Symmetry breaking in fast bifurcational transitions

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It is observed that besides the conventional "stochastic" scenario of bifurcational transition to one of two equivalent (probabilistically symmetric) final states in nonlinear systems, a different—"dynamic"—scenario can be realized, having strong probability symmetry breaking due to the high speed of the transition. In a model example (the first period doubling bifurcation in the logistic mapping) the boundary is found dividing the stochastic (probabilistically symmetric) regime from the dynamic regime (having broken probability symmetry) of bifurcational transitions. It is shown that the critical (limiting) noise level is expressed in terms of the speed of the transition by a power law with a rather high exponent (around seven).
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1. INTRODUCTION

In bifurcations a nonlinear system acquires new stable equilibrium states. We consider the frequently encountered situation, in which two equivalent final states arise in the system, that is, states having identical energies, but differing in some non-energetic aspect, say, the phase (period-doubling bifurcations[1] in mappings, and parametric generators[2]) or polarization (polarization states in nonlinear optics[3]).

It is customary to assume that as a result of the action of noise one of these states is chosen according to the law of chance, so that the equivalent states 1 and 2 turn out to be symmetric in their probability:

$$P_1 = P_2 = 1/2. \tag{1}$$

We will call a bifurcational transition with probability symmetry a stochastic scenario. In practice, this scenario is realized for very slow (adiabatic) variation of the parameters of the system.

Historically it turned out that the stochastic scenario, presupposing probability symmetry of the final states, strongly overshadowed the other possible scenario of a bifurcational transition, specifically a fast bifurcational transition with symmetry breaking, wherein the probability of one of the final states turns out to be higher than that of the other: $P_1 > P_2$ or $P_1 < P_2$. An indication of the existence of transitions with probability symmetry breaking as a consequence of rapid variation of the controlling parameter of the system was given by Shishkova[4] and Neshtadt.[5] They noted that in the absence of noise the final state of a system with continuous parameters is completely determined by the initial conditions and that a "noise-free" bifurcational transition with finite speed is completely regular and completely predictable. We shall call such a transition scenario dynamic. According to the dynamic scenario, the probabilities P_1 and P_2 of the transitions to states 1 and 2 take, depending on the initial conditions, the extreme values

$$P_1 = 1, \quad P_2 = 0 \quad \text{or} \quad P_1 = 0, \quad P_2 = 1. \tag{2}$$

Of course, a purely dynamic and a purely stochastic scenario enter as limiting cases of the behavior of real systems with bifurcations, which, on the one hand, obey dynamic regularities and, on the other, are subject to the action of noise.

The striking difference in the behavior of nonlinear systems near a bifurcation in the absence and in the presence of noise stimulated us to analyze intermediate regimes, where the contributions of dynamic and stochastic factors are comparable in magnitude. Such an analysis has allowed us to estimate the ratio between the noise level and the rate of variation of the control parameter of the system for which the symmetry of the probability of winding up in a definite final state is broken and the stochastic scenario (1) gives way to the dynamic scenario (2).

Let us consider the question in the particular instance of the first period-doubling bifurcation in a dynamic system described by the logistic mapping. The particular nature of the treatment does not prevent us from seeing through to the general pattern of the problem. The main result of the analysis consists in establishing a boundary between the stochastic and dynamic scenarios. We will show that the boundary (critical) value of the noise level σ_c^2 is given by a power law: $\sigma_c^2 = CS^{\alpha}$, where S is the dimensionless rate of variation of

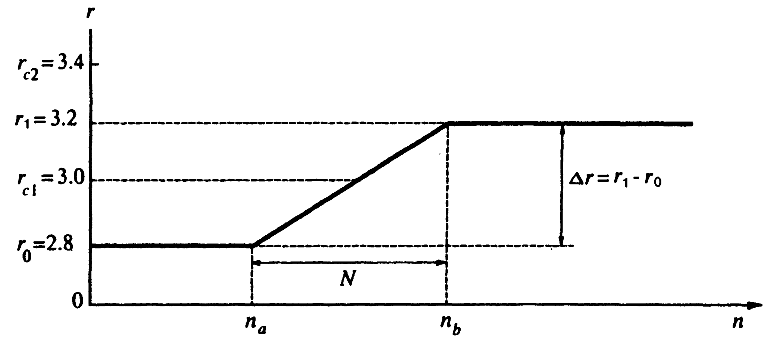

FIG. 1. Dependence of the control parameter r_n on the discrete time n, adopted for illustrative calculations.

the control parameter of the system, and C and α are constants. For $\sigma \ll \sigma_c$ the purely stochastic scenario (1) is realized, whereas for $\sigma \gg \sigma_c$ the bifurcational transition is controlled by the purely dynamic scenario (2) with strong probability symmetry breaking.

It seems to us that the proposed dynamic–stochastic model of bifurcational transitions should be useful in the analysis of many systems with symmetry breaking, including the excitation of oscillations with a definite phase in parametric oscillators and polarization effects in lasers.

2. MODEL OF A NOISY DYNAMIC SYSTEM WITH VARIABLE CONTROLLING PARAMETER

As a simple model of a dynamic system with bifurcation, we will use the standard example of the logistic mapping

$$x_{n+1}=rx_n(1-x_n), \quad n=1,\ldots,N. \tag{3}$$

We will assume the control parameter r in this mapping to be a variable quantity that depends on the discrete time n. In the numerical estimates, we assign the dependence of r_n on n by the piecewise-continuous function

$$r_n=\begin{cases} r_0, & n<n_a, \\ r_0+Sn, & n_a<n<n_b, \\ r_0+SN=r_1, & n>n_b, \end{cases} \tag{4}$$

where S, as above, is the rate of variation of the control parameter r and $\Delta r=r_1-r_0$ is the total variation of r. The quantity $N\equiv[\Delta r/S]=n_b-n_a$ denotes the integer part of the fraction $\Delta r/S$. We choose the initial value r_0 to be somewhat smaller than the first critical value $r_{c1}=3$, which corresponds to the first period-doubling bifurcation, and the final value r_1 to be greater than r_{c1} but less than the second critical value $r_{c2}=3.4$, at which the second period-doubling takes place. Thus, the indicated parameters satisfy the inequalities $r_0<r_{c1}<r_1<r_{c2}$. A graph of the dependence of r_n on n is shown in Fig. 1.

In addition, we introduce additive noise into the logistic mapping by replacing x_n by x_n+f_n. As the final result, the iteration procedure is prescribed by the relation

$$x_{n+1}=r_n(x_n+f_n)(1-x_n-f_n), \quad n=1,\ldots,N. \tag{5}$$

We assume the noise process f_n with zero mean $\langle f_n\rangle=0$ to be stationary. We assume the values of f_n at neighboring points in time to be uncorrelated: $\langle f_i f_j\rangle=\sigma^2\delta_{ij}$, where σ^2 is the variance of the process f_n.

In the absence of noise, after reaching the final state r_1 the sequence x_n can belong to one of two final "antiphase" states,

$$x_n^{1f}=\ldots,x^+,x^-,x^+,x^-,\ldots, \tag{6}$$

or

$$x_n^{2f}=\ldots,x^-,x^+,x^-,x^+,\ldots\,. \tag{7}$$

Here x^- and x^+ are the lesser and greater values of x_f in the limit cycle arising after the first period doubling (Fig. 2b). The antiphase character of the sequences x_n^{1f} and x_n^{2f} means that when x^{1f} takes the larger value x^+, x^{2f} takes the smaller value x^-, and conversely. This antiphase character is reflected in Fig. 2c. The phase difference between the sequences x^{1f} and x^{2f} can be expressed as a time delay $x_n^{2f}=x_{n-1}^{1f}$.

These sequences are depicted in Fig. 2c by a series with a bifurcation diagram (Fig. 2a) and the logistic mapping (Fig. 2b) possessing a limit cycle $x^f=\ldots,x^-,x^+,x^-,x^+,\ldots$.

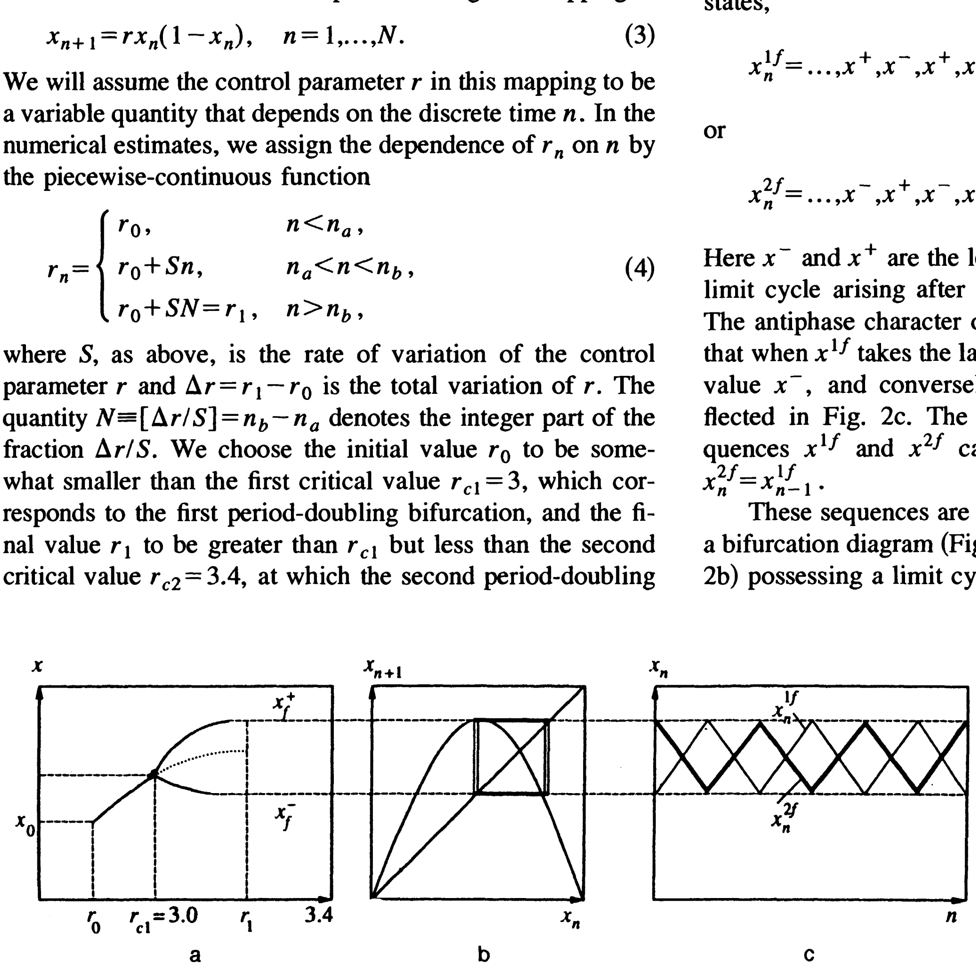

FIG. 2. Bifurcation diagram of the first period doubling (a), the limit cycle (b), and the antiphase sequences x_n^{1f} and x_n^{2f} for logistic mapping.

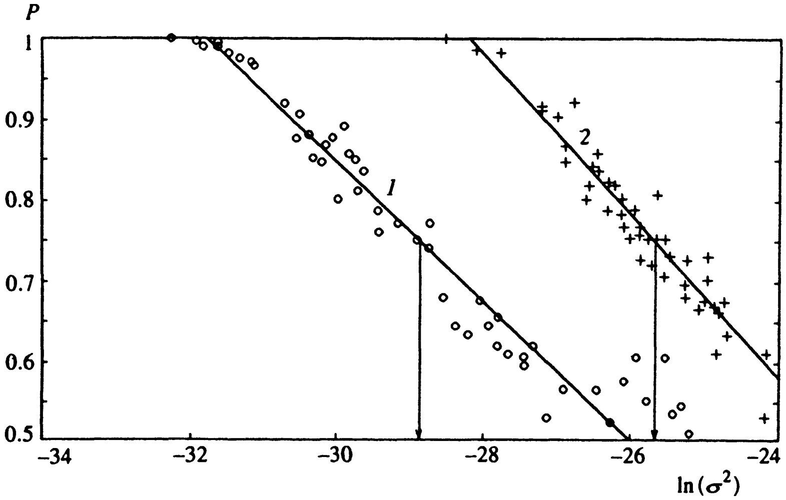

FIG. 3. Probability of reaching the first final state x_n^{1f} versus the logarithm of the noise intensity $\ln(\sigma^2)$ for two values of the normalized rate of change S of the controlling parameter: *1)* $S=0.004$, *2)* $S=0.008$.

3. PROBABILITY OF TRANSITION TO A PRESCRIBED FINAL STATE IN THE PRESENCE OF NOISE

The system described above was subjected to numerical analysis with the starting value $r_0=2.8$ and final value $r_1=3.2$ and various values of the speed S lying within the limits from 0.002 to 0.01. The iteration procedure was begun from the fixed point defined by the condition $x_0=1-1/r_0$. To obtain the random quantity f_n, we used a random number generator which produced values f normally distributed with variance σ^2.

For the first term of the sequence, x_1, we chose the value $x_1=x_0+f_1$. Subsequent iterations $x_2, x_3, \ldots$, etc., were determined by means of the mapping (5).

The calculations were aimed at determining the probability P_1 of winding up in the final state x_n^{1f} [Eq. (6)], which is realized in the given system in the absence of noise. The dependence of P_1 on the noise level for two values of the rate of change of the controlling parameter, 0.004 and 0.008, are shown in Fig. 3. Each point in the graph was obtained from 200 realizations of the process (5).

In the absence of noise, i.e., for $\sigma^2=0$, the investigated nonstationary system continuously approaches the sequence x_n^{1f}, so that $P_1=1$ and $P_2=0$. When noise is introduced, the probability P_1 begins to decrease, and at a high enough value of the noise intensity σ^2 it tends asymptotically to 1/2. The probability $P_2=1-P_1$ of the appearance of the antiphase state x_n^{2f} tends to the same value, 1/2, but from below.

The plots of P_1 vs the logarithm of the noise intensity σ^2 reveal a dependence that is nearly linear. From these curves we can estimate the critical noise level σ_c^2 at which the probability P_1 falls to the level 0.75, intermediate between the dynamic and stochastic regimes. The critical values $\ln(\sigma_c^2)$ are indicated in Fig. 3 by the arrows.

The dependence of the critical value of the noise level σ_c^2 on the normalized values of the speed $S=S/\Delta r=1/N$ is shown by the points in Fig. 4. The quantity $1/S=N$ tells how many steps it takes to cover the difference $\Delta r=r_1-r_0$. This dependence can be approximated by a power-law (the solid line in the figure) $\sigma_c^2=CS^\alpha$ with coefficient $C=1820$ and fairly large exponent $\alpha\simeq 7$.

For $\sigma^2 \ll \sigma_c^2$, noise has only a weak effect and the probability P_1 is essentially equal to unity. The dynamic scenario (2) is realized in this case. On the contrary, for $\sigma^2 \gg \sigma_c^2$ the stochastic, probabilistically symmetric scenario (1) is realized. It follows from what has been said that for the system to reliably wind up in the post-bifurcation state the transition must take place quite rapidly, in order to satisfy the inequality $\sigma^2 \ll \sigma_c^2 = CS^\alpha$.

4. POSSIBLE APPLICATIONS

In the literature, reports frequently appear about nonlinear systems in which the probability symmetry of the final equivalent states is broken. We will mention only two examples of physical phenomena. One is the breaking of polarization symmetry attendant on the passage of laser radiation through an active medium (see the review in Ref. 3 and the literature cited therein). It cannot be ruled out that the reason for marked polarization asymmetry in a number of cases can be transitional dynamic processes taking place during the formation of the polarization state of the generated secondary laser radiation.

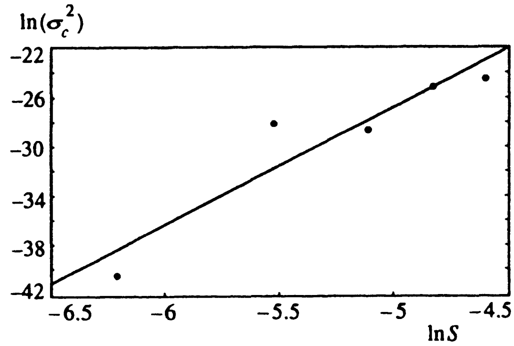

FIG. 4. Dependence of the critical noise level corresponding to the probability of realization of the first final state $P_1=0.75$ on the normalized rate of change S of the controlling parameter. The region above the curve corresponds to the stochastic bifurcation regime, and below—to the dynamic bifurcation regime.

Another example pertains to the formation of two equivalent phase states in degenerate parametric oscillators. Stable equilibrium states in such generators have identical amplitudes, but differ in phase by 180° (Ref. 1). Goto in 1959 (Ref. 6) and Akhmanov and Roshal' in 1964 (Ref. 7) proposed using the phase difference in the oscillations to create parametrons, i.e., phase memory cells.

At high noise levels the phase states of a parametron are probabilistically equivalent, but experiments have frequently recorded them as being probabilistically non-equivalent, especially when the parametron is switched on rapidly.[7] Such an asymmetry is directly connected with the dynamic scenario discussed above. If the "residue" of the previous oscillation in the parametron exceeds the noise level, then this residue will impose its phase on the new oscillation arising in the parametron after the bifurcational transition. This corresponds exactly to the dynamic scenario with broken symmetry of the phase states. If the amplitude of the residue is small in comparison with the noise, then the stochastic scenario with equiprobable phase states will be realized. The parametron weak-signal detector proposed in Ref. 7 makes use of symmetry breaking of the phase states as an indication of the presence of a signal.

It is not ruled out that the speed of the transition may play an important role (or, as they say, be a predisposing factor) in other bifurcation phenomena with symmetry breaking.[8–10]

5. CONCLUSION

In spite of the simplicity of the model chosen for analysis (first bifurcation of period doubling), the analysis performed here reflects the general regularities of fast bifurcational transitions in noisy nonlinear systems. Such regularities include the following:

1. The transition probability to a given final state is determined not only by the noise intensity, as was assumed earlier, but also by the speed of the transition. Depending on the ratio between the factors, a regime is realized in practice approximating either the stochastic scenario with probability symmetry of the final states, or the dynamic scenario with strong symmetry breaking. The conventional stochastic scenario with equiprobable final states corresponds in truth only to slow bifurcational transitions.

2. The boundary between the stochastic and dynamic regimes in the plane of the noise intensity σ and the speed S of the transition is defined by the power-law dependence $\sigma_c^2 = CS^\alpha$ with a fairly large exponent α (for first bifurcation of period doubling, $\alpha \simeq 7$).

The regularities we have identified can shed additional light on the nature of bifurcational transitions in which the symmetry of the final states is broken.

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