One-loop radiative corrections in a gauge with improved infrared and ultraviolet properties

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A new gauge for the photon propagator is suggested in which the once subtracted mass operator and the vertex function are free from ultraviolet divergences and the low-frequency asymptotic behavior of these diagrams contain additional suppression in comparison to the Feynman gauge. © *1996 American Institute of Physics.* [S1063-7761(96)00110-2]

1. INTRODUCTION

Two essentially different types of divergences arise in calculations of the matrix elements of quantum-electrodynamic processes. Primarily these are ultraviolet divergences originating in the high-momentum range of virtual particles. For the divergent expressions to acquire a formal meaning one is forced to introduce an ultraviolet cutoff parameter, say, a cutoff momentum Λ. Integration over the virtual quanta with momenta close to zero also leads to divergences if the free electron lines in the Feynman diagram under consideration lie on the mass surface. The common way to remove these infrared divergences is to introduce a small photon mass λ. Naturally, the final result of calculations depends neither on Λ nor on λ, but the same cannot be said about the contributions of individual Feynman diagrams.

In view of conservation of electromagnetic current no physical results are changed by the following substitution of the photon propagator:

$$D_{\alpha\beta}(q)\rightarrow D_{\alpha\beta}(q)+q_\alpha\chi_\beta+\chi_\alpha q_\beta, \tag{1}$$

where the χ_α are any functions of q_0 and $\mathbf{q}$. This statement is true for the complete gauge-invariant set of Feynman diagrams, whereas the contributions of individual diagrams are not invariant under substitution (1). A well-chosen gauge of the photon propagator may simplify the calculation of matrix elements considerably. For instance, at high virtual-photon momenta $|q_i|$ the diagrams with several successive logarithmic integrations contain in the Landau gauge[1] one power of the logarithm less than they do in the Feynman gauge. In the Landau gauge the mass operator and the first-order vertex function contain no range of integration that is logarithmic as $|q|\rightarrow\infty$, with the result that the one-loop renormalization constants Z_1 and Z_2 are ultraviolet-finite. On the other hand, with infrared divergences it is convenient to use the Fried–Yennie gauge[2,3] or the Coulomb gauge. In these gauges the radiative corrections have a softer low-energy asymptotic behavior, so that we can ignore the finite photon mass λ. If canceling out divergent contributions is not too difficult, the most convenient gauge from the standpoint of calculations is the Feynman gauge. However, in some problems (say, in calculating the radiative corrections to bound states) removing the ultraviolet and infrared divergences is not so simple. What would be convenient in this case is a gauge having the merits of both the Landau gauge and the Fried–Yennie or Coulomb gauge. It is also desirable that such a gauge contain no additional dimensional parameter hindering specific calculations.

A gauge with such properties is presented in this paper. Section 2 is devoted to the derivation of the expression for the photon propagator. Sections 3 and 4 discuss the behavior of radiative corrections. Some additional properties of the gauge and its relation to other gauges are discussed in the Appendix.

The relativistic system of units, in which $\hbar=1$ and $c=1$, is used throughout the paper, and all notation is standard.[4]

2. CHOICE OF PROPAGATOR

The simplest way to find a gauge that satisfies the above requirements is to examine the following vertex function in different regimes:

$$\Lambda_\mu(p,p')=\frac{\alpha}{4\pi}\int\frac{d^4q}{\pi^2 i}\frac{\gamma^\alpha(\hat p+\hat q+m)\gamma_\mu(\hat p'+\hat q+m)\gamma^\beta}{D(p+q)D(p'+q)}\times D_{\alpha\beta}(q). \tag{2}$$

Here and in what follows we use the following notation for the electron denominators:

$$D(p)=p^2-m^2+i\varepsilon. \tag{3}$$

In the case of the photon propagator of the general form

$$D_{\alpha\beta}(q)\equiv\frac{G_{\alpha\beta}(q)}{q^2+i\varepsilon}, \tag{4}$$

both infrared and ultraviolet divergences are inherent in (2).

We start by examining the terms in the integrand that are potentially dangerous from the standpoint of divergences in the infrared range. In this respect it is sufficient to know the vertex function at zero transfer momentum and near the mass surface: $p'=p\simeq(m,0)$. Then for small q we have

$$\Lambda_\mu^{IR}(p,q)\simeq\frac{\alpha}{4\pi}\gamma_\mu\int\frac{d^4q}{\pi^2 i}\frac{4p^\alpha p^\beta}{(2pq+i\varepsilon)^2}\frac{G_{\alpha\beta}(q)}{q^2+i\varepsilon}\simeq\frac{\alpha}{4\pi}\gamma_\mu\int\frac{d^4q}{\pi^2 i}\frac{1}{(q_0+i\varepsilon)^2}\frac{G_{00}(q)}{q^2+i\varepsilon}. \tag{5}$$

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In the Feynman gauge the tensor $G_{\alpha\beta}=g_{\alpha\beta}$ is independent of the momentum q and the analytic properties of the vertex function are determined entirely by the two denominators in (5). When integrating in the complex q_0 plane we find the integration contour squeezed between two poles, $q_0=-i\varepsilon$ and $q_0=-|\mathbf{q}|+i\varepsilon$, which leads to a singularity. There are two ways of resolving this difficulty. The first consists in removing the pole $q^2=0$ by choosing the photon propagator in the Coulomb gauge $D_{00}=-1/\mathbf{q}^2$. Here $G_{00}=q^2/\mathbf{q}^2$ and there is no more $q^2=0$ pole in (5). The other approach consists in selecting $G_{00}(q)$ in such a way that the residue of $G_{00}/q_0^2q^2$ at the pole $q^2=0$ vanishes. The simplest way to do this is to put

$$G_{00}(q)=1+\frac{2q_0^2}{q^2}. \tag{6}$$

Knowing G_{00}, we can recover the spatial components of the tensor $G_{\alpha\beta}$ in a purely covariant way. As a result we arrive at the well-known Fried–Yennie gauge:

$$D_{\alpha\beta}(q)=\frac{1}{q^2+i\varepsilon}\left(g_{\alpha\beta}+\frac{2q_\alpha q_\beta}{q^2}\right). \tag{7}$$

The first to point out the special properties of the Fried–Yennie gauge (7) was Abrikosov.[2] He proved that the electron Green's function has a simple pole at $p^2=m^2$ only in the gauge (7), while in other gauges at a zero photon mass λ the value $p^2=m^2$ corresponds to a branch point. Gor'kov[5] showed that the appearance of an additional singularity in the Green's function of a charged particle when the particle interacts with an electromagnetic field is related solely to the classical properties of the electric current generated by the particle in uniform motion. In this way the result holds for zero-spin particles, too. The gauge (7) proved to be extremely useful in the theory of bound states. Fried and Yennie[3] demonstrated that if the photon propagator is chosen in the form (7), the principal contribution to the Lamb shift is provided solely by the two diagrams corresponding to the first two terms in the expansion of the Coulomb Green's function in powers of the external field strength. Later the infrared properties of the Fried–Yennie gauge were repeatedly used in calculations of one-loop radiative corrections.[6–10] As for higher-order corrections, the Fried–Yennie gauge was first used to calculate the two-loop contributions (in the radiative photon) to the Lamb shift and hyperfine splitting of the ground state of hydrogen.[11–14] The problem of removing the infrared divergences in the Fried–Yennie gauge can be approached from another angle if one allows for the fact that the photon propagator is transverse in the coordinate representation (see the Appendix). Representing the electron propagator in the first line of Eq. (5) as

$$\frac{4p^\alpha p^\beta}{(2pq)^2}=-\frac{\partial}{\partial q_\alpha}\left(\frac{2p^\beta}{2pq}\right)\simeq-\frac{\partial}{\partial q_\alpha}\left(\frac{2p^\beta}{q^2+2pq}\right), \tag{8}$$

we can write the infrared-dangerous part of the vertex function in terms of a total derivative as follows:

$$\Lambda_\mu^{IR}(p,q)=\frac{\alpha}{4\pi}\gamma_\mu\int\frac{d^4q}{\pi^2 i}\frac{\partial}{\partial q_\alpha}\left[-\frac{2p^\beta}{q^2+2pq}D_{\alpha\beta}^{FY}(q)\right], \tag{9}$$

which is possible because of Eq. (A12). As a result Eq. (9) vanishes under integration over the surface of a hypersphere of infinite radius.

The Fried–Yennie gauge (7) is not the only possible way of extending Eq. (6) to the spatial components. If explicit covariance is not required, the following choice of the propagator is possible:

$$D_{\alpha\beta}(q)=\frac{1}{q^2+i\varepsilon}\left[g_{\alpha\beta}+2\xi\frac{q_\alpha q_\beta}{q^2}+(1-\xi)\frac{\eta q}{q^2}(q_\alpha\eta_\beta+\eta_\alpha q_\beta)\right], \tag{10}$$

where $\eta=(1,0)$ is a unit time-like vector.

The remaining undefined parameter ξ can be used to remove the ultraviolet divergences. To this end we again take the vertex function (2), but this time we examine its behavior at high momenta $|q|\gg m$:

$$\begin{aligned}\Lambda_\mu^{UV}&\simeq\frac{\alpha}{4\pi}\int\frac{d^4q}{\pi^2 i}\frac{\gamma^\alpha\hat{q}\gamma_\mu\hat{q}\gamma^\beta}{q^4}D_{\alpha\beta}(q)\\&=\frac{\alpha}{4\pi}\int\frac{d^4q}{\pi^2 iq^6}[q^2\gamma_\mu+2\xi q^2\gamma_\mu\\&\quad+(1-\xi)(\eta q)(\hat{\eta}\hat{q}\gamma_\mu+\gamma_\mu\hat{q}\hat{\eta})]\\&\simeq\frac{\alpha}{4\pi}\gamma_\mu\int_{m^2}^{\Lambda^2}\frac{dq^2}{q^2}\left[1+2\xi+\frac{1-\xi}{2}\eta^2\right].\end{aligned} \tag{11}$$

This expression vanishes if

$$\xi=-1. \tag{12}$$

Note that in finding the asymptotic forms (11) we used only the commutation relations for the Dirac matrices and the covariance of all the denominators but not the properties of the wave functions or the explicit form of vector η. The condition (12) also follows if we examine the self-energy operator, but in this case it appears at a later stage due to the need to renormalize the mass.

Thus, the desired gauge has the form

$$D_{\alpha\beta}(q)=\frac{1}{q^2+i\varepsilon}\left[g_{\alpha\beta}-\frac{2q_\alpha q_\beta}{q^2}+\frac{2(\eta q)}{q^2}(q_\alpha\eta_\beta+\eta_\alpha q_\beta)\right]. \tag{13}$$

Another example of a gauge possessing enhanced infrared and ultraviolet properties is

$$D_{\alpha\beta}(q)=\frac{1}{q^2+i\varepsilon}\left[g_{\alpha\beta}+\frac{2q_\alpha q_\beta}{q^2}-\frac{3q_\alpha q_\beta}{q^2-\mu^2}\right]. \tag{14}$$

In the limiting cases $|q|\ll\mu$ and $|q|\gg\mu$, where μ is a parameter with dimensions of mass, the expression (14) reproduces the Fried–Yennie and Landau gauges, respectively. The chief merit of the proposed gauge (13) is the absence of a dimensional parameter. The loss of explicit Lorentz-invariance is not so important in the case of bound states. More than that, in calculating the one-loop radiative corrections in Secs. 3 and 4 we will see that the vector η can be chosen in a covariant manner.

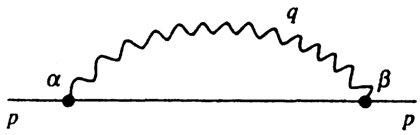

FIG. 1. The one-loop self-energy operator.

3. THE ONE-LOOP SELF-ENERGY OPERATOR

We start our study of radiative corrections in the gauge (10) by calculating the one-loop self-energy of an electron. The initial expression (the notation is clear from Fig. 1) is

$$\Sigma(p)=\frac{\alpha}{4\pi}\int\frac{d^4q}{\pi^2 i}\frac{\gamma^\alpha(\hat{p}+\hat{q}+m)\gamma^\beta}{(q^2+i\varepsilon)D(p+q)}\times\left[g_{\alpha\beta}+2\xi\frac{q_\alpha q_\beta}{q^2}+(1-\xi)\frac{\eta q}{q^2}(q_\alpha\eta_\beta+\eta_\alpha q_\beta)\right]. \tag{15}$$

The integration with respect to momentum q is done in the standard way by merging the denominators via the Feynman parameter x and by performing a Euclidean rotation of the integration contour. It must be noted at this point that in all the gauges except the Fried–Yennie gauge ($\xi=1$) Eq. (15) contains a linearly divergent part, so that the shift $q\to q-px$ of the integration variable results in an additive constant (related to the surface term) appearing in the integral (15). This leads to the following explicit expression for the unrenormalized mass operator:

$$\Sigma(q)=\delta m-\frac{3}{2}(1+\xi)(\hat{p}-m)\ln\frac{\Lambda^2}{m^2}+(1-\xi)[-\hat{p}+(\eta p)\hat{\eta}]+3m(p^2-m^2)\int_0^1 dx\frac{x}{\Delta}+3(1+\xi)(\hat{p}-m)p^2\times\int_0^1 dx\frac{x}{\Delta}\left(-1+\frac{x}{2}\right)+2(1-\xi)m[p^2-(\eta p)^2]\times\int_0^1 dx\frac{x(1-x)}{\Delta}, \tag{16}$$

where

$$\Delta=m^2-p^2(1-x). \tag{17}$$

In deriving (16) we performed an additional integration by parts with respect to x to remove logarithmic terms of the form $\ln\Delta$.

The quantity

$$\delta m=\frac{3\alpha}{4\pi}m\left(\ln\frac{\Lambda^2}{m^2}+\frac{1}{2}\right), \tag{18}$$

which diverges logarithmically as $\Lambda\to\infty$, is gauge-invariant and cancels out the mass counterterm in the Lagrangian. The Euclidean cutoff momentum Λ is chosen symmetric in all the components of momentum q, which is possible because all the denominators are covariant.

The expression (16) simplifies considerably at two special values of the gauge parameter, $\xi=1$ and $\xi=-1$.

In the first case, $\xi=1$ (the Fried–Yennie gauge) we arrive at the well-known result[17,18,8]

$$\Sigma^{FY}(p)=\delta m+(\hat{p}-m)(1-Z_2^{-1})+(\hat{p}-m)^2\left(-\frac{3\alpha\hat{p}}{4\pi}\right)\int_0^1 dx\frac{x}{\Delta}, \tag{19}$$

where

$$1-Z_2^{-1}=\frac{3\alpha}{4\pi}\left(-\ln\frac{\Lambda^2}{m^2}+\frac{1}{2}\right). \tag{20}$$

The last equality shows that the renormalization constant Z_2 in the Fried–Yennie gauge is infrared-finite but diverges logarithmically as the ultraviolet cutoff momentum Λ tends to infinity.

In the case $\xi=-1$ Eq. (16) implies that the once subtracted mass operator

$$\Sigma(p)-\delta m\equiv\bar{\Sigma}(p) \tag{21}$$

does not depend on the cutoff momentum Λ. At $\xi=-1$ it is convenient to write Eq. (16) in the form

$$\bar{\Sigma}(p)=\frac{\alpha}{4\pi}\left\{(\hat{p}-m),\frac{\hat{Q}}{m}\right\}+\frac{\alpha}{\pi}(\hat{p}-m)\times\left[\frac{3}{2}+2Q^2\int_0^1 dx\frac{(1-x)^2}{\Delta}\right]+\frac{\alpha}{4\pi}(\hat{p}-m)^2\times\int_0^1\frac{dx}{\Delta}\left[\frac{4Q^2(1-x)^2}{m}+3mx+3(1-x)(2\hat{p}+m)\right]. \tag{22}$$

This expression depends on two variables, the "virtuality" variable $\rho=1-p^2/m^2$ and the space-like vector

$$Q=p-(\eta p)\eta,\quad(\eta Q)=0, \tag{23}$$

whose square is

$$Q^2=(pQ)=p^2-(\eta p)^2. \tag{24}$$

When the unit time-like vector η is selected in the form (1,0), $-Q^2$ coincides with the square of the three-dimensional momentum, $\mathbf{p}^2$, and the expression (22) can be interpreted as a function of the variables ρ and $\mathbf{p}^2$ or equivalently, of the variables p_0^2 and $\mathbf{p}^2$. However, wishing to retain the freedom in choosing the vector η, below we use the variables ρ and Q^2.

A characteristic feature of the mass operator in the gauge (10) that sets the operator apart from the corresponding expression in any covariant gauge is the presence of a term with an anticommutator. A similar term appears in other noncovariant gauges, for instance, in the Coulomb gauge,[19] with the numerical coefficient in front of the anticommutator depending on the specific form of the propagator. Let us now examine the behavior of the mass operator (22) near the pole $p^2\simeq m^2$ (the virtuality variable ρ tends to zero in this case) and for an arbitrary value of momentum Q. If we are interested only in the terms proportional to the first power of $\hat{p}-m\sim m\rho$, we can write (22) in the form

$$\bar{\Sigma}(p)=\frac{\alpha}{4\pi}\left\{(\hat{p}-m),\frac{\hat{Q}}{m}\right\}+\frac{\alpha}{\pi}(\hat{p}-m)F(\rho,Q^2)$$

$$+\frac{\alpha}{\pi}(\hat{p}-m)^2R(\rho,Q^2), \qquad (25)$$

where we have introduced the following function of the variables ρ and Q^2:

$$F(\rho,Q^2)=\frac{3}{2}+\frac{Q^2}{m^2}\left(2\ln\frac{1}{\rho}-3\right). \qquad (26)$$

Here we are not interested in the explicit form of the function $R(\rho,Q^2)$ in (25). It is enough to know that it behaves no worse than $\ln\rho$ as $\rho\to 0$.

Note the logarithmic singularity of $F(\rho,Q^2)$ as $\rho\to 0$. Such a singularity before all term linear in $(\hat{p}-m)$ is present in every gauge except the Fried–Yennie gauge, in which the coefficient of the logarithmic term vanishes. It would seem that the attempt to find a gauge combining the merits of the Landau and Fried–Yennie gauges has been defeated, since we were unable to remove the term of the form $(\hat{p}-m)\ln\rho$ in the asymptotic expression for the mass operator. However, the infrared behavior of the mass operator improves the coefficient Q^2/m^2 of the logarithm. For instance, in the case of bound states, where the most important momenta are those of order of atomic momenta, the coefficient of $\ln\rho$ proves to be small, $Q^2/m^2\sim\mathbf{p}^2/m^2\sim(Z\alpha)^2$, and the corresponding contribution to the energy coincides in order of magnitude with that of the term quadratic in $\hat{p}-m$ in (25). On the other hand, for a number of corrections to the Lamb shift and the hyperfine splitting,[6–8] the most important momenta are the high ones $|p|\sim m$. This makes it possible to ignore the binding energy and the momenta of the wave functions in all the propagators,[8] as a result of which we arrive at the relation $Q^2/m^2=-\rho$, i.e., again the contributions of the terms with Q^2 and $(\hat{p}-m)^2$ are of the same order. Finally, by choosing the vector η in the form $\eta=p/\sqrt{p^2}$ we can make Q^2 identically zero and, in general, remove the terms of the form $(\hat{p}-m)\ln\rho$ from the mass operator.

Thus, if in a specific physical situation Q^2 does lead to an additional smallness, the worst term from the infrared standpoint is the term $\frac{3}{2}$ in $F(\rho,Q^2)$ [in the Fried–Yennie gauge the similar term is $-\frac{3}{4}\ln(\Lambda^2/m^2)+\frac{3}{8}$]. The corresponding term $(3\alpha/2\pi)(\hat{p}-m)$ in the mass operator can lead to infrared divergences, but without logarithmic enhancement. In calculating the matrix elements it is convenient to use the mass operator in the form (22) with previously subtracting the term $(3\alpha/2\pi)(\hat{p}-m)$. According to the Ward identity, this term cancels out with the term $-(3\alpha/2\pi)\gamma_\mu$ in the vertex function. Of course this procedure is not renormalization in the ordinary sense of the word, but it is sufficient from a practical standpoint.

In order to carry out the standard renormalization procedure we must calculate the derivative $\partial\Sigma(p)/\partial p_\mu$ rigorously for $p=\bar{p}$, where $\bar{p}$ stands for momentum on the mass surface. This derivative diverges in proportion to $\ln\rho$, so that a small photon mass λ is introduced to formally remove the divergences. In the gauge (13) considered here the introduction of a photon mass may prove unjustified computationally. Indeed, as noted earlier, in some cases the quantity $-Q^2/m^2$ coincides with the virtuality variable ρ or has the same order of smallness. By introducing a photon mass λ we obtain a finite term multiplied by $\ln\lambda$ in the renormalization part and the same with the opposite sign in the renormalized mass operator $\Sigma_R(p)$. To avoid the emergence of fictitious contributions that cancel out in the final result we will attempt to carry out the renormalization procedure without introducing a photon mass.

We define the renormalized mass operator $\Sigma_R(p)$ in the following way:

$$\Sigma(p)=\Sigma(\bar{p})+(p-\bar{p})^\mu\frac{\partial}{\partial p^\mu}\Sigma(p)|_{p\to\bar{p}}+\Sigma_R(p). \qquad (27)$$

After differentiating with respect to p and subtracting the constant $\delta m=\Sigma(\bar{p})$ we obtain

$$\bar{\Sigma}(p)=\frac{\alpha}{4\pi}\left\{(\hat{p}-m),\frac{\hat{Q}}{m}\right\}$$

$$+\frac{\alpha}{\pi}(\hat{p}-m)\left(1+\rho\frac{\partial}{\partial\rho}\right)F(\rho,Q^2)|_{\rho\to 0}+\Sigma_R(p). \qquad (28)$$

In contrast to the standard renormalization procedure, where the virtuality variable ρ is strictly zero, the coefficient of the structure $(\hat{p}-m)$ in (28) is not a constant. Nevertheless, we write the once subtracted mass operator (28) in a form that is as close to the standard form as possible:

$$\bar{\Sigma}(p)=\frac{\alpha}{4\pi}\left\{(\hat{p}-m),\frac{\hat{Q}}{m}\right\}$$

$$+(\hat{p}-m)[1-Z_2^{-1}(\rho,Q^2)]+\Sigma_R(p). \qquad (29)$$

The introduced function

$$Z_2^{-1}(\rho,Q^2)=\left\{1-\frac{\alpha}{\pi}\left(1+\rho\frac{\partial}{\partial\rho}\right)F(\rho,Q^2)|_{\rho\to 0}\right\}^{-1}$$

$$\simeq 1+\frac{\alpha}{\pi}\left[\frac{3}{2}+\frac{Q^2}{m^2}\left(2\ln\frac{1}{\rho}-5\right)\right] \qquad (30)$$

at $Q^2=0$ coincides with the ordinary renormalization constant of the wave function, $Z_2=1+3\alpha/2\pi$.

Next we examine the behavior of the exact electron Green's function $S'(p)$ near the pole $p^2=m^2$. To first order in α we have

$$[S'(p)]^{-1}=\hat{p}-m_0-\Sigma(p)=\hat{p}-m-\bar{\Sigma}(p)$$

$$\simeq-\frac{\alpha}{4\pi}\left\{(\hat{p}-m),\frac{\hat{Q}}{m}\right\}+(\hat{p}-m)Z_2^{-1}(\rho,Q^2). \qquad (31)$$

We can remove the matrices in the denominator of $S'(p)$ by multiplying and dividing the function by the expression (31) in which $\hat{p}$ and $\hat{Q}$ are taken with signs reversed:

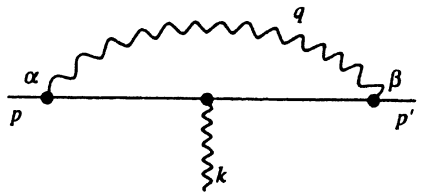

FIG. 2. The one-loop vertex function.

$$S'(p)\simeq\frac{Z_2^2(\rho,Q^2)}{p^2-m^2}\left[\frac{\alpha}{4\pi}\left\{(\hat{p}+m),\frac{\hat{Q}}{m}\right\}+(\hat{p}+m)Z_2^{-1}(\rho,Q^2)\right]$$
$$\simeq\frac{Z_2(\rho,Q^2)}{p^2-m^2}\left(1+\frac{\alpha}{4\pi}\frac{\hat{Q}}{m}\right)(\hat{p}+m)\left(1+\frac{\alpha}{4\pi}\frac{\hat{Q}}{m}\right). \quad (32)$$

The function $S'(p)$ at $p^2\simeq m^2$ can be written in the usual form

$$S'(p)\simeq\sum_{s=1,2}\frac{\Psi(p,s)\bar{\Psi}(p,s)}{p^2-m^2}. \quad (33)$$

To this end we use the expression

$$(\hat{p}+m)=\sum_{s=1,2}u(p,s)\bar{u}(p,s) \quad (34)$$

to introduce the "renormalized" wave function

$$\Psi(p,s)=\sqrt{Z_2(\rho,Q^2)}\left(1+\frac{\alpha}{4\pi}\frac{\hat{Q}}{m}\right)u(p,s). \quad (35)$$

The function (35) coincides in form with the wave function in the Coulomb gauge,[19,20] differing from the later only in numerical coefficients. In the particular case of $-Q^2=\mathbf{p}^2\ll m^2$ Eq. (30) implies

$$Z_2(\rho,-\mathbf{p}^2)\simeq1+\frac{\alpha}{\pi}\left[C+2\left(-\frac{\mathbf{p}^2}{m^2}\right)\ln\frac{1}{\rho}\right]. \quad (36)$$

After simple transformations the renormalization factor in the Coulomb gauge can be reduced to the same form. Here the factor 2 in front of the structure $(-\mathbf{p}^2/m^2)\ln(1/\rho)$ in (36) is replaced by $\frac{2}{3}$, and the constant $C=\frac{3}{2}$ in the Coulomb gauge corresponds to a quantity dependent on the ultraviolet cutoff parameter Λ. In addition, the factor $\alpha/4\pi$ in the matrix structure $1+(\alpha/4\pi)(\gamma\mathbf{p}/m)$ must be replaced by $\alpha/6\pi$. Thus, the wave function $\Psi(p,s)$ exhibits similar infrared behavior in both gauges. The main merit of the gauge (13) is the ultraviolet finiteness of the renormalization constant.

4. THE VERTEX FUNCTION

The initial expression for the one-loop vertex function (Fig. 2) in the gauge (13) has the form

$$\Lambda_\mu(p,p')=\frac{\alpha}{4\pi}\int\frac{d^4q}{\pi^2 i}\frac{\gamma^\alpha(\hat{p}+\hat{q}+m)\gamma_\mu(\hat{p}'+\hat{q}+m)\gamma^\beta}{(q^2+i\varepsilon)D(p+q)D(p'+q)}$$
$$\times\left[g_{\alpha\beta}-2\frac{q_\alpha q_\beta}{q^2}+\frac{(\eta q)}{q^2}(q_\alpha\eta_\beta+\eta_\alpha q_\beta)\right]. \quad (37)$$

Let us study this expression at zero momentum transfer $k=p'-p$ and small values of the virtuality variables of external electron momenta $\rho=1-p'^2/m^2=1-p^2/m^2$. For the case $\xi=-1$ we have

$$\Lambda_\mu(p,p)\simeq-\frac{\alpha}{4\pi}\left\{\gamma_\mu,\frac{\hat{Q}}{m}\right\}-\frac{\alpha}{\pi}\gamma_\mu\left[\frac{3}{2}+\frac{Q^2}{m^2}\left(2\ln\frac{1}{\rho}-5\right)\right]. \quad (38)$$

Comparing this asymptotic behavior with the one of the mass operator [Eq. (25)], we see that the Ward identity is valid:

$$\Lambda_\mu(p,p)=\frac{\partial\Sigma(p)}{\partial p^\mu}. \quad (39)$$

Equation (38) can be written as follows:

$$\Lambda_\mu(p,p)\simeq-\frac{\alpha}{4\pi}\left\{\gamma_\mu,\frac{\hat{Q}}{m}\right\}+\gamma_\mu[Z_1^{-1}(\rho,Q^2)-1], \quad (40)$$

where $Z_1(\rho,Q^2)$ coincides with the function $Z_2(\rho,Q^2)$ defined in (30). As expected, in the limit $\rho\to0$ the total transition current $\bar{\Psi}(p)\Gamma_\mu(p,p)\Psi(p)=\bar{\Psi}(p)\times[\gamma_\mu+\Lambda_\mu(p,p)]\Psi(p)$ becomes equal to the free current $\bar{u}(p)\gamma_\mu u(p)$.

Next, following the results of Adkins,[20] we examine the terms in the vertex function that contribute to the electron magnetic moment. In covariant gauges the origin of the anomalous magnetic moment is trivial: it appears because of the gauge-invariant term $(\alpha/2\pi m)i\sigma_{\mu\nu}k^\nu$ in the vertex function $\Lambda_\mu(p,p')$. In noncovariant gauges, however, not only does the vertex function contribute to the anomalous moment but so does the self-energy correction to the wave function.

To calculate the electron magnetic moment it is sufficient to retain only the terms linear in momentum k in the matrix element $\bar{\Psi}(p')\Gamma_\mu(p,p')\Psi(p)$. The simplest way to find the vertex function with the given accuracy is to add to (40) the term $(\alpha/2\pi m)i\sigma_{\mu\nu}k^\nu$ and to replace the anticommutator $\{\gamma_\mu,\hat{Q}\}$ by the sum $\hat{Q}'\gamma_\mu+\gamma_\mu\hat{Q}$. As a result we obtain

$$\Gamma_\mu(p,p')\simeq\gamma_\mu Z_1^{-1}+\frac{\alpha}{2\pi}\frac{i\sigma_{\mu\nu}(p'-p)^\nu}{2m}$$
$$-\frac{\alpha}{8\pi}(1-\xi)\left[\frac{\hat{Q}'}{m}\gamma_\mu+\gamma_\mu\frac{\hat{Q}}{m}\right]. \quad (41)$$

Here for the sake of generality we have retained the undefined parameter ξ, with the functions Z_1 and Z_2 (the arguments are dropped) containing ultraviolet divergences of the type $(1+\xi)\ln(\Lambda/m)$ and the matrices $\hat{Q}$ multiplied by $1-\xi$. Simple calculations lead us to the ordinary gauge-invariant result:

$$
\begin{aligned}
\bar{\Psi}(p')\Gamma_\mu(p,p')\Psi(p)
&\simeq Z_2\bar{u}(p')\left[1+\frac{\alpha}{8\pi}(1-\xi)\frac{\hat{Q}'}{m}\right]\\
&\quad\times\Gamma_\mu(p,p')\left[1+\frac{\alpha}{8\pi}(1-\xi)\frac{\hat{Q}}{m}\right]u(p)\\
&\simeq\bar{u}(p')\left[\gamma_\mu+\frac{\alpha}{2\pi}\,\frac{i\sigma_{\mu\nu}(p'-p)^\nu}{2m}\right]u(p). \qquad (42)
\end{aligned}
$$

Equations (41) and (42) show that in the Fried–Yennie gauge ($\xi=1$) the wave function $\Psi(p)$ contains no additional matrix structure and the entire anomalous moment is determined by the second term on the right-hand side of Eq. (41). On the other hand, in the gauge (13) at $\hat{Q}=-\gamma\mathbf{p}$ the magnetic part of the last two terms in (20) cancels out and the entire correction to the magnetic moment originates in the additional matrix structure in the wave function $\Psi(p)$.

5. CONCLUSION

Thus, we have found that in the gauge (13) the once subtracted mass operator and the unrenormalized vertex function are free of ultraviolet divergences. The infrared asymptotic expressions for these diagrams contain the characteristic term $(Q^2/m^2)\ln(1/\rho)$, where ρ is the small virtuality variable of the external electron momenta. The softness of the infrared behavior of the radiative corrections is determined by the coefficient of $\ln(1/\rho)$. For the majority of gauges, which are not suited for use in the infrared range, the coefficient of the logarithm is of order unity (the common factor $\alpha/4\pi$ is dropped). Both the Fried–Yennie gauge and the Coulomb gauge are especially suited for use in the infrared range. In the Fried–Yennie gauge the coefficient is proportional to ρ, which makes it possible to carry out the renormalization procedure rigorously on the mass surface $\rho=0$ without introducing a photon mass λ. In the Coulomb gauge the coefficient of $\ln(1/\rho)$ is proportional to $\mathbf{p}^2/m^2$. For bound states the factor $\mathbf{p}^2/m^2$ leads to additional softening in the infrared range, but standard renormalization can be done only at $\mathbf{p}=0$ (see Ref. 21). In the gauge (13) the coefficient of the logarithm contains $Q^2=p^2-m^2$, which depends on the unit time-like vector η. The freedom in choosing η makes it possible to select any simplifying assumption valid in a specific physical situation. In the simplest case of $\eta=(1,0)$ the coefficient of the infrared logarithm proves to be proportional (as it is in the Coulomb gauge) to $\mathbf{p}^2/m^2$. Choosing η in the form of $p/\sqrt{p}$, in which case $Q=0$ holds, allows us to remove the term $(Q^2/m^2)\ln(1/\rho)$ from the infrared asymptotic expansion and to carry out the renormalization procedure rigorously on the mass surface. For η we can take any combination of the initial (p) and final (p') electron momenta, which in the rest frame becomes $\eta=(1,0)$.

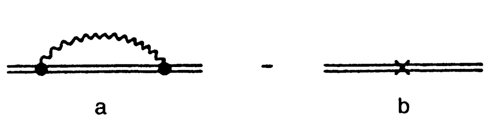

FIG. 3. The self-energy of an electron in a Coulomb field in the lowest order (the double line corresponds to the exact propagator).

The gauge (13) is intended primarily for use in the bound-state problem. For instance, when calculating the self-energy operator (in Fig. 3 the double line corresponds to the exact electron propagator in the Coulomb field), we must subtract the self-mass δm (the diagram b) in an appropriate manner from the main diagram a. Various ways of subtracting δm have been used in Refs. 22–26 and 9. In addition to these approaches there is the one based on the ultraviolet and infrared properties of the gauge (13). Using the Dirac equation in the Coulomb field, we represent the once subtracted mass operator in the form of the sum of three diagrams (Fig. 4). Then the sum of the diagrams $a1$ and b is reduced to the finite expression (22) averaged over the wave functions. The diagram $a2$ is also ultraviolet-finite, since at high momenta of the radiative photon the Coulomb Green's function can be replaced by the free Green's function and then the asymptotic expression (11) can be employed.

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APPENDIX: SOME ADDITIONAL PROPERTIES OF THE GAUGE (13)

The photon propagator (13) can be written in the symmetric form

$$D_{\alpha\beta}(q)=\frac{1}{q^2+i\varepsilon}\left(g_{\alpha\beta}+\frac{q_\alpha\bar{q}_\beta+\bar{q}_\alpha q_\beta}{q^2}\right) \qquad (A1)$$

if we introduce a vector $\bar{q}$ defined as follows:[27]

$$\bar{q}\equiv2(\eta q)\eta-q,\quad \bar{q}^2=q^2,\quad (\eta\bar{q})=(\eta q). \qquad (A2)$$

In the particular case $\eta=(1,0)$ the vectors q and $\bar{q}$ differ only in the sign of the spatial part.

Representing in a similar way the propagator in the Coulomb gauge

$$D^C_{\alpha\beta}(q)=\frac{1}{q^2+i\varepsilon}\left(g_{\alpha\beta}-\frac{q_\alpha\bar{q}_\beta+\bar{q}_\alpha q_\beta}{2[(\eta q)^2-q^2]}\right), \qquad (A3)$$

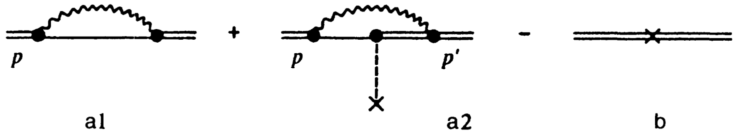

FIG. 4. Transformation of the diagrams of Fig. 3 via the Dirac Equation.

we see that the difference of (A1) and (A2) is

$$D_{\alpha\beta}(q)-D^{C}_{\alpha\beta}(q)=(q_{\alpha}\bar{q}_{\beta}+\bar{q}_{\alpha}q_{\beta})g(q_0,\mathbf{q}), \tag{A4}$$

with the residue of the function $g(q_0,\mathbf{q})$ at the pole $q^2=0$ vanishing.

Let us examine the properties of the propagator (A1) in the coordinate representation. We define the Fourier transform as follows:

$$D_{\alpha\beta}(x)=\int \frac{d^4q}{(2\pi)^4}e^{-iqx}D_{\alpha\beta}(q), \tag{A5}$$

where we write x instead of $x-x'$, assuming one of the vector potentials of the electromagnetic field in the corresponding vacuum average taken at point $x'=0$. Plugging (A1) into (A5) and introducing a vector

$$\bar{x}\equiv 2(\eta x)\eta-x, \quad \bar{x}^2=x^2, \quad (\eta\bar{x})=(\eta x) \tag{A6}$$

similar to the vector $\bar{q}$ [Eq. (A2)] in momentum space, we obtain

$$D_{\alpha\beta}(x)=\frac{i}{2\pi^2}\frac{1}{x^2-i\varepsilon}\left(\eta_{\alpha}\eta_{\beta}-\frac{x_{\alpha}\bar{x}_{\beta}+\bar{x}_{\alpha}x_{\beta}}{2x^2}\right). \tag{A7}$$

By a direct check we see that the operator (A7) satisfies the relationship

$$D_{\alpha\lambda}(x)D^{\lambda}_{\beta}(x)\sim D_{\alpha\beta}(x) \tag{A8}$$

and hence has no inverse. The propagator in the Fried–Yennie gauge,

$$D^{FY}_{\alpha\beta}(x)=\frac{i}{2\pi^2}\frac{1}{x^2-i\varepsilon}\left(g_{\alpha\beta}-\frac{x_{\alpha}x_{\beta}}{x^2}\right), \tag{A9}$$

possesses the same property. The propagator (A7) was chosen from considerations of infrared finiteness, with the result that the components D_{00} in (A7) and (A9) coincide.

It is interesting to compare the propagator (A9) with the propagator in the Landau gauge,

$$D^{L}_{\alpha\beta}(x)=\frac{i}{8\pi^2}\frac{1}{x^2-i\varepsilon}\left(g_{\alpha\beta}+\frac{2x_{\alpha}x_{\beta}}{x^2}\right). \tag{A10}$$

From the above Fourier transforms (A9) and (A10) we obtain at $q^2\neq 0$ and $x^2\neq 0$ the following four symmetric relationships:

$$q^{\alpha}D^{L}_{\alpha\beta}(q)=0, \quad \frac{\partial}{\partial x_{\alpha}}D^{L}_{\alpha\beta}(x)=0, \tag{A11}$$

$$\frac{\partial}{\partial q_{\alpha}}D^{FY}_{\alpha\beta}(q)=0, \quad x^{\alpha}D^{FY}_{\alpha\beta}(x)=0. \tag{A12}$$

Thus, the Fried–Yennie gauge at large values of x^2 is similar to the Landau gauge at large values of q^2. More than that, the second relationship in (A12) implies that the Fried–Yennie gauge in the coordinate representation corresponds to the well-known Fock–Schwinger fixed-point gauge

$$x^{\mu}A_{\mu}(x)=0. \tag{A13}$$

This gauge was suggested by Fock back in 1937 (Ref. 28) and was used by Schwinger[29] to investigate the infrared properties of the electron Green's function.

The vector potential $A_{\mu}(x)$ satisfying (A13) can be expressed solely in terms of the gauge-invariant electromagnetic field tensor $F_{\mu\nu}$:

$$A_{\mu}(x)=\int_0^1 d\lambda\ \lambda x^{\alpha}F_{\alpha\mu}(\lambda x), \tag{A14}$$

with $\partial A_{\nu}/\partial x^{\mu}-\partial A_{\mu}/\partial x^{\nu}=F_{\mu\nu}$.

For soft photons with a wavelength considerably larger than the characteristic size of the interaction region the field $F_{\mu\nu}$ can be assumed uniform, so that we can write

$$A_{\mu}(x)\simeq\frac{x^{\alpha}}{2}F_{\alpha\mu}. \tag{A15}$$

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