Helical scaling in turbulence

S. S. Moiseev and O. G. Chkhetiani

Institute of Space Research, Russian Academy of Sciences, 117810 Moscow, Russia
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The influence of the mean helicity on the turbulence spectra in a stratified incompressible fluid and a compressible nonentropic gas is considered. The helicity is taken into account by analyzing the transformation properties of a Hopf-like equation for a characteristic functional. It is shown that a self-similar region with cascade transfer of the helicity over the energy spectrum, i.e., a region of helical scaling that can be observed in natural and laboratory experiments, forms in the large-scale region. An $E \propto k^{-7/3}$ energy spectrum and an arbitrary two-point semi-invariant are obtained in the asymptotic limit. An analog of the Corrsin–Obukhov spectrum is obtained in the helical case. A quasisound energy spectrum is obtained in the compressible case for an arbitrary adiabatic exponent and transforms into an $E \propto k^{-3}$ spectrum in the shock-wave limit.
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1. INTRODUCTION

The concept of helical scaling, i.e., the possibility of the self-similar cascade transfer of a helicity flux over the spectrum, was first introduced in Ref. 1. The following two basic scenarios were analyzed in that work from the standpoint of the dimensionality method: a) the simultaneous transfer of energy and helicity with constant fluxes over the spectra of both parameters; b) a constant helicity flux determining the energy distribution. The authors considered such a possibility to be highly speculative and unlikely under real conditions. This was due to a lack of sources for generating helicity.

However, several observations of an $E \propto k^{-7/3}$ energy spectrum in the mesoscale region,[2–4] which can be unequivocally associated with a helicity flux, appeared 20 years later. Generally, an $E \propto k^{-5/3}$ spectrum is observed in the small-scale region. On the other hand, $E \propto k^{7/3}$ is often observed in the small-scale region in magnetohydrodynamic turbulence.[3] It can be confidently stated that there is a broad class of effects that generate both helicity itself and large helicity fluctuations under terrestrial and astrophysical conditions. In particular, the simultaneous presence of such factors as density and temperature gradients, large temperature fluctuations, shearing flows, and nonuniform rotation is sufficient. We note that helicity can also be generated at moderate latitudes.

Investigations of large-scale helical structures in a nonconducting continuous medium have been actively pursued during the last 12 years. It was shown in Ref. 5 that the coherent generation of a mean velocity $\langle \mathbf{V} \rangle$ (the averaging is generally performed over a chaotic ensemble) with $\langle \mathbf{V} \rangle \cdot (\nabla \times \langle \mathbf{V} \rangle) \neq 0$ is possible in a compressible medium with small-scale helical turbulence $\mathbf{V}^t$ when $\langle \mathbf{V}^t \rangle \cdot (\nabla \times \langle \mathbf{V}^t \rangle) \neq 0$, i.e., the mean helicity of the small-scale velocity field is nonzero. It subsequently became understood that the generation of a mean field $\langle \mathbf{V} \rangle$ is also possible in an incompressible medium, if additional conditions hold when the mean helicity of the field $\mathbf{V}^t$ is nonzero. The set of these additional conditions (convective or shearing flow and an appreciable number of impurities in the medium; see, for example, Ref. 6) restricts the applicability of this important model. On the other hand, the requirement of nonzero mean helicity for a small-scale field $\mathbf{V}^t$ is typical of terrestrial conditions, since rotation and even weak temperature–density stratification is sufficient for satisfying it (see, for example, Ref. 6). However, for the reasons just stated, large-scale helicity (i.e., a typhoon-like cyclone) develops in the tropics, regions of strongly inhomogeneous flows, and areas with a large number of impurities. Therefore, for example, while satisfactorily explaining such a phenomenon as a tropical cyclone,[7,8] the model of helical turbulence cannot lead to the generation of a large-scale vortex under average terrestrial conditions because of the quasi-inhomogeneity of the atmosphere, i.e., the fluctuations in $\langle \mathbf{V} \rangle$ that appear decay with time. Does this mean that the role of the helicity is negligible in such systems? No, it does not. First, only the possibility of long-wavelength instability was considered for the generation of the mean velocity $\langle \mathbf{V} \rangle$, and nothing was said about the behavior of the higher moments in the large-scale region. Second, the properties of the turbulence itself should vary strongly in the presence of chaotic screw motions. The present paper is devoted to an examination of the latter question.

The influence of helicity is obvious from a physical standpoint. It is sufficient to note that two helical vortices with strong axial motion in one direction have a tendency to merge because of the Bernoulli effect. In other words, helicity results in redistribution of the chaotic energy; a helicity flux that characterizes the variation of the mean helicity $\langle \mathbf{V}^t \cdot (\nabla \times \mathbf{V}^t) \rangle$ also appears. The importance of such an examination of the role of helicity can be seen from the following remarks. We stress again that there is actually a mean nonzero helicity in any region of the geophysical environment. This, in turn, means that the variations are geographically universal. Furthermore, not only the movement of heat, but also the profile of its surface must be noted. For example, it is to be expected that helical properties will be manifested

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more strongly over a hilly terrestrial surface than over flat relief.

Above all, helicity has an effect on the spectral features of turbulence. However, the correlation characteristics are subject to strong variation. Not only the asymptotes in **r** space, but also the relations between their moments vary.

As for the spectra, variations occur in incompressible, compressible, and stratified turbulent media, as we shall see below. For example, the tendency inherent in helical media, i.e., energy transfer to the long-wavelength region (due to the tendency of helical vortices to merge) is maintained in a compressible medium (with consideration of the discontinuities). Furthermore, helicity has a significant influence on the buoyancy range and the corresponding characteristic scales in a stratified medium. We now proceed to investigate the problem at hand.

2. TURBULENT HELICAL SCALING IN A STRATIFIED MEDIUM

Stratification has a significant influence on the dynamics of turbulence and leads to qualitative differences from a homogeneous medium. In an unstratified medium the Kolmogorov energy spectrum $E\propto k^{-5/3}$, which is obtained theoretically using the experimentally confirmed hypothesis of self-similarity and locality, has been verified over a broad range of wave numbers **k**.

The determination of the transformation properties of the equation for a characteristic functional with respect to scaling transformations made it possible to establish a similarity theory in unstratified and stratified media, which makes it possible to find the exact form of turbulence spectra without invoking dimensionality arguments, either in the inertial range or in the buoyancy range,[9,10] and furthermore with consideration of the existing anisotropy of the directions, which is usually very difficult to take into account when the dimensionality method is used. Below we analyze the influence of helicity on the spectra in a temperature-stratified medium.

In this context we recall several assumptions of the work in Refs. 9 and 10. We consider the equations of motion, continuity, and heat conduction with external random forces in the Boussinesq approximation:

$$\frac{\partial u_i}{\partial t}+u_k\frac{\partial u_i}{\partial x_k}=-\frac{\partial p'}{\partial x_i}+g\beta\theta e_i+\nu\Delta u_i+f_i(\mathbf{x},t), \quad (2.1)$$

$$\frac{\partial \theta}{\partial t}+u_k\frac{\partial \theta}{\partial x_k}=\frac{\partial \bar{T}}{\partial z}u_je_j+\chi\Delta\theta+q(\mathbf{x},t), \quad (2.2)$$

$$\frac{\partial u_k}{\partial x_k}=0, \quad (2.3)$$

where the u_i are the components of the velocity field, p' and θ are the deviations of the pressure and the temperature from their equilibrium values, ν is the kinematic viscosity, χ is the thermal diffusivity, $g\beta$ is the buoyancy parameter, $\partial\bar{T}/\partial z$ is the mean constant temperature gradient, and **e** is the unit vector in the vertical direction, i.e., $\mathbf{e}=(0,0,1)$. The turbulent kinetic energy and the temperature pulsations are maintained by the external forces $f_i(\mathbf{x},t)$ and $q(\mathbf{x},t)$, which we assume to be Gaussian, uniform, and delta-correlated in time with zero mean values:

$$\langle f_i(\mathbf{x},t)\rangle=\langle q(\mathbf{x},t)\rangle=0, \quad (2.4)$$

$$\langle f_i(\mathbf{x}_1,t_1)f_j(\mathbf{x}_2,t_2)\rangle=B_{ij}(\mathbf{x}_2-\mathbf{x}_2,t_2-t_1)\,\delta(t_2-t_1), \quad (2.5)$$

$$\langle q(\mathbf{x}_1,t_1)q(\mathbf{x}_2,t_2)\rangle=G(\mathbf{x}_2-\mathbf{x}_2,t_2-t_1)\,\delta(t_2-t_1). \quad (2.6)$$

The statistical properties of the velocity and temperature fields are described by the characteristic functional

$$\Phi=\left\langle \exp\left[i\int_{-\infty}^{\infty}\{y_i(\mathbf{x})u_i(\mathbf{x},t)+z(\mathbf{x})\theta(\mathbf{x},t)\}d\mathbf{x}\right]\right\rangle, \quad (2.7)$$

where $y_i(\mathbf{x})$ and $z(\mathbf{x})$ are arbitrary smooth functions. The averaging is carried out over the probability distribution of the external forces.

It follows from the continuity condition that

$$\Phi\{\mathbf{y}(\mathbf{x})+\nabla\psi;z(\mathbf{x})\}=\Phi\{\mathbf{y}(\mathbf{x});z(\mathbf{x})\}, \quad (2.8)$$

where ψ is an arbitrary function whose gradient tends rapidly to zero at infinity. Differentiating (2.7) with respect to time and taking into account the equations of motion (2.1)–(2.3), we obtain

$$\frac{\partial\Phi}{\partial t}=i\int_{-\infty}^{\infty}y_i(\mathbf{x})\left\{D_k\frac{\partial}{\partial x_k}D_i+ig\beta e_iD_\theta\right\}\Phi d\mathbf{x}+i\int_{-\infty}^{\infty}z(\mathbf{x})$$
$$\times\left\{D_k\frac{\partial}{\partial x_k}D_\theta+i\frac{d\bar{T}}{dz}e_jD_j-i\chi\Delta D_\theta\right\}\Phi d\mathbf{x}+\hat{I}. \quad (2.9)$$

Here D_i and D_θ are the variational differentiation operators

$$D_i=\frac{\delta}{\delta y_i(\mathbf{x})},\qquad D_\theta=\frac{\delta}{\delta z(\mathbf{x})},$$

and $\hat{I}$ is the source describing the influence of external forces on the fluid:

$$\hat{I}=-\frac{1}{2}\int_{-\infty}^{\infty}\left\{y_i(\mathbf{x})\left\langle f_i(\mathbf{x},t)\exp\left[i\int_{-\infty}^{\infty}(y_ku_k+z\theta)d\mathbf{x}'\right]\right\rangle\right.$$
$$\left.+z(\mathbf{x})\left\langle q(\mathbf{x},t)\exp\left[i\int_{-\infty}^{\infty}(y_ku_k+z\theta)d\mathbf{x}'\right]\right\rangle\right\}d\mathbf{x}. \quad (2.10)$$

We assume that the forces start to act at time $t=0$, and that at $t=0$ the fluid is assumed to be at rest:

$$\mathbf{u}|_{t=0}=\theta|_{t=0}=0,\quad \text{i.e., } \Phi_{t=0}=1. \quad (2.11)$$

Equation (2.9) with the initial condition (2.11) is Cauchy's problem for the characteristic functional in a temperature stratified medium.

Equation (2.9) for Gaussian forces that are delta-correlated in time can be closed. The source $\hat{I}$ can then be expressed in terms of Φ:

$$\hat{I}=-\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}y_i(\mathbf{x})y_j(\mathbf{x})B_{ij}(\mathbf{x}-\mathbf{x}_1)\Phi d\mathbf{x}d\mathbf{x}_1$$
$$-\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}z(\mathbf{x})z(\mathbf{x}_1)G(\mathbf{x}-\mathbf{x}_1)\Phi d\mathbf{x}d\mathbf{x}_1. \quad (2.12)$$

After averaging the energy balance equation of the fluid and the square of the temperature inhomogeneity, we obtain

$$\langle f_i uij\rangle = 2\nu\left\langle \frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_i}\right\rangle - g\beta\langle \theta u_3\rangle, \tag{2.13}$$

$$\langle q\theta\rangle = 2\chi\left\langle\left(\frac{\partial\theta}{\partial x_j}\right)^2\right\rangle - \frac{\partial\bar{T}}{\partial z}\langle\theta u_3\rangle. \tag{2.14}$$

External random pumping of the helicity leads to an additional equilibrium condition

$$\langle f_i w_i\rangle = \nu\left\langle\frac{\partial u_i}{\partial x_j}\frac{\partial wi}{\partial x_j}\right\rangle - \frac{1}{2}g\beta\langle\theta w_3\rangle, \tag{2.15}$$

where

$$\mathbf{w} = \nabla\times\mathbf{u}.$$

The left-hand sides of Eqs. (2.13)–(2.15) can be expressed in terms of the corresponding correlators of the external forces according to the Furutsu–Novikov equation,[11] yielding

$$\frac{1}{2}B_{ij}(0) = \bar{\varepsilon} - g\beta\langle\theta u_3\rangle, \tag{2.16}$$

$$\frac{1}{2}G(0) = N - \frac{\partial\bar{T}}{\partial z}\langle\theta u_3\rangle, \tag{2.17}$$

$$\frac{1}{2}C(0) = \bar{\eta} - \frac{1}{2}g\beta\langle\theta w_3\rangle, \tag{2.18}$$

where $\bar{\varepsilon} = \nu\langle(\partial u_i/\partial x_j)^2\rangle$ is the mean energy dissipation rate, $C(0) = \langle\mathbf{f}\cdot(\nabla\times\mathbf{f})\rangle$ specifies the helicity pumping on the source scale, $\bar{\eta} = \nu\langle(\partial u_i/\partial x_j)(\partial w_i/\partial x_j)\rangle$ is the mean helicity dissipation rate, $N = \chi\langle(\partial\theta/\partial x_j)^2\rangle$ is the dissipation rate of the temperature inhomogeneities, and $g\beta\langle\theta u_e\rangle$ is the mean work of the buoyancy force in the turbulent displacements of elements of the fluid.

We note one important consequence. We turn our attention to the equilibrium condition for the helicity flow. It is clear that in the stratified fluid the buoyancy force can serve as a source of helicity. Attention was focused on this circumstance already in Refs. 12–14 (in the nonturbulent case). An additional source is not needed to generate helicity; intensity temperature fluctuations suffice. This occurs, in particular, in regions of intense vapor condensation and wherever the turbulent region has initial angular momentum, which always appears under the action of shearing flows and the Coriolis force.

Equation (2.9) with the source (2.12) and the initial condition (2.11) are invariant under the following group of transformations:

$$\alpha\mathbf{x} = \mathbf{x}', \quad \alpha L = L', \quad \alpha L_T = L_T', \quad \alpha^{1-\gamma}t = t',$$

$$\alpha^{-(\gamma+3)}\mathbf{y}(\mathbf{x}) = \mathbf{y}'(\mathbf{x}'), \quad \alpha^{-(\delta+3)}z(\mathbf{x}) = z'(\mathbf{x}'),$$

$$\alpha^{1+\gamma}\nu = \nu', \quad \alpha^{1+\gamma}\chi = \chi', \quad \alpha^{2\gamma-\delta-1}g\beta = (g\beta)',$$

$$\alpha^{\delta-1}\frac{\partial\bar{T}}{\partial z} = \left(\frac{\partial\bar{T}}{\partial z}\right)', \quad \alpha^{3\gamma-1}B_{ij} = B_{ij}',$$

$$\alpha^{3\gamma-2}C = C', \quad \alpha^{2\delta+\gamma-1}G = G',$$

$$\alpha^{3\gamma-1}\bar{\varepsilon} = \bar{\varepsilon}', \quad \alpha^{3\gamma-2}\bar{\eta} = \bar{\eta}', \quad \alpha^{2\delta+\gamma-1}N = N', \tag{2.19}$$

where L is the external turbulence scale, L_T is the temperature scale, and α, γ, and δ are arbitrary parameters ($0<\alpha<\infty$, $\gamma,\delta<\infty$). The transformation (2.19) performs the transition to new measurement scales of the physical quantities. The uniqueness of the solution of the Cauchy problem for Eq. (2.9) leads to the similarity theorem

$$\Phi\left\{\mathbf{y}(\mathbf{x}),z(\mathbf{x}),t,\bar{\varepsilon},\bar{\eta},N,\frac{\partial\bar{t}}{\partial z},g\beta,\nu,\chi,L,L_T\right\}$$
$$=\Phi\left\{\mathbf{y}'(\mathbf{x}')',z'(\mathbf{x}'),t,\bar{\varepsilon}',\bar{\eta}',N',\left(\frac{\partial\bar{T}}{\partial z}\right)',\right.$$
$$\left.\times(g\beta)',\nu',\chi',L',L_T'\right\}. \tag{2.20}$$

Unlike the similarity hypothesis, Eq. (2.20) is exact.

We restrict ourselves below to consideration of stationary turbulence spectra for the scales $l^{-1}\gg k\gg L^{-1}$, where L is the external turbulence scale, and l is a certain internal scale that specifies the upper boundary of the dissipation range. It can be shown that $l \simeq \max\{\nu^{3/4}(\bar{\varepsilon})^{-/14},\chi^{3/4}(\bar{\varepsilon})^{-1/4}\}$.

Investigations of helical turbulence have shown that there is a boundary scale

$$l_h = \frac{\bar{\varepsilon}}{\bar{\eta}}, \tag{2.21}$$

that separates the inertial range into two regions. The Kolmogorov spectrum $E\propto k^{-5/3}$ is observed on scales $k>l_h^{-1}$, and an $E\propto k^{-7/3}$ spectrum is observed on scales $k<l_h^{-1}$ (Refs. 2–4). This scale is of the order of 7 km for mesoscale atmospheric turbulence in pretyphoon situations.[4]

Let us consider the special case of a vanishing temperature gradient in the medium, i.e., $\partial\bar{T}/\partial z=0$. We consider the kinetic energy spectrum

$$E = -\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{|k|=k} e^{-i\mathbf{kx}}\{D_iD_i\Phi\}_{\mathbf{y}=z=0}d\mathbf{x}dS(\mathbf{k}), \tag{2.22}$$

where $dS(\mathbf{k})$ is an area element of the surface of sphere $|\mathbf{k}|=k$. Going over in (2.22) to the functional that depends on the primed variables and setting $\alpha=k$, we obtain

$$E(k) = \frac{f(k^{1+\gamma}\nu,k^{1+\gamma}\chi,k^{3\gamma-1}\bar{\varepsilon},k^{3\gamma-2}\bar{\eta},k^{2\delta+\gamma-1}N,k^{2\gamma-\delta-1}g\beta)}{k^{2\gamma+1}}. \tag{2.23}$$

It follows from the arbitrariness of γ and δ that $\partial E/\partial\gamma=0$ and $\partial E/\partial\delta=0$. From the condition $\partial E/\partial\delta=0$ we obtain the equation

$$2\xi_1\frac{\partial f}{\partial\xi_1} - \xi_2\frac{\partial f}{\partial\xi_2} = 0,$$

where

TABLE I.

	Helical scaling	Kolmogorov scaling
	$\bar{\eta}$	$\bar{\varepsilon}$
$E(k)$	$\bar{\eta}^{2/3}k^{-7/3}\Psi((kL_*)^{1/2},kl_h)$	$\bar{\varepsilon}^{2/3}k^{-5/3}\Psi'(kL_{\rm BO})$
$E_{\rm TT}(k)$	$N\bar{\eta}^{-1/3}k^{-4/3}\Psi_{\rm TT}((kL_*)^{1/2},kl_h)$	$N\bar{\varepsilon}^{-1/3}k^{-5/3}\Psi'_{TT}(kL_{\rm BO})$
$E_{{\rm TU}_3}(k)$	$N\bar{\eta}^{-1/6}k^{-7/6}\Psi_{{\rm TU}_3}((kL_*)^{1/2},kl_h)$	$N\bar{\varepsilon}^{-1/6}k^{-5/3}\Psi'_{{\rm TU}_3}(kL_{\rm BO})$
	$L_*=\bar{\eta}^{-5}N^3(g\beta)^6$	$L_{\rm BO}=\bar{\varepsilon}^{5/4}(g\beta)^{-3/2}N^{-3/4}$

$$\xi_1=k^{2\delta+\gamma-1}N, \qquad \xi_2=k^{2\gamma-\delta-1}g\beta,$$

whose solution is

$$f=f(k^{1+\gamma}\nu,k^{1+\gamma}\chi,k^{3\gamma-1}\bar{\varepsilon},k^{3\gamma-2}\bar{\eta},\xi_1^{1/2}\xi_2)$$
$$\equiv f(k^{1+\gamma}\nu,k^{1+\gamma}\chi,k^{3\gamma-1}\bar{\varepsilon},k^{3\gamma-2}\bar{\eta},g\beta N^{1/2}k^{5\gamma/2-3/2}).$$

From the condition $\partial E/\partial\gamma=$ we obtain the equation

$$3\zeta_1\frac{\partial f}{\partial\zeta_1}+3\zeta_2\frac{\partial f}{\partial\zeta_2}+\frac{5}{2}\zeta_3\frac{\partial f}{\partial\zeta_3}+\zeta_4\frac{\partial f}{\partial\zeta_4}+\zeta_5\frac{\partial f}{\partial\zeta_5}=2f,$$

$$\zeta_1=k^{3\gamma-1}\bar{\varepsilon}, \qquad \zeta_2=k^{3\gamma-2}\bar{\eta}, \qquad \zeta_3=k^{5\gamma/2-3/2}g\beta N^{1/2},$$

$$\zeta_4=k^{1+\gamma}\nu, \qquad \zeta_5=k^{1+\gamma}\chi.$$

The solution of this and the analogous equations provides spectra of the energy and the square of the temperature fluctuations, as well as a mixed spectrum of the temperature and the vertical component of the velocity. We present the results in Table I. The third column contains the classical (nonhelical) values of the spectra.[10] In the case of helical turbulence, we have the characteristic scale $L_*=\bar{\eta}^{-5}N^3(g\beta)^6=l_h(l_h/L_{\rm BO})^4$, which is an analog of the Bolgiano–Obukhov scale, whose classical value is $L_{\rm BO}=\bar{\varepsilon}^{5/4}(g\beta)^{-3/2}N^{-3/4}$. Here and in the following we neglect the influence of viscosity on the spectra, since we are considering turbulent motion on the scales $k<l_h^{-1}\ll l^{-1}$. The $E_{\rm TT}(k)\propto k^{-4/3}$ spectrum is the analog of the Corrsin–Obukhov spectrum in the helical range. The exponent $-5/3$ is replaced by $-4/3$. This "anomalous" exponent should be a clear indicator of a helicity cascade.

No assumptions regarding the nature of the transfer of energy over the spectrum were used to obtain the spectrum. In the absence of a gravitational force that causes stratification, the spectra from Table I should be converted into the usual expressions for locally isotropic turbulence. When $g\beta\to 0$, the scale $L_*\to 0$, i.e.,

$$E(k)=\bar{\eta}^{2/3}k^{-7/3}\Psi^*(kl_h). \tag{2.24}$$

An $E\propto k^{-7/3}$ spectrum is obtained under the additional hypothesis that the asymptotic expansion (2.24) in $kl_h\ll 1$ exists. When $kl_h\gg 1$, but $kl\ll 1$, the assumption that $\Psi\sim C(kl_h)^{2/3}$ transforms (2.24) into $E(k)=C\bar{\varepsilon}^{2/3}k^{-5/3}$.

Using a similar procedure, we can easily obtain the expression for a two-point semi-invariant of arbitrary rank,

$$S_{i_1,\dots,i_n}(\mathbf{r})=(-1)^n\frac{\delta^n\ln\Phi}{\delta y_{i_1}\dots\delta y_{i_n}}, \tag{2.25}$$

which has the following form in the helical subrange:

$$S_{i_1,\dots,i_n}(\mathbf{r})=\mathrm{const}\cdot(\bar{\eta}r^2)^{n/3}\Theta_{i_1,\dots,i_n}\left(\frac{\mathbf{r}}{|r|}\right), \tag{2.26}$$

where $\Theta_{i_1,\dots,i_n}(\mathbf{r}/r)$ is the angular part of the spectral tensor. In particular, an exact relation has been obtained[15] for the isotropic part of the third-rank semi-invariant, as in the inertial range, so that the antisymmetric part of the correlation tensor has the form

$$\langle v_i(\mathbf{x})v_j(\mathbf{x})v_k(\mathbf{x}+\mathbf{r})\rangle^{\rm AS}=\frac{\bar{\eta}}{60}(\varepsilon_{jkl}r_lr_i+\varepsilon_{ikl}r_lr_j). \tag{2.27}$$

When $k\ll L_{\rm BO}^{-1}$ the energy in this range is utilized to oppose the buoyant forces, and the spectrum is determined mainly by $g\beta$, N, and $\partial\bar{T}/\partial z$. The temperature gradient determines one more characteristic scale

$$L'_*=N^{1/2}(g\beta)^{-1/4}\left(\frac{\partial\bar{T}}{\partial z}\right)^{3/4},$$

which characterizes the minimum size of the inhomogeneities in which the influence of the characteristics of the averaged temperature field becomes significant. Under real conditions $L_{\rm BO}$ is always smaller than L'_*, and in the range of scales $L'_*\gg k^{-1}\gg L_{\rm BO}$ we observed the Bolgiano–Obukhov spectrum[10] with the density

$$F(k_z,k_\perp)=C_1N^{2/5}(g\beta)^{4/5}k^2k_z^{-11/5}k_\perp^{-4}. \tag{2.28}$$

It would also be interesting to investigate behavior in the absence of temperature fluctuations. This corresponds, in particular, to dry cyclones. In the buoyancy range ($k^{-1}\gg L_*$) it is already necessary to take into account the anisotropy of the problem, which expands the group of scaling transformations. The anisotropy also makes the stationary state corresponding to the determination of the corresponding group of scaling transformations that leave the Hopf equation invariant possible only when horizontal and vertical components with different transformation laws can be separated in the energy and helicity dissipation fields. We then obtain the following group:

$$\alpha_z z=z', \qquad \alpha r_\perp=r'_\perp, \qquad \alpha_z\alpha^{-\gamma}t=t',$$

$$\alpha^{-(\gamma+3)}y_{1,2}(\mathbf{x})=y'_{1,2}(\mathbf{x}'), \qquad \alpha_z^{-1}\alpha^{-(\gamma+2)}y_3(\mathbf{x})=y'_3(\mathbf{x}'),$$

$$\alpha_z^{-\mu}\alpha^{2\gamma-\delta-1}(g\beta)=(g\beta)', \qquad \alpha_z^{\mu-2}\alpha^{\delta+1}\left(\frac{\partial\bar{T}}{\partial z}\right)=\left(\frac{\partial\bar{T}}{\partial z}\right)',$$

$$\alpha_z^{-3}\alpha^{3\gamma+2}\bar{\varepsilon}_\perp=\bar{\varepsilon}'_\perp, \qquad \alpha_z^{-1}\alpha^{3\gamma}\bar{\varepsilon}_z=\bar{\varepsilon}'_z,$$

$$\alpha_z^{-4}\alpha^{3\gamma+2}\bar{\eta}_\perp=\bar{\eta}'_\perp, \qquad \alpha_z^{-2}\alpha^{3\gamma}\bar{\eta}_z=\bar{\eta}'_z, \tag{2.29}$$

A similarity theorem that permits determination of the anisotropic spectral characteristics follows from (2.29). The absence of a dependence of the spectrum on the arbitrary parameters γ, μ, and δ leads to a spectral density of the form

$$F(k_z,k_\perp)$$
$$=\frac{\partial\bar{T}}{\partial z}g\beta k^2k_z^{-3}k_\perp^{-4}\Psi(k_zL_{\bar{\varepsilon}_z},k_\perp L_{\bar{\varepsilon}_\perp};k_zL_{\bar{\eta}_z},k_\perp^2k_z^{-1}L_{\bar{\eta}_\perp};), \tag{2.30}$$

which corresponds to the Lumley–Shur spectrum.[10] In (2.30) we have introduced scales that characterize the minimum size of the inhomogeneities in which the effects of buoyancy and the gradient of the mean temperature are significant:

$$L_{\bar{\varepsilon}_z}=\bar{\varepsilon}_z^{1/2}[g\beta(\partial\bar{T}/\partial z)]^{-3/4}, \qquad L_{\bar{\varepsilon}_\perp}=\bar{\varepsilon}_\perp^{1/2}[g\beta(\partial\bar{T}/\partial z)]^{-3/4},$$

$$L_{\bar{\eta}_z}=\bar{\eta}_z[g\beta(\partial\bar{T}/\partial z)]^{-3/2}, \qquad L_{\bar{\eta}_\perp}=\bar{\eta}_\perp[g\beta(\partial\bar{T}/\partial z)]^{-3/2}. \tag{2.31}$$

It is noteworthy that there is no intermediate spectral range similar to the Bolgiano–Obukhov range observed in the absence of an independent source of temperature fluctuations. There are two ranges: a Kolmogorov (helical) range with $E(k)\propto k^{-5/3}$ $(k^{-7/3})$ and a Lumley–Shur range with $E(k)\propto k^{-3}$.

Thus, when there is a helicity flux in a stratified medium, spectral differences are observed in the range $l_h \ll k^{-1} \ll L_{\mathrm{BO}}$. In the buoyancy range both energy dissipation and helicity dissipation have little influence on the form of the turbulence spectra, but in the absence of a source of temperature fluctuations they specify a characteristic scale that distinguishes different spectral ranges.

3. TURBULENT HELICAL SCALING IN A COMPRESSIBLE MEDIUM

Let us consider the turbulence spectra in the presence of a helicity flux in a compressible medium. In an ideal nonentropic gas, as in the incompressible case, the helicity is invariant. Various relations have been obtained for turbulent scaling, depending on the physical suppositions. Zakharov and Sagdeev[16] showed that in the case of a weak deviation of the sound dispersion law from a linear dependence and a weak level of excitation, the universal turbulence spectrum in the inertial range is proportional to $k^{-3/2}$. Kadomtsev and Petviashvili noted[17] that under a linear sound dispersion law the spectrum is specified by shock waves, i.e., discontinuities in the density of the medium, and is proportional to k^{-2}.

At the same time, an evaluation of the characteristic times for the formation of a shock wave and the transfer of energy over the spectrum, which prevents the formation of shock waves, shows that they are of the same order,[18] so that the form of the spectrum in a compressible medium remains not entirely clear. In fact, acoustic, vortical, and entropic modes coexist and interact with one another in a compressible medium. Since sound energy can be converted into heat, it was theorized by Moiseev *et al.*[9,19] that compressibility plays the role of a kinetic energy sink, along with viscosity and thermal conductivity. An analysis of the group of scaling transformations of the corresponding equation for a characteristic functional made it possible to obtain an energy spectrum in the form

$$E(k)=\mathrm{const}\cdot \rho_0 C_0^{2/(1-3\gamma)}\bar{\varepsilon}^{2\gamma/(3\gamma-1)}k^{-(5\gamma-1)/(3\gamma-1)}, \tag{3.1}$$

where ρ_0 is the density of the fluid in the unperturbed state, C_0 is the velocity of sound at ρ_0, and γ is the adiabatic exponent. The spectrum transformed into $E\propto k^{-2}$, i.e., a shock-wave spectrum, when $\gamma\to 1$ and into the Kolmogorov spectrum of an incompressible fluid $E\propto k^{-5/3}$ when $\gamma\to\infty$. It was noted in Ref. 20 that the exponent of the density in the spectrum (3.1) was improperly defined, which precludes a fully correct transition to the Kolmogorov spectrum $E\propto k^{-5/3}$. Shivamoggi[20] corrected the error by invoking the Zakharov–Sagdeev hypothesis regarding the kinetic energy density in acoustic turbulence for this purpose. Below we show, in particular, that the correct exponent of the density was potentially present in Ref. 9, but was omitted in the final stage.

We next investigate helical scaling in a compressible medium. We consider the spectral characteristics in the range of $\mathbf{k}$ space where the dissipation coefficients and, therefore, the variation of the entropy with time can be neglected. As in Ref. 19 we consider a model equation of state

$$P=A_0\rho^\gamma, \qquad A_0=\rho_0^{1-\gamma}C_0^2. \tag{3.2}$$

In this case the variation of the entropy is a third-order term, and the flux can be considered nonentropic when the quadratic effects are considered. The basic system of equations in this case contains the Navier–Stokes equation and the continuity equation:

$$\frac{\partial u_i}{\partial t}+u_k\frac{\partial u_i}{\partial x_k}=-\frac{A_0}{\rho}\frac{\partial \rho^\gamma}{\partial x_i}+\frac{\zeta}{\rho}\Delta u_i+\frac{\zeta/3+\xi}{\rho}\frac{\partial}{\partial x_i}\frac{\partial u_k}{\partial x_k}+\frac{F_i(\mathbf{x},t)}{\rho}, \tag{3.3}$$

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_k}(u_k\rho)=0. \tag{3.4}$$

Here $\mathbf{F}$ is a random force that excites turbulence, and ζ and ξ are the first and second viscosities. The statistical properties of the fluid and the random force can be described by the characteristic functional

$$\Phi=\left\langle \exp\left[i\int_{-\infty}^{\infty}\{y_i(\mathbf{x})u_i(\mathbf{x},t)+y_\rho(\mathbf{x})\rho(\mathbf{x},t)\}d\mathbf{x}\right]\right\rangle, \tag{3.5}$$

where $y_i(\mathbf{x})$ and $y_\rho(\mathbf{x})$ are smooth arbitrary functions. The averaging is carried out over the distribution probability of the external forces. Differentiating (3.5) with respect to time and taking the equation of motion into account, we obtain

$$\frac{\partial\Phi}{\partial t}=i\int_{-\infty}^{\infty}y_i(\mathbf{x})\left\{D_k\frac{\partial}{\partial x_k}D_i-(i)^{1-\gamma}A_0D_\rho^{-1}\frac{\partial}{\partial x_k}D_\rho^\gamma+\zeta D_\rho^{-1}\left(\Delta D_i+\frac{1}{3}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k}D_k\right)+\xi D_\rho^{-1}\times\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k}D_k\right\}\Phi d\mathbf{x}+i\int_{-\infty}^{\infty}y_\rho(\mathbf{x})\frac{\partial}{\partial x_k}D_\rho D_k\Phi d\mathbf{x}+\hat{I}. \tag{3.6}$$

Here D_i and D_ρ are the variational differentiation operators

$$D_i=\frac{\delta}{\delta y_i(\mathbf{x})}, \qquad D_\rho=\frac{\delta}{\delta y_\rho(\mathbf{x})},$$

and $\hat{I}$ is a source that describes the influence of the external forces on the fluid:

$$\hat{I}=-\int_{-\infty}^{\infty}y_i(\mathbf{x})D_\rho^{-1}\left\langle f_i(\mathbf{x},t)\exp\left[i\int_{-\infty}^{\infty}(y_k u_k+y_\rho\rho)d\mathbf{x}'\right]\right\rangle d\mathbf{x}. \quad (3.7)$$

We assume that the forces start to act at time $t=0$ and that the fluid is at rest and has a uniform density distribution at $t=0$. A closed description of the statistical properties of the fluid in terms of the functional Φ is possible only when the source can be expressed in terms of Φ. As was shown in Refs. 9 and 11, this is possible, in particular, if the random force can be represented in the form

$$\mathbf{F}(\mathbf{x},t,\rho)=\rho(\mathbf{x},t)\mathbf{g}(\mathbf{x},t), \quad (3.8)$$

where $\mathbf{g}(\mathbf{x},t)$ has a Gaussian distribution law and is delta-correlated with respect to the time:

$$\langle g_i(\mathbf{x},t)g_j(\mathbf{x}_1,t_1)\rangle=B_{ij}(\mathbf{x}-\mathbf{x}_1)\delta(t-t_1). \quad (3.9)$$

The use of the Furutsu–Novikov equation makes it possible to obtain an expression for the source and to close Eq. (3.6):

$$\hat{I}=-\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}y_i(\mathbf{x})y_j(\mathbf{x})B_{ij}(\mathbf{x}-\mathbf{x}_1)\Phi d\mathbf{x}d\mathbf{x}_1. \quad (3.10)$$

Other methods of perturbing a compressible fluid that lead to nonlocal energy transfer that depends on density correlations at the turbulence scales were also discussed in Ref. 9.

When turbulence is excited by random Gaussian acceleration, which corresponds to a relatively low pumping level, the characteristic functional satisfies Eq. (3.6) with the source (3.10) and the initial condition of a fluid at rest at $t=0$:

$$\rho|_{t=0}=\langle\rho\rangle=\rho_0, \qquad \mathbf{u}|_{t=0}=0. \quad (3.11)$$

Let us consider the invariance properties of the Cauchy problem for the functional Φ. The constant $A_0=1^{1-\gamma}C_0^2$ would seem to make the group of scaling transformations a three-parameter group. Equation (3.6) with the source (3.10) is invariant under the following group of transformations:

$$\alpha x=x', \quad \alpha^{1-\beta}t=t', \quad \alpha L=L', \quad \delta\alpha^{1+\beta}\zeta=\zeta',$$
$$\delta\alpha^{1+\beta}\xi=\xi', \quad \alpha^{-(\beta+3)}\mathbf{y}(\mathbf{x})=\mathbf{y}'(\mathbf{x}'),$$
$$\alpha^{-(\delta+3)}y_\rho(\mathbf{x})=y'_\rho(\mathbf{x}'), \quad \alpha^{3\beta-1+\delta}\bar{\varepsilon}=\bar{\varepsilon}',$$
$$\alpha^{3\beta-2+\delta}\bar{\eta}=\bar{\eta}', \quad \alpha^{2\beta+\delta(1-\gamma)}C=C'(C=\gamma\rho_0^{1-\gamma}C_0^2), \quad (3.12)$$

where L is the correlation scale of the external exciting force. Here $\bar{\varepsilon}$ is the energy dissipation in a compressible fluid, and $\bar{\eta}$ is the helicity dissipation in a compressible fluid normalized to the density of the fluid, so that as in the case of an impressible fluid, the ratio $\bar{\varepsilon}/\bar{\eta}$ specifies the scale that divides the Kolmogorov range into two subranges. The similarity theorem is formulated as invariance relative to the group of transformations (3.12). We require that $\alpha^{2\beta+\delta(1-\gamma)}C\equiv 1$, i.e.,

$$\alpha^\delta=\rho^{-1}C_0^{2/(\gamma-1)}\alpha^{2\beta/(\gamma-1)}\gamma^{1/(\gamma-1)}.$$

This enables us to eliminate the velocity of sound from the nonlinear terms in the Navier–Stokes equation during renormalization of the source and sink and corresponds to acoustic mechanisms of energy and helicity dissipation. Then the renormalization of the energy and helicity dissipation (and the viscosity coefficients, which we shall not write out below) returns us once again to a two-parameter group of scaling transformations:

$$\bar{\varepsilon}^*=\rho_0^{-1}\gamma^{1/(\gamma-1)}C_0^{2/(\gamma-1)}\bar{\varepsilon},$$
$$\bar{\eta}^*=\rho_0^{-1}\gamma^{1/(\gamma-1)}C_0^{2/(\gamma-1)}\bar{\eta}, \quad \alpha x=x', \quad \alpha^{1-\beta}t=t',$$
$$\alpha L=L', \quad \alpha^{-(\beta+3)}\mathbf{y}(\mathbf{x})=\mathbf{y}'(\mathbf{x}'),$$
$$\alpha^{-(2\beta/(\gamma-1)+3)}y_\rho^*(\mathbf{x})=y_\rho^{*\prime}\mathbf{x}', \quad \alpha^{3\beta-1+2\beta/(\gamma-1)}\bar{\varepsilon}^*=\bar{\varepsilon}^{*\prime},$$
$$\alpha^{3\beta-2+2\beta/(\gamma-1)}\bar{\eta}^*=\bar{\eta}^{*\prime}. \quad (3.13)$$

The transformation (3.13) transfers us to new measurement scales for the parameters. The uniqueness of the solution of the Cauchy problem for Eq. (3.6) leads to the similarity theorem

$$\Phi\{\mathbf{y}(\mathbf{x}),y_\rho^*(\mathbf{x}),t,\bar{\varepsilon}^*,\bar{\eta}^*,L\}=\Phi\{\mathbf{y}'(\mathbf{x}'),y_\rho^{*\prime}(\mathbf{x}'),t,\bar{\varepsilon}^{*\prime},\bar{\eta}^{*\prime},L'\}. \quad (3.14)$$

Unlike similarity hypotheses, (3.14) is an exact relation. Let us examine the correlation function

$$\langle\rho(\mathbf{x}_1,t)u_i(\mathbf{x}_1,t)u_i(\mathbf{x}_2,t)\rangle. \quad (3.15)$$

Its spectral density can be expressed in terms of Φ:

$$E=-\frac{i}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{|k|=k}e^{-i\mathbf{k}(\mathbf{x}_2-\mathbf{x}_1)}\times\{D_\rho(\mathbf{x}_1)D_i(\mathbf{x}_1)D_i(\mathbf{x}_2)\Phi\}_{\substack{y=0\\y_\rho=0}}d\mathbf{x}dS(\mathbf{k}), \quad (3.16)$$

where $dS(\mathbf{k})$ is an area element of the surface of the sphere $|\mathbf{k}|=k$. It can be shown that

$$\left\langle\frac{\rho u^2}{2}\right\rangle=\frac{1}{2}\rho_0 C_0^{-2/(\gamma-1)}\int_0^\infty E(k,t)dk. \quad (3.17)$$

Transforming in (3.16) to a functional that depends on the primed variables and setting $\alpha=k$, we have

$$E(k)=\frac{f_1(k^{3\beta-1+2\beta/(\gamma-1)}\bar{\varepsilon}^*,k^{3\gamma-2+2\beta/(\gamma-1)}\bar{\eta}^*,kL)}{k^{2\beta\gamma/(\gamma-1)+1}}. \quad (3.18)$$

It follows from the arbitrariness of β that $\partial E/\partial\beta=0$, whence we obtain the following spectrum in the helical range:

$$E(k)=\gamma^{2\gamma/(\gamma-1)(3\gamma-1)}\rho^{(\gamma-1)/(3\gamma-1)}C_0^{-2/(3\gamma-1)}\bar{\eta}^{2\gamma/(3\gamma-1)}\times k^{-(7\gamma-1)/(3\gamma-1)}\Psi(kl_h,kL). \quad (3.19)$$

When $\bar{\varepsilon}$ was renormalized in Refs. 9 and 19, the factors proportional to the density and the adiabatic exponent were omitted, and a totally incorrect result was consequently obtained. To determine the form of the spectrum we need an additional hypothesis regarding the asymptotic behavior of Ψ in the range $l_h\ll k^{-1}\ll L$. The locality hypothesis allows us to replace Ψ by its asymptote.

We note that in the limit $\gamma\rightarrow 1$ the spectrum that we obtained becomes

$$E(k)=e\bar{\eta}C_0^{-1}k^{-3}\Psi(kl_h,kL), \tag{3.20}$$

where e is the base of the natural logarithms. The form of the spectrum suggests that the shock sink weakens in the helical case. When we move over to the Kolmogorov range, Ψ is specified by the leading term in the asymptotic behavior at $kl_h \gg q$, and we obtain

$$E(k)=\text{const}\cdot\bar{\varepsilon}C_0^{-1}k^{-2}. \tag{3.21}$$

In the limit of an incompressible fluid ($\gamma\rightarrow\infty$) we have

$$E(k)=\rho_0^{1/3}\bar{\eta}^{2/3}k^{-7/3}\Psi(kl_h,kL). \tag{3.22}$$

The hypothesis that compressibility can serve as a sink for turbulent motions leaves only two parameters in the group of scaling transformations of the equation for the characteristic functional, making it possible to obtain spectral dependences both in the Kolmogorov range and in the helical range. This hypothesis was confirmed by an analysis of the turbulence of drift waves.[20]

4. CONCLUSIONS

Thus, the mechanism that generate the mean helicity lead to a second cascade range in addition to the Kolmogorov range in the system. The constant that does not depend on the scale of the helicity here is its flux. Nevertheless, note that this requirement, like the requirement that the energy flux be constant in the Kolmogorov range, is not inflexible. The spectral exponents can be obtained for an arbitrary transformation group parameter β, while the requirements that the fluxes be constant hold only for fixed values of β (Ref. 21). The spectral characteristics undergo significant changes, which are associated, as we understand, with at least a partial reverse cascade into the large-scale region.

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