An atom in an ultrastrong laser field

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A consistent quantum mechanical approach is used to derive the analytical solution of the problem of multiphoton ionization of an atom by an ultrastrong laser field. The phenomenon of stabilization of an atom in an ultrastrong laser field is shown to be related to spectral singularities in the submatrix of the atom's dipole moment operator. © *1996 American Institute of Physics.* [S1063-7761(96)00508-2]

1. INTRODUCTION

Ionization processes in laser fields have been actively studied in recent years by theoreticians and experimenters.[1–15] While a correct theoretical description for the case of a weak laser field can be constructed by choosing an appropriate variant of perturbation theory, in the case of an (ultra)strong field any perturbation theory yields at best a qualitative description. For this reason, attempts have been made to derive a theory of atomic ionization for narrow ranges of parameters by drawing on quasiclassical and semiclassical ideas or by reducing theoretical analysis to numerical experiments.[1–15] Up till now, however, no consistent theory for describing the ionization of atoms in an ultrastrong laser field has been suggested. This paper is an attempt to build such a theory by exploiting the methods of Refs. 16 and 17.

Let us formulate the terminology. Problems of the type considered here, such as photoionization of an atom (including multiphoton ionization), photodetachment of an electron from a negatively charged ion, decay of a molecule initiated by a laser field, and photoionization in the presence of external static fields, contain parameters that determine the dynamics of the process: the effective Rabi parameter R of the external field (expressed in frequency units), the ionization potential M (or its analog), the frequency ω of the external field. With proper choice of units all these parameters have the same dimensions. An external field is said to be strong if R and M are of the same order and ultrastrong if $R \gg M, \omega$. It is this situation that is discussed in the present paper. Of course, the restrictions imposed on R can be more stringent or less stringent, depending on the actual problem. (Obviously, in the problem of electron photodetachment the value of M is much lower than in the problem of ionization of an atom.)

The analytical approach to this problem that we develop differs considerably from the known approaches and makes possible a consistent study of the problem under extremely broad assumptions about the atom and the external field. We begin with the "quantum-optics" representation (or the "bare" states representation), in which the solution of the Schrödinger equation is sought. Thus, we examine the problem in consistent quantum mechanical terms. In this representation the Schrödinger equation is written in the form of an (infinite) linear system of differential equations. This system can be reduced to two integro-differential equations, which naturally incorporate a large parameter (proportional to the amplitude of the external field). By employing asymptotic methods we construct the asymptotic solution of an initial-value problem corresponding to the multiphoton ionization of the atom in an ultrastrong laser field. Similar approaches were used in Refs. 16 and 17 to study the dynamics of simple level–band and level–continuum models. Here, however, the problem is more complicated, and it actually breaks down into two parts. First, we must develop a general scheme for the asymptotics of the problem, at least in simple situations. Second, we must apply the scheme to real atoms. Here we derive only the leading term of the asymptotic expansion of the solution of the problem, i.e., find the solution to within "small" corrections. Nevertheless, the results concerning the dynamics of the simpler models discussed in Refs. 16 and 17 (such models allow calculating the first "small" corrections explicitly) make it possible to estimate the role of the next terms in the asymptotic expansion qualitatively. Then we discuss the ranges of applicability of our approaches to real atoms. As a whole, these results provide a new explanation of the known effect of atom stabilization in an ultrastrong field: the reduction in the atom ionization rate as the amplitude of the external field grows (a detailed discussion of this effect can be found in Ref. 18). More precisely, we relate this effect to spectral singularities in the submatrix of the atom's dipole moment operator.

2. DESCRIPTION OF THE BASIC FORMALISM

Let $|0\rangle$ be the initially populated level (it is assumed that only one such level can exist), and let $|E\rangle$ be the other energy levels of the atom (both discrete states and states in the continuous spectrum). We seek the atom's wave function $\Psi(t)$ in the form of a linear combination (the quantum-optics representation):

$$\Psi(t)=a(t)|0\rangle+\int b(E,t)|E\rangle dE. \tag{1}$$

Here and in what follows the integral with respect to E (we drop the limits) also incorporates (when necessary) a sum over discrete states. By a shift in the energy scale we can put the energy of the zeroth level equal to zero, so that E is the energy of the state $|E\rangle$.

Suppose that the atom in placed in an ultrastrong laser field with an amplitude $A(t)=\rho q(t)$. Here

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$$q(t)=\int_{\omega_1}^{\omega_2}\mu(\omega)\cos[\omega t+\varphi(\omega)]d\omega,$$

with the optical frequencies ω_2 and ω_1 being of the same order, and ρ and $q(t)$ chosen so that $|q(t)|=1$. To be more definite we assume, for instance, that $q(t)=\cos\omega t$. Next, we choose the time scale so that $\omega=1$, where generally $\omega=\frac{1}{2}(\omega_1+\omega_2)$. In terms of the amplitudes $a(t)$ and $b(E,t)$ the Schrödinger equation has the form

$$\frac{da(t)}{dt}=i\rho q(t)\int g(E)b(E,t)dE, \tag{2}$$

$$\frac{db(E,t)}{dt}=-iEb(E,t)+i\rho q(t)\times\left[g(E)a(t)+\int P(E,E_1)b(E_1,t)dE_1\right]. \tag{3}$$

Here $g(E)$ and $P(E,E_1)$ are the respective matrix elements of the atom's dipole moment operator. Equations (2) and (3) constitute an infinite system of ordinary linear differential equations: the parameter E assumes an infinite set of values. For brevity, on the right-hand side of Eq. (2) we have dropped the term proportional to $a(t)$, since this term can be taken into account trivially in our approach.

The parameter ρ in the above equations has yet to be specified: we can, for instance, multiply ρ and divide $g(E)$ by an arbitrary factor. We fix ρ in the following manner. Let us assume that

$$\int g^2(E)dE=1.$$

We call the corresponding value of the parameter ρ the effective Rabi parameter of our problem, and to obtain the dimensional value of R it must be multiplied by ω.

We seek a solution of Eqs. (2) and (3) that satisfies the initial conditions

$$a(0)=1, \tag{4}$$

$$b(E,0)=0. \tag{5}$$

The information about the potential of the atom in our model is contained in the spectrum of the atom and the functions $g(E)$ and $P(E,E_1)$. Suppose that these functions satisfy the following conditions:

$$g(E)\sim\theta E^{-\alpha-1},\quad E\to\infty,$$

$$P(E,E_1)\sim\nu|E-E_1|^{-1-\alpha},\quad |E-E_1|\to\infty, \tag{6a}$$

$$0<\alpha<1/2.$$

We assume $\rho\gg 1$ and call such a laser field ultrastrong. We also assume that the functions $g(E)$ and $P(E,E_1)$ are of order unity. This means that the moments of these functions that are not infinite and the parameter θ and ν are $O(1)$. Mathematically this means that the initial parameters ω (the frequency of the external radiation) and M (the ionization potential of the atoms) are quantities of the same order. From our assumption about the behavior of the functions $g(E)$ and $P(E,E_1)$ it follows that the "essential" part (from the standpoint of interaction with a level) lies near $A\omega$, where A is a quantity of order unity. This is an important condition, since it determines the relationship between the Rabi parameter and the structure of the atom and makes it possible to speak of an "effective width of the atomic spectrum." In practical terms, this quantity coincides in order of magnitude with the ionization potential M.

As noted earlier, first we must develop a procedure for constructing the asymptotic solution of our problem for a model situation, i.e., we assume that

$$\int dE\,dE_1P^2(E,E_1)<\infty,\quad g(E),P(E,E_1)>0. \tag{6b}$$

These conditions imply that the integral operator P with the kernel $P(E,E_1)$ is a Hilbert–Schmidt operator[19] and has only a discrete spectrum. We call this operator the submatrix of the atom's dipole moment operator. The unrealistic assumption (6b) will enable us to avoid complications associated with the structure of the spectrum of operator P. Then we will discuss in detail the structure of the spectrum of P for real atoms and the problem of extending the results to this case. Note that P is not of the atom dipole moment operator, although it is closely related to that operator. Section 6 is devoted to its description.

Let us introduce a function $\Xi(E,x,t)$ in the following way. Assuming the initial condition (5) holds, we put

$$b(E,t)=i\rho\int_0^t a(x)q(x)\Xi(E,x,t)dx. \tag{7}$$

Then, plugging (7) into (3), we obtain an equation for $\Xi(E,x,t)$:

$$\frac{\partial\Xi(E,x,t)}{\partial t}=-iE\Xi(E,x,t)+i\rho q(t)\times\int P(E,E_1)\Xi(E_1,x,t)dE_1, \tag{8}$$

$$\Xi(E,t,t)=g(E). \tag{9}$$

Using (8), we can reduce Eq. (2) to the following integro-differential equation:

$$\frac{da}{dt}=-\rho^2q(t)\int_0^t a(x)q(x)\Phi(x,t)dx, \tag{10}$$

$$\Phi(x,t)=\int g(E)\Xi(E,x,t)dE. \tag{11}$$

We have therefore reduced the problem of multiphoton ionization of an atom in an ultrastrong laser field to the problem of finding the solution of the initial-value problem (10), (4), (8), and (9). These equations contain a large parameter ρ, which suggests using an asymptotic procedure to find the solution. The main idea of our further reasoning, which is a modification of the approaches developed in Refs. 16 and 17, consists in the following. We perform an asymptotic expansion of the kernel of the integral operator on the right-hand side of Eq. (10) using the procedure of integrating by parts. To determine the leading term in the asymptotic expansion of the solution of our initial-value problem it is sufficient to construct the two leading terms in an approximation of the

integral operator on the right-hand side of Eq. (10). These approximations (which are integral operators) have a simple structure, which makes it possible to find the asymptotic behavior of the solution of our initial-value problem.

Note that the external laser field in our model can be modulated in an arbitrary manner. It may also contain several modulated harmonics with frequencies of the same order. Thus, we examine multiphoton processes outside the scope of the rotating wave approximation.

In the next section we describe the procedure of performing the asymptotic expansion of the integral operator in the right-hand side of Eq. (10) and write a "truncated" equation containing the two leading terms of this expansion. In Sec. 4 we construct the solution of the truncated equation. In Sec. 5 we show that stabilization of the atom in the model (6b) can be related to the spectral behavior of the operator P (eigenvalues). Section 6 is devoted to studying the operator P for real atoms. There we find the spectrum of P and discuss its features. In the Conclusion we discuss the results. Note that if in our model the parameter ρ is small, we can use the regular perturbation theory approach of Ref. 20 to derive an asymptotic expansion of the solution up to any order in ρ.

3. DERIVATION OF THE TRUNCATED EQUATION

Following the terminology of Refs. 16 and 17, we say that a function is fast (slow) if differentiation raises (leaves unchanged) its asymptotic order.

The procedure for constructing an asymptotic expansion of an integral of the product of a fast function and a slow function is well-known.[21] One needs only to integrate by parts, shifting differentiation from the fast function to the slow. Properly modified, these ideas were used in Refs. 16 and 17 to describe the dynamics of level–band and level–continuum systems in strong laser fields. However, the right-hand side of Eq. (10) contains the product of two fast functions in the integrand, $a(x)$ and $\Phi(x,t)$; the fact that these functions are fast follows from Eqs. (10) and (8). Thus, as a result of integrating by parts and shifting the derivative from $a(x)$ to $\Phi(x,t)$ we obtain terms of equal asymptotic order. Integrating by parts an infinite number of times leads to an infinite series of terms of the same asymptotic order. The main idea of our approach consists in the following. If we collect terms of the leading asymptotic order, we arrive at an integro-differential equation with an extremely simple structure. Collecting terms of the next asymptotic order, which appear as a result of an infinite iteration of the integration-by-part procedure, we get a correction for the right-hand side, which is another integral operator with a kernel that can be calculated explicitly. These ideas make it possible to determine the leading terms in the asymptotic expansion of the solution of the initial-value problem.

Here is an analogy with the ordinary WKB-expansion of the solution of an ordinary differential equation with a large parameter. The asymptotic behavior of the solution has the following structure

$$u(t)=\exp[i\rho\mu_1(t)+\mu_2(t)+\mu_3(t)/\rho+\ldots].$$

Our goal is to find the analogs of $\mu_1(t)$ and $\mu_2(t)$, which together determine what we call the leading term in the asymptotic expansion of the solution, a solution to within a small correction term.

In integrating by parts we will need to calculate quantities of the form $\partial^k\Phi(x,t)/\partial x^k$, $k=0, 1, 2,\ldots$. Let us write Eqs. (8) and (9) in a more convenient form. Clearly, these equations are equivalent to the following equation:

$$\Xi(E,x,t)=g(E)\exp[-iE(t-x)]+i\rho\int_x^t q(s)\times\exp[-iE(t-s)]ds\times\int P(E,E_1)\Xi(E_1,x,s)dE_1. \tag{12}$$

When combined with Eq. (11), this equation makes it possible to effectively calculate the derivatives of $\Phi(x,t)$.

To illustrate the new procedure, we perform a double integration of Eq. (10) by parts. For brevity we introduce a new operator D as follows:

$$Dm(t)=[i\rho q(t)]^{-1}\frac{dm(t)}{dt}. \tag{13}$$

If we now introduce a new variable,

$$p(t)=i\rho\int_0^t q(s)ds, \tag{14}$$

then

$$v(t)=[Dm](p)=\frac{dm}{dp}. \tag{15}$$

Here we assume that

$$[D^{-1}v](x)=\int_0^x v(p)dp. \tag{16}$$

From Eq. (13) it follows that

$$i\rho q(t)v(t)=\frac{d(D^{-1}v)}{dt}. \tag{17}$$

Thus, according to (17), Eq. (10) can be written as

$$\frac{da(t)}{dt}=i\rho q(t)\int_0^t\left\{\frac{d}{dx}[D^{-1}a]\right\}(x)\Phi(x,t)dx.$$

Integrating by parts, we obtain

$$\frac{da}{dt}=i\rho q(t)\left[[D^{-1}a](t)\Phi(t,t)-\int_0^t[D^{-1}a](x)\frac{\partial\Phi(x,t)}{\partial x}dx\right]. \tag{18}$$

Here we employed the fact that, as Eq. (16) implies, $[D^{-1}a](0)=0$. Combining (11) and (12) we find

$$\Phi(t,t)=G_1=\int g^2(E)dE. \tag{19}$$

Calculating $\partial\Xi(E,x,t)/\partial x$ by Eq. (12), we get

$$\frac{\partial \Xi(E,x,t)}{\partial x}=\phi_1(E,x,t)+i\rho q(x)\Psi_1(E,x,t), \tag{20}$$

where

$$\phi_1(E,x,t)=iEg(E)\exp[-iE(t-x)],$$

$$\Psi_1(E,x,x)=-\int P(E,E_1)g(E_1)dE_1.$$

Using Eqs. (20) and (11), we can now calculate the integrand on the right-hand side of Eq. (18). Then, integrating the term containing $\Psi_1(E,x,t)$ by parts in the resulting integral, we arrive at

$$\frac{da}{dt}=i\rho q(t)\Bigg[[D^{-1}a](t)G_1+[D^{-2}a](t)G_2-i\times\int_0^t[D^{-1}a](x)\psi_1(x,t)dx+\int_0^t[D^{-2}a](x)\frac{\partial\hat{\Psi}_1(x,t)}{\partial x}dx\Bigg],$$

$$\hat{\Psi}_1(x,t)=\int g(E)\Psi_1(E,x,t)dE,$$

$$\psi_1(x,t)=\int Eg^2(E)\exp[-iE(t-x)]dE,$$

$$G_2=\int\int dE\,dE_1 g(E)P(E,E_1)g(E_1).$$

We continue this procedure bearing in mind the following:

(a) integration of a fast function lowers its asymptotic order; and

(b) if $a(x)$ is a fast function and $q(x)$ and $\psi(x)$ are slow functions, then

$$\int_0^t dx\,a(x)\int_x^t ds\,q(s)\psi(s)=\int_0^t dx\,a(x)q(x)\int_x^t ds\,\psi(s)[1+o(1)].$$

As a result, collecting the two leading terms of the asymptotic expansion of the integral operator on the right-hand side of Eq. (10), we arrive at the following truncated equation:

$$\frac{da}{dt}=i\rho q(t)\Bigg[\sum_{k=1}^{\infty}[D^{-k}a](t)G_k-i\times\int_0^t dx\sum_{k=1}^{\infty}[D^{-k}a](x)\psi_k(x,t)\Bigg], \tag{21}$$

where

$$G_k=\int\ldots\int\left(\prod_{s=1}^{k}dE_s\right)g(E_1)\left(\prod_{s=1}^{k-1}P(E_s,E_{s+1})\right)g(E_k), \tag{22}$$

$$\psi_k(x,t)=\int\ldots\int\left(\prod_{s=1}^{k}dE_s\right)g(E_1)\times\left(\prod_{s=1}^{k-1}P(E_s,E_{s+1})\right)g(E_k)\times\exp[-iE_1(t-x)]\sum_{s=1}^{k}E_s.$$

Note that the above two terms are indeed two successive terms in the asymptotic expansion of the integral operator on the right-hand side of Eq. (10). The reason for this is that the second term, in comparison to the first, contains one more integration with respect to x, which lowers the asymptotic order. To solve the initial-value problem (10), (4) it is sufficient to construct a solution of the truncated equation (21) that satisfies the condition (4).

4. SOLVING THE TRUNCATED EQUATION

We start with Eq. (21) in the leading term of the asymptotic expansion, i.e., the equation

$$\frac{da}{dt}=i\rho q(t)\sum_{k=1}^{\infty}[D^{-k}a](t)G_k. \tag{23}$$

Let us construct a solution of this equation that satisfies the condition $a(0)=1$. If according to (14) we shift to the variable p and integrate this equation once with allowance for the initial value, we arrive at a Volterra integral equation of the second kind, which has the form

$$a(p)-\sum_{k=1}^{\infty}[D^{-k-1}a](p)G_k=1,$$

with D^{-1} defined in (16). Here we will not write the kernel of the integral part of this equation explicitly. Instead we use the Laplace transformation and find $\hat{a}(r)$, the Laplace transform of the function $a(p)$ (see Ref. 19):

$$\hat{a}(r)=\{r[1-\hat{K}(r)]\}^{-1},\quad \hat{K}(r)=\sum_{k=1}^{\infty}G_k r^{-k-1}.$$

Let us write this function in a more explicit form. Suppose that λ_n and $\mu_n(E)$ are the eigenvalues and normalized eigenfunctions of the operator P,

$$(P\mu)(E)=\int P(E,E_1)\mu(E_1)dE_1$$

[in view of Eq. (6b), P has a purely discrete spectrum]. The function $P(E,E_1)$ is symmetric, so that $\lambda_n=\mathrm{Re}\,\lambda_n$, and the eigenfunctions form a orthonormal basis in the respective function space. We expand the function $g(E)$ in these base functions:

$$g(E)=\sum_n g_n\mu_n(E).$$

Then

$$G_k=\sum_n g_n^2\lambda_n^{k-1}.$$

Thus

$$\hat{K}(r)=\sum_n g_n^2\{r[r-\lambda_n]\}^{-1}.$$

As a result we arrive at the following expression:

$$\hat{a}(r)=\left[r-\sum_n g_n^2(r-\lambda_n)^{-1}\right]^{-1}. \tag{24}$$

Let us now discuss the position of the singularities of the function $\hat{a}(r)$. They coincide with the solutions of the equation

$$r=\sum_n g_n^2(r-\lambda_n)^{-1}.$$

The operator P is positive (we have chosen this property exclusively to simplify matters), and all its eigenvalues are positive and converge to zero. We assume that they are in descending order. When r varies from λ_{n+1} to λ_n, the sum on the right-hand side of the above equation decreases from $+\infty$ to $-\infty$, and this variation is monotonic since the g_n^2 are positive. Thus, each interval $(\lambda_{n+1},\lambda_n)$ contains a single pole of the function $\hat{a}(r)$. The function $\hat{a}(r)$ has two more poles: the rightmost pole is on the real axis to the right of all the eigenvalues λ_n, and the leftmost pole is in the negative real axis. By ζ_n we denote the successive poles of $\hat{a}(r)$ in decreasing order of their value, and by κ_n the corresponding residues of $\hat{a}(r)$. Inverting the Laplace transformation according to Ref. 19, we find

$$a(p)=\sum_n \kappa_n\exp[\zeta_n p].$$

Here the sum is over all the poles of $\hat{a}(r)$. Returning to the original variables, we get

$$a(t)=\sum_n \kappa_n \exp\left[i\zeta_n\rho\int_0^t q(s)ds\right]. \tag{25}$$

This relationship describes the solution of Eq. (23) satisfying the condition $a(0)=1$.

We seek the solution of Eq. (21) in the form

$$\begin{aligned} &a(t)=\sum_n \kappa_n \exp[R_n(t)],\\ &R_n(t)=i\zeta_n\rho\int_0^t q(s)ds+\int_0^t m_n(s)ds. \end{aligned} \tag{26}$$

Here the initial condition (4) is satisfied automatically. We introduce an equation for the function $m_n(s)$ by plugging (26) into Eq. (21). Now we can consider each exponential in (26) separately and assume that the functions $m_n(s)$ are slow: integration with respect to s lowers the asymptotic order of the fast term. We have

$$\begin{aligned} &[i\rho\zeta_n q(t)+m_n(t)]\exp[R_n(t)]\\ &=i\rho q(t)\left\{\int_0^t i\rho q(x)Z\left[i\rho\int_x^t q(s)ds\right]\right.\\ &\left.\times\exp[R_n(x)]dx+\int_0^t M(x,t)\exp[R_n(x)]dx\right\}. \end{aligned} \tag{27}$$

Here $Z(p-h)$ is the kernel of the integral operator on the right-hand side of Eq. (23), and $M(x,t)$ is the kernel of another integral operator, the second term on the right-hand side of Eq. (21). The following properties of Z, the operator with the kernel $Z(p-h)$, are important:

$$Z\exp(\zeta_n p)=\zeta_n\exp(\zeta_n p).$$

Integration by parts in the first integral on the right-hand side of Eq. (27) yields

$$\begin{aligned} &i\rho q(t)\int_0^t i\rho q(x)Z\left[i\rho\int_x^t q(s)ds\right]\exp[R_n(x)]dx\\ &=i\rho\zeta_n q(t)\exp[R_n(t)]-i\rho\zeta_n q(t)\\ &\times\int_0^t dx\, m_n(x)\exp[R_n(x)]. \end{aligned}$$

Plugging this expression into (27) and allowing for the fact that $R_n(x)$ is a fast function and $m_n(t)$ and $q(t)$ are slow functions, we arrive at an equation for $m_n(x)$:

$$2m_n(t)=i\rho q(t)\exp[-R_n(t)]\int_0^t M(x,t)\exp[R_n(x)]dx. \tag{28}$$

It is sufficient to calculate the integral on the right-hand side of Eq. (28) to the leading order of the asymptotic expansion, i.e., by dropping the term $m_n(x)$ in the expression for $R_n(x)$ and ignoring the dependence on the slow variables in the integral operator $M(x,t)$. Bearing all this in mind, we get

$$m_n(t)=\text{const}=-iV(\zeta_n)/2.$$

The function $V(s)$ is specified in the following manner. For s with a large positive real part we define this function via a series:

$$\begin{aligned} &V(s)=\sum_{k=1}^{\infty}\delta_k s^{-k-1},\quad \delta_1=\int Eg^2(E)dE,\\ &\delta_k=\int\ldots\int\left(\prod_{s=1}^{k}dE_s\right)g(E_1)\\ &\times\left[\prod_{s=1}^{k-1}P(E_s,E_{s+1})\right]g(E_k)\sum_{s=1}^{k}E_s, \end{aligned}$$

and for the other values of s we use the result of analytic continuation of this function. Here $\delta_k=O(1)$, according to our initial assumptions about the functions $g(E)$ and $P(E,E_1)$. From their definition it directly follows that the δ_k are positive. The quantities $1/2V(\zeta_n)$ are analogs of the high-frequency Stark shift. Their explicit expression in terms of the eigenvalues and eigenfunctions of the operator P can easily be derived, and all their values can be shown to be finite. We, however, will avoid the tedious details.

These results make it possible to write the final asymptotic expression for the solution of Eq. (21) satisfying the initial condition (4):

$$a(t)=\sum_n \kappa_n \exp\left[i\zeta_n\rho\int_0^t q(s)ds-\frac{itV(\zeta_n)}{2}\right]. \tag{29}$$

This relationship describes the solution of the initial-value problem to within corrections in terms of the functions $g(E)$ and $P(E,E_1)$, i.e., in terms of the atomic potential.

Let us briefly discuss the other terms in the asymptotic expansion of the solution of the initial-value problem. In order to construct these terms we must derive a truncated equation that is more refined than Eq. (21). If we continue to integrate the initial equation by parts, we finally get moments of the functions $g(E)$ and $P(E,E_1)$ of orders higher than δ_k, for instance, $\int E^2 g^2(E)dE$. As the assumptions (6a) imply, such moments are infinite; in other words, a singularity appears, which we tentatively call a weak singularity of integro-differential equations with a large parameter. Within our model the problem of properly accounting for the corresponding terms appears to be extremely complicated. However, in Ref. 16 the present author used a simpler model (one-photon decay of a level into the continuum without continuum–continuum transitions) and discussed this weak singularity. It was found that the presence of such a singularity leads to exponential decay of the average level population, the rate of this decay being of order $\rho^{-2\alpha}$. Qualitative ideas suggest that within the scope of the rather general model under discussion here the results should be similar.

Another feature of our problem that emerges in the next term of the asymptotic expansion has to do with the presence in our integro-differential equations of turning points, the zeros of the function $q(t)$. (We assume that the function has only simple zeros.) A detailed study of this singularity requires further refinement of the truncated equation. However, it is possible to qualitatively estimate the contribution of the singularity on the basis of the results of Ref. 17, where the singularity was studied for the simplest situation, i.e., for a level–band system in a quasiresonant highly modulated field. There it was also found that the presence of such a singularity leads (in the general case) to an exponential decay of the average level population at a rate ρ^{-1}. Qualitative reasoning suggests that with our model this phenomenon leads to similar results.

5. STABILIZATION OF AN ATOM IN AN ULTRASTRONG FIELD

The result obtained in the present work and in Refs. 16 and 17 suggest the following pattern of ionization of an atom in a ultrastrong field. The process consists of two parts, so to speak. First, the ultrastrong field initiates the formation of an electron wave packet described by Eq. (29). Because of the zeros in the function $q(t)$ the wave packet is distorted and decays at a rate of order ρ^{-1}. (For the simplest situation this distortion is described in Ref. 17.) We call this phenomenon "oscillations in the finite part of the spectrum." It is accompanied by the transfer of excitations to a distant region of the spectrum (for the simplest case this situation is described in Ref. 16). This transfer of excitations leads to a decay in the population of the initially populated level at a rate of order $O(\rho^{-2\alpha})$.

Let us show that our results provide a new explanation, differing from the one given in Ref. 18, for stabilization of an atom in an ultrastrong field and of the relation of this effect to the presence of singularities in the spectrum of the operator P. Employing the solution (29), we calculate the population $n(t)$ of the initially populated level of the atom. We get

$$n(t)=|a(t)|^2=\sum_n \kappa_n^2+\sum_{n\neq m}\kappa_n\kappa_m\exp\left\{i\rho(\zeta_n-\zeta_m)\times\int_0^t q(s)ds-\frac{1}{2}it[V(\zeta_n)-V(\zeta_m)]\right\}. \quad (30)$$

Since we have $\rho\gg 1$, when calculating the average of $n(t)$ over one period (which in our case is equal to unity), the second sum yields an asymptotically small contribution: the rapidly oscillating terms in the expression in braces "kill" the slowly varying terms. What is important here is that the ζ_n assume a discrete set of values. (If the function $a(t)$ were represented in the form of an integral with respect to the variable ζ, the level population would stabilize at a value asymptotically close to zero.) We get

$$\int_t^{t+1} dx\ n(x)=\sum_n \kappa_n^2[1+o(1)]>0. \quad (31)$$

Hence the average population of the initially populated level is time-independent in the leading term of the asymptotic expansion. To allow for decay of the average population of the level in the next order of the asymptotic expansion we must allow for transfer of excitations to a distant region of the spectrum. The results of Ref. 16 imply that the rate of excitation transfer to a distant region of the spectrum also decreases as ρ increases. The physical reason for this is quite clear. Indeed, the Rabi parameter determines the range of states most effectively interacting with the initially populated level. As ρ grows, these states move into the high-energy region and (in accordance with the asymptotic behavior of the matrix elements of the atom's dipole moment, the functions $g(E)$ and $P(E,E_1)$) the interaction of the level and these states weakens. Together these two phenomena lead to stabilization of the atom. But why does an atom in an ultrastrong field becomes stabilized? The answer from the viewpoint of our approach is as follows. First, because the operator P has a discrete spectrum (or, in general, spectral singularities), which constitutes a universal fact, as we will see shortly. Second, because an increase in the Rabi parameter leads to a slowing down in the transfer of excitations from the level to a distant region of the continuum.

6. THE REAL ATOM

Actually, the solution (25) or the more general solution (29) can be considered a representation of the solution of the initial-value problem in the form of an integral over the spectrum of operator P. If conditions (6b) are met, the spectrum is discrete and the integrals are reduced to sums over contributions of eigenvalues. But what is the spectrum of P for real atoms and can our results be applied to this case? The discussion of these topics given below is basically of a heuristic nature.

If the spectrum of P contains a continuous part, the solution (29) acquires an integral for this part of the spectrum. The asymptotic behavior of integrals of rapidly oscillating

functions implies[21] that the contribution of the integral over the continuous part of the spectrum to the average population $n(t)$ in this case is asymptotically small. For real atoms the kernel $P(E,E_1)$ has a strong singularity of the form $\delta'(E-E_1)$ and the spectrum of P contains a continuous part. In addition, P may have spectral singularities (or even eigenvalues). We will now describe such features. The atom is assumed one-dimensional.

We go back to the definition of the operator P. Here it is convenient to operate in the spatial representation. Let $\psi_0=|0\rangle$ be the normalized wave function of the initially populated level. Let S be the projection operator on the one-dimensional space associated with this function. Then, according to our definitions,

$$P\psi(x)=(1-S)x(1-S)\psi(x),$$

where x is the operator of multiplication by the variable x, and 1 is the identity operator. We wish to find the wave functions of the continuous spectrum of P. Such functions must satisfy the following equations:

$$x\psi(\mu,x)=\mu\psi(\mu,x)+\kappa(\mu)\psi_0(x), \tag{32}$$

$$\int \psi(\mu,x)\psi_0^*(x)dx=0. \tag{33}$$

Equation (32) immediately yields

$$\psi(\mu,x)=\delta(x-\mu)+\kappa(\mu)\psi_0(x)(x-\mu)^{-1}, \tag{34}$$

where $-\infty<\mu<\infty$. Plugging this into (33) yields

$$\kappa(\mu)=-\psi_0^*(\mu)\left[\int |\psi_0(x)|^2(x-\mu)^{-1}dx\right]^{-1}. \tag{35}$$

Clearly, the wave functions (34) satisfy the condition

$$\int \psi(\lambda,x)\psi^*(\mu,x)dx\propto\delta(\lambda-\mu).$$

A special case emerges when

$$\int |\psi_0(x)|^2(x-\mu)^{-1}dx=0. \tag{36}$$

In this case Eq. (32) becomes invalid. We call the solutions of Eq. (36) the spectral singularities of the operator P.

Equation (36) is a transcendental equation and has roots (or apparently has roots) for almost all physically real atoms and all initial levels. Suppose, for instance, that the wave function $\psi_0(x)$ is symmetric with respect to the point $x=0$. Then Eq. (36) has the root $\mu=0$. Thus, at certain points of the continuous spectrum the operator P may have spectral singularities. More than that, if we assume that the function $\psi_0(x)$ is symmetric and has a root at $x=0$, the function $\psi_0(x)x^{-1}$ is a true eigenfunction of P with a zero eigenvalue. As shown above in the example of a model atom, the presence of discrete eigenvalues of P leads to stabilization of the atom in a ultrastrong laser field. Note that for a one-dimensional symmetric atom with a realistic potential (which has a standard singularity at zero) the fact that $\psi_0(0)=0$ is always true.

Of course, consistent generalization of the asymptotic technique developed on the basis of a model atom to the case of a real atom requires additional analytical research, including the derivation of formulas of the Parseval type for the set of functions (32) and (35). Here, apparently, it is natural to start immediately from relationships of the type (29). A study of the problems emerging in this connection lies outside the scope of the present work.

7. CONCLUSION

We have attempted to give a consistent description of the dynamics of an atom in an ultrastrong laser field on the basis of asymptotic methods. The atom was examined in the quantum-optics representation. This allowed reducing the initial-value problem to a linear integro-differential equation, with the reduced amplitude of the external field acting as a large parameter. Thus, this apparatus (the asymptotic analysis of solutions of integro-differential equations with a large parameter) is extremely convenient for studying the dynamics of atomic systems in an ultrastrong laser field. Similar approaches can be developed for the case of a strong field, where the common approach to excluding the optical frequency is to use the resonance approximation. Note in this connection the similarity between the statements of the problem and the results in Refs. 3 and 16.

Here are the main results of the present work. For a model atom we derived the analytical solution of the problem of multiphoton ionization of an atom in an ultrastrong laser field. We found that atom stabilization is related to the existence of spectral singularities in the submatrix of the dipole moment operator of the atom (for the model atom stabilization is related to the discrete spectrum of the submatrix of the atom's dipole moment operator). Under physically natural assumptions about real atoms this submatrix always has spectral singularities in the continuous spectrum. Occasionally these singularities can even be eigenvalues.

One more fact is worth mentioning. We assumed that the frequency ω of the external field and the ionization potential M are quantities of the same order. This assumption, however, is unnecessary in our approach. For instance, we can assume that $\omega\ll M$, introduce a time scale via M, and repeat our calculations, only to find that the results are the same. A remark is in order here, however. Our results describe the asymptotic behavior of the solution over times of order $\rho^{-2\alpha}\max(M,\omega)^{-1}$; here it is assumed that reduction of ρ to dimensionless form is done via $\max(M,\omega)$. Hence for low ω our relationships cannot be used to describe the behavior of the solution on times of order ω^{-1}.

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