Wave–turbulence mechanism for the relaxation of nonequilibrium in shear flows

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(Submitted 26 December 1995)
Zh. Éksp. Teor. Fiz. **110**, 163–176 (July 1996)

A general principle is formulated, stating that nonequilibrium physical systems, irrespective of their degree of complexity, will tend to relax as rapidly as possible to the best equilibrated of all possible states. The acceleration of relaxation of vertical velocity shear in plane-parallel fluid flows with statically stable density stratification by the development of wave–turbulence instability is investigated as an example. © *1996 American Institute of Physics.* [S1063-7761(96)01307-8]

1. INTRODUCTION

One of the most timely problems in modern physics is how to describe the evolution of complex physical systems with a source of nonequilibrium and, in particular, to find criteria suitable for the development of methods to achieve such a description in systems of diverse physical nature. A general principle can be formulated on the basis of thermodynamic considerations: regardless of the complexity of a particular physical system, it tends to evolve in such a way as to achieve the fastest possible relaxation to the state of maximum equilibrium in the set of all states possible under the given conditions.

This principle applies to systems of extremely diverse nature. The following phenomena can be considered as examples of this principle: the relaxation of multiparameter nonequilibrium in a magnetized inhomogeneous plasma with an electron beam through the development of secondary instability when the primary instability in the relaxation of one source of nonequilibrium (the electron beam) accelerates the relaxation of the other source—the inhomogeneity of the plasma across the magnetic field;[1] the acceleration of relaxation of inhomogeneity of a plasma in the ionosphere through spontaneous breaking of the initial one-dimensional symmetry of the system[2,3]; the acceleration of relaxation of shear inhomogeneity of the velocity of plane-parallel flow of a stably stratified fluid in a gravity field through the development of wave–turbulence instability.[4–11] The last in this list of accelerated relaxation mechanisms can appear not only in hydrodynamic flows, but also in plasma–beam systems.[12,13] In the present article we give the results of an analysis of its occurrence in the hydrodynamics of a fluid situated in a homogeneous gravity field.

The presence of such a field in many cases causes the properties of the fluid in the direction of the field (vertical) to change far more rapidly than in directions perpendicular to it (horizontal). This, in turn, creates conditions for the motions of the fluid to be separated into large-scale and small-scale components relative to one of the characteristic vertical scales. Here the large-scale and small-scale motions can be distinguished by type: the former can appear as regular waves, and the latter as fully developed small-scale turbulence. Such physical systems can therefore be analyzed by means of a model comprising three components: 1) average plane-parallel fluid flow with statically stable stratification of the density $\rho_0(z)$ and a vertically nonuniform distribution of one (the horizontal) component of the velocity $U(z)$ (z is the upward-directed vertical coordinate); 2) large-scale, regular wave motions; 3) small-scale turbulence.

The presence of stable density stratification, which occurs for real values of the Brunt–Väisälä frequency N determined from the expression[14]

$$N^2=-\frac{g}{\rho_0}\frac{d\rho_0}{dz}-\frac{g^2}{c_s^2}, \qquad (1)$$

where g is the acceleration of gravity, and c_s is the sound velocity, enables the flow to remain laminar at indefinitely high Reynolds numbers if the vertical velocity shear $U'\equiv dU/dz$ is not sufficient to overcome the stabilizing influence of stratification and if horizontal velocity shear and external forces capable of maintaining turbulence are nonexistent. The vertical velocity shear required for the development of turbulence is expressed in terms of the dimensionless parameter known as the dynamic Richardson number: $\mathrm{Ri}=N^2/U'^2$ (Ref. 15). If $\mathrm{Ri}>1/4$, infinitesimally small disturbances in the flow cannot grow exponentially; otherwise this restriction is lifted.[16]

Turbulent fluctuations as disturbances of small but finite amplitude can have a threshold Richardson number $\mathrm{Ri}_{\mathrm{cr}}$ close to 1/4, imposing an upper bound on the range of values for which the fluctuations can grow. The deviation of $\mathrm{Ri}_{\mathrm{cr}}$ from 1/4 can be of either sign, depending on the nature of the nonlinearity. If the Richardson number exceeds this critical value throughout the entire flow, relaxation of the nonuniform velocity distribution is suppressed. One way to accelerate such relaxation was proposed earlier.[4] It ties in with the fact that the corresponding relaxation process can involve regular large-scale (relative to the characteristic space and time scales of the turbulence) disturbances, or internal waves. In the course of its propagation such a wave modulates the initial distribution $\mathrm{Ri}(z)$, and if the stability reserve is small in a certain layer $z_0-h/2<z<z_0+h/2$, i.e., if $\mathrm{Ri}(z)=\mathrm{Ri}_{\mathrm{cr}}+\delta\mathrm{Ri}$ and $\delta\mathrm{Ri}\ll 1$, it can become smaller than $\mathrm{Ri}_{\mathrm{cr}}$ in the part of the wave period where $\mathrm{Ri}(z_0)$ decreases, and turbulence can grow in this layer.

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In the simplest model, which enables us to avoid issues such as the nature of the transition from the laminar to the turbulent state, we can assume that the system is subjected to a small external force capable of maintaining a certain minimum possible level of turbulence in the given layer in the part of the wave period where it causes $\mathrm{Ri}(z_0)$ to increase and thus further stabilizes the flow. If the other, flow-destabilizing part of the wave period is long enough for the turbulence energy to rise well above this minimum possible level, the feedback effect of turbulence on the wave is determined mainly by its parameters in the destabilizing phase of the wave. In this case the turbulence represents a periodic sequence of patches traveling along the diminished-stability layer together with the destabilizing parts of each wave period. The influence of such turbulence on the wave can differ significantly from the influence of spatially homogeneous turbulence, which causes the wave to be damped by turbulent viscosity.

Several authors[4–9] have investigated analytically and numerically the conditions under which such influence can result in growth of the wave amplitude and, with it, the level of the turbulence induced by it (wave–turbulence instability). However, present-day models of turbulence in a stratified fluid afford the possibility of describing more accurately the influence of stratification on the development of turbulence than the model used in the cited papers.[17,18] In particular, a numerical model proposed in Ref. 19 can be adapted to the analytic investigation of the conditions underlying the development of wave–turbulence instability. The results of such an adaptation are briefly summarized below in Sec. 2. They are described in greater detail in Refs. 10 and 11. In Sec. 3 we use the adapted model to refine the conditions for the development of wave–turbulence instability.

2. DYNAMICS OF HOMOGENEOUS TURBULENCE IN PLANE-PARALLEL, STRATIFIED, SHEAR FLOW AND CHARACTERISTICS OF THE RELAXATION OF VERTICAL INHOMOGENEITY IN THE NEAR-THRESHOLD REGIME

The simplest interaction of a wave with the turbulence induced by it can be described when the turbulence dynamics is determined by local wave fields, and the influence of the resulting vertical inhomogeneity of its parameters can be disregarded. To determine the conditions of validity of this approximation, we begin by investigating the turbulence dynamics in the spatially homogeneous case (when the flow parameters N, U' and hence Ri do not depend on the coordinates) and then follow up with the specific characteristics of the relaxation of vertical inhomogeneity of the turbulence and flow parameters in the near-threshold case, when the Richardson number throughout the entire flow differs very little from its critical value ($\mathrm{Ri}=\mathrm{Ri}_{\mathrm{cr}}+\delta\mathrm{Ri}(z)$, where $|\delta\mathrm{Ri}|\ll 1$) on the basis of the above-selected model of the description of turbulence,[19] in which turbulence is characterized by two independent parameters: the turbulent energy density b and its dissipation rate ε. The basic turbulence equations in the given situation can be written in the form

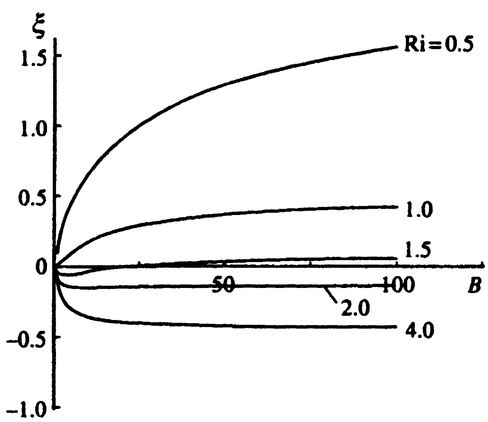

FIG. 1. Graphs of $\xi(B)$ for various values of the Richardson number Ri.

$$\frac{\partial b}{\partial t}=\frac{\partial}{\partial z}\left(\nu_T\frac{db}{dz}\right)+\nu_T U'^2(1-\mathrm{Rf})-\varepsilon, \tag{2}$$

$$\frac{\partial \varepsilon}{\partial t}=\frac{\partial}{\partial z}\left(\frac{\nu_T}{\sigma_\varepsilon}\frac{\partial\varepsilon}{\partial z}\right)+c_{\varepsilon 1}\nu_T\frac{\varepsilon}{b}U'^2-c_{\varepsilon 2}\frac{\varepsilon^2}{b}, \tag{3}$$

$$\frac{\partial U'}{\partial t}=\frac{\partial^2(\nu_T U')}{\partial z^2}, \tag{4}$$

$$\frac{\partial N^2}{\partial t}=\frac{\partial}{\partial z}\left(\frac{\nu_T}{\sigma_T}\frac{\partial^2 N^2}{\partial z^2}\right). \tag{5}$$

Here σ_ε, $c_{\varepsilon 1}$, and $c_{\varepsilon 2}$ are empirical constants of the model, ν_T is the turbulent viscosity, σ_T is the ratio of the turbulent viscosity and thermal conductivity conditions (the analog of the Prandtl number), and $\mathrm{Rf}=\mathrm{Ri}/\sigma_T$ is the dynamic Richardson number. Previously[19] the best match of the results of numerical calculations with experimental data has been attained for the following choice of constants:

$$\sigma_\varepsilon=1.3,\quad c_{\varepsilon 1}=1.45,\quad c_{\varepsilon 2}=1.9. \tag{6}$$

In the model ν_T and σ_T are determined by means of algebraic relations, which in their original form[19] can be used to close the system of equations (2)–(5) in each step of the numerical calculations. We can use them for the analytic derivation of an algebraic equation, which closes the system (2)–(5), interrelates the parameters Ri, $\xi=\nu_T U'^2(1-\mathrm{Rf}/\varepsilon$, and $B\equiv N^2b^2/\varepsilon^2$ (Ref. 10), and involves empirical constants of the model as parameters. Graphs of $\xi(B)$ determined from this equation for several values of Ri are shown in Fig. 1.

In the spatially homogeneous case, when N, U', b, and ε do not depend on z, the system (2)–(5) reduces to a system of two autonomous ordinary differential equations in b and ε, since N and U' remain constant and influence of the behavior of b and ε only parametrically, where the form of the phase trajectories in the $b\varepsilon$ plane is determined by the value of one parameter—the gradient Richardson number—and the dimensioned values of N and U' determine the time scale of motion along these phase trajectories.

For any nonnegative value of Ri the evolution of turbulence can be divided into two stages: a transition stage and a self-similar stage. The second stage characterizes the behavior of turbulence in the limit $t\rightarrow\infty$. For small values of Ri in

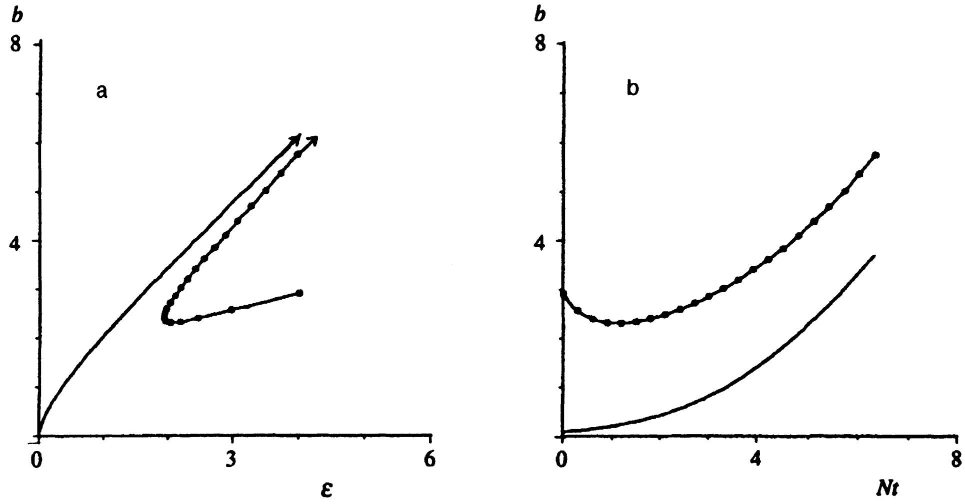

FIG. 2. Phase trajectories in the $b\varepsilon$ plane (a) and turbulence energy vs. time $b(t)$ (b) for $\mathrm{Ri}=0.1<\mathrm{Ri}_{\mathrm{cr}}$. The arrows indicate the direction of motion of the system along the phase trajectory.

the second stage, once a self-similar relation between b and ε has been established, the turbulence energy increases exponentially, although it can decay in the first stage if the initial value of ε is specified too high, corresponding exactly to the limiting case of an unstratified fluid ($\mathrm{Ri}=0$).[20] Typical examples of phase trajectories and time variations of the turbulence energy for this case are shown in Fig. 2.

As Ri increases, the growth rate of turbulence in the second stage decreases, reaching zero upon attainment of the critical value $\mathrm{Ri}=\mathrm{Ri}_{\mathrm{cr}}$, which depends on the choice of empirical constants of the model. If the constants are chosen according to the recommendations of Ref. 19, we have $\mathrm{Ri}_{\mathrm{cr}}\approx 0.225$, i.e., the threshold value is slightly lower than 1/4. In the case $\mathrm{Ri}=\mathrm{Ri}_{\mathrm{cr}}$, b and ε tend monotonically in the self-similar stage to certain limits, which depend on the initial conditions. If the ratio of the initial values of b and ε is equal to the self-similar value, both parameters remain equal to their initial values. Corresponding examples of the phase trajectories and $b(t)$ curves are shown in Fig. 3.

When Ri becomes greater than $\mathrm{Ri}_{\mathrm{cr}}$, the turbulence energy tends to zero in the self-similar stage, where three turbulence decay regimes can be discerned, one superseding the other in succession as Ri increases. These regimes are further separated by two special values of the Richardson number Ri_1 and Ri_2, which are also dictated by the choice of constants of the model. For the above-indicated choice we have $\mathrm{Ri}_1\approx 0.745$ and $\mathrm{Ri}_2\approx 1.8$. If $\mathrm{Ri}_{\mathrm{cr}}<\mathrm{Ri}<\mathrm{Ri}_1$, the turbulence energy can increase in the transition stage for sufficiently low initial values of ε and can enter into self-similar decay after attaining a maximum value far in excess of the initial value. Examples of phase trajectories and $b(t)$ curves for this case are shown in Fig. 4. For $\mathrm{Ri}>\mathrm{Ri}_1$ the decay becomes monotonic for any initial values of b and ε (see Fig. 5); in this event, when $\mathrm{Ri}>\mathrm{Ri}_2$, the exponential decay in the self-similar stage is superseded by power-law decay: $b(t) \propto t^{-\alpha(\mathrm{Ri})}$, where the power exponent α varies nonmonotonically with the Richardson number and reaches a minimum value $\alpha_{\min}\approx 17$ at $\mathrm{Ri}\approx 5$.

For values of Ri such that the turbulence energy in the self-similar stage increases or decreases exponentially, its dissipation rate varies in direct proportion to its value, with a proportionality coefficient that depends on Ri. Here the turbulence macroscale, which is specified by the expression

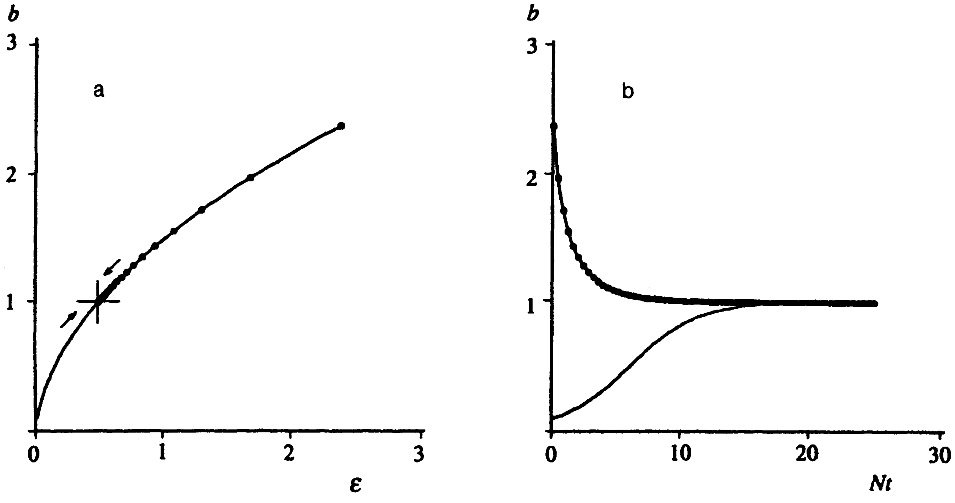

FIG. 3. The same as Fig. 2, for $\mathrm{Ri}=\mathrm{Ri}_{\mathrm{cr}}\approx 0.225$. The large + symbol indicates the steady state to which the system tends in motion along the plotted phase trajectories.

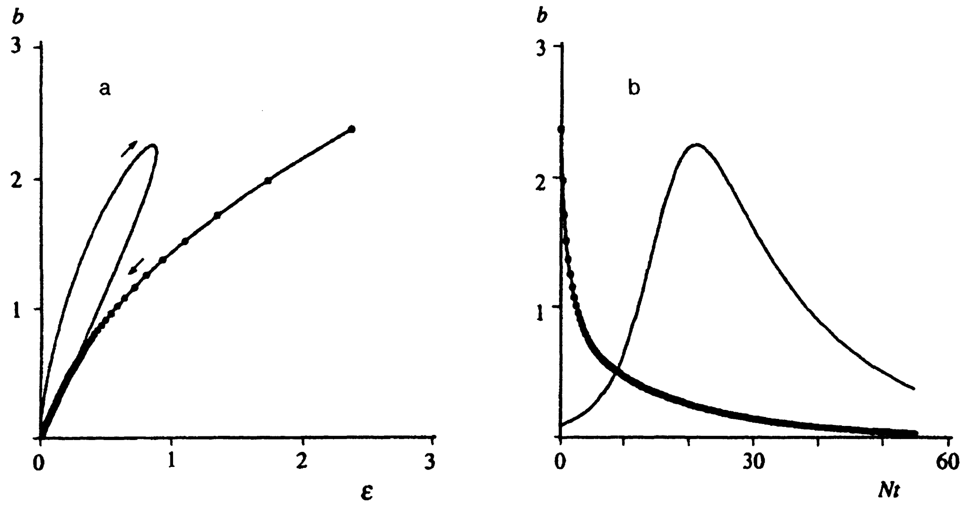

FIG. 4. The same as Fig. 2, for $\mathrm{Ri}=0.3>\mathrm{Ri}_{cr}$.

$L\equiv\sqrt{b^3/\varepsilon^2}$ and characterizes the size of the largest vortices, also increases or decreases exponentially together with the turbulence energy and its dissipation rate in proportion to the square root of the turbulence energy. For small deviations of Ri from Ri_{cr} the characteristic time constant of such growth or decay is much longer than the duration of the transition stage $\tau^*\sim 1/N$. If Ri varies with a characteristic time $\tau\gg\tau^*$ in this case, the evolution of turbulence corresponds to the self-similar regime for the instantaneous value of Ri and is described by the single equation

$$\frac{db}{dt}=-\zeta bN(\mathrm{Ri}-\mathrm{Ri}_{cr}), \tag{7}$$

where ζ is determined by the values of the model constants and is close to unity for the above-indicated choice of constants.

Assuming that the variations $\delta\,\mathrm{Ri}\equiv\mathrm{Ri}-\mathrm{Ri}_{cr}$ are induced by a wave disturbance with a local frequency $\Omega\ll N$, with a period $T\equiv 2\pi/\Omega\gg\tau^*$, it is readily inferred from Eq. (7) that even for a small but finite amplitude of this disturbance $\tau^*/T\ll|\delta\,\mathrm{Ri}|\ll 1$ it can lead to appreciable variation of the turbulence energy: $b_{\max}/b_{\min}\sim\exp(|\delta\mathrm{Ri}|T/\tau^*)\gg 1$.

We now consider the relaxation dynamics of vertical inhomogeneity in the distribution of b and δRi with a view toward determining the conditions under which such relaxation does not lead to appreciable deviations from the above-described local turbulence dynamics in the field of wave-induced disturbances δ Ri.

According to Eqs. (4) and (5), vertically inhomogeneous perturbations of the Richardson number, i.e., at least one of the two parameters N and U', disrupt the steady state of these parameters under the influence of the resulting divergence of the turbulent mass or momentum fluxes. These fluxes are caused to diverge both by vertical inhomogeneity of the flow parameters and by the vertical inhomogeneity induced by them in the turbulent transport coefficients.

We compare the relative influence of these two factors in the example of the momentum flux:

$$\frac{\delta(\nu_T U')}{\nu_T U'}=\frac{\delta\nu_T}{\nu_T}+\frac{\delta U'}{U'}. \tag{8}$$

In the self-similar regime the algebraic relations of the model for the determination of ν_T can be reduced to the form

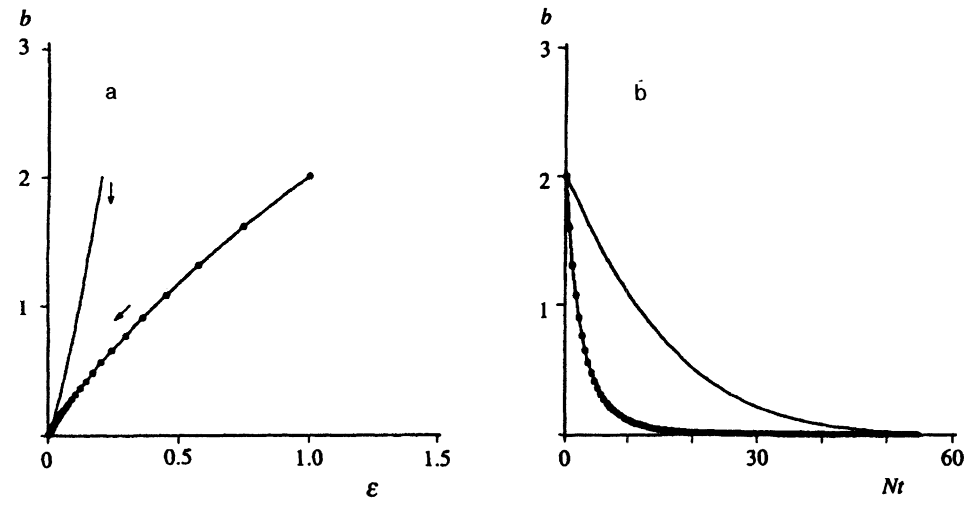

FIG. 5. The same as Fig. 2, for $\mathrm{Ri}=1>\mathrm{Ri}_{cr}$.

$$\nu_T = c_\nu(\mathrm{Ri})b/N, \tag{9}$$

where the form of the function $c_\nu(\mathrm{Ri})$ is determined by the choice of constants of the model. For the above-indicated choice of these constants we have $c_* \equiv c_\nu(\mathrm{Ri}_{cr}) \approx 0.15$. Taking Eq. (7) into account, we deduce the following estimate from (9) (Ref. 10):

$$\frac{\delta\nu_T}{\nu_T} = 2\mathrm{Ri}_{cr}\frac{\delta U'}{U'}\left(\zeta NT - \frac{dc_\nu}{c_\nu d\mathrm{Ri}}\right). \tag{10}$$

In the given situation $NT \gg 1$ we have $\delta\nu_T/\nu_T \gg \delta U'/U'$. This means that the perturbations of the turbulent fluxes are determined mainly by the inhomogeneity of the turbulent transport coefficients and not by the wave field, hence one should not expect the feedback effect of such fluxes to be reducible to ordinary dissipation.

Keeping only the first of the above-mentioned components of the turbulent fluxes in Eqs. (4) and (5) and replacing Eqs. (2) and (3) by the self-similar equation (7), for the case of small deviations from the homogeneous steady state, i.e., for $\mathrm{Ri}(z,t) = \mathrm{Ri}_{cr} + \delta\,\mathrm{Ri}(z,t)$, $b(z,t) = b_0 + \delta b(z,t)$ [from which it follows that $\nu_T(z) = \nu_0 + \delta\nu_T(z,t)$], and $|\delta\,\mathrm{Ri}|/\mathrm{Ri}_{cr}, |\delta b|/b_0, |\delta\nu|/\nu_0 \ll 1$ we can obtain a closed system of equations describing the relaxation of perturbations of the homogeneous vertical distribution of the Richardson number and the turbulent viscosity coefficient:

$$\frac{\partial\delta\,\mathrm{Ri}}{\partial t} = \left(\frac{1}{\sigma_T} - 2\right)\mathrm{Ri}_{cr}\frac{\partial^2\nu_T}{\partial z^2}, \tag{11}$$

$$\frac{\partial\nu_T}{\partial t} = -\zeta\nu_0 N(\mathrm{Ri} - \mathrm{Ri}_{cr}). \tag{12}$$

Equations (11) and (12) are reducible to the single equation

$$\frac{\partial^2\nu_T}{\partial t^2} = \zeta\nu_0 N\left(2 - \frac{1}{\sigma_T}\right)\mathrm{Ri}_{cr}\frac{\partial^2\nu_T}{\partial z^2}. \tag{13}$$

It follows from this equation that under the given conditions large-scale perturbations of the homogeneous turbulence field do not spread out diffusively but propagate in the form of waves traveling vertically upward or downward with the phase velocity

$$c_p = \sqrt{\zeta\nu_0 N\left(2 - \frac{1}{\sigma_T}\right)\mathrm{Ri}_{cr}}, \tag{14}$$

which does not depend on the scale of the disturbance, i.e., is free of dispersion distortions. This is true when the turbulent transport of turbulence energy and its dissipation rate can be disregarded in comparison with its generation or decay by variation of the Richardson number. Comparing the characteristic time constants of these processes for a periodic disturbance with wave number K, i.e., $\tau_{\mathrm{dis}} = (\nu_0 K^2)^{-1}$ for the first and $\tau_{\mathrm{dyn}} = (c_p K)^{-1}$ for the second, we can obtain the condition for the wave propagation of disturbances to prevail over diffusion propagation:

$$K^2 L^2 \ll 1. \tag{15}$$

Consequently, wave propagation occurs for disturbances with a characteristic scale much greater than the turbulence macroscale. Since the latter, as mentioned, is proportional to $\sqrt{b}$ in the self-similar regime, condition (15) can be expected to hold for small values of b.

The wave relaxation of inhomogeneity can also take place when the unperturbed turbulence distribution is energized by an external driving force that offsets its decay for a slight excess of the uniformly distributed Richardson number over Ri_{cr}: $\mathrm{Ri} = \mathrm{Ri}_{cr} + \delta\,\mathrm{Ri}_0$, $0 < \delta\,\mathrm{Ri}_0 \ll \mathrm{Ri}_{cr}$. Now, however, not only is the range of wave numbers limited above by condition (15), but also below by the condition

$$K^2 L^2 \gg (\delta\,\mathrm{Ri}_0)^2, \tag{16}$$

whose satisfaction ensures a small oscillation period in comparison with the relaxation time of homogeneous turbulence to the unperturbed level determined by the balance of generation and decay. Adding the generation rate P_0 to Eq. (7) and characterizing the unperturbed turbulence level by the corresponding value of the turbulent viscosity ν_0, which is related to the turbulence energy level by Eq. (9), we obtain

$$\nu_0 = \frac{c_* P_0}{\zeta N^2 \delta\,\mathrm{Ri}_0}. \tag{17}$$

Equation (17) can be used to estimate the minimum value of P_0 needed to maintain turbulence: it must be sufficient for ν_0 to remain much larger than the molecular viscosity ν. In this case the same value of P_0 can be sufficient to maintain turbulence in the diminished-stability layer, where $\delta\,\mathrm{Ri}_0$ is small, and insufficient in other layers, where $\delta\,\mathrm{Ri}_0/\mathrm{Ri}_{cr} = O(1)$. We assume that the thickness of the diminished-stability layer (h), which governs the characteristic scale of vertical inhomogeneity of the distribution of unperturbed values of $\delta\,\mathrm{Ri}_0$ and ν_0, is much greater than the turbulence macroscale corresponding to the lowest possible turbulence energy level. Relaxation of the inhomogeneity of the wave-induced disturbances can then take place predominantly as a result of variation of the average flow by turbulent transport, and the self-diffusion of turbulence plays a secondary role. Consequently, the condition for locality of the development of turbulence in the wave field can be obtained from Eq. (11). In order for the average flow not to vary appreciably in one wave period, it is necessary that the buildup of turbulent viscosity in the destabilizing phase of the wave be limited by the condition

$$\int_t^{t+T} \nu_T(t)\,dt \ll \frac{h^2\delta\,\mathrm{Ri}_0}{\left(2 - \frac{1}{\sigma_T}\right)\mathrm{Ri}_{cr}}. \tag{18}$$

3. INFLUENCE OF A WAVE ON THE TURBULENCE DYNAMICS AND FEEDBACK EFFECT OF TURBULENCE ON THE WAVE

When condition (18) is satisfied, the evolution of turbulence in a wave field with a spatial period much greater than the thickness of the diminished-stability layer can be described by the following equation in a reference system where $U(z_0) = 0$:

$$\frac{db}{dt} = -\zeta bN(\mathrm{Ri}-\mathrm{Ri}_{\mathrm{cr}}) + P_0, \tag{19}$$

where the rate of generation of turbulence energy by the external force P_0 is assumed to be sufficient for maintaining turbulence in the stabilizing phase of the wave and, at the same time, low enough not to significantly influence the evolution of turbulence in the destabilizing phase. These constraints can be satisfied simultaneously under the condition

$$\frac{\delta\,\mathrm{Ri}_0}{\mathrm{Ri}_{\mathrm{cr}}} \gg \frac{\nu T}{h^2}\left(2-\frac{1}{\sigma_T}\right),$$

which, in turn, is consistent with the assumption of smallness of $\delta\,\mathrm{Ri}_0$ if the thickness of the diminished-stability layer and the wave period are related in such a way as to ensure a sufficiently large value of the corresponding effective Reynolds number $\mathrm{Re}_{\mathrm{eff}} = h^2/\nu T$.

Under these conditions it can be assumed that the wave field perturbations induced by turbulent transport are small and that the wave-induced deviation $\delta\,\mathrm{Ri}_w$ of the Richardson number is determined by the spatial structure of the wave, undistorted by the influence of turbulence. We assume, in addition, that the wave amplitude is sufficiently small that it does not have time to develop nonlinear self-action during interaction with the turbulence induced by it, the unperturbed wave structure is determined by the solution of the linear boundary-value problem, and the influence of turbulence can be represented by a small correction. In this case the internal wave can be described by the distribution of vertical fluid displacements induced by it in the form

$$\eta(x,z,t) = [A(t)F(z) + \eta_1(t,z)]\exp[i(\omega t - kx)] + \text{c.c.}, \tag{20}$$

where $F(z)$ describes the vertical mode structure of the unperturbed wave, $\eta_1(t,z)$ is a small correction associated with the influence of turbulence, $A(t) \equiv |A|\exp(i\varphi)$ is the complex wave amplitude, which can vary under the influence of turbulence with a characteristic time constant much greater than the wave period $T = 2\pi/\omega$, and x is the horizontal coordinate in the direction parallel to the vector $\mathbf{U}(z)$, whose sign is chosen to satisfy the condition $U'(z_0) > 0$. In addition to the foregoing restrictions on the frequency and the wave number, we also assume that the local frequency $\Omega(z) = \omega - kU(z)$ neither changes sign nor vanishes anywhere along the entire thickness of the flow.

The wave-induced deviation of the Richardson number from the unperturbed value in the diminished-stability layer for such a wave can be written in the form

$$\delta\,\mathrm{Ri}_w = -2|A|S(z)\cos(\omega t - kx + \varphi), \tag{21}$$

where

$$S(z) = 2\frac{U'(z)}{c-U(z)}\mathrm{Ri}_{\mathrm{cr}}F(z) - F'(z). \tag{22}$$

For a constant amplitude A the turbulence energy equation (19) has a periodic solution:

$$b(t) = \frac{P_0}{1-\exp(\zeta NT\delta\,\mathrm{Ri}_0)}\int_{t-T}^{t}\exp\left\{-\zeta N\int_{t'}^{t}[\delta\,\mathrm{Ri}_w(t'') + \delta\,\mathrm{Ri}_0]dt''\right\}dt', \tag{23}$$

which can be used to evaluate all the turbulence parameters, including the turbulent viscosity ν_T, for the instantaneous value of the complex wave amplitude $A(t)$ when it varies slowly under the influence of turbulence.

This influence can be assessed by a standard asymptotic "slow" perturbation procedure (see Refs. 7 and 9 for the details). This procedure makes it possible to obtain an equation for the modulus of the complex wave amplitude from the boundedness condition on the correction η_1:

$$2J\frac{d|A|}{dt} = \int_{z_1}^{z_2}\mathrm{Re}\{e^{-i\varphi}\nu_T^{(\omega,k)}\}U'(z)G(z)dz, \tag{24}$$

where

$$J = 2\int_{z_1}^{z_2}\Omega\left[F^2 + \left(\frac{F'}{k}\right)^2\right]dz, \tag{25}$$

z_1 and z_2 are the lower and upper boundaries of the flow, respectively,

$$\nu_T^{(\omega,k)}(z) = \frac{\omega k}{(2\pi)^2}\int_t^{t+T}dt'\int_x^{x+2\pi/k}\nu_T(x',z,t') \times\exp[i(kx'-\omega t')]dx' \tag{26}$$

is the wave-resonance component of the perturbation of the turbulent viscosity, and

$$G(z) = \frac{U'}{\Omega}\left[\mathrm{Ri}_{\mathrm{cr}}\left(1+\frac{1}{\sigma_T}\right)\frac{U'F}{c-U} - \left(2-\frac{\mathrm{Ri}_{\mathrm{cr}}}{\sigma_T}\right)F'\right]. \tag{27}$$

Consequently, the condition for growth of the wave amplitude is a positive value of the integral on the right-hand side of Eq. (24). It follows from (21), (23), and (26) that the sign of the quantity $\mathrm{Re}\{\exp(-i\varphi)\nu_T^{(\omega,k)}\}$ is determined by the sin of the function $S(z)$. If the thickness of the diminished-stability layer is small in comparison with the characteristic space scale of the vertical structure of the wave, the sign of the integral is determined by the sign of the integrand at $z = z_0$. Assuming for definiteness that $\Omega > 0$ (in this case the direction of wave propagation is specified by the sign of the wave number k and the phase velocity c), we can write the condition of wave growth by interaction with the turbulence induced by it, i.e., the wave–turbulence instability condition, in the form

$$\left(\frac{F'}{F} - 2\frac{U'}{c-U}\mathrm{Ri}_{\mathrm{cr}}\right)\left[\mathrm{Ri}_{\mathrm{cr}}\left(1+\frac{1}{\sigma_T}\right)\frac{U'}{c-U} - \left(2-\frac{\mathrm{Ri}_{\mathrm{cr}}}{\sigma_T}\right)\frac{F'}{F}\right] > 0, \tag{28}$$

where all functions are evaluated at $z = z_0$.

This condition is readily established if the function $F(z)$ has at least two extrema of opposite signs and crosses zero between them. Between the extremum points the ratio F'/F then runs through all values from $-\infty$ to $+\infty$, including those which satisfy condition (28). If the diminished-

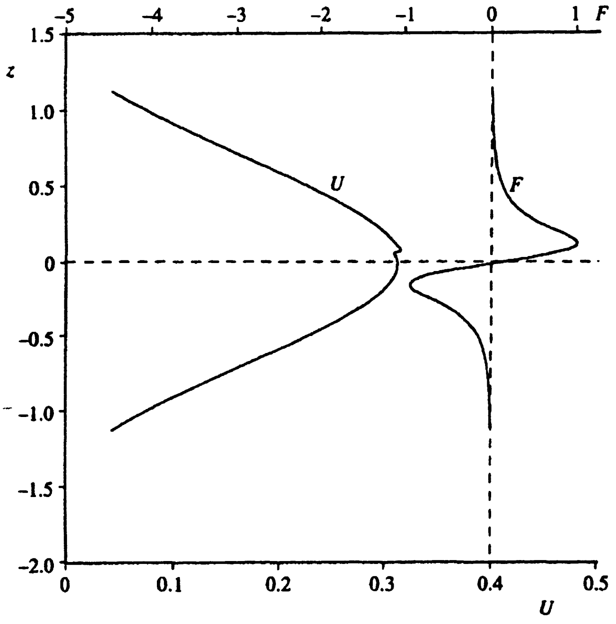

FIG. 6. Normalized values of the average-flow velocity U and the vertical wave displacements F vs. the normalized vertical coordinate z at a constant Brunt–Väisälä frequency N. The normalization scales of the velocity U_0 and the coordinate H are related by the equation $U_0=NH$.

stability layer is located in a level where F'/F runs through the interval of values defined by condition (28), the wave amplitude will grow.

Figures 6 and 7 show two different pairs of $U(z)$ and $N(z)$ profiles for which the development of wave–turbulence instability is possible. In the first case the necessary internal wave structure is provided by a jet-type average-velocity profile at a constant Brunt–Väisälä frequency. Such plane-parallel jet flows are encountered in the atmosphere, specifically in the upper troposphere during the formation of typhoons. The second case represents the modeling of oceanic conditions, when the necessary internal wave structure is created by a stepped profile of the Brunt–Väisälä frequency, and the presence of a diminished-stability layer is provided by shear flow.

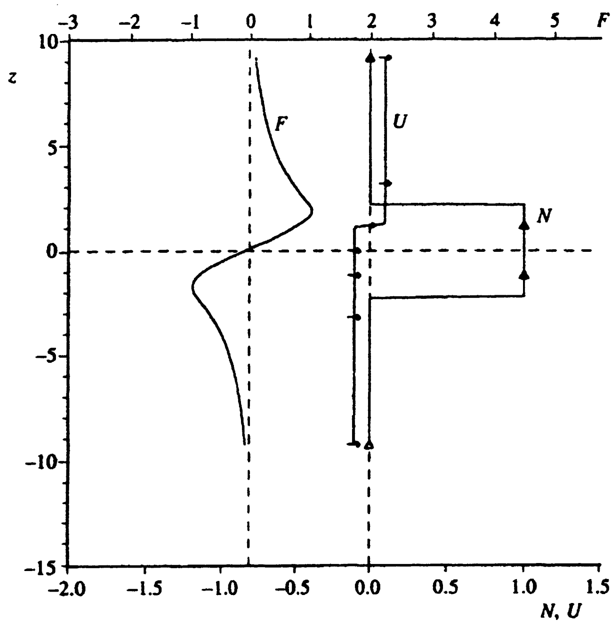

FIG. 7. Normalized values of the average-flow velocity U, the Brunt–Väisälä frequency N, and the vertical wave displacements F vs. the normalized vertical coordinate z. The normalization scales of the velocity U_0, the Brunt–Väisälä frequency N_0, and the coordinate H are related by the equation $U_0=N_0H$.

Comparing Eqs. (11) and (24), we infer that the breakup of the diminished-stability layer by wave-induced turbulence is a much faster process than the growth of the wave amplitude. However, if the wave motions are localized in a wave packet spatially bounded along the x axis, this packet can advance along the diminished-stability layer until it enters the intact part of the layer and be further amplified. This process can be most efficient when the quantity $\delta\,\mathrm{Ri}_0$ increases in the direction of wave propagation, establishing the most favorable relation between the growth rate of the wave and the decay rate of the diminished-stability layer.

4. CONCLUSION

In summary, the foregoing analysis shows that the application of a more rigorous turbulence model confirms the possibility of the onset of wave–turbulence instability and provides a means for refining the conditions of its development in stratified shear flows. In cases where such conditions occur, internal waves can accelerate the relaxation of vertical velocity shear in a stably stratified fluid and, in so doing, be amplified themselves, confirming the principle stated in the Introduction, that physical systems of any degree of complexity evolve toward minimizing the relaxation time to the best equilibrated of all possible states.

This work has received financial support from the Russian Fund for Fundamental Research (Grant 94-01-01241), and the International Science Foundation (Grant JC6100), and INTAS (Grant 93-1194).

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Translated by James S. Wood