The Lorentz–Dirac equation in light of quantum theory

A. I. Nikishov

P. N. Lebedev Institute of Physics, Russian Academy of Sciences, 117924 Moscow, Russia
(Submitted 26 January 1996)
Zh. Éksp. Teor. Fiz. **110**, 510–525 (August 1996)

To high accuracy, an electron in ultrarelativistic motion "sees" an external field in its rest frame as a crossed field ($E=H$, $\mathbf{E}\cdot\mathbf{H}=0$). In this case, quantum expressions allow the introduction of a local intensity of the radiation, which determines the radiative term of the force of radiative reaction. For $\gamma=(1-v^2)^{-1/2}\gg 1$ this term is much larger than the mass term, i.e., the term with $\dddot{x}$. Under these conditions, the reduced Lorentz–Dirac equation, which is obtained from the full Lorentz–Dirac equation by eliminating the terms $\dddot{x}$ and $\ddot{x}$ on the right side using the equation of motion without taking into account the force of radiative reaction, is equivalent to good accuracy to the original Lorentz–Dirac equation. Exact solutions to the reduced Lorentz–Dirac equation are obtained for a constant field and the field of a plane wave. For $\gamma\sim 1$ a local expression for the radiative term cannot be obtained quantitatively from the quantum expressions. In this case the mass (Lorentz–Dirac) terms in the original and reduced Lorentz-Dirac equations are not small compared to the radiative term. The predictions of these equations, which depend appreciably on the mass terms, are therefore less reliable. © *1996 American Institute of Physics.* [S1063-7761(96)01208-5]

1. INTRODUCTION

In Ref. 1, Nikishov and Ritus reported the discovery of a relation between the emission of scalar quanta by an accelerated mirror in $1+1$ space and the emission of photons by a classical electron moving along an analogous path. This discovery accentuates the need for additional and better understanding of the properties of the Lorentz–Dirac equation. In Ref. 1 we found that the renormalized energy-momentum tensor of the field created by an accelerated mirror is related to the force of radiation reaction. In particular, the vanishing of the radiative reaction force when an electron moves along a hyperbolic path corresponds to vanishing of the energy-momentum tensor of the field created by a mirror undergoing such motion. This implies that although particles are radiated, the energy density of the field everywhere equals zero. This is a manifestation of the nontrivial nature of the relation between energy-momentum of a field and particles: the presence of particles does not imply the presence of energy (see Ref. 2). For an electron, there is the further difficulty that radiation during hyperbolic motion takes place without radiative reaction.[3]

In this paper we will discuss the status of the $\dddot{x}$ problem in the Lorentz–Dirac equation. We begin the discussion with a more general question: under what circumstances may we expect that the predictions of the Lorentz–Dirac equation will be the classical limit of quantum physics? To address this question, we will deal not only with the Lorentz–Dirac equation but also the reduced equation obtained by using the equation of motion without taking into account the force of radiative reaction to eliminate the $\dddot{x}$ and $\ddot{x}$ terms from the right side (see Sec. 76 in Ref. 4). We will see that these equations are equivalent to good accuracy in the region of applicability of the classical theory, and in particular in the ultrarelativistic case. This is important because exact solutions to the Lorentz–Dirac equations are almost unknown in the relativistic region, while such solutions can be found easily for the reduced Lorentz–Dirac equations. In particular, we present solutions for a constant field and the field of a plane wave.

Finally, we will discuss one special case for which an exact solution to the Lorentz–Dirac equation is known.[5] This solution explicitly reveals which terms exhibit a difference from the reduced Lorentz–Dirac equation. This example is important for a better understanding of the connection between classical and quantum expressions for the radiation.

The introduction of a radiative reaction force is based on the possibility of determining the instantaneous intensity of the radiation in classical theory. The concept of an instantaneous intensity is foreign to quantum physics, because the measurement of energy requires time. This is in agreement with the fact that we cannot always even approximately find the intensity of radiation by interpreting the spectrum of the radiation, as is done in the classical limit of quantum theory.[6,7] Thus, for one-dimensional motion, where we clearly have a connection with the radiation of a mirror in $1+1$ space, only the order of magnitude of the intensity of the radiation is determined. This is related to the fact that for any interval of time in which we attempt to determine the average intensity, the duration of the interval turns out to equal the length over which the radiation is generated in order of magnitude,[6] i.e., this interval, and with it the intensity of the radiation, is ill-defined.

By analogy we may conclude that the renormalized energy-momentum tensor of the field of a mirror also corresponds to the possibility of determining the instantaneous intensity of the radiation. This does not agree with the finiteness of the length over which the main portion of the radiation is generated. Furthermore, for paths with a finite interval of acceleration we can determine the energy-momentum tensor of the field of the mirror by using normal ordering. This

 1063-7761/96/080274-08$10.00

method, which gives the energy-momentum tensor of the field including acceleration, does not require regularization. The expression obtained does not coincide with the regularized one.[8] However, the same total energy of the field is given by both tensors. For an electron moving along the same path, the total radiated energy is also well defined.

In Sec. 2 we argue that the intensity of the radiation obtained from the Lorentz–Dirac equation coincides to good accuracy with the quantity given by the reduced Lorentz–Dirac equation. In Secs. 3–6 we present exact solutions to the reduced Lorentz–Dirac equation for several configurations of external fields. It is shown that the change in the momentum of a particle due to radiation coincides with the expected change corresponding to the classical limit of the quantum expressions for radiation when $\gamma \gg 1$. With mass terms (i.e., with $\dddot{x}$ terms in the Lorentz–Dirac equation or terms derived from $\dddot{x}$ in the reduced Lorentz–Dirac equation) the situation is less satisfactory: there are arguments that indicate that no quantitative analogs to these terms exist in quantum electrodynamics.

In Sec. 7 we discuss a plane wave propagating along a constant electric field. For this configuration a special solution to the Lorentz–Dirac equations is known. As far as we know, this is the only exact solution to this equation. A comparison with solutions of the reduced Lorentz–Dirac equation show that the parameters of these solutions differ very little.

2. BASIC PROPERTIES OF THE LORENTZ-DIRAC EQUATION AND THE REDUCED LORENTZ-DIRAC EQUATION

In this section we reproduce some key elements of the theory of these equations,[4,9,10] while emphasizing that they have clear analogs in quantum physics. One unusual property of the Lorentz–Dirac equation is the fact that physically reasonable solutions are obtained by specifying the condition that acceleration in the future be absent. This was first shown for general one-dimensional motion in Ref. 11. These solutions exhibit a violation of causality (pre-acceleration) at distances of the same order as the classical electron radius $r_0 = e^2/m$. The unusual properties of the Lorentz–Dirac equations cause them to be viewed as unsatisfactory.[3,12,13] In Ref. 14, an attempt was made to find solutions without pre-acceleration. The solutions found there are incorrect, and are easily shown to include runaway solutions; in the most general formulation, a particle expends more energy over microscopic intervals than it can obtain from work done by an external force.[15]

All physically reasonable solutions should satisfy an integrodifferential equation that takes into account the condition that there should be no acceleration after the external field is turned off.[10,11,16] It is possible to have an external field that does not switch off, but in order to check whether a solution satisfies the integrodifferential equation in this case on some interval, it is necessary to know the solution for somewhat longer times (in fact, over an interval longer than several r_0).

Thus, in the integrodifferential equation[10]

$$m\dot{u}^i = \int_0^\infty K^i(\tau + \alpha\tau_0) e^{-\alpha} d\alpha,$$

$$K^i = F^{i\,\mathrm{ext}} - Ru^i, \quad u^i = \frac{dx^i}{d\tau}, \quad \dot{u}^i = \frac{du^i}{d\tau}, \tag{1}$$

$$R = \frac{2}{3} e^2 \dot{u}_i \dot{u}^i, \quad \tau_0 = \frac{2}{3}\frac{e^2}{m} \approx 6.27 \times 10^{-24}\ \mathrm{s}$$

a central role is played by the classical electron radius $r_0 = e^2/m$, which is 137 times smaller than the Compton length $\hbar/mc$ which determines the "size" of an electron in quantum physics. We would like the error in the electron position, which is much larger than $\hbar/mc$, to have almost no effect on the law of motion of the electron in the classical limit. Then, recalling that

$$K^i(\tau) \equiv K^i(x(\tau)),$$

we can obtain only two "reliable" terms from the right side of (1):

$$m\dot{u}^i = F^{i\,\mathrm{ext}} - u^i R + \dots . \tag{2}$$

The quantity R transforms according to the following rule (see Sec. 73 in Ref. 4):

$$\dot{u}^2 \equiv \dot{u}_i \dot{u}^i \to m^{-2} F^{i\,\mathrm{ext}} F_i^{\mathrm{ext}}. \tag{3}$$

The arrow implies that the replacement can be made with good accuracy. This rule agrees with quantum electrodynamics in the sense that during the generation of radiation we can neglect attenuation of the original state due to radiation because the fine structure constant is much smaller than unity: $\alpha = e^2/\hbar c \ll 1$. Furthermore, in the general case, an ultrarelativistic particle "sees" the external field as almost a crossed field ($E = H$, $\mathbf{EH} = 0$). The distance over which radiation is emitted is small compared to the distance over which the change in momentum of the particle is the same order as the momentum itself. In this case, we can introduce the intensity of the radiation into quantum electrodynamics, which is determined by the quantity standing on the right side of (3), and corresponds in the classical limit to the classical expression.[6,7]

Unfortunately, we cannot limit ourselves only to the terms written in (2). We must also satisfy the purely geometric condition $u^2 = -1$ and the condition $u\dot{u} = 0$ that follows from it. Then from (2) and (3) we obtain the reduced Lorentz–Dirac equation:[4]

$$\dot{u}^i = \frac{e}{m} F^{ik} u_k + \tau_0 \left(\frac{e}{m} \frac{\partial F^{ij}}{\partial x^k} u^k u_j + \frac{e^2}{m^2} F^{ij} F_{jk} u^k \right) - \tau_0 \frac{e^2}{m^2} (F^{lj} u_j)(F_{lk} u^k) u^i,$$

$$F^{i\,\mathrm{ext}} = e F^{ik} u_k. \tag{4}$$

This does not lead to a violation of causality or to runaway solutions. Hence, there is no longer any basis for assuming that this equation is less accurate than the Lorentz–Dirac equation.

As we already mentioned in the Introduction, the reduced Lorentz–Dirac equation was obtained in Ref. 4 by eliminating $\ddot{u}$ and $\dot{u}$. The derivation we present here emphasizes that the radiative term (the last term in (4)) follows from (1) under mild requirements on electron localization, whereas the mass terms should either be inserted by hand, so that the condition $u\dot{u}=0$ is satisfied, or obtained from the next term of the expansion of (1). In the latter case we are obliged to accept the existence of preacceleration and the possibility of the position of an electron varying over intervals of order τ_0. Therefore, predictions connected with the mass terms are less reliable. Now let us write down the next term of the expansion (2):

$$\tau_0\left[e\frac{d}{d\tau}(F^{ik}u_k)-\tau_0 m\frac{d}{d\tau}(u^i\dot{u}^2)\right]. \tag{5}$$

Whether a term with τ_0 can be observed, along with the other terms in the expansion, is a subject for future experimental and theoretical research. Here we limit ourselves to the term with τ_0. This term describes the electromagnetic "cloud" around a classical electron. Combining it with the left side converts the mass into a mass tensor, and the electron velocity u^i into the velocity of an electron together with its cloud:

$$U^i=u^i-\tau_0\frac{e}{m}F^{ik}u_k. \tag{6}$$

Then (2) leads to the semireduced Lorentz–Dirac equation

$$\dot{U}^i=\frac{e}{m}F^{ik}u_k-\tau_0 u^i\frac{e}{m}F^{jk}u_k\dot{u}_j. \tag{7}$$

This discussion shows that a classical electron together with its cloud is described by the kinetic momentum

$$m\dot{U}^i=\pi^i=(m\delta^{ik}-\tau_0 eF^{ik})u_k. \tag{8}$$

Now, the quantitative analog of this quantity is not obvious in quantum theory. If, nevertheless, (8) has any relation to reality, then it is possible that this quantity plays the role of a momentum during the collision of particles in the field, and that the intensity of radiation of an electron is determined not by the quantities $\dot{u}_i\dot{u}^i$, but rather the quantities $\dot{U}_i\dot{U}^i$, if we take bremsstrahlung into account. In this case (3) naturally remains valid: the two vectors on the right side of (7) are orthogonal and the square of the second is much smaller than the square of the first in the classical region, where the Lorentz–Dirac equation can be used.

In the following sections we obtain exact solutions to the reduced Lorentz–Dirac equation for a number of external field configurations, and show the terms cause the rule (3) to be inaccurate in these cases.

In what follows we assume $a_i=(\mathbf{a},a_4)$, $a_4=ia_0$.

3. CONSTANT ELECTRIC FIELD

Assuming the field $\mathbf{E}$ is directed along the 3-axis, we have the following components of the reduced Lorentz–Dirac equation:

$$\dot{u}_1=-\kappa\zeta u_1,\quad \dot{u}_2=-\kappa\zeta u_2,$$

$$\dot{u}_3=\varepsilon u_0+\kappa(1-\zeta)u_3,\quad \dot{u}_0=\varepsilon u_3+\kappa(1-\zeta)u_0,$$
$$\zeta=1+u_\perp^2=u_0^2-u_3^2,\quad \kappa=\tau_0\varepsilon^2,\quad \varepsilon=\frac{eE}{m},\quad \tau_0=\frac{2}{3}\frac{e^2}{m}. \tag{9}$$

Multiplying the first component of the equation by u_1, the second by u_2, and adding them, we find

$$\frac{1}{2}\frac{d}{d\tau}\zeta=-\kappa(\zeta-1)\zeta. \tag{10}$$

After integration we have

$$\zeta(\tau)=[1-c\exp(-2\kappa\tau)]^{-1},\quad c=1-\zeta^{-1}(0). \tag{11}$$

Adding and subtracting the 0- and 3-components of the equation, we obtain

$$\dot{u}_+=[\varepsilon+\kappa(1-\zeta)]u_+,\quad \dot{u}_-=[-\varepsilon+\kappa(1-\zeta)]u_-,$$
$$u_\pm=u_0\pm u_3. \tag{12}$$

Substituting $\zeta(\tau)$ from (11) into (12) and (9) and integrating, we find

$$u_+=u_{+in}\exp(\varepsilon\tau)[\zeta_{in}-(\zeta_{in}-1)\exp(-2\kappa\tau)]^{-1/2},$$
$$u_-=u_{-in}\exp(-\varepsilon\tau)[\zeta_{in}-(\zeta_{in}-1)\exp(-2\kappa\tau)]^{-1/2},$$
$$u_1=u_{1in}[\zeta_{in}(\exp(2\kappa\tau)-1)+1]^{-1/2},$$
$$u_2=u_{2in}[\zeta_{in}(\exp(2\kappa\tau)-1)+1]^{-1/2},\quad \zeta_{in}\equiv\zeta(0). \tag{13}$$

Directly from (9) we have

$$\dot{u}^2=\dot{\mathbf{u}}^2-\dot{u}_0^2=\varepsilon^2\zeta+\kappa^2\zeta(\zeta-1). \tag{14}$$

The degree to which the rule (3) is violated is measured by the ratio of the second term on the right side of (14) to the first. The order of magnitude of this ratio is as follows:

$$\frac{\kappa^2\zeta}{\varepsilon^2}=\left(\frac{2}{3}\right)^2\alpha^2\chi^2,\quad \alpha=\frac{1}{137},\quad \chi^2=\left(\frac{\varepsilon}{m}\right)^2\zeta. \tag{15}$$

The quantum expression for the intensity of the radiation passes to the classical limit under the conditions[7,17,18]

$$\chi^2\equiv\left(\frac{e\hbar E}{m^2c^3}\right)^2\zeta\ll 1.$$

Besides this factor, in (15) there is a still smaller factor of $\alpha^2=(e^2/\hbar c)^2$, so that (3) is well fulfilled.

We can expect that inclusion of the force of radiative reaction corresponds in quantum theory to inclusion of the decay of the initial state due to radiation. Then the intensity of the radiation should decrease. However, the sign of the correction in (14) is positive. This once more suggests that perhaps when the force of radiative reaction is included the intensity is determined by the rate of change of the kinetic velocity (7), i.e., essentially the square of the right side of (14) without the mass terms.

In quantum theory, the total probability of radiation is determined by the quantum numbers of the initial electron state. In classical theory these correspond to integrals of motion. We can choose them to be the components of the momentum at any instant of time. In the case under discussion, we pick p_1, p_2, p_3 as quantum numbers.[7] The radiation of a photon with momentum k' takes place with conservation of 3-momentum

$$\mathbf{p}-\mathbf{p}'=\mathbf{k}', \tag{16}$$

where $\mathbf{p}'$ are the quantum numbers of the final state of the electron. When $\zeta \gg 1$ holds, in which case the motion of an electron is significantly three-dimensional, the intensity of the radiation is well defined.[6,7] In the classical limit the recoil is small, and we have from (16) that

$$dp_1=-k_1', \quad \frac{d\bar{p}_1}{d\tau}=-\frac{\bar{k}_1'}{\tau_{\text{eff}}}. \tag{17}$$

The mean values of $\bar{k}_1'$ and τ_{eff} are determined from the relations

$$\bar{k}_1' W(\mathbf{p})=\frac{dP_1(\mathbf{p})}{d\tau}, \quad \tau_{\text{eff}}\, W(\mathbf{p})=1. \tag{18}$$

Here $W(\mathbf{p})$ is the total probability of radiation per unit proper time, and $dP_1(p)/d\tau$ is the intensity of radiation of the 1-component of the 4-momentum. In the classical limit we have[4]

$$\frac{dP_1}{d\tau}=\frac{2}{3}\frac{e^4}{m^2}(F_{ik}u^k)(F^{ij}u_j)u_1. \tag{19}$$

Then from (17)–(19) we obtain

$$m^{-1}\frac{d\bar{p}_1}{d\tau}=-\kappa\zeta u_1. \tag{20}$$

This coincides with the first equation in (9). The same situation obtains for the second equation in (9). These two equations are recovered exactly because the mass terms give no contribution to them.

That the mass terms contribute to the 3- and 0- components of (9) is indicated by the fact that they contain $1-\zeta$ instead of $-\zeta$. When $\zeta \gg 1$ holds this is unimportant. Besides, u_3 and u_0 change even without taking radiation reaction into account. The change in u_3 due to radiation is explained in the same way as for u_1, u_2. The change due to the mass term cannot be explained in this way. We turn to this question in the next section.

4. CONSTANT MAGNETIC FIELD

For a field $\mathbf{H}$ directed along the 3-axis, we have the following reduced Lorentz–Dirac equation:

$$\begin{aligned} &\dot{u}_1=\eta u_2-\kappa'\zeta u_1, \quad \dot{u}_2=-\eta u_1-\kappa'\zeta u_2, \\ &\dot{u}_3=-\kappa'(\zeta-1)u_3, \quad \dot{u}_0=-\kappa'(\zeta-1)u_0. \end{aligned} \tag{21}$$

Here

$$\eta=eH/m, \quad \kappa'=\tau_0\eta^2, \quad \zeta=1+u_\perp^2=u_0^2-u_3^2. \tag{22}$$

From the first two equations of (21) we have

$$\frac{1}{2}\frac{d}{d\tau}\zeta=-\kappa'\zeta(\zeta-1). \tag{23}$$

Integrating, we obtain

$$\zeta(\tau)=\zeta_{in}[\zeta_{in}-(\zeta_{in}-1)\exp(-2\kappa'\tau)]^{-1}. \tag{24}$$

Substituting $\zeta(\tau)$ into the third and fourth equations of (21) and integrating, we find

$$\begin{aligned} &u_3(\tau)=u_{3in}[\zeta_{in}-(\zeta_{in}-1)\exp(-2\kappa'\tau)]^{-1/2}, \\ &u_0(\tau)=u_{0in}[\zeta_{in}-(\zeta_{in}-1)\exp(-2\kappa'\tau)]^{-1/2}. \end{aligned} \tag{25}$$

From the first and second equations of (21) we obtain

$$u_2\dot{u}_1-u_1\dot{u}_2=\eta u_\perp^2. \tag{26}$$

This prompts us to use the same analysis as for $\kappa'=0$, i.e.,

$$\begin{aligned} &u_1=u_\perp(\tau)\cos(\eta\tau), \quad u_2=-u_\perp(\tau)\sin(\eta\tau), \\ &u_\perp(\tau)=\sqrt{u_1^2(\tau)+u_2^2(\tau)}. \end{aligned} \tag{27}$$

For simplicity we set $u_2(0)=0$.

We have directly from (21) that

$$\begin{aligned} &\dot{u}_1^2+\dot{u}_2^2=\eta^2u_\perp^2+\kappa'^2\zeta^2u_\perp^2, \quad \dot{u}_\pm=-\kappa'(\zeta-1)u_\pm, \\ &\dot{u}_+\dot{u}_-=\kappa'^2(\zeta-1)^2u_+u_-. \end{aligned} \tag{28}$$

From this we find

$$\dot{u}^2=\dot{u}_1^2+\dot{u}_2^2-\dot{u}_+\dot{u}_-=\eta^2(\zeta-1)+\kappa'^2\zeta(\zeta-1). \tag{29}$$

Here the situation is analogous to the case of an electric field; cf. (14).

It is interesting to note that it is easy to obtain an analog of Pomeranchuk's result from the 0-component of (21) (see Sec. 76 in Ref. 4). Setting $u_3=0$, for $\zeta \gg 1$ we have approximately that

$$\begin{aligned} &\dot{u}_0=-\kappa'u_0^3, \quad \frac{du_0}{u_0^2}=-\kappa'u_0d\tau=-\kappa'dt, \\ &\frac{1}{u_0}-\frac{1}{u_{0in}}=\kappa't. \end{aligned} \tag{30}$$

Thus, for $u_{0in}^{-1} \ll \kappa't$ we have $u_0^{-1} \approx \kappa't$. In other words, for sufficiently large initial energies, the finite but still ultrarelativistic energy is determined only by the quantity $(\kappa't)^{-1}$, and does not depend on the initial energy.

We now perform a calculation analogous to Eqs. (16)–(20). For the required choice of quantum numbers of the electron state, the conservation laws have the form

$$p_1-p_1'=k_1', \quad p_3-p_3'=k_3', \quad p_0-p_0'=k_0'. \tag{31}$$

In the classical limit, starting from the last equation we have

$$m\dot{u}_0=\frac{d\bar{p}_0}{d\tau}=-\frac{\bar{k}_0'}{\tau_{\text{eff}}}, \quad \tau_{\text{eff}}\,W=1, \quad \bar{k}_0'W=\frac{dP_0}{d\tau}. \tag{32}$$

From this we obtain

$$\dot{u}_0=-m^{-1}\frac{dP_0}{d\tau}=-\kappa'(\zeta-1)u_0, \tag{33}$$

which agrees with the 0-component of (21). The same thing applies to the 3-component of (21) as well. The mass terms do not contribute to these components.

The component u_1 (and u_2) changes even without including the radiation reaction. The additional change caused by radiation is described naturally by analogy with (33):

$$m^{-1}\frac{d\bar{p}_1}{d\tau}=-\kappa'(\zeta-1)u_1. \quad (34)$$

However, it is ζ that enters into the first equation of (21), not $-\zeta$. The difference is caused by the contribution of the mass term. We might think that the mass terms of the Lorentz–Dirac equation or the reduced Lorentz–Dirac equation could arise from higher-order perturbation theory of quantum electrodynamics,[19] However, this possibility is doubtful, because a careful investigation of the mass operator by Lorentz–Dirac[20] shows that the mass remains unchanged in a constant magnetic field, and that the mass shift in a constant electric field is only qualitatively similar to the 0-component of the Lorentz–Dirac mass tensor in the classical limit; see expressions (6) and (8). These facts are sufficient to convince us that in those cases where the mass terms in the Lorentz–Dirac equations are comparable to the radiative terms, the predictions of the equations will be degraded. In particular, this applies to one-dimensional motion.

5. CONSTANT ELECTROMAGNETIC FIELD

Assuming that the fields **E** and **H** are directed along the 3-axis, we have for the reduced Lorentz–Dirac equation

$$\begin{aligned}
&\dot{u}_1=\eta u_2-(\kappa'+\kappa)\zeta u_1,\\
&\dot{u}_2=-\eta u_1-(\kappa'+\kappa)\zeta u_2,\\
&\dot{u}_3=\varepsilon u_0-(\kappa'+\kappa)(\zeta-1)u_3,\\
&\dot{u}_0=\varepsilon u_3-(\kappa'+\kappa)(\zeta-1)u_0,\\
&\eta=eH/m,\quad \varepsilon=eE/m,\quad \kappa'=\tau_0\eta^2,\quad \kappa=\tau_0\varepsilon^2.
\end{aligned} \quad (35)$$

Proceeding in the same way as before, we find that $\zeta(\tau)$ is given by Eq. (11), in which we must replace κ by $\kappa+\kappa'$. Analogously, u_+ and u_- are obtained from the corresponding expressions in (13) by the same replacement. The components u_1 and u_2 have the form (27) with $u_\perp=[\zeta(\tau)-1]^{-1}$. The square of the invariant acceleration equals

$$\dot{u}^2=\dot{u}_\perp^2-\dot{u}_+\dot{u}_-=\eta^2(\zeta-1)+\varepsilon^2\zeta+(\kappa'+\kappa)^2\zeta(\zeta-1). \quad (36)$$

The violation of the rule (3) is measured by the ratio of the terms depending on $\kappa+\kappa'$ to the sum of the first two terms on the right side of (36). This ratio is always much smaller than unity in the classical region.

Note that in a quantum description of the sort given here only the two conservation laws $p_2-p_2'=k'$ and $p_3-p_3'=k'$ introduce the quantum numbers p_2 and p_3, which besides participating in the conservation laws play another role: they determine the center of the wave function with respect to x_1 and t. This implies that the 2- and 3-components of the reduced Lorentz–Dirac equation contain a contribution from the mass terms. Therefore, these components, as in Secs. 3 and 4, cannot be exactly recovered by using the considerations given here.

6. PLANE MONOCHROMATIC WAVE

Here the new element is the dependence of the field on coordinates and time. Let us write the vector potential of the field in the form

$$A_i(\varphi)=a_i^{(1)}\cos\varphi+a_i^{(2)}\sin\varphi,\quad \varphi=kx=-\omega x_-. \quad (37)$$

Then for the field we have

$$\begin{aligned}
&F_{ij}(\varphi)=\partial_iA_j-\partial_jA_i=-F_{ij}^{(1)}\sin\varphi+F_{ij}^{(2)}\cos\varphi,\\
&F_{ij}^{(1,2)}=k_ia_j^{(1,2)}-k_ja_i^{(1,2)},\quad k=(0,0,\omega,i\omega),\\
&a^{(1)}=(a_1,0,0,0),\quad a^{(2)}=(0,a_2,0,0).
\end{aligned} \quad (38)$$

The wave propagates along the 3-axis. The components of the regularized Lorentz–Dirac equation have the form

$$\begin{aligned}
&\dot{u}_1=-\eta_1u_-\sin\varphi+\rho u_-\eta_1\cos\varphi-\tau_0u_1u_-^2\mathbf{H},\\
&\dot{u}_2=\eta_2u_-\cos\varphi+\rho u_-\eta_2\sin\varphi-\tau_0u_2u_-^2\mathbf{H},\\
&\dot{u}_3=(-\eta_1u_1+\rho\eta_2u_2)\sin\varphi+(\eta_2u_2+\rho\eta_1u_1)\cos\varphi\\
&\qquad+\tau_0u_-(1-u_-u_3)\mathbf{H},\\
&\dot{u}_0=(-\eta_1u_1+\rho\eta_2u_2)\sin\varphi+(\eta_2u_2+\rho\eta_1u_1)\cos\varphi\\
&\qquad+\tau_0u_-(1-u_-u_0)\mathbf{H},\\
&\mathbf{H}=\eta_1^2\sin^2\varphi+\eta_2^2\cos^2\varphi,\quad \eta_1=ea_1\omega/m,\\
&\eta_2=ea_2\omega/m,\quad \rho=\tau_0\omega u_-.
\end{aligned} \quad (39)$$

From (39) we find

$$\dot{u}_-=-\tau_0u_-^3\mathbf{H}, \quad (40)$$

$$\begin{aligned}
\dot{u}_+=&2(\eta_1u_1+\rho\eta_2u_2)\sin\varphi+2(\eta_2u_2+\rho\eta_1u_1)\cos\varphi\\
&+\tau_0u_-(2-u_+u_-)\mathbf{H}.
\end{aligned} \quad (41)$$

From these equations we have

$$\begin{aligned}
\dot{u}^2=\dot{u}_1^2+\dot{u}_2^2-\dot{u}_+\dot{u}_-=&u_-^2[(1+\rho^2)\mathbf{H}+\rho(\eta_2^2\\
&-\eta_1^2)\sin(2\varphi)]+\tau_0^2u_-^2\mathbf{H}^2.
\end{aligned} \quad (42)$$

This is the left side of (3). The right side of (3) equals $u_-^2\mathbf{H}$. In the classical region we have $\rho\ll 1$, and the rule (3) is satisfied according to the same considerations as for a constant field.

In order to integrate Eq. (39) it is convenient to go from the proper time τ to the variable $\varphi=kx$:

$$\dot{u}_i=\frac{du_i}{d\varphi}\frac{d\varphi}{d\tau}=\frac{du_i}{d\varphi}ku=-\omega u_-\frac{du_i}{d\varphi}. \quad (43)$$

Then from (40) we obtain

$$\frac{1}{u_-(\varphi)}=\frac{1}{u_-(0)}-\frac{\tau_0}{\omega}\left[\frac{\eta_1^2+\eta_2^2}{2}\varphi+\frac{\eta_2^2-\eta_1^2}{4}\sin(2\varphi)\right]. \quad (44)$$

This is the analog of the Pomeranchuk formula: compare (30) and the remarks after it. Knowing $u_-(\varphi)$, we find $u_1(\varphi)$ and $u_2(\varphi)$ from (39) and (44), and then $u_0(\varphi)$.

In the remainder of this section we will be interested only in the case of a circularly polarized wave.

Setting $a_1=a_2=a,\eta_1=\eta_2=\eta$, we obtain

$$u_1(\varphi)=u_-(\varphi)\left\{-\eta\left[\frac{\sigma}{2\omega^2u_-^2(0)}(\sin\varphi-\varphi\cos\varphi)+\frac{\cos\varphi}{\omega u_-(0)}\right]-\tau_0\eta\sin\varphi+\frac{u_1(0)}{u_-(0)}+\frac{\eta}{\omega u_-(0)}\right\},$$

$$\sigma=2\tau_0\eta^2u_-^2(0),$$

$$u_2(\varphi)=u_-(\varphi)\left\{\eta\left[\frac{\sigma}{2\omega^2u_-^2(0)}(\cos\varphi+\varphi\sin\varphi)-\frac{\sin\varphi}{\omega u_-(0)}\right]+\tau_0\eta\cos\varphi+\frac{u_2(0)}{u_-(0)}-\tau_0\eta-\frac{\eta\sigma}{2\omega^2u_-^2(0)},\right. \quad (45)$$

$$x_-(\tau)=\frac{2u_-(0)}{\sigma}(\sqrt{1+s}+-1), \quad s=\sigma\tau.$$

In a quantum description of the radiation process, the conservation laws have the form[17,18]

$$p_1-p_1'=k_1', \quad p_2-p_2'=k_2', \quad p_--p_-'=k_-', \quad p_-\equiv p_0-p_3'. \quad (46)$$

Just as in (16)–(20), we find

$$\dot{u}_-=m^{-1}\frac{d\bar{p}_-}{d\tau}=-\tau_0u_-^3\eta^2, \quad (47)$$

which agrees with (40). Analogous expressions are obtained for the additional changes in u_1 and u_2: one power of u_- on the right side of (47) must be replaced by u_1 or u_2 respectively. Of course, we cannot explain the change due to the mass term containing ρ in (39) by this method. If the force of radiation reaction is not included, u_1 is unchanged for a suitable choice of the constant of motion. It would be interesting to experimentally track u_3 through its evolution under conditions where the contribution of the mass term is important.

When bremsstrahlung is included, the magnetic field of the wave accelerates the electron in the direction of propagation of the wave. As a result, the electron in its rest frame "sees" the frequency and intensity of the plane wave decrease. This leads to a decrease in its interaction with the wave. The acceleration of the electron along the direction of wave propagation can be stopped by applying a constant electric field. Thus, we are led to the field configuration discussed in the next section.

7. CIRCULARLY POLARIZED WAVE PROPAGATING ALONG A CONSTANT ELECTRIC FIELD

First let us consider the motion without including the force of radiative reaction. This is necessary if we are to understand the rule (3) in this relatively complicated case. We will take into account the force of radiative reaction only in the special case of a stationary circular orbit, for which the exact solution to the Lorentz–Dirac equation is known.[5] This allows us to compare the solutions to the original and reduced Lorentz–Dirac equations in detail.

Let us assume that a plane wave propagates along the 3-axis, as in the previous case. In addition, a constant electric field **E** is directed along this axis. The equations of motion without including the radiative reaction force have the form

$$\dot{u}_1=-\eta u_-\sin\varphi, \quad \dot{u}_2=\eta u_-\cos\varphi,$$
$$\dot{u}_3=\eta(u_2\cos\varphi-u_1\sin\varphi)+\varepsilon u_0, \quad (48)$$
$$\dot{u}_0=\eta(u_2\cos\varphi-u_1\sin\varphi)+\varepsilon u_3,$$
$$\eta=ea\omega/m, \quad \varepsilon=eE/m, \quad \varphi=-\omega x_-.$$

For this motion the square of the invariant acceleration equals

$$\dot{u}^2=(eF_{ik}u_k/m)^2=\eta^2u_-^2+\varepsilon^2u_+u_-+2\eta\varepsilon u_-(u_2\cos\varphi-u_1\sin\varphi). \quad (49)$$

A feature that is specific to this expression is the fact that the third term on the right side of (49), which is proportional to $\eta\varepsilon$, can cancel an appreciable part of the first two terms.

By virtue of the equation of motion (48) we have

$$u_1(\tau)=\bar{u}_1-\xi\cos\varphi, \quad u_2(\tau)=\bar{u}_2-\xi\sin\varphi, \quad \xi=ea/m, \quad (50)$$

where $\bar{u}_1$ and $\bar{u}_2$ are constants of the motion. Therefore,

$$u_2\cos\varphi-u_1\sin\varphi=\bar{u}_2\cos\varphi-\bar{u}_1\sin\varphi. \quad (51)$$

From (48) we have

$$\dot{u}_-=-\varepsilon u_-, \quad u_-=u_{-in}e^{-\varepsilon\tau}, \quad x_-=\frac{u_{-in}}{\varepsilon}(1-e^{-\varepsilon\tau}),$$
$$\varphi(\tau)=\frac{\omega u_{-in}}{\varepsilon}(e^{-\varepsilon\tau}-1). \quad (52)$$

Analogously,

$$\dot{u}_+(\tau)=\varepsilon u_++2\eta(u_2\cos\varphi-u_1\sin\varphi),$$
$$u_+=(1+u_\perp^2)/u_-. \quad (53)$$

Assuming the charge of the particle is positive, we see that $u_-\rightarrow 0$ when $\tau\rightarrow\infty$, and, thus, according to (49),

$$\dot{u}^2\rightarrow\varepsilon^2u_{+in}u_{-in}, \quad (54)$$

i.e., as $\tau\rightarrow\infty$ a particle moving along the wave with a velocity close to the velocity of light ceases to "feel" the wave, because the frequency and the field of the wave in the rest frame of the particle reduce to zero.

It is clear from (49) that the field of the wave, the electric field, and the cross term proportional to $\eta\epsilon$ all contribute to the radiation intensity. The last term can be reduced to zero by setting $\bar{u}_1=\bar{u}_2=0$, i.e., motion without drift in the 1-2 plane [see (51)]. In the next case of interest to us, the cross term plays an important role by strongly decreasing the radiation in the far-relativistic limit.

From (52) and (53) we have the τ-dependence of the quantity

$$u_3=(u_+-u_-)/2. \quad (55)$$

Thus, for large negative τ a positively charged particle moving opposite the electric field is braked and then accelerated along the electric field. For large negative τ the intensity of the radiation is large, since $u_- \gg 1$. The inclusion of bremsstrahlung radically changes the character of the particle motion in this region.

A negatively charged particle ($e = -|e|$) arriving from $x_3 = -\infty$ is initially braked by the constant electric field, and feels the effect of the wave field more and more strongly in its rest frame. The particle then begins to move opposite the momentum of the plane wave, and its radiation continues to increase throughout the motion. This leads to braking of the motion in the 1–2 plane, a result of which is that the magnetic field of the wave opposes the constant electric field and the acceleration counter to the 3-axis ceases. Thus, when radiation reaction is included, a stable circular orbit can exist for an electron moving in a field with this configuration. The losses due to radiation are balanced by the work done by the wave electric field. The solution to the Lorentz–Dirac equation for such an orbit is found from simple considerations of symmetry.[5] This solution is valuable to us because it allows us to see explicitly how much the original and reduced Lorentz–Dirac equations differ and by what terms, and to understand why the rule (3) is satisfied even when a stable orbit is formed with appreciable participation of the radiative reaction force, which is comparable to the Lorentz–Dirac force in a strong wave.

Let us look for the solution to the Lorentz–Dirac equation in the form

$$u_1 = -v\gamma\cos(\varphi-\psi), \quad u_2 = -v\gamma\sin(\varphi-\psi),$$
$$u_3 = 0, \quad u_0 = \gamma = \text{const}, \quad \varphi = -\omega\gamma\tau. \qquad (56)$$

Substitution into the right side of the Lorentz–Dirac equation gives

$$\dot u_1 = -\eta\gamma\sin\varphi + \tau_0 v\gamma^3\omega^2(1+v^2\gamma^2)\cos(\varphi-\psi),$$
$$\dot u_2 = \eta\gamma\cos\varphi + \tau_0 v\gamma^3\omega^2(1+v^2\gamma^2)\sin(\varphi-\psi),$$
$$\dot u_3 = \eta v\gamma\sin\psi - \varepsilon\gamma, \qquad (57)$$
$$\dot u_0 = \eta v\gamma\sin\psi - \tau_0\omega^2 v^2\gamma^5, \quad \gamma = (1-v^2)^{-1/2}.$$

Here, and in what follows, we will set $e = |e|$ for convenience. The use of (56) on the left side as well gives[5]

$$\cos\psi = \frac{v\gamma}{\xi}, \quad \sin\psi = \frac{\tau_0\omega v\gamma^4}{\xi}, \quad \tan\psi = \tau_0\omega\gamma^3,$$
$$\frac{\varepsilon}{\eta} = v\sin\psi, \quad \xi = \frac{ea}{m}. \qquad (58)$$

The Lorentz factor γ is determined from the condition $\sin^2\psi + \cos^2\psi = 1$, which leads to the relation

$$\left(\frac{v\gamma}{\xi}\right)^2 + \left(\frac{\tau_0\omega v\gamma^4}{\xi}\right)^2 = 1, \quad \tau_0 = \frac{2}{3}\frac{e^2}{m}, \qquad (59)$$

which can be rewritten in the form

$$\left(\frac{v\gamma}{\xi}\right)^2(1+\tan^2\psi) = 1.$$

From this it is clear that

$$v\gamma \approx \xi \quad \text{for } \tan\psi \ll 1,$$
$$\gamma^4 \approx \frac{\xi}{\tau_0\omega} \quad \text{for } \tan\psi \gg 1. \qquad (60)$$

In the factor $1 + v^2\gamma^2 = \gamma^2$ that appears in the first two equations of (57), the one comes from $\ddot u$, and the $v^2\gamma^2$ comes from the radiative term.

We now see how things stand with rule (3). Omitting the radiation reaction terms on the right side of (57), we obtain for the right side of (3)

$$\dot u_1^2 + \dot u_2^2 - \dot u_0^2 = \eta^2\gamma^2(1-v^2\sin^2\psi)$$
$$= \eta^2\gamma^2[\cos^2\psi + (1-v^2)\sin^2\psi]. \qquad (61)$$

Taking into account the expression for $\cos\psi$ in (58), it is easy to verify that the term with $\cos^2\psi$ in (61) gives the left side of (3), i.e., $\ddot u^2$, in agreement with the solution to the Lorentz–Dirac equation. Consequently, the amount of violation of rule (3) is the following:

$$\frac{\tan^2\psi}{\gamma^2} = (\tau_0\omega\gamma^2)^2. \qquad (62)$$

In the classical region

$$\tau_0\omega\gamma \equiv -\tau_0 ku \ll 1, \qquad (63)$$

and we might think that the discrepancy (62) can lead to a rather large amount of scatter in γ, even if we remain in region (63). However, in this problem γ is a function of the field, and even in a strong field, according to the second term in (60), we have

$$\tau_0^2\omega^2\gamma^4 = \tau_0\omega\xi = \frac{2}{3}e^2\frac{eB}{m^2}$$

(where B is the amplitude of the wave field), which is small in the classical region.

Note that the second term in square brackets in (61) is positive, i.e., the instantaneous intensity of the radiation according to the Lorentz–Dirac equation is smaller than the intensity when radiation reaction is neglected.

It remains for us to show that the parameters of the stationary orbit (56) for the reduced Lorentz–Dirac equation agree closely with the parameters determined in (58) and (59). The reduced Lorentz–Dirac equation has the form

$$\dot u_1 = -\eta u_-(1+\tau_0)\sin\varphi + \rho\eta u_-\cos\varphi - R_1,$$
$$\dot u_2 = \eta u_-(1+\tau_0)\cos\varphi + \rho\eta u_-\sin\varphi - R_2,$$
$$\dot u_3 = \eta(u_2+\rho u_1)\cos\varphi - \eta(u_1-\rho u_2)\sin\varphi - \varepsilon u_0$$
$$+ \tau_0[\eta^2 u_- + \varepsilon^2 u_3 + \varepsilon\eta(u_1\sin\varphi - u_2\cos\varphi)] - R_3, \qquad (64)$$
$$\dot u_0 = \eta(u_2+\rho u_1)\cos\varphi - \eta(u_1-\rho u_2)\sin\varphi - \varepsilon u_3$$
$$+ \tau_0[\eta^2 u_- + \varepsilon^2 u_0 + \varepsilon\eta(u_1\sin\varphi - u_2\cos\varphi)] - R_0,$$
$$R_i = u_i\tau_0\left(\frac{eF_{ik}u_k}{m}\right)^2, \quad \eta = \frac{eB}{m}, \quad \varepsilon\frac{eE}{m}, \quad \rho = \tau_0\omega u_-.$$

The difference between the 0- and 3-components of the reduced Lorentz–Dirac equation give

$$\dot u_- = \varepsilon u_- + \tau_0\varepsilon^2 u_- - R_-, \quad R_- = R_0 - R_3.$$

However, R_- now depends on all the components u_j (see (49)), which hinders our search for an exact solution.

In order to determine the parameters of the stationary orbit, we substitute (56) into (64). The first equation of (64) (or the second) gives

$$(1+\tau_0\varepsilon)\cos\psi=\frac{v\gamma}{\xi}-\rho\sin\psi,\quad \xi=\frac{ea}{m}=\frac{\eta}{\omega},\tag{65}$$

$$(1+\tau_0\varepsilon)\sin\psi=\rho\cos\psi+\frac{\tau_0 v}{\eta}\left(\frac{eF_{ik}u_k}{m}\right)^2.\tag{66}$$

From the third equation of (64) we obtain the condition that there be no motion along the 3-axis. Writing it in the form

$$(1+\tau_0\eta v\sin\psi-\tau_0\varepsilon)\frac{\varepsilon}{\eta}=v\sin\psi+\tau_0\eta\left(1-\frac{v\gamma}{\xi}\cos\psi\right),\tag{67}$$

we see that we need retain only the one in the brackets on the left side. The term in brackets on the right side is small for $\psi\ll 1$, since in this case it follows from (65) that $v\gamma/\xi\approx 1$. For $\psi\sim 1$ the second term on the right side of (67) is small compared with the first. Therefore, the ratio ε/η is the same as in (58).

Finally, from (66), (49), (60), and (61) we have for $\psi\ll 1$

$$\sin\psi=\tau_0\omega\gamma(1+v^2\gamma^2)=\tau_0\omega\gamma^3,$$

which agrees with the $\sin\psi$ in (58) when (60) is taken into account. For $\psi\sim 1$, i.e., in the ultrarelativistic limit, the mass terms do not play a role; therefore, the reduced Lorentz–Dirac equation almost coincides with the original Lorentz–Dirac equation. For $\gamma\sim 1$ the equivalence of these equations is ensured by the smallness of the radiative reaction force compared to the Lorentz–Dirac force. To sum up, we claim that both equations have the same status. If, nevertheless, it turns out that the Lorentz–Dirac equation is more accurate than the reduced Lorentz–Dirac equation, then the solution to the latter can be used as a first approximation, which can be refined when necessary.

I am deeply grateful to M. V. Berry for fruitful discussions and the opportunity to work in the unique atmosphere of the Isaac Newton Institute (Cambridge, England). I am also grateful to V. I. Ritus and S. L. Lebedev for fruitful discussions and valuable information. This work was supported by the Russian Fund for Fundamental Research (Grant No. 95-02-04219-a).

[1] A. I. Nikishov and V. I. Ritus, Zh. Éksp. Teor. Fiz. **108**, 1121 (1995) [JETP **81**, 615 (1995)].

[2] N. D. Birrell and P. C. W. Davies, *Quantized Fields in Curved Space*. Cambridge Univ. Press, Oxford, New York, 1984; Russ transl. Mir, Moscow, 1984.

[3] L. Herrera, Phys. Lett. A **145**, 14 (1990).

[4] L. D Landau and E. M. Lifshits, *The Classical Theory of Fields*, Nauka, Moscow, 1988 (Eng. transl. of 6th edition by Pergamon Press, Oxford, 1975).

[5] Ya. B. Zel'dovich, *Problems of Theoretical Physics*, p. 281, Nauka, Moscow, 1972 [in Russian].

[6] A. I. Nikishov and V. I. Ritus, Zh. Éksp. Teor. Fiz. **56**, 2035 (1969) [Sov. Phys. JETP **29**, 1093 (1969)].

[7] A. I. Nikishov, P. N. Lebedev, Inst. of Phys., USSR Acad. Sci. **111**, 152 (1979) [in Russian].

[8] A. I. Nikishov, Zh. Éksp. Teor. Fiz. **109**, 1889 (1996) [JETP **82**, 1015 (1996)].

[9] V. L. Ginzburg, *Problems of Theoretical Physics and Astrophysics*, Nauka, Moscow, 1981 [in Russian].

[10] F. Rohrlich, in *The Physicist's Conception of Nature*, J. Mehra ed., D. Reidel publ., Dordrecht, Holland and Boston, USA, 1973.

[11] D. Ivanenko and A. Sokolov, *The Classical Theory of Fields*, Gostekhizdat, Moscow, 1951 [in Russian].

[12] T. C. Mo and C. H. Papas, Phys. Rev. D **4**, 3566 (1971).

[13] W. B. Bonnor, Proc. Roy. Soc. A **337**, 591 (1974).

[14] A. O. Barut, Phys. Lett. A **145**, 387 (1990).

[15] G. N. Plass, Rev. Mod. Phys. **33**, 37 (1961).

[16] R. Haag, Z. Naturforsch. **10a**, 752 (1955).

[17] N. B. Narozhnyĭ, A. I. Nikishov, and V. I. Ritus, Zh. Éksp. Teor. Fiz. **47**, 931 (1964) [Sov. Phys. JETP **20**, 622 (1964)].

[18] V. I. Ritus, P. N. Lebedev, Inst. of Phys., USSR Acad. Sci. **111**, 5 (1979) [in Russian].

[19] V. S. Krivitskiĭ and V. N. Tsytovich, Usp. Fiz. Nauk **161**, 125 (1991) [Sov. Phys. Usp. **34**, 167 (1991)].

[20] V. I. Ritus, P. N. Lebedev, Inst. of Phys., USSR Acad. Sci. **168**, 5 (1986) [in Russian].

Translated by Frank J. Crowne