Dynamic instability of a membrane tube

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We consider the early stage of the Rayleigh instability of a membrane tube, taking the surface tension to be comparable with the Helfrich energy. We find the spectra of the natural modes and discuss some special cases. © *1996 American Institute of Physics.* [S1063-7761(96)01708-8]

INTRODUCTION

In recent years considerable attention has been paid to the investigation of dilute lyotropic phases.[1,2] The phases are solutions of different structures constructed from membranes, which are bilayer lipid films. Note that membranes are widely encountered in biological systems. One type of structure is cylindrical membrane tubes. In recent experiments[3] the relaxation of the tubes following optical excitation has been investigated. The relaxation has different stages and is characterized by long times, and can consequently be described by means of a hydrodynamic approach.

It is well known that a liquid tube is unstable,[4,5] which is related to the decrease in surface area in deviations from a cylindrical shape. The same is true for a tube made from a film, the instability being associated with the surface tension of the film. However, a free membrane has zero surface tension, which is the equilibrium condition corresponding to variations of the membrane area. Bending fluctuations of such membranes are governed by the Helfrich energy,[6] which depends on the curvature of the membrane. A tube made from a membrane with zero surface tension is stable. Furthermore, in experiments membrane tubes can have finite surface tension, since as a rule they are parts of a complex structure and consequently are not free. Then the surface tension and the curvature energy compete, and at a critical value of the surface tension the tube becomes unstable. A subcritical value of the surface tension can be induced by an external influence. In experiment it is done by «laser tweezers».[3]

In this paper we theoretically examine the initial stage of the instability, assuming that both the surface tension and the curvature energy are relevant. Our aim is to find the spectra of natural modes of the tube for arbitrary ratio of the tube radius to the wavelength.

1. INSTABILITY: STATIC ANALYSIS

Let us consider the energy associated with a membrane immersed in water. We will take into account the energy associated both with the curvature of the membrane and with its elasticity. The curvature energy is the following surface integral

$$F_{\text{curv}}=\int dS\left(\frac{\kappa}{2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)^2+\frac{\bar{\kappa}}{R_1R_2}\right). \tag{1.1}$$

Here R_1 and R_2 are local curvature radii of the membrane and κ, $\bar{\kappa}$ are the bending moduli introduced by Helfrich.[6] The term proportional to $\bar{\kappa}$ in (1.1) is the topological invariant. We will be interested in variations of the membrane shape which did not change its topology, so the term is constant and consequently does not influence fluctuations of the membrane.

Besides (1.1) we should take into account also the elastic energy associated with variations of the density n_s of the membrane molecules per unit area. Fluctuations of n_s are faster than the bending fluctuations of the membrane.[7] Thus, in treating the bending fluctuations we can take n_s to be a constant: $n_s=N/A$, where N is the number of molecules of the membrane and A is its area. Thus, n_s varies with A; for small relative variations of A we can take $\delta n_s \propto \delta A$. In the same approximation the variation of the elastic energy can be written $\delta F_{el}=\alpha\delta A$, where α is the surface tension of the membrane. For a free membrane we have $\alpha=0$, since in equilibrium $\delta F/\delta A$ should be equal to zero. However, we have in mind the situation where the membrane is stretched between bulky aggregates of membranes and consequently is not free. Then the surface tension α can be nonzero.

Note that the value of the surface tension is strongly renormalized by fluctuations. Consequences of this fact are treated elsewhere.1) Here we should stress only that everything said above is attributed to the renormalized surface tension, the value of which strongly differs from its bare value. Only the former controls the instability, so in what follows we are referring to the renormalized value.

We are going to investigate a state of the tube which differs only slightly from the cylindrical one. For this purpose it is suitable to rewrite the expression for the energy in the following form

$$F=\int dz\, d\varphi\, r\sqrt{1+(\nabla u)^2}\times\left(\alpha+\frac{\kappa}{2}(\nabla_i l_i)^2+\frac{\bar{\kappa}}{2}((\nabla_i l_i)^2-\nabla_i l_j \nabla_j l_i)\right), \tag{1.2}$$

incorporating both the curvature energy and the elastic energy. The latter is taken in the approximation linear in δA, since we will assume the value of δA to be much smaller than A. In (1.2) r, φ and z are cylindrical coordinates with the z-axis along the axis of the cylinder, $u(\varphi,z)$ is the displacement of the membrane in the radial direction from the radius R corresponding to the uniform cylinder: $r=R+u$; and $\mathbf{l}$ is the unit vector normal to the membrane with components

$$\begin{aligned} l_r&=(1+(\nabla u)^2)^{-1/2},\\ l_\varphi&=-(\partial u/\partial\varphi)r^{-1}(1+(\nabla u)^2)^{-1/2},\\ l_z&=-(\partial u/\partial z)(1+(\nabla u)^2)^{-1/2}. \end{aligned} \tag{1.3}$$

The displacement u can be written as the following Fourier series

$$u(\varphi,z)=\sum_{q,m} u_{q,m}\exp(im\varphi+iqz), \tag{1.4}$$

where the integer m is the number of the angular harmonic and the longitudinal wave vector q equals $2\pi n/L_z$ where n is an integer and L_z is the length of the tube. We will that due to the incompressibility of the water the fluctuations of the tube shape do not change its volume, which leads to the following constraint

$$u_{0,0}=-\frac{1}{2R}\sum_{q,m}|u_{q,m}|^2. \tag{1.5}$$

This condition excludes $u_{0,0}$ from the set of independent variables.

Substituting the expressions (1.4), (1.5) into (1.2) we can obtain the free energy F in the form of a series of powers of $u_{q,m}$. The zeroth term determines the energy of the unperturbed tube and the higher order terms determine the energy associated with perturbations of the cylindrical shape. The first nonvanishing term is the second order term

$$F^{(2)}=\frac{\pi L_z}{R}\sum_{q,m}\alpha(q,m)|u_{q,m}|^2, \tag{1.6}$$

$$\alpha(q,m)=\alpha(\mu-1)+\frac{\kappa}{2}(2\mu^2-\mu+3-4m^2)R^{-2}. \tag{1.7}$$

Here we have introduced the dimensionless parameter

$$\mu=m^2+(qR)^2. \tag{1.8}$$

From (1.7) we conclude that for small α the cylinder is stable, while for large enough α it is absolutely unstable (which corresponds to the Rayleigh instability). The cylindrical shape becomes absolutely unstable for

$$\alpha>\alpha_{\rm cr}=3\kappa/2R^2. \tag{1.9}$$

The instability is associated with small μ, which implies $m=0$ and small wave vectors q.

Below we will examine the initial stage of the instability, which can be treated in terms of the second order term (1.6). To treat the fully developed stage of the instability one should take into account both higher-order terms of the Landau expansion and fluctuation effects. This is the subject of a separate work.

2. DYNAMICAL EQUATIONS

To investigate the dynamics of a membrane tube one should use the hydrodynamic equations together with the boundary conditions on a membrane. The conditions are determined by the surface tension and by the curvature of the membrane. Their derivation can be found in Refs. 7 and 8. Here we merely reproduce the final results.

The membrane tube can be described in terms of the displacement u in the radial direction (introduced in the preceding section) and the surface density n_s of molecules. The dynamical equations for the quantities have the following form

$$\frac{\partial}{\partial t}u=v_r-\mathbf{v}_s\cdot\nabla u, \tag{2.1}$$

$$\frac{\partial}{\partial t}n_s+\mathbf{v}_s\cdot\nabla n_s+n_s\nabla^\perp\cdot\mathbf{v}_s=0. \tag{2.2}$$

Here $\mathbf{v}_s$ is the velocity of the membrane, v_r is the radial component of the velocity and we define $\nabla_i^\perp=\nabla_i-l_il_k\nabla_k$, where $\mathbf{l}$ is the unit vector (1.3) normal to the membrane. Equation (2.1) means simply that the membrane moves with the velocity $\mathbf{v}_s$, and Eq. (2.2) is the conservation law of the number of membrane molecules. Their conservation means that we neglect the solubility of the membrane, which seems to agree with experimental situation. We stress that (2.1), (2.2) are exact nonlinear equations. The quantities u and n_s are defined on the surface. We will assume that they are functions of φ, z. Note that any two coordinates can be taken as arguments of u and n_s since Eqs. (2.1), (2.2) actually contain derivatives only along the membrane.[8]

Next, we formulate boundary conditions for the hydrodynamic equations. The temperature and the velocity should be continuous on the membrane, that is the temperature T and the velocity $\mathbf{v}$ of the membrane should coincide with the temperature and the velocity of the surrounding water. The continuity of the velocity means that the membrane is advected by the water without being permeated. The pressure P and the velocity gradient change discontinuously on the membrane. The jumps are determined by the equation

$$g^{-1}\frac{\partial}{\partial t}(g\rho_s v_{si})+g^{-1}\nabla_k(g(\rho_s v_{si}v_{sk}+T_{ik}^{(s)}))=(P_1-P_2)l_i+(\Pi_{1ik}-\Pi_{2ik})l_k, \tag{2.3}$$

where $g=\sqrt{1+(\nabla u)^2}$ and the subscripts 1 and 2 label the regions inside and outside the tube. In (2.3) $\mathbf{l}$ is the unit vector (1.3) normal to the membrane, ρ_s is the surface mass density of the membrane, ρ is the density of the water, $T_{ik}^{(s)}$ is the surface stress tensor, and Π_{ik} is the viscous stress tensor entering the bulk equation

$$\frac{\partial(\rho v_i)}{\partial t}=-\nabla_k(P\delta_{ik}+\rho v_k v_i+\Pi_{ik}). \tag{2.4}$$

We will consider slow processes. This means that the condition $T=\mathrm{const}$ is satisfied together with the incompressibility condition $\nabla_i v_i=0$. In this approximation the viscous stress tensor entering (2.3), (2.4) is

$$\Pi_{ik}=-\eta(\nabla_i v_k+\nabla_k v_i), \tag{2.5}$$

where η is the viscous coefficient of the water. The surface stress tensor is[7,8]

$$T_{ik}^{(s)}=-\left(\alpha+\frac{\kappa}{2}(\nabla\mathbf{l})^2\right)\delta_{ik}^{\perp}+\kappa\nabla_n l_n\delta_{ij}^{\perp}\nabla_j l_k$$
$$-\kappa l_i\delta_{kj}^{\perp}\nabla_j\nabla_n l_n, \tag{2.6}$$

where $\delta_{ij}^{\perp}=\delta_{ij}-l_i l_j$.

We are going to treat normal modes of the membrane tube. To find their dispersion laws we must treat the linearized hydrodynamic equations and boundary conditions. The linearized equation (2.4) has the form

$$\rho\frac{\partial\mathbf{v}}{\partial t}=-\nabla P+\eta\nabla^2\mathbf{v}. \tag{2.7}$$

We will solve Eq. (2.7) in polar coordinates. Then we must add to the boundary conditions imposed on the motion of the liquid at the membrane also two conditions at $r\to 0$ and $r\to\infty$. Namely, values of all dependent variables must remain finite at $r\to 0$ and tend to zero at $r\to\infty$. From (2.1) it follows that the linearized boundary condition is

$$\frac{\partial u}{\partial t}=v_r, \tag{2.8}$$

which should be imposed on the velocity at $r=R$.

We assume the surface tension α to be of the order of the critical value $\alpha_{\mathrm{cr}}=3\kappa/2R^2$. Then the dynamics will be fairly soft, and we can regard the membrane as incompressible. Thus, the boundary condition (2.2) is reduced to $\nabla_i^{\perp}v_{si}=0$. Because of the incompressibility of water, $\nabla_i v_i=0$, this condition reduces to $l_i l_k\nabla_i v_k=0$. In the linear approximation in polar coordinates it may be rewritten as

$$\frac{\partial v_r}{\partial r}=0. \tag{2.9}$$

The linearized boundary conditions following from (2.3) are

$$-\frac{B}{n_s}\frac{\partial n_s'}{\partial z}=\eta_1\left(\frac{\partial v_{r1}}{\partial z}+\frac{\partial v_{z1}}{\partial r}-\frac{v_{z1}}{R}\right)$$
$$-\eta_2\left(\frac{\partial v_{r2}}{\partial z}+\frac{\partial v_{z2}}{\partial r}-\frac{v_{z2}}{R}\right), \tag{2.10}$$

$$-\frac{B}{n_s}\frac{\partial n_s'}{\partial\varphi}=\eta_1\left(\frac{\partial v_{r1}}{\partial\varphi}+\frac{\partial v_{\varphi 1}}{\partial r}-\frac{v_{\varphi 1}}{R}\right)$$
$$-\eta_2\left(\frac{\partial v_{r2}}{\partial\varphi}+\frac{\partial v_{\varphi 2}}{\partial r}-\frac{v_{\varphi 2}}{R}\right), \tag{2.11}$$

$$-\alpha(\nabla_{\perp}^2u+u/R^2)+\kappa(\nabla_{\perp}^4u+\nabla_{\perp}^2u/R^2)=P_1'-P_2'. \tag{2.12}$$

Here the prime denotes the variable part of the pressure P or of the surfactant concentration n_s, and

$$\nabla_{\perp}^2=\partial^2/\partial z^2+R^{-2}\partial^2/\partial\varphi^2. \tag{2.13}$$

The parameter B in (2.10), (2.11) is $B=-n_s\partial\alpha/\partial n_s$. We expect it to be of the order of the conventional magnitude of the surface tension so that $B\gg\alpha$, since the surface tension $\alpha\sim\alpha_{\mathrm{cr}}$ is anomalously small. In equations (2.10)–(2.12) we omitted the terms with $\partial v_s/\partial t$, which make negligible contributions to the dispersion laws due to the softness of the process. In this expressions (2.10)–(2.12) we keep only terms with derivatives of α. The point is that we have $n_s\partial\kappa/\partial n_s\sim\kappa$, whereas $B\gg\alpha$. This together with the estimate $\alpha\sim\kappa/R^2$ justifies neglecting derivatives of κ.

The velocity $\mathbf{v}$ in the general case can be written as

$$v_r=\frac{\partial(\psi+\phi)}{\partial r}-\frac{1}{r}\frac{\partial\chi}{\partial\varphi}, \tag{2.14}$$

$$v_\varphi=\frac{1}{r}\frac{\partial(\psi+\phi)}{\partial\varphi}+\frac{\partial\chi}{\partial r}, \tag{2.15}$$

$$v_z=\frac{\partial\psi}{\partial z}-\left(\frac{\partial}{\partial z}\right)^{-1}\left(\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\varphi^2}\right). \tag{2.16}$$

The first terms containing ψ determine the irrotational part of the velocity. From (2.7) it follows that the variable part of the pressure P' equals

$$P'=-\rho\frac{\partial\psi}{\partial t}. \tag{2.17}$$

The incompressibility condition means that the function ψ satisfies the Laplace equation

$$\nabla^2\psi=0. \tag{2.18}$$

The terms in (2.14)–(2.16) containing ϕ and χ determine the rotational part of the velocity. The dynamical equations for ϕ and χ can be derived from (2.7). They are

$$\rho\frac{\partial\phi}{\partial t}=\eta\nabla^2\phi,\quad\rho\frac{\partial\chi}{\partial t}=\eta\nabla^2\chi. \tag{2.19}$$

Now we can begin to solve these linearized equations.

3. DISPERSION RELATIONS

Since we seek solutions of linear equations, whose coefficients do not explicitly depend on time, we can consider a solution $\propto\exp(-i\omega t)$, where the frequency ω is generally a complex number. We will also expand all quantities in Fourier series like (1.4). Passing to the Fourier representation we can make the replacement $-\nabla_{\perp}^2\to q^2+m^2/R^2=\mu/R^2$.

A solution of the equation (2.18) with given values q and m behaves as

$$\psi\propto\begin{cases}I_m(qr)\exp(im\varphi+iqz), & r<R,\\ K_m(qr)\exp(im\varphi+iqz), & r>R,\end{cases} \tag{3.1}$$

where $I_m(qr)$ and $K_m(qr)$ are Bessel functions of imaginary argument. Solutions of Eqs. (2.19) behave as

$$\phi,\chi \propto \begin{cases} I_m(kr)\exp(im\varphi+iqz), & r<R, \\ K_m(kr)\exp(im\varphi+iqz), & r>R, \end{cases} \tag{3.2}$$

where

$$k^2=q^2-i\omega\rho/\eta. \tag{3.3}$$

The coefficients of proportionality in (3.1), (3.2) are different in different regions. The coefficients are related by the boundary conditions (2.8)–(2.12). The solubility condition of the system of equations gives us the dispersion law determining the frequency ω as a function of q and m.

Below we derive the general dispersion relation and also derive the dispersion laws in some particular cases corresponding to soft long-scale modes. In these cases the general dispersion law can be simplified considerably.

Note also that in this section we will suppose that the liquids inside and outside the tube can be different, that is, they have different densities and viscosity coefficients.

3.1. General dispersion relation

In this subsection we derive the general dispersion relation describing modes associated with variations of the membrane shape. For this we must find the matching conditions for the velocity determined by (2.14)–(2.16), (3.1), (3.2) at the membrane, that is, at $r=R$. The first of these are the continuity conditions for the velocity,

$$v_{r1}=v_{r2}, \quad v_{z1}=v_{z2}, \quad v_{\varphi 1}=v_{\varphi 2}. \tag{3.4}$$

Then we must also use (2.10)–(2.12). Eliminating the variable n_s from (2.10), (2.11) we obtain

$$\eta_1\left(m\frac{\partial v_{z1}}{\partial r}-q\left(R\frac{\partial v_{\varphi 1}}{\partial r}-v_{\varphi 1}\right)\right)$$

$$=\eta_2\left(m\frac{\partial v_{z2}}{\partial r}-q\left(R\frac{\partial v_{\varphi 2}}{\partial r}-v_{\varphi 2}\right)\right). \tag{3.5}$$

Substituting (2.17) into (2.12) we find

$$\frac{\alpha(q,m)}{R^2}u=i\omega(\rho_1\psi_1-\rho_2\psi_2), \tag{3.6}$$

where $\alpha(q,m)$ is introduced by (1.7). The displacement u here is related to $\mathbf{v}$ through (2.8), which is rewritten as $-i\omega u=v_r$.

The relations (2.9), (3.4), (3.5), (3.6) constitute the full system of boundary conditions for the six variables $\mathbf{v}$, $\partial\mathbf{v}/\partial r$ (the derivatives $\partial\mathbf{v}/\partial z$ and $\partial\mathbf{v}/\partial\varphi$ are continuous at $r=R$ due to the continuity of $\mathbf{v}$). To avoid misunderstanding we note that the relation (2.9), which should be satisfied at both the inner and outer sides of the vesicle actually gives only one condition. The point is that $\partial v_{r1}/\partial r=0$ implies $\partial v_{r2}/\partial r=0$ because of the continuity of $\partial\mathbf{v}/\partial z$, $\partial\mathbf{v}/\partial\varphi$, and the incompressibility conditions $\nabla_i v_{1i}=0$, $\nabla_i v_{2i}=0$.

The above boundary conditions can be rewritten in terms of the functions ψ, φ and χ. Using now (3.1), (3.2) we find the general dispersion law. For simplicity we write it for the case when liquids inside and outside the tube are identical:

$$\frac{\alpha(q,m)}{\omega^2\rho R^3}\left(c_1c_2-\frac{q^2}{k^2}c_2a_2b_1+\frac{q^2}{k^2}c_1a_1b_2\right)$$

$$=I_m(qR)c_2-K_m(qR)c_1, \tag{3.7}$$

where $\alpha(q,m)$ is determined by (1.7), k^2 is defined by (3.3) and

$$c_1=qR\dot{I}_m(qR)+\frac{(qR)^2\ddot{I}_m(qR)I_m(\widetilde{kR})}{I_m(kR)-kR\dot{I}_m(kR)}-\frac{q^2}{k^2}a_1d_1$$

$$+\frac{q^2}{k^2}a_2b_1,$$

$$c_2=qR\dot{K}_m(qR)+\frac{(qR)^2\ddot{K}_m(qR)K_m(kR)}{K_m(kR)-kR\dot{K}_m(kR)}+\frac{q^2}{k^2}a_2d_2$$

$$-\frac{q^2}{k^2}a_1b_2,$$

$$a_1=kR\dot{I}_m(kR)+\frac{(kR)^2\ddot{I}_m(kR)I_m(kR)}{I_m(kR)-kR\dot{I}_m(kR)},$$

$$a_2=kR\dot{K}_m(kR)+\frac{(kR)^2\ddot{K}_m(kR)K_m(kR)}{K_m(kR)-kR\dot{K}_m(kR)},$$

$$b_1=CqR\dot{I}_m(qR)I_m(kR)-kRI_m(qR)\dot{I}_m(kR),$$

$$b_2=CqR\dot{K}_m(qR)K_m(kR)-kRK_m(qR)\dot{K}_m(kR),$$

$$d_1=CqR\dot{I}_m(qR)K_m(kR)-kRI_m(qR)\dot{K}_m(kR),$$

$$d_2=CqR\dot{I}_m(qR)K_m(kR)-kRK_m(qR)\dot{I}_m(kR).$$

We have set $C=(m^2+(kR)^2)/(m^2+(qR)^2)$ in the above expressions. A dot over a function denotes a derivative with respect to its argument.

3.2. Squeezing mode

Let us first consider the squeezing mode, which is associated with the Rayleigh instability. This mode corresponds to the motion of the membrane with $m=0$. From (3.7) one derives the following dispersion law:

$$\omega=-i\frac{\alpha(q,m)}{2R}X^{-1}, \tag{3.8}$$

where

$$X=\eta_1\frac{I_0(qRI_0-I_1)}{qR(I_1^2-I_0^2)+2I_1I_0}+\eta_2\frac{K_0(qRK_0+K_1)}{qR(K_0^2-K_1^2)+2K_0K_1}. \tag{3.9}$$

Here we have written $I_0=I_0(qR)$ etc. In the long-wavelength limit $qR\ll 1$ this mode has the asymptotic dispersion law

$$\omega=-i\frac{\alpha(q,m)R}{8\eta_1}q^2. \tag{3.10}$$

We see that in accordance with the static analysis the squeezing mode is unstable for $\alpha>\alpha_{\mathrm{cr}}$. It is interesting to note that in the long-wavelength limit the dispersion law of this mode depends only on the viscosity of the liquid inside the tube.

In the short-wavelength limit $qR\gg 1$ the asymptotic dispersion law has the form

$$\omega=-i\,\frac{\kappa}{2(\eta_1+\eta_2)}\,q^3. \tag{3.11}$$

In this limit the mode spectrum does not depend on the shape of tube. The result (3.11) is well known for a flat membrane.[9,10]

3.3. Squeezing mode near the threshold

The critical value of α is $\alpha_{\mathrm{cr}}=3\kappa/2R^2$. Let us consider the case $\alpha>\alpha_{\mathrm{cr}}$ but with $\alpha-\alpha_{\mathrm{cr}}\ll\alpha_{\mathrm{cr}}$. Then the instability growth rate of the squeezing mode (3.10) can be rewritten as

$$\gamma=-i\omega=-\frac{R\kappa}{8\eta_1}\,q^2(q^2-2q_c^2), \tag{3.12}$$

where

$$q_c=\sqrt{\frac{\alpha-\alpha_{\mathrm{cr}}}{2\kappa}}. \tag{3.13}$$

Suppose that an external force is applied to the membrane. Then the equation for u at $m=0$ (which corresponds to the squeezing mode) can be written as

$$\frac{\partial u}{\partial t}=\hat{\gamma}u(z)+w(z), \tag{3.14}$$

where w is the radial velocity induced by the force and $\hat{\gamma}$ is the linear operator corresponding to (3.12). A solution of (3.14) can be written as

$$u(t,z)=\int dt_1dz_1G(t-t_1,z-z_1)w(t_1,z_1). \tag{3.15}$$

Here $G(t,z)$ is the Green's function. From (3.12) it follows that

$$G(t,z)=\int\frac{dq}{2\pi}\exp\left(\frac{R\kappa}{8\eta_1}\,q^2(2q_c^2-q^2)t\right)\exp(iqz). \tag{3.16}$$

For $t\gg\eta/\kappa Rq_c^4$ the integral in (3.16) is evaluated by the saddle point method and we find

$$G(t,z)=\sqrt{\frac{2\eta}{\pi R\kappa q_c^2t}}\cos(q_cz)\exp\left(\frac{\kappa Rq_c^4t}{8\eta_1}-\frac{\eta_1z^2}{2tR\kappa q_c^2}\right). \tag{3.17}$$

This expression represents the behavior of u caused by any influence localized near the origin and localized in time near $t=0$. We see that (3.17) is an oscillating function with amplitude depending on time. Note that any point with a fixed amplitude moves with the velocity

$$v_f=\frac{\kappa Rq_c^3}{2\eta_1}, \tag{3.18}$$

which does not depend on the amplitude. This result agrees at least qualitatively with the experiment.[3]

3.4. Bending mode

The next mode which is soft in the long-wavelength limit is the bending mode corresponding to $m=1$. Its dispersion law may be obtained from (3.7). Unfortunately, in the general case it is very involved. Therefore we present here only asymptotic expressions. In the long-wavelength limit $qR\ll1$ it is

$$\omega=-i\,\frac{3Rq^2\ln(qR)}{2\eta_2}\left(\alpha+\frac{3\kappa}{2R^2}+\kappa q^2\right). \tag{3.19}$$

Note that this expression is determined only by the viscosity of the liquid outside the tube. The result (3.19) agrees with the corresponding formula of Ref. 11. In the short-wavelength limit $qR\gg1$ the dispersion law of the bending mode has the same form (3.11) as for the squeezing mode. This coincidence is natural because in this limit both the bending mode and the squeezing mode can be treated as perturbations of a flat membrane.

3.5. Concentration mode

Let us consider the mode associated with the density variations of the membrane tube. We have already mentioned that in the case under consideration the inequality $B\gg\alpha$ holds. This condition enables us to treat the concentration mode and modes related to deformations of the tube shape independently. Investigating this mode we may neglect the deformation of the membrane, that is, consider the tube as a cylinder. We also need to take into account only the term in the surface stress tensor (2.6) depending on n_s, which is proportional to $B=-n_s\partial\alpha/\partial n_s$.

The most interesting is the case with $m=0$. Using the linearized boundary conditions (2.10), (2.11) it is easy to obtain the following dispersion law for the concentration mode:

$$\omega=-i\,\frac{B}{2}\,q^2X^{-1}, \tag{3.20}$$

where

$$X=\eta_1\frac{(k^2-q^2)I_1(qR)I_1(kR)}{kI_0(kR)I_1(qR)-qI_0(qR)I_1(kR)}+\eta_2\frac{(k^2-q^2)K_1(qR)K_1(kR)}{kK_0(kR)K_1(qR)-qK_0(qR)K_1(kR)}. \tag{3.21}$$

In the long-wavelength limit $qR\ll1$ one obtains

$$\omega=-i\,\frac{BRq^2}{4\eta_1}. \tag{3.22}$$

The thermodynamic stability condition leads to $\partial\alpha/\partial n_s<0$, so this mode is a purely relaxational one. It is interesting that the asymptotic dispersion law is determined only by the viscosity of the liquid inside the tube as in the case of the squeezing mode.

In the short-wavelength limit $qR\gg1$ the dispersion law of the concentration mode is

$$\omega=\frac{\pm\sqrt{3}-i}{2}\,\frac{B^{2/3}q^{4/3}}{(\sqrt{\eta_1\rho_1}+\sqrt{\eta_2\rho_2})^{2/3}}. \tag{3.23}$$

This dispersion law was first obtained in Ref. 12 for Langmuir films.

4. CONCLUSION

We have found equations describing the normal modes of the tube taking into account both the curvature energy and the surface tension of the membrane and examines the asymptotic behavior of the dispersion laws. There are three modes: squeezing, bending and concentration, which are soft in the long-wavelength limit. The softness of the squeezing and of the concentration modes is related to the conservation laws of the water inside the tube and of the number of its molecules; the softness of the bending mode is related to the homogeneity of the liquid. The squeezing mode is associated with the Rayleigh instability occurring at a subcritical value of the surface tension. Our analysis enables us to examine the early (linear) stage of the instability: by expanding any initial perturbation into Fourier series we can find its dynamics using these dispersion laws. It is interesting that for small subcritical values of the surface tension in growing perturbations initially localized near a point, points with a given amplitude move with a constant velocity which depends critically on the surface tension. The investigation of the nonlinear stage of the instability is a subject of a separate work. Note only that to describe the nonlinear stage one should seriously take into account fluctuations of the membrane shape.

5. ERRATUM

In a previous paper[7] we made a mistake. Equation (39) of the paper should have the form

$$\frac{\partial n_s'}{\partial t}+\frac{n_s}{R\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta v_\theta^s)+\frac{\partial v_\varphi^s}{\partial\varphi}+2v_r\sin\theta\right)=0. \tag{5.1}$$

The dispersion relation for the bending modes should be also corrected. Instead of Eq. (49) we have

$$\omega=\left(\frac{\alpha_l}{R^3}\frac{(l-1)l(l+1)(l+2)}{(l+1)\rho_1+l\rho_2}\right)^{1/2}\left(1-\frac{i}{2R\sqrt{2\omega}}X\right), \tag{5.2}$$

where

$$X=\frac{(l^2-1)\sqrt{\eta_1\rho_1}+(l^2+2l)\sqrt{\eta_2\rho_2}}{(l+1)\rho_1+l\rho_2}. \tag{5.3}$$

Equation (53) should have the form

$$\omega=-i\,\frac{\alpha_l(l-1)l(l+1)(l+2)}{R[\eta_1(2l+3)(l^2-1)+\eta_2(2l-1)(l^2+2l)]}. \tag{5.4}$$

These equations do not coincide with the corresponding results of Ref. 13 because the author has used our erroneous equation instead of (5.1). They also differ from the equations of Ref. 14 because the authors of Ref. 14 have considered the case of nonzero excess area, i.e., they studied fluctuations near a nonspherical surface.

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