to the function $\hat{\lambda}(t)$ in the exponent. As a result we obtain $\int_{t'>t}\Delta(t,t')\exp(t-t')dtdt'$. This integral equals zero for any value of $\Delta(t,t')$. Therefore in the adiabatic limit we should set $\Delta(t,t')$ equal to zero.

Thus, the equation for Δ mentioned above is

$$\Delta_{t,t'}=\langle m'^2\rangle\hat{\Delta}_{t,t'}+\langle m''^2\rangle+\frac{1}{2}\,\hat{\Delta}_{t,t'}(\hat{q}_t+\hat{q}_{t'})\langle m'm'''\rangle + \langle m'^3\rangle\int \hat{\Delta}_{t,1}\hat{\Delta}_{1,t'}d1+\langle mm'm''\rangle\hat{\Delta}_{t,t'} \times\int(\hat{\Delta}_{t,1}+\hat{\Delta}_{t',1})d1. \tag{30}$$

The equation for q is

$$q_{t,t'}=\langle m'^2\rangle\hat{q}_{t,t'}+\frac{1}{2}\langle mm''\rangle(\hat{q}_t+\hat{q}_{t'}) + \langle m^2m'\rangle\int(\hat{\Delta}_{t',1}+\hat{\Delta}_{t'1})d1+\langle m''^2\rangle\left[\frac{1}{2}\hat{q}^2_{t,t'} + \frac{1}{4}\hat{q}_t\hat{q}_{t'}\right]+\frac{1}{2}\langle m'm'''\rangle\hat{q}_{t,t'}(\hat{q}_t+\hat{q}_{t'}) + \frac{1}{8}\langle mm^{(4)}\rangle(\hat{q}_t^2+\hat{q}_{t'}^2)+\langle m'^3\rangle\int(\hat{\Delta}_{t,1}\hat{q}_{t',1} + \hat{\Delta}_{t',1}\hat{q}_{t,1})d1+\frac{1}{2}\langle m'''m^2\rangle\int(\hat{\Delta}_{t,1}\hat{q}_t + \hat{\Delta}_{t',1}\hat{q}_{t'})d1+\langle mm'm''\rangle\left\{\frac{1}{2}\int[\hat{\Delta}_{t,1}(\hat{q}_{t'}+\hat{q}_1) + \hat{\Delta}_{t',1}(\hat{q}_t+\hat{q}_1)]d1+\int(\hat{\Delta}_{t,1}\hat{q}_{t,1}+\hat{\Delta}_{t',1}\hat{q}_{t',1})d1 + \int\hat{q}_{t,t'}(\hat{\Delta}_{t,1}+\hat{\Delta}_{t',1})d1\right\}+\frac{1}{2}\langle m''m^3\rangle \times\int(\hat{\Delta}_{t,1}\hat{\Delta}_{t,2}+\hat{\Delta}_{t',1}\hat{\Delta}_{t',2})d1d2+\langle m'^2m^2\rangle \times\left[\hat{\Delta}_{t,1}\hat{\Delta}_{t',2}d1d2+\int(\hat{\Delta}_{t,1}\hat{\Delta}_{1,2}+\hat{\Delta}_{t',1}\hat{\Delta}_{1,2})\right]. \tag{31}$$

Here

$$m^n=\frac{d^n}{dz^n}\tanh(z+\beta b).$$

Several remarks should be made about the behavior of the functions $q(t,t')$ and $\Delta(t,t')$. If the parameter q varies smoothly on the cooling trajectory, these functions vary also smoothly on time scales order t_0. If q jumps discontinuously at the critical temperature, we should also expect discontinuous jump in the function $q(t,t')$ at $t'\sim t_c$. This means that the correlation function $C(t,t')$ varies for $t'\sim t_c$ on a time scales shorter than the characteristic cooling time t_0. Thus, the functions $q(t,t')$ and $\Delta(t,t')$ for t' near t_c should be related by the generalized FDT:

$$\partial_{t'}q(t,t')=\Delta(t,t'), \tag{32}$$

and consequently

$$\partial_{t'}\hat{q}(t,t')=\hat{\Delta}(t,t'),$$

which is consistent with the Eqs. (30) and (31). This condition should be also satisfied if $t-t'\ll t_0$. The same assumptions were used in Ref. 17, where the dynamics of the spherical model was considered. In other words, the functions $G(t,t')$ and $C(t,t')$ at $t-t'\ll t_0$ and $t-t'\ll t_c$ can be regarded as δ-function and θ-function respectively.

Now we show how one can obtain some of the results of the previous section with the help of this generalized FDT and equations (30) and (31). For example let us find tricritical point. Suppose that the cooling trajectory crosses the transition line near this point for $b<b_{\mathrm{tr}}$. If we put $t=t'=t_c+0$, then

$$q=\langle m^2\rangle+\langle m'^2\rangle\hat{q}+\hat{q}^2\langle(1-m^2)^2(3m^2-1)\rangle. \tag{33}$$

Above T_c, q satisfies the equation $q=\langle m^2\rangle$ and for $b=b_{\mathrm{tr}}$ the jump in the order parameter δq becomes zero. Expansion of (33) in δq gives the marginal stability condition (15) in first order and equation (17) in second order.

Consider now the case of $b=0$ and $t=t'=t_c+0$. The equations become

$$1=(p-1)\mu q^{p-2}(1-2\hat{q}),\quad q=\hat{q}(1-2\hat{q})+\hat{q}^2.$$

This yields $q_c=\varepsilon$, $\mu_c=1+\varepsilon-\varepsilon\ln\varepsilon$ as in (21).

5.2. The anomalous response function at $b=0$

Let us now consider the solution for q and Δ in the spin-glass state below the transition point but at zero external field b. Equations (30) and (31) reduce to

$$\Delta_{t,t'}=\hat{\Delta}_{t,t'}(1-\hat{q}_t-\hat{q}_{t'})+\int \hat{\Delta}_{t,1}\hat{\Delta}_{1,t'}d1,$$

$$q_{t,t'}=\hat{q}_{t,t'}(1-\hat{q}_t-\hat{q}_{t'})+\hat{q}_{t,t_c}\hat{q}_{t',t_c}+\int (\hat{\Delta}_{t,1}\hat{q}_{t',1}+\hat{\Delta}_{t',1}\hat{q}_{t,1})d1. \tag{34}$$

Here and below integration interval does not contain t_c. The solution for $t\geqslant t'$, $\mu=\mu_c+2\tau$, and $\tau\ll\varepsilon\ll 1$ is

$$q(t,t')=(\varepsilon+\tau+\tau')\theta(t-t'),$$

$$\Delta(t,t')=(\varepsilon+\tau)\delta(t'-t_c). \tag{35}$$

For τ, $\tau'\geqslant\varepsilon$,

$$q(t,t')=\sqrt{\tau\tau'}\,\theta(t-t'),\quad \Delta(t,t')=\sqrt{\tau\varepsilon}\,\delta(t-t_c),$$

$$q(t,t_c)=\sqrt{\tau\varepsilon}. \tag{36}$$

As can be seen, anomalous response function Δ consists of the δ-function contribution only. As will be discussed below, such a solution is in agreement with the one-step replica symmetry breaking solution obtained previously for the same model in the static approach.[11,13] However, it turns out that these results holds only approximately, and more accurate calculations lead to an appearance of the regular (smooth) part in the anomalous response function.

To find the smooth part of the anomalous response function Δ we should expand $\hat{J}$ in (29) to third order in q and Δ. We also can put $\Delta\propto\delta(t-t_c)$ in the third-order terms. Thus, the equation for Δ is

$$\Delta_{t,t'}=\hat{\Delta}_{t,t'}[1-\hat{q}_t-\hat{q}_{t'}+2(\hat{q}_t^2+\hat{q}_{t'}^2+\hat{q}_{t,t'}^2)+\hat{q}_t\hat{q}_{t'}-\hat{q}_{t,t_c}^2-\hat{q}_{t',t_c}^2-2\hat{q}_{t,t_c}\hat{q}_{t',t_c}]+\int \hat{\Delta}_{t,1}\hat{\Delta}_{1,t'}d1. \tag{37}$$

The equation for q is

$$q_{t,t'}=\hat{q}_{t,t'}\left[1-\hat{q}_t-\hat{q}_{t'}+2\left(\hat{q}_t^2+\hat{q}_{t'}^2+\frac{1}{3}\hat{q}_{t,t'}^2\right)+\hat{q}_t\hat{q}_{t'}\right]+\hat{q}_{t,t_c}\hat{q}_{t,t_c}-(\hat{q}_t^2+\hat{q}_{t'}^2)(\hat{q}_{t,t_c}^2+\hat{q}_{t',t_c}^2)-\hat{q}_{t,t_c}\hat{q}_{t,t_c}\hat{q}_{t,t'}+\int (\hat{q}_{t,1}\hat{\Delta}_{t',1}+\hat{q}_{t',1}\hat{\Delta}_{t,1})d1. \tag{38}$$

Consider the region $\tau\ll\varepsilon$, and introduce the notation $q(t,t')=\varepsilon+\tau+\tau'+w(t,t')$. Then

$$\varepsilon\Delta(t,t')w(t,t')+\int \Delta_{t,1}\Delta_{1,t'}d1=0,$$

$$\frac{1}{2}w^2(t,t')+\int (w_{t,1}\Delta_{t',1}+w_{t',1}\Delta_{t,1})d1+\frac{10}{3}\varepsilon\tau\tau'=0. \tag{39}$$

The solution of this equation has a degeneracy: we can change the sign of the functions Δ and w simultaneously. However, we should choose the positive value of Δ, since it corresponds to higher magnetization:

$$\Delta(t,t')=(\varepsilon+\tau)\delta(t'-t_c)+\frac{10}{3}\sqrt{\varepsilon}\frac{d\tau'}{dt'}\theta(t-t'),$$

$$w(t,t')=-\frac{10}{3}|\tau-\tau'|\sqrt{\varepsilon}. \tag{40}$$

The above choice of solution is based on the following physical arguments: the very presence of the anomalous response function is due to the possibility that the system can choose between different configuration-space valleys (which come into existence during the cooling) in order to minimize the free energy of the final state.[2] In particular, when the cooling procedure is done at some constant magnetic field, the valleys with higher magnetization along the applied field direction are certainly preferable—which means that the irreversible contribution to the susceptibility should always be positive.

The result (40) can be compared with the analogous one in the spherical model.[17] There the smooth part of the function Δ at small τ and ϵ is also a constant proportional to $\sqrt{\varepsilon}$. However, there is a family of special cooling trajectories where Δ has no smooth part and cooling at zero field belongs to this family. It is not clear whether such a family exists in the Ising model.

Consider now the anomalous response function for τ, $\tau'\gg\varepsilon$. The equations for q and Δ in this region are

$$\Delta(t,t')\left(\varepsilon\ln\frac{q_{t,t'}}{\sqrt{\tau\tau'}}+q_{t,t'}^2-\tau^2-\tau'^2\right)+\int \Delta_{t,1}\Delta_{1,t'}d1=0,$$

$$q_{t,t'}\left(\varepsilon\ln\frac{q_{t,t'}}{\sqrt{\tau\tau'}}+\frac{2}{3}q_{t,t'}^2-\tau^2-\tau'^2\right)+q_{t,t_c}q_{t',t_c}+\int (q_{t,1}\Delta_{t',1}+q_{t',1}\Delta_{t,1})d1=0,$$

$$q_{t,t_c}\left(\varepsilon\ln\frac{q_{t,t_c}}{\sqrt{\tau\varepsilon}}\right)+\int \Delta_{t,1}q_{1,t_c}d1=0. \tag{41}$$

For τ, $\tau'\gg\sqrt{\varepsilon}$ the terms containing ε in (41) are small, and the solution is closed to that for the SK model:

$$\Delta(t,t')=2\tau'\frac{d\tau'}{dt'}\theta(t-t'),\quad q(t,t')=2\tau. \tag{42}$$

In the opposite limit, τ, $\tau'\ll\sqrt{\varepsilon}$, we can introduce the notation

$$\Delta(t,t')=\sqrt{\frac{\tau\varepsilon}{\tau'}}\,\Theta(t,t')\quad\text{and}\quad q(t,t')=\sqrt{\frac{\tau\tau'}{\varepsilon}}\,w(t,t')$$

and write these equations as

$$\Theta(t,t')w(t,t')+\int \Theta_{t,1}\Theta_{1,t'}d1=0,$$

$$\frac{1}{2}w^2(t,t')+\int(w_{t,1}\Theta_{t',1}+w_{t',1}\Theta_{t,1})d1$$

$$+\frac{2}{3}\tau\tau'-\tau^2-\tau'^2=0. \qquad (43)$$

We have not found analytical solutions of the Eqs. (43), but it seems evident that the solutions (which may be found numerically) interpolate smoothly between the results (40) and (42) corresponding to the regions τ, $\tau'\ll\varepsilon$ and τ, $\tau'\gg\sqrt{\varepsilon}$, respectively. In another words, there is no qualitative difference between the solutions in these two regions. This result looks quite surprising: Gardner[11] found using the static-replica approach that a second phase transition should exist in this model (for $\tau\sim\sqrt{\varepsilon}$), which is characterized by full-replica symmetry breaking. In the dynamic approach we have not found any evidence for such a transition.

5.3. Susceptibility and heat capacity

Now it is easy to find observable quantities such as finite-field cooled and zero-field cooled susceptibilities and heat capacity. Consider the region $\tau\ll\varepsilon$. The value of χ_{FC} is

$$\chi_{FC}=\int_{-\infty}^{t}G(t,t')\beta(t')dt'=\frac{1}{T_c}$$

to within $\tau\sqrt{\varepsilon}$, and hence it has a jump in the derivative with respect to temperature. Zero-field cooled susceptibility χ_{ZFC} is determined by the integral of the "fast" part of the response function:

$$\chi_{ZFC}=\int_{-\infty}^{t}[G(t,t')-\Delta(t,t')]\beta(t')dt'=\frac{1-\varepsilon-\tau}{T_c},$$

and has a jump at the transition point.

The heat capacity $C=dE/dT$ also has a jump; in order to derive it, the internal energy should be written in terms of functions $C(t,t')$ and $G(t,t')$:[17]

$$\beta(t)E(t)=-\int_{-\infty}^{t}C^{p-1}(t,t')G(t,t')dt'.$$

The derivative of this equation with respect to temperature yields a negative discontinuity in the heat capacity

$$\Delta C=C_{T_c-0}-C_{T_c+0}=-\varepsilon$$

(note that in the standard Landau theory of second-order phase transitions the heat capacity shows a positive discontinuity when the temperature decreases). Static theory[12] also predicts a jump in the heat capacity, but at lower (static transition) temperature.

6. CONCLUSIONS

We have studied the dynamics of the p-spin interaction Ising spin glass ($p=2+\varepsilon$) above and below the dynamic-transition temperature T_d [implicitly defined by Eq. (21)]. The discontinuous transition is found at zero and weak external magnetic fields, $b\leqslant b_{tr}\sim\varepsilon$, whereas at higher fields, $b\geqslant b_{tr}$, the transition is continuous and resembles (though definitely is not identical to) the SK model transition. Near the "tricritical point" $b=b_{tr}$ the dynamic exponent determining the rate of long-time relaxation right at the transition approaches zero: $\nu_1\propto\sqrt{b-b_{tr}}$.

In the glassy phase the history dependence is described quantitatively by the anomalous response and correlation functions $\Delta(t,t')$ and $q(t,t')$. We have derived equations for these functions and (in the case of zero external field) solved them in several regions of reduced temperature $\tau\ll1$. Very close to T_d, at $\tau\ll\varepsilon$, the main contribution to the anomalous response function $\Delta(t,t')$ comes from the δ-function term [cf. Eq. (35)], in agreement with the replica-theory solution.[11] Indeed, the one-step replica symmetry breaking found in Ref. 11 is commonly interpreted[9] as a signature of the instantaneous "appearance" of an exponentially large number of valleys right at the transition i.e., of the extensive configurational entropy $S_{con}\propto N$); those states do not acquire additional "fine structure" (and their number does not grow) as temperature decreases further. This structure of the equilibrium valleys would precisely agree with the "δ-function only" solution for $\Delta(t,t')$: the physical idea behind the slow-cooling approach is that the anomalous response is nonzero when the number of valleys grows (usually when T decrease) making it possible to lower the free energy by proper choice of the valley.

However, we found that even in the smallest-τ range the $\Delta(t,t')$ response contains also the smooth part [cf. Eq. (40)] corresponding, by the same logic, to continuous splitting of the valleys below T_d. The same conclusion was reached by a different route in a recent preprint.[9] The above results provide additional [cf. [17]] evidence that the structure of the valleys most relevant to slow dynamics is different from those responsible for the equilibrium Gibbs partition function (which is reflected already in the fact that the dynamic transition temperature is higher than the static one); for a discussion of additional aspects of the relation between dynamic and static quantities see Ref. 22.

Heat capacity and zero field cooled susceptibility have a downward jump at the transition temperature. Note that a similar jump in heat capacity has been observed experimentally in real liquid-glass transitions.[23]

At lower temperatures ($\tau\gg\sqrt{\varepsilon}$) the solution of the slow-cooling equations approaches that of the SK model,[2,5] whereas in the intermediate region $\varepsilon\ll\tau\ll\sqrt{\varepsilon}$ it interpolates between the above limiting cases. Since the solutions are qualitative similar in the regions $\tau\ll\varepsilon$ and $\tau\gg\sqrt{\varepsilon}$, we do not expect any additional phase transitions in the slow-cooling approach. Thus we found that the low-temperature properties of the SK ($p=2$) and $p=2+\varepsilon$ Ising glass models are rather similar, in spite of the drastic difference known to exist between the corresponding phase transitions.

Real experiments on glasses as well as Monte Carlo simulations of glassy systems are always done on a limited time scale, which makes it virtually impossible to observe Gibbs equilibrium properties which are the subject of static-replica theory; on the other hand, the slow-cooling approach seems to be most suited to describe finite-time experiments. Unfortunately, it does not seem possible to compare directly the present analytical results with Monte Carlo simulation since our calculations are done for small $\varepsilon\ll1$; however we expect qualitative features of our solution to survive, e.g., for $p=3$ Ising-glass model which could be simulated directly.

Let us note finally that potentially interesting problem which we have not studied here is the anomalous response behavior close to the tricritical point $b=b_{\rm cr}$ (where the dynamic exponents ν_1 and ν_2 tend to zero).

We are grateful to P. Chandra and L. Ioffe for many useful comments. This work was supported by International Science Foundation (Grant M6M000), Russian Fund for Fundamental Research (Grant 95-02-05720) and INTAS Grant 93-2492 (within the research program of the International Center for Fundamental Physics in Moscow).

APPENDIX 1

Let us first derive the generating functional $Z[\hat{\lambda}]$ for one Ising spin in an arbitrary field. The distribution function for (5) obeys the equation

$$\partial_t P\{\sigma,t\}+\hat{L}P\{\sigma,t\}=0, \qquad (44)$$

where

$$\hat{L}P\{\sigma,t\}=-\sum_{\{\sigma'\}}[w\{\sigma\to\sigma'\}P\{\sigma,t\}-w\{\sigma'\to\sigma\}P\{\sigma',t\}].$$

We will be also interested in the function $P(\sigma,t,\sigma',t')$, the probability of finding spin σ at time t provided that at time t' its value was σ'; $P(\sigma,t,\sigma',t')=0$, $t<t'$. This probability is the Green function of the operator $\partial_t+\hat{L}_\sigma$:

$$(\partial_t+\hat{L}_\sigma)P(\sigma,t,\sigma',t')=\delta(t-t')\,\delta_{\sigma,\sigma'}.$$

The solution of this equation is

$$P(\sigma,t,\sigma',t')=\frac{1}{2}[1+\sigma m_t(\sigma',t')], \quad t>t',$$

where

$$m_t(\sigma',t')=\sigma'\exp(t'-t)+\int_{t'}^{t}\exp(\tau-t)\tanh(\beta b_\tau)d\tau.$$

We also can write the solution of (44), i.e., the distribution function:

$$P(\sigma,t)=\frac{1}{2}[1+\sigma m_t(m_0,t_0)].$$

The process described by the equation (5) is Markovian. For this reason, the correlation function $C=\langle\sigma_{t_1}\sigma_{t_2}\cdots\sigma_{t_n}\rangle$ is determined only by the distribution and probability functions:

$$C=\sum_{\{\sigma_1,\sigma_2\cdots\sigma_n\}}\sigma_1 P(\sigma_1,t_1,\sigma_2,t_2)\sigma_2 \times P(\sigma_2,t_2,\sigma_3,t_3)\sigma_{n-1}P(\sigma_{n-1},t_{n-1},\sigma_n,t_n)\sigma_n \times P(\sigma_n,t_n) \qquad (45)$$

for $t_1>t_2>\cdots>t_n>t_0$.

Later it will be convenient to introduce the auxiliary functions

$$A(t)=\exp\left(\int_{-\infty}^{t}\hat{\lambda}(\tau)m(\tau)d\tau\right)$$

and

$$B(t)=m(t)\exp\left(\int_{\infty}^{t}\hat{\lambda}(\tau)m(\tau)d\tau\right),$$

where $m(t)$ obeys the equation

$$\partial_t m(t)=i\hat{\lambda}(t)[1-m^2(t)]-[m(t)-\tanh(\beta b)],$$
$$m(t_0)=m_0. \qquad (46)$$

These functions are connected by the relation

$$B(t)=\int_{-\infty}^{t}\exp(\tau-t)[\hat{\lambda}_\tau+\tanh(\beta b_\tau)]A(\tau)d\tau.$$

We assume that

$$\langle\sigma_2\cdots\sigma_n\rangle=\hat{\delta}_2\cdots\hat{\delta}_n Z_0\Big|_{\hat{\lambda}=0}, \quad t_2>\cdots>t_n,$$

where

$$Z_0[\hat{\lambda}]=\left\langle\exp\left(\int_{t_0}^{\infty}\hat{\lambda}(t)\sigma(t)dt\right)\right\rangle =\exp\left(\int_{t_0}^{\infty}\hat{\lambda}(t)m(t)dt\right), \qquad (47)$$

and prove the same equation for the n-point correlator.

On the one hand we can find using (45)

$$\langle\sigma_1\sigma_2\cdots\sigma_n\rangle=\langle\sigma_3\cdots\sigma_n\rangle\exp(t_2-t_1)+\langle\sigma_2\cdots\sigma_n\rangle \times\int_{t_2}^{t_1}\exp(\tau-t_1)\tanh(\beta b_\tau)d\tau. \qquad (48)$$

On the other hand, for $t>t_1>\cdots>t_n>t_0$

$$\begin{aligned}\hat{\delta}_1\hat{\delta}_2\cdots\hat{\delta}_n Z_0[\hat{\lambda}]\Big|_{\hat{\lambda}=0} &= \hat{\delta}_1\hat{\delta}_2\cdots\hat{\delta}_n A(t)\Big|_{\hat{\lambda}=0}\\ &=\hat{\delta}_2\cdots\hat{\delta}_n B(t_1)\Big|_{\hat{\lambda}=0}\\ &=\hat{\delta}_3\cdots\hat{\delta}_n\Big[\exp(t_2-t_1)A(t_2)+\int_{t_2}^{t_1}\exp(\tau-t_1)[\hat{\lambda}_\tau+\tanh(\beta b_\tau)]\hat{\delta}_2 A(\tau)d\tau\Big]\Big|_{\hat{\lambda}=0}\\ &=\hat{\delta}_3\cdots\hat{\delta}_n\Big[\exp(t_2-t_1)A(t)+\int_{t_2}^{t_1}\exp(\tau-t_1)\tanh(\beta b_\tau)\times\hat{\delta}_2 A(t)d\tau\Big]\Big|_{\hat{\lambda}=0},\end{aligned}$$

which is identical to (48).

The procedure for the generalizing the functional Z_0 to the case of interacting spins is described in Refs. 19 and 20. Let us write $C=\langle\sigma_1\sigma_2\ldots\sigma_k\rangle$, where the indices are the unions of the time and space arguments. Now one can introduce auxiliary fields h and $\hat{h}$ and expand in J the term with the interaction. The correlator C is

$$C=\sum_{n=0}^{\infty}\int\prod_{i=1}^{N}\mathscr{D}h_i\mathscr{D}\hat{h}_i\exp\left[\sum_{i=1}^{N}-i\int\hat{h}_i(h_i-\beta b)dt\right]\times\left\langle\frac{1}{n!}K_n\right\rangle, \tag{49}$$

where

$$K_n=\sigma_1\sigma_2\ldots\sigma_k\left(\sum_{i_1<i_2<\cdots<i_p}\int dtJ_{i_1\ldots i_p}i\hat{h}_{i_1}\sigma_{i_2}\cdots\sigma_{i_p}\right)^n,$$

and $\langle\ldots\rangle$ is the average over the dynamics of the noninteracting spins in the field h_i. $\langle K_n\rangle$ can be written as a variation of Z_0:

$$\sum_{n=0}^{\infty}\left\langle\frac{1}{n!}K_n\right\rangle=\frac{1}{n!}\sum_{n=0}^{\infty}\hat{\delta}_1\hat{\delta}_2\ldots\hat{\delta}_k\times\left(\sum_{i_1<i_2<\cdots<i_p}\int dtJ_{i_1\ldots i_p}i\hat{h}_{i_1}\hat{\delta}_{i_2}\cdots\hat{\delta}_{i_p}\right)^n\times Z_0\bigg|_{\lambda=0}=\hat{\delta}_1\hat{\delta}_2\ldots\hat{\delta}_k\exp\left(\sum_{i_1<i_2<\cdots<i_p}\times\int dtJ_{i_1\ldots i_p}i\hat{h}_{i_1}\hat{\delta}_{i_2}\cdots\hat{\delta}_{i_p}\right)Z_0\bigg|_{\lambda=0},$$

where $\hat{\delta}_t=\delta/\delta\hat{\lambda}(t)$. In expression (49) the integration over h and $\hat{h}$ can be performed, which replaces h by βb and $i\hat{h}$ by $\delta/\delta\beta b$. Thus, the generating functional Z is

$$Z=\hat{J}Z_0\bigg|_{h=\beta b},$$

where $\hat{J}$ is

$$\hat{J}=\exp\left(\sum_{i_1<i_2<\cdots<i_p}\int dtJ_{i_1\ldots i_p}\delta_{i_1}\hat{\delta}_{i_2}\cdots\hat{\delta}_{i_p}\right).$$

APPENDIX 2

We show how to derive the adiabatic generating functional (26), starting from the Langevin dynamics of soft spins:

$$\Gamma_0^{-1}\partial_t\sigma_i(t)=-\frac{\partial\beta H}{\partial\sigma_i(t)}+r_0\sigma_i-u\sigma_i^3+\xi_i(t). \tag{50}$$

Here H is the Hamiltonian (3), and $\xi_i(t)$ is a white noise with zero mean and variance

$$\langle\xi_i(t)\xi_j(t')\langle\rangle=2\Gamma_0^{-1}\delta_{ij}\delta(t-t').$$

The constants r_0 and u should tend to infinity in the Ising limit provided that $r_0/u=1$.

As was shown in Refs. 12 and 17, the average of this Langevin equation over disorder yields

$$\partial_t\sigma(t)=r_0\sigma(t)-u\sigma^3(t)+\int_{-\infty}^{t}dt'\hat{G}(t,t')\sigma(t')+h(t)+\xi(t) \tag{51}$$

with nonlocal noise

$$\langle\xi(t)\xi(t')\rangle=2\Gamma_0^{-1}\delta(t-t')+\hat{C}(t,t'). \tag{52}$$

The next step is to divide the functions C and G and noise ξ into slow and fast parts:[2,21]

$$C(t,t')=\tilde{C}(t-t')+q(t,t'),\quad G(t,t')=\tilde{G}(t-t')+\Delta(t,t'),\quad \xi(t)=\tilde{\xi}(t)+z(t), \tag{53}$$

$$\langle z(t)z(t')\rangle=\sqrt{\mu(t)\mu(t')}q^{p-1}(t,t').$$

Integration over fast noise leads to an equation for the slow magnetization[2] which replaces the Langevin equation in the adiabatic limit:

$$\langle\sigma\rangle_{\tilde{\xi}}=m(t)=\tanh(H_{\text{eff}}(t)), \tag{54}$$

where in the case of Ising spins we have

$$H_{\text{eff}}(t)=z(t)+\beta b(t)+\int_{-\infty}^{t}dt'\hat{\Delta}(t,t')m(t'). \tag{55}$$

Now we can easily construct the adiabatic generating functional $Z[\hat{\lambda}]$. Note that the correlation function can be written

$$C(t_1,\ldots,t_n)=\langle m(t_1)\ldots m(t_n)\rangle_z.$$

Thus,

$$Z[\hat{\lambda}]=\left\langle\int\mathscr{D}h\mathscr{D}\hat{h}J\exp(S)\exp\left[\int\hat{\lambda}\tanh(\beta b)\right]\right\rangle_z, \tag{56}$$

where

$$S=-i\int\hat{h}_t\left(h_t-z_t-\beta b_t-\int\hat{\Delta}_{t,t'}\tanh h_{t'}dt'\right)dt,$$

and the Jacobian is

$$J=\frac{\partial(h_t-z_t-\beta b_t-\int\hat{\Delta}_{t,t'}\tanh h_{t'}dt')}{\partial h_{t'}}=\exp\left[-\int\hat{\Delta}_{t,t}(1-\tanh^2 h_t)dt\right]=1,$$

since we should put $\Delta(t,t)=0$ in the adiabatic limit, as noted in Sec. 5.1. The average over z yields

$$S=-i\int\hat{h}(h-\beta b)-\frac{1}{2}\int\hat{h}_1\hat{h}_2\sqrt{\mu_1\mu_2}q_{1,2}^{p-1}+i\int\hat{h}_1\Delta_{1,2}\tanh h_2.$$

Then we expand $Z[\hat{\lambda}]$ over q and Δ and rewrite each term of the expansion as a variation of $Z_0=\exp(\int\hat{\lambda}\tanh h)$. After some algebra we obtain formula (26).

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