

Quantum theory of radiative damping of a relativistic electron

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(Submitted 23 April 1996)

Zh. Ėksp. Teor. Fiz. **110**, 1629–1640 (November 1996)

An exact quantum mechanical expression for the generalized radiative damping force in the stochastic equation for a relativistic electron is derived. The effect of saturation of the radiative damping coefficient at frequencies ω higher than mc^2/\hbar is established. Finally, it is shown that the Lorentz–Langevin quantum equation satisfies the causality principle and does not contain unstable solutions inherent in the classical theory. © 1996 American Institute of Physics. [S1063-7761(96)00711-1]

1. INTRODUCTION

The classical expression for the radiative damping force,

$$F_j(t) = \frac{2}{3} \frac{e^2}{c^2} \frac{d^3}{dt^3} r_j(t), \quad j=1,2,3 \quad (1)$$

leads to instability of electron motion and to the self-acceleration paradox.^{1,2} Actually this means that the causality principle is violated. Indeed, the equation

$$m \frac{d^2}{dt^2} r_j(t) - \frac{2}{3} \frac{e^2}{c^2} \frac{d^3}{dt^3} r_j(t) = eE_j(\omega)e^{-i\omega t} + eE_j^*(\omega)e^{i\omega t} \quad (2)$$

in accordance with the definition

$$\langle r_j(\omega) \rangle = \chi(\omega) eE_j(\omega), \quad (3)$$

yields the following expression for the linear susceptibility:

$$\chi(\omega) = \frac{1}{m} [-\omega^2(1+i\tau_0\omega)]^{-1}, \quad \tau_0 = \frac{2}{3} \frac{e^2}{mc^3}, \quad (4)$$

which has a pole in the upper half-plane of the complex variable ω , and this contradicts the causality principle.³ In addition, the fact that the radiative damping force (1) is zero when the electron acceleration is constant contradicts the Larmor formula, and hence the energy conservation law.² Finally, within classical theory, it is impossible to consistently derive a relativistic expression for the radiative damping force. The generally accepted procedure for transition from the approximate formula (1) to the relativistic range via Lorentz transformations (see Refs. 1 and 2) cannot be considered satisfactory.

All these questions, which emerged at the beginning of the 20th century, still remain at the foreground.^{4–9} Nevertheless, the unflagging interest did not result in any solution within the classical electrodynamics setting. Physically speaking, the problem of radiative damping must be solved with allowance for quantum effects.² The main reason for the paradoxes of the classical theory of radiative damping stems from the fact that the theory does not allow for vacuum fluctuations of the electromagnetic field and quantum properties of the electron. Moreover, an important as-

pect in solving the problem of radiative damping is the allowance of the time lag in the interaction between the electron and the quantized field.

To solve the stated problem, we employ the method of non-Markovian Langevin equations for nonlinear quantum systems suggested in Ref. 10 and developed in Refs. 11 and 12. In contrast to the results of a number of works in non-Markovian relaxation,^{13–18} the kinetics and fluctuations in nonlinear quantum systems interacting with dissipative surroundings, a heat bath, were studied in Refs. 10–12 within a unified setting. In Refs. 4 and 7 the method of Langevin equation was actively used to analyze the problem of radiative damping without resorting to the microscopic approach.

The present paper fills this gap. It is devoted to developing a microscopic theory of the Brownian motion of an electron in a photon heat bath. Consistently deriving the stochastic equations for the variables of a relativistic electron in a microscopic setting makes it possible to obtain a rigorous expression for the generalized radiative damping force and fluctuation sources with a definite procedure for calculating correlation functions of any order. The theory of radiative damping developed in this paper is believed to be free of the paradoxes inherent in classical electrodynamics.

2. BROWNIAN MOTION OF A RELATIVISTIC ELECTRON IN A PHOTON HEAT BATH

Below we give a consistent microscopic derivation of the expression for the radiative damping force on a relativistic electron that interacts with a quantized electromagnetic field, a photon heat bath. We begin with the Hamiltonian of the dynamic subsystem consisting of an electron in an external field $V(\mathbf{r})$:

$$H = c\boldsymbol{\alpha}\mathbf{p} + \beta mc^2 + V(\mathbf{r}), \quad (5)$$

where $\boldsymbol{\alpha}$ and β are the Dirac matrices.

The photon heat bath, whose Hamiltonian we denote by F , acts as a dissipative system. The relativistic nature of the dissipative system determines the relativistic setting of the problem of radiative damping. Using the single-particle approach in describing the dynamic subsystem without resorting to second quantization methods makes it possible to employ, if necessary, the classical picture in describing the

motion of charged particles in external fields and to use the expression for the radiative damping force obtained in quantum theory.

In analyzing the interaction of an electron with the photon heat bath it is convenient to fix the gauge symmetry by selecting the transverse gauge for the field potentials:

$$\operatorname{div} \mathbf{A}(\mathbf{r}, t) = 0. \quad (6)$$

The scalar potential $A_0(\mathbf{r}, t)$, responsible in this case for the Coulomb interaction, leads to no observable effects in the single-particle problem and can be ignored in the system Hamiltonian.

Thus, the initial Hamiltonian of the complete system can be written as

$$H = c\alpha \left[\mathbf{p} - \frac{e}{c} \mathbf{A}(\mathbf{r}, t) \right] + \beta m c^2 + V(\mathbf{r}) + F. \quad (7)$$

The problem of the theory of radiative damping consists in eliminating the heat bath variables from the Heisenberg equations for the dynamical variables of the electron and obtaining, as a result, an explicit expression for the radiative damping force. For the sake of definiteness, we write the Heisenberg equation for the projections of momentum

$$\pi_j(t) = p_j(t) - \frac{e}{c} A_j(\mathbf{r}(t), t)$$

in the form

$$\begin{aligned} \frac{d\pi_j(t)}{dt} + \nabla_j V(\mathbf{r}(t)) = & -\frac{e}{c} \frac{d}{dt} A_j(\mathbf{r}(t), t) \\ & + \frac{e}{c} \dot{r}_\alpha(t) \nabla_j A_\alpha(\mathbf{r}(t), t), \end{aligned} \quad (8)$$

whose classical relativistic analog is the well-known Lorentz equation

$$\begin{aligned} \frac{d}{dt} \left[\frac{\dot{\mathbf{r}}(t)}{\sqrt{1 - v^2(t)/c^2}} \right] + \nabla_j V(\mathbf{r}(t)) \\ = -\frac{e}{c} \frac{\partial}{\partial t} A_j(\mathbf{r}(t), t) + \frac{e}{c} \dot{r}_\alpha(t) [\nabla_j A_\alpha(\mathbf{r}(t), t) \\ - \nabla_\alpha A_j(\mathbf{r}(t), t)]. \end{aligned} \quad (9)$$

The remarkable thing about the right-hand side of Eq. (8) is that instead of a partial derivative of the vector potential, it contains the total derivative, which makes it possible to write the radiative damping force in a more compact form. In the Heisenberg representation the field potentials are functions of the operator of the electron's coordinate. It is therefore convenient to write the field potentials in the form of Fourier expansions:

$$A_j(\mathbf{r}(t), t) = \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\mathbf{r}(t)} A_j(\mathbf{k}, t). \quad (10)$$

The Fourier components $A_j(\mathbf{k}, t)$ do not explicitly contain the electron operators and act as the variables of the photon heat bath.

We use the expression (7) for the Hamiltonian to find the variables of the dynamical subsystem that are the canonical conjugates of $A_j(\mathbf{k}, t)$ via the following relationship:

$$-\frac{\delta H}{\delta A_j(\mathbf{k}, t)} = \frac{e}{c} \dot{r}_j(t) e^{i\mathbf{k}\mathbf{r}(t)}, \quad (11)$$

where

$$\dot{r}_j(t) = \frac{1}{i\hbar} [r_j(t), H] \quad (12)$$

are the components of the electron velocity operator.

Next we assume that the interaction of the electron and the photon heat bath is turned on adiabatically in an infinitely distant point in time. Prior to turn-on of this interaction, the photon heat bath was in thermodynamic equilibrium at a temperature T . In particular, $T=0$ corresponds to an electromagnetic vacuum state. In this case the unperturbed potentials $A_j^0(\mathbf{k}, t)$ can be interpreted as Gaussian variables. Then, according to the general theory,¹¹ the full evolution in time of the potentials $A_j(\mathbf{k}, t)$ can be written as

$$\begin{aligned} A_j(\mathbf{k}, t) = A_j^0(\mathbf{k}, t) + \frac{e}{c} \int_{-\infty}^{\infty} dt_1 D_{je}(\mathbf{k}, t \\ - t_1) e^{-i\mathbf{k}\mathbf{r}(t_1)} \dot{r}_e(t_1), \end{aligned} \quad (13)$$

where

$$\begin{aligned} \left\langle \frac{i}{\hbar} [A_j^0(\mathbf{k}, t), A_j^0(\mathbf{k}_1, t_1)] \right\rangle \theta(t - t_1) = D_{je}(\mathbf{k}, t - t_1) \\ \times (2\pi)^3 \delta^3(\mathbf{k} + \mathbf{k}_1) \end{aligned} \quad (14)$$

is the so-called photon Green's function. In the adopted gauge (6) the photon Green's function has the form¹⁹

$$\begin{aligned} D_{je}(\mathbf{k}, \omega) = \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} D_{je}(\mathbf{k}, \tau) \\ = \frac{4\pi}{k^2 - (\omega/c)^2 (1 + i\epsilon \operatorname{sgn} \omega)} \left(\delta_{je} - \frac{k_j k_e}{k^2} \right), \end{aligned} \quad (15)$$

where $k = |\mathbf{k}|$. The presence of $i\epsilon \operatorname{sgn} \omega$ ($\epsilon > 0$) ensures the correct traversal of the pole in the retarded Green's function.

The effect of the heat bath on electron motion is determined by the vector potential (10), which is convenient to express in symmetric form:

$$A_j(\mathbf{r}(t), t) = \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{1}{2} [e^{i\mathbf{k}\mathbf{r}(t)}, A_j(\mathbf{k}, t)]_+. \quad (16)$$

Plugging the fundamental solution (13) into (16) yields

$$\begin{aligned} A_j(\mathbf{r}(t), t) = \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{1}{2} [e^{i\mathbf{k}\mathbf{r}(t)}, A_j^0(\mathbf{k}, t)]_+ \\ + \frac{e}{c} \int \frac{d\mathbf{k}}{(2\pi)^3} \int_{-\infty}^{\infty} dt_1 D_{je}(\mathbf{k}, t \\ - t_1) \frac{1}{2} [e^{i\mathbf{k}\mathbf{r}(t)}, e^{-i\mathbf{k}\mathbf{r}(t_1)} \dot{r}_e(t_1)]_+. \end{aligned} \quad (17)$$

The unperturbed heat bath variables $A_j^0(\mathbf{k}, t)$ in (17), which have a fixed parametric effect on the electron, contribute to the nonlinear dynamics and at the same time determine the fluctuation sources.

Using the basic assumption that the unperturbed potentials $A_j^0(\mathbf{k}, t)$ are Gaussian makes it possible to clearly distinguish the nonlinear dynamics in the parametric terms, and provides a rigorous definition of fluctuation sources on the basis of a quantum fluctuation analog of the Furutsu–Novikov formula¹¹

$$\begin{aligned} & \frac{e}{c} \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{1}{2} [e^{i\mathbf{k}\mathbf{r}(t)}, A_j^0(\mathbf{k}, t)]_+ \\ &= \frac{e}{c} \int \frac{d\mathbf{k}}{(2\pi)^3} \int_{-\infty}^{\infty} dt_1 M_{je}(\mathbf{k}, t-t_1) \frac{\delta(e^{i\mathbf{k}\mathbf{r}(t)})}{\delta A_l^0(-\mathbf{k}, t_1)} \\ &+ \xi_j^A(t), \end{aligned} \quad (18)$$

where the expression for the functional derivative

$$\frac{\delta(e^{i\mathbf{k}\mathbf{r}(t)})}{\delta A_l^0(-\mathbf{k}, t_1)} = \frac{e}{c} \frac{i}{\hbar} [e^{i\mathbf{k}\mathbf{r}(t)}, e^{-i\mathbf{k}\mathbf{r}(t_1)} \dot{r}_l(t_1)] \theta(t-t_1) \quad (19)$$

is the differential response to an external potential.¹¹

The correlation function $M_{jl}(\mathbf{k}, t-t_1)$ is determined by the following relationship:

$$\begin{aligned} & \left\langle \frac{1}{2} [A_j^0(\mathbf{k}, t) A_l^0(\mathbf{k}_1, t_1)]_+ \right\rangle \\ &= M_{jl}(\mathbf{k}, t-t_1) (2\pi)^3 \delta^3(\mathbf{k}+\mathbf{k}_1). \end{aligned} \quad (20)$$

In accordance with the Källén–Welton fluctuation–dissipation theorem, the imaginary part of the photon Green’s function,

$$D''_{jl}(\mathbf{k}, \omega) = 4\pi^2 \delta\left(\mathbf{k}^2 - \frac{\omega^2}{c^2}\right) \left(\delta_{jl} - \frac{k_j k_l}{k^2} \right) \text{sgn } \omega, \quad (21)$$

determines the spectral density of the fluctuations of the field potentials:

$$\begin{aligned} S_{jl}(\mathbf{k}, \omega) &= \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} M_{jl}(\mathbf{k}, \tau) \\ &= \hbar \coth\left(\frac{\beta\omega}{2}\right) D''_{jl}(\mathbf{k}, \omega), \quad \beta = \frac{\hbar}{T}, \end{aligned} \quad (22)$$

where $T=0$ automatically corresponds to an electromagnetic vacuum state. The definition (18) of the fluctuation sources $\xi_j^A(t)$ actually means that all pairing between the factors $\exp[i\mathbf{k}\mathbf{r}(t)]$ and $A_j^0(\mathbf{k}, t)$ has been eliminated. The rules of calculating the mathematical expectations of Gaussian operators established in Ref. 11 make it possible to find the correlation functions of fluctuation sources of any order.

Combining (18) and (19) with the expression (17) for the vector potential, we obtain

$$\begin{aligned} A_j(\mathbf{r}(t), t) &= \frac{e}{c} \int_{-\infty}^{\infty} dt_1 \int \frac{d\mathbf{k}}{2\pi} \left\{ D_{jl}(\mathbf{k}, t \right. \\ &\quad \left. - t_1) \frac{1}{2} [e^{i\mathbf{k}\mathbf{r}(t)}, e^{-i\mathbf{k}\mathbf{r}(t_1)} \dot{r}_l(t_1)]_+ + M_{jl}(\mathbf{k}, t \right. \end{aligned}$$

$$\begin{aligned} & \left. - t_1) \frac{i}{\hbar} [e^{i\mathbf{k}\mathbf{r}(t)}, e^{-i\mathbf{k}\mathbf{r}(t_1)} \dot{r}_l(t_1)] - \theta(t-t_1) \right\} \\ &+ \xi_j^A(t). \end{aligned} \quad (23)$$

The expression for $\dot{r}_\alpha(t) A_\alpha(\mathbf{r}(t), t)$ on the right-hand side of Eq. (8) can be written in a similar manner. Thus, by employing the Gaussian statistics for the unperturbed variables of the heat bath, the field potentials $A_j^0(\mathbf{k}, t)$, we can eliminate the heat bath variables from the equations of motion for the electron variables and find the expression for the radiative damping force. In particular, Eq. (8) assumes the form

$$\begin{aligned} \frac{d\pi_j(t)}{dt} + \nabla_j V(\mathbf{r}(t)) &= -\left(\frac{e}{c}\right)^2 \frac{d}{dt} \int_{-\infty}^{\infty} dt_1 \int \frac{d\mathbf{k}}{2\pi} \left\{ D_{jl}(\mathbf{k}, t \right. \\ &\quad \left. - t_1) \frac{1}{2} [e^{i\mathbf{k}\mathbf{r}(t)}, e^{-i\mathbf{k}\mathbf{r}(t_1)} \dot{r}_l(t_1)]_+ \right. \\ &\quad \left. + M_{jl}(\mathbf{k}, t-t_1) \frac{i}{\hbar} [e^{i\mathbf{k}\mathbf{r}(t)}, \right. \\ &\quad \left. \times e^{-i\mathbf{k}\mathbf{r}(t_1)} \dot{r}_l(t_1)] - \theta(t-t_1) \right\} \\ &+ \left(\frac{e}{c}\right)^2 \nabla_j \int_{-\infty}^{\infty} dt_1 \int \frac{d\mathbf{k}}{2\pi} \left\{ D_{\alpha l}(\mathbf{k}, t \right. \\ &\quad \left. - t_1) \frac{1}{2} [e^{i\mathbf{k}\mathbf{r}(t)} \dot{r}_\alpha(t), e^{-i\mathbf{k}\mathbf{r}(t_1)} \dot{r}_l(t_1)]_+ \right. \\ &\quad \left. + M_{\alpha l}(\mathbf{k}, t-t_1) \frac{i}{\hbar} [e^{i\mathbf{k}\mathbf{r}(t)} \dot{r}_\alpha(t), \right. \\ &\quad \left. \times e^{-i\mathbf{k}\mathbf{r}(t_1)} \dot{r}_l(t_1) - \theta(t-t_1) \right\} + \xi_j(t) \\ &\equiv F_j(t) + \xi_j(t), \end{aligned} \quad (24)$$

where the radiative damping force $F_j(t)$ allows for both the response of the photon heat bath to electron motion, which is given by the Green’s function $D_{je}(\mathbf{k}, t-t_1)$, and the parametric effect on the electron of vacuum and thermal fluctuations of the heat bath, which is given by the correlation function $M_{je}(\mathbf{k}, t-t_1)$. The expressions for the fluctuation sources $\xi_j^A(t)$ are similar to that for the $\xi_j^A(t)$ in (18).

3. THE GENERALIZED RADIATIVE DAMPING FORCE

In the chosen model of the photon heat bath, which does not allow for the interaction of the heat bath and the electron–positron vacuum, the expression (24) for the generalized radiative damping force $F_j(t)$ is exact. The classical limit of the problem of radiative damping of a relativistic electron is of special interest and requires a separate discussion. Here we note once more that the paradox of a self-accelerating electron cannot be resolved in classical electrodynamics. Thus, the radiative damping force must be calculated with allowance for quantum effects. The quantum stochastic equation (24) is most suitable for this purpose. Using Eq. (24) in analyzing physical effects requires additional assumptions that simplify the explicit expression for $F_j(t)$.

We assume that the electron interacts with a uniform electric field $E_j(t)$ and that the energy of this interaction is

$$V(t) = -r_j(t)eE_j(t) \equiv -r_j f_j(t). \quad (25)$$

We write Eq. (24) with the perturbation (25) in the reference frame linked to the electron:

$$\ddot{r}_j(t) = \frac{f_j(t)}{m} + \frac{F_j(t)}{m} + \frac{\xi_j(t)}{m}, \quad (26)$$

where $F_j(t)$ is strictly determined by (24).

The solution of Eq. (26), $r_j(t)$, can be represented in the form of a series expansion in powers of the external force:

$$r_j(t) + r_j^0 + \int_{-\infty}^{\infty} dt_1 \varphi_{jl}(t, t_1) f_l(t_1) + \dots \quad (27)$$

In accordance with (27), the response to the external force $f_l(t)$ conjugate to $r_l(t)$ is²⁰

$$\varphi_{jl}(t, t_1) = \frac{\delta r_j(t)}{\delta f_l(t_1)} = \tilde{\varphi}_{jl}(t, t_1) \theta(t - t_1), \quad (28)$$

where the Heaviside unit-step function $\theta(t - t_1)$ automatically takes into account the causality principle, and

$$\tilde{\varphi}_{jl}(t, t_1) = \frac{i}{\hbar} [r_j(t), r_l(t_1)]_- \quad (29)$$

is the quantum Poisson bracket for the coordinate projection operators $r_j(t)$ and $r_l(t_1)$. The nonlinear nature of Eq. (26) is determined entirely by the nonlinear dependence of the radiative damping force $F_j(t)$ in (24) on the coordinate operators $r_j(t)$ and $r_l(t_1)$. Since $F_j(t)$ is proportional to the fine structure constant $\alpha = e^2/\hbar c$, the nonlinearity of Eq. (26) is effectively small and hence, in accordance with nonlinear fluctuation-dissipation theorems,^{21,22} we can ignore the fluctuations of the responses (28) in comparison to the average values of the responses.

We use the above reasoning in calculating the radiative damping force $F_j(t)$ in (24), assume that the Poisson brackets (29) are c -numbers, and replace them by their average values in the expression (24) for $F_j(t)$, i.e.,

$$\tilde{\varphi}_{jl}(t, t_1) = \frac{i}{\hbar} [r_j(t), r_l(t_1)]_- \rightarrow \left\langle \frac{i}{\hbar} [r_j(t), r_l(t_1)]_- \right\rangle. \quad (30)$$

In view of homogeneity of time and isotropy of space, for (30) we have

$$\left\langle \frac{i}{\hbar} [r_j(t), r_l(t_1)]_- \right\rangle = \delta_{jl} \tilde{\varphi}(t - t_1). \quad (31)$$

Using condition (30) with (31), we can write the products of the exponential factors in $F_j(t)$ in the following manner:

$$\begin{aligned} e^{i\mathbf{k}\mathbf{r}(t)} e^{-i\mathbf{k}\mathbf{r}(t_1)} &= e^{-ibk^2} e^{i\mathbf{k}\Delta\mathbf{r}}, \\ e^{-i\mathbf{k}\mathbf{r}(t_1)} e^{i\mathbf{k}\mathbf{r}(t)} &= e^{ibk^2} e^{i\mathbf{k}\Delta\mathbf{r}}, \end{aligned} \quad (32)$$

$$b = \frac{\hbar}{2} \tilde{\varphi}(t - t_1), \quad \Delta\mathbf{r} = \mathbf{r}(t) - \mathbf{r}(t_1).$$

In calculating $F_j(t)$ we also employ the fact that the operators $\dot{r}_l(t_1)$, $\dot{r}_j(t)$, and $\dot{r}_\alpha(t)$ commute with the factors $\exp\{i\mathbf{k} \cdot \mathbf{r}(t)\}$ and $\exp\{-i\mathbf{k} \cdot \mathbf{r}(t_1)\}$, which follows from (28) and (29) and the gauge conditions for the Green's function and the correlation function,

$$k_j D_{jl}(\mathbf{k}, t - t_1) = 0, \quad k_l M_{jl}(\mathbf{k}, t - t_1) = 0, \quad (33)$$

in accordance with (6).

Thus, the assumption (30) leads to the following expression for the radiative damping force:

$$\begin{aligned} F_j(t) = & - \left(\frac{e}{c} \right)^2 \frac{d}{dt} \int_{-\infty}^{\infty} dt_1 \int \frac{d\mathbf{k}}{(2\pi)^3} \left\{ D_{je}(\mathbf{k}, t \right. \\ & - t_1) \cos(bk^2) + \frac{2}{\hbar} M_{je}^{\text{ret}}(\mathbf{k}, t \\ & - t_1) \sin(bk^2) \left. \right\} e^{i\mathbf{k}\Delta\mathbf{r}} \dot{r}_l(t_1) \\ & + \left(\frac{e}{c} \right)^2 \nabla_j \int_{-\infty}^{\infty} dt_1 \int \frac{d\mathbf{k}}{(2\pi)^3} \left\{ \left[\frac{1}{2} D_{\alpha l}(\mathbf{k}, t - t_1) \right. \right. \\ & + \frac{i}{\hbar} M_{je}^{\text{ret}}(\mathbf{k}, t - t_1) \left. \right\} e^{-ibk^2} \dot{r}_\alpha(t) \dot{r}_l(t_1) \\ & + \left. \left[\frac{1}{2} D_{\alpha l}(\mathbf{k}, t - t_1) - \frac{i}{\hbar} M_{je}^{\text{ret}}(\mathbf{k}, t \right. \right. \\ & \left. \left. - t_1) \right] e^{ibk^2} \dot{r}_l(t_1) \dot{r}_\alpha(t) \right\} e^{i\mathbf{k}\Delta\mathbf{r}}, \end{aligned} \quad (34)$$

where we have introduced the so-called retarded correlation function

$$M_{je}^{\text{ret}}(\mathbf{k}, t - t_1) = M_{je}(\mathbf{k}, t - t_1) \theta(t - t_1), \quad (35)$$

which allows for the causality principle in the parametric terms $F_j(t)$. Introducing the function (35) makes it possible to exclude the unit-step function $\theta(t - t_1)$ from the exponential factors in (34).

We define the spectral density of the retarded correlation function (35) as follows:

$$\begin{aligned} S_{je}^{\text{ret}}(\mathbf{k}, \omega) &= \int_{-\infty}^{\infty} d\tau M_{je}(\mathbf{k}, \tau) \theta(\tau) e^{i\omega\tau} \\ &= \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \frac{i}{\omega - \Omega + i\varepsilon} S_{je}(\mathbf{k}, \Omega), \end{aligned} \quad (36)$$

where the spectral density $S_{je}(\mathbf{k}, \omega)$ can be determined from the Källén-Welton fluctuation-dissipation theorem, and $i(\omega - \Omega + i\varepsilon)^{-1}$ is the spectral representation of the Heaviside unit-step function $\theta(\tau)$. Using Cauchy's residue theorem and (36), we arrive at an extremely useful formula:¹²

$$\begin{aligned} S_{je}^{\text{ret}}(\mathbf{k}, \omega) &= \frac{\hbar}{2} D_{je}''(\mathbf{k}, \omega) \coth \frac{\beta\omega}{2} \\ & - \frac{i\hbar\omega}{2ck} \frac{4\pi}{[k^2 - (\omega/c)^2]} \left(\delta_{jl} - \frac{k_j k_l}{k^2} \right) \coth \frac{\beta ck}{2}. \end{aligned} \quad (37)$$

Thus, the assumption (30) considerably simplifies the expression for the generalized radiative damping force.

4. THE RADIATIVE DAMPING COEFFICIENT

The stochastic equation (24) or (26) with allowance for formula (34) obtained in the approximation (30) makes it possible to study a broad range of effects of a similar physical nature: renormalization of mass with allowance for temperature dependence, the frequency and Lamb shifts, radiative damping, and fluctuation processes.

In the present paper we limit our discussion to an analysis of the important features of radiative damping that stem from the quantum properties of the electron and the photon heat bath. To this end we simplify the expression (34) for the radiative damping force $F_j(t)$ still further. Ignoring the contribution of the nonlinear terms containing ∇_j and discarding the factor $\exp(i\mathbf{k} \cdot \Delta\mathbf{r})$ responsible for the fluctuations of the electron velocity in the reference frame linked to the electron, we obtain the following expression for the radiative damping force:

$$\frac{F_j(t)}{m} = -\frac{d^2}{dt^2} \int_{-\infty}^{\infty} dt_1 \gamma(t-t_1) r_j(t_1), \quad (38)$$

where

$$\begin{aligned} \gamma(t-t_1) = & \frac{e^2}{mc^2} \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \int \frac{d\mathbf{k}}{(2\pi)^3} e^{-i\Omega(t-t_1)} \\ & \times \left\{ e^{-ibk^2} \left[\frac{1}{2} D(\mathbf{k}, \Omega) + \frac{i}{\hbar} S^{\text{ret}}(\mathbf{k}, \Omega) \right] \right. \\ & \left. + e^{ibk^2} \left[\frac{1}{2} D(\mathbf{k}, \Omega) - \frac{i}{\hbar} S^{\text{ret}}(\mathbf{k}, \Omega) \right] \right\}. \quad (39) \end{aligned}$$

Since in the given approximation the radiative damping is isotropic, Eq. (39) is determined by the following functions:

$$\begin{aligned} D(\mathbf{k}, \Omega) = & \frac{1}{3} \sum_j D_{jj}(\mathbf{k}, \Omega), \\ S^{\text{ret}}(\mathbf{k}, \Omega) = & \frac{1}{3} \sum_j S_{jj}^{\text{ret}}(\mathbf{k}, \Omega). \end{aligned} \quad (40)$$

Combining Eq. (26) with (38), we arrive at the equation

$$\ddot{r}_j(t) + \frac{d^2}{dt^2} \int_{-\infty}^{\infty} dt_1 \gamma(t-t_1) r_j(t_1) = \frac{f_j(t)}{m} + \frac{\xi_j(t)}{m}. \quad (41)$$

If we now apply the Fourier transformation to Eq. (41),

$$-\omega^2 [1 + \gamma(\omega)] r_j(\omega) = \frac{f_j(\omega)}{m} + \frac{\xi_j(\omega)}{m}, \quad (42)$$

we obtain the following expression for the linear susceptibility:

$$\chi(\omega) = \frac{1}{m} [-\omega^2 (1 + \gamma(\omega))]^{-1}, \quad (43)$$

where

$$\gamma(\omega) = \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} \gamma(\tau) = \gamma'(\omega) + i\gamma''(\omega) \quad (44)$$

is called the radiative damping coefficient.

After renormalization of the mass, the real part $\gamma'(\omega)$, i.e., $\tilde{\gamma}'(\omega) = \gamma'(\omega) - \gamma'(0)$, determines the frequency shifts, and the imaginary part $\gamma''(\omega)$ is responsible for radiative damping.

Below we give the expression for the imaginary part of the radiative damping coefficient, $\gamma''(\omega)$, which is determined to first order in the fine structure constant. In this case, when calculating the quantum Poisson bracket (29), we assume that the electron motion is free.

If we ignore the rapidly oscillating relativistic terms, we arrive at the following expression for the Poisson bracket (29):

$$\begin{aligned} \tilde{\varphi}_{ij}(t-t_1) = & \frac{i}{\hbar} [r_i(t), r_j(t_1)] = \frac{t-t_1}{m} \delta_{ij}, \\ b = & \frac{\hbar}{2m} (t-t_1). \end{aligned} \quad (45)$$

Plugging (45) into (39) and allowing for the definition (44), we arrive at an intermediate expression for the imaginary part of the radiative damping coefficient:

$$\begin{aligned} \gamma''(\omega) = & \frac{2}{3} \alpha \int_0^{\infty} dx x \left\{ \delta\left(\frac{x^2}{2} + x - \frac{\omega}{\Omega_0}\right) \right. \\ & \left. - \delta\left(\frac{x^2}{2} + x + \frac{\omega}{\Omega_0}\right) \right\}, \end{aligned} \quad (46)$$

where $\alpha = e^2/\hbar c$ and $\Omega_0 = mc^2/\hbar$.

After integration with respect to x we get

$$\gamma''(\omega) = \frac{2}{3} \alpha [\sigma(\omega)\theta(\omega) - \sigma(-\omega)\theta(-\omega)], \quad (47)$$

where

$$\sigma(\omega) = 1 - \left(1 + \frac{2\omega}{\Omega_0}\right)^{-1/2}.$$

Finally, plugging (47) into (43), we arrive at the following expression for the linear susceptibility:

$$\begin{aligned} \chi(\omega) = & -\frac{1}{m\omega^2} \left\{ 1 + \frac{2i}{3} \alpha [\sigma(\omega)\theta(\omega) - \sigma(-\omega) \right. \\ & \left. \times \theta(-\omega)] \right\}^{-1}. \end{aligned} \quad (48)$$

5. CONCLUSIONS

A fundamental feature of (47) is that at frequencies ω exceeding Ω_0 the radiative damping coefficient becomes saturated. More precisely, the limit of $\gamma''(\omega)$ does not exceed $2\alpha/3$:

$$\gamma''(\omega) \approx \frac{2}{3} \alpha \operatorname{sgn} \omega, \quad \omega \gg \Omega_0. \quad (49)$$

Due to this the susceptibility (48) contains no poles in the upper half-plane of the complex variable ω , which is in

full agreement with the causality principle. In accordance with this, the Lorentz–Langevin equation (41) with allowance for (47) has no unstable solutions inherent in classical theory.

Let us now discuss the possible approximations of the solutions for frequencies $\omega \ll \Omega_0$. What is important here is the order of the two small parameters α and ω/Ω_0 in the given problem. First we must use the fact that α is small.

Due to the saturation effect, Eq. (48) to first order in α can be written as

$$\chi(\omega) \approx -\frac{1}{m} \frac{1}{\omega^2} \left\{ 1 - \frac{2i}{3} \alpha \left[\left(1 - \left(1 + \frac{2\omega}{\Omega_0} \right)^{-1/2} \right) \theta(\omega) - \left(1 - \left(1 - \frac{2\omega}{\Omega_0} \right)^{-1/2} \right) \theta(-\omega) \right] \right\}. \quad (50)$$

Next we assume that $|\omega/\omega_0| \ll 1$. Equation (50) then yields the classical expression for the susceptibility (not containing the Planck constant):

$$\chi(\omega) \approx -\frac{1}{m\omega^2} \left[1 - \frac{2i}{3} \frac{e^2}{c^3} \omega \right]. \quad (51)$$

If we now go back to the Lorentz equation, we arrive at a classical equation well-known in practical calculations:

$$m\ddot{r}_j(t) = \left(1 + \tau_0 \frac{d}{dt} \right) eE_j(t), \quad \tau_0 = \frac{2}{3} \frac{e^2}{mc^3}. \quad (52)$$

Here we have given a rigorous substantiation of this equation based on the quantum-theory approach. Note that using the small parameters in reverse order leads to the well-known paradox of classical theory. Strictly speaking, there is no way in which Eq. (52) can be justified from the classical viewpoint, since the radiative damping force (1) becomes arbitrarily large in the course of an extremely short time $\tau \sim \tau_0 = 10^{-23}$ s.

This, we have suggested a possible approach to solving the radiative damping problem based on a rigorous microscopic derivation of the generalized radiative damping force $F_j(t)$ in Eq. (24), which is fundamentally relativistic. The nonrelativistic approximation was used in calculating the fre-

quency dependence of the radiative damping coefficient (47) and is therefore valid for $\omega < \Omega_0$. Nevertheless, the approximate formula (47), which allows for quantum effects, resolves the main paradox of classical theory. In a more precise calculation of radiative damping in the ultrahigh frequency range $\omega > \Omega_0$ one must discard the nonrelativistic approximation and at the same time allow for the contribution (determined by the factor $\exp(ik \cdot \Delta r)$) of fluctuations to the radiative damping force.

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Translated by Eugene Yankovsky