Transition from a kinetic to a nonlocal hydrodynamic description for a relativistic collisional plasma

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This paper discusses the dynamics of weak perturbations in a relativistic nondegenerate collisional plasma in the absence of chemical reactions. It is shown that, in a certain class of external actions, a description based on kinetic theory is equivalent to a description in terms of nonlocal hydrodynamics, when the constitutive relations are nonlocal in space and time. The hydrodynamic model consistently combines the properties of dissipativity and causality. The conditions are indicated for the coefficients in the constitutive relations associated with covariance, dissipativity and reversibility at the microscopic level (an analog of the Onsager relations). © *1996 American Institute of Physics.* [S1063-7761(96)01511-9]

It is well known that describing dissipative processes in the relativistic theory based on the traditional Eckart scheme[1] and the Landau–Lifshitz scheme[2] results in superluminal speeds of propagation of small perturbations.[3] References 4 and 5 proposed a method of going from a kinetic description of a relativistic gas to a hydrodynamic description in which it is possible to consistently combine dissipativity with causality. The equations of hydrodynamics are nonlocal in space and time. The transition from kinetics to hydrodynamics is exact, since, for any solution of the hydrodynamic equations, it is possible to reconstruct the corresponding solution of the kinetic equation. The nonrelativistic version of this method is explained in Refs. 6 and 7.

In this paper, the results of Refs. 4 and 5 are extended to the case of a classical relativistic collisional plasma without chemical reactions. In formulating the kinetic theory for a plasma, we use the approach and notation of Refs. 8 and 5.

A system of measurement units is used in which the speed of light in vacuum c, Planck's constant $\hbar$, and Boltzmann's constant k equal unity. The measurement units of the electromagnetic quantities are defined in accordance with Gauss's approach. Greek indices run through the values 0, 1, 2, 3, corresponding to a certain inertial measurement system x^α, where x^0 is time. The 4-momenta of the plasma particles are denoted as p^α. Latin indices a, b, c, d run through the values 1, 2, 3, corresponding to the spatial coordinates. The latin indices A and B run through the values $0, \ldots, (K+3)$, where K is the number of components of the plasma. The following notation is adopted for derivatives: $\partial_\alpha = \partial/\partial x^\alpha$ and, $D_\alpha = \partial/\partial p^\alpha$. The space–time indices are lowered and raised by means of the Minkowsky metric $(\eta_{\alpha\beta}) = \mathrm{diag}(1,-1,-1,-1)$. Summation is carried out over repeated indices, unless otherwise specified.

The state of the plasma at the point of space–time x^α is characterized by the single-particle distribution function $f = f(x^\alpha, p^\beta, r)$, where the 4-momenta p^α lie on the mass shell

$$\eta_{\alpha\beta} p^\alpha p^\beta = m^2(r), \quad p^0 \geqslant 0, \qquad (1)$$

$m(r)$ is the rest mass of the plasma particles, and r is a collective parameter that characterizes degrees of freedom other than translational. More specifically, the parameter r has the component form (i, r'), where i runs through the values $1, \ldots, K$ corresponding to the labels of the components, while the parameter r' is associated with internal degrees of freedom (for example, with vibrational or rotational degrees of freedom).

The following measure is defined in p^α, r parameter space:

$$d\zeta = d\nu_r d\mu(r),$$

where $d\nu_r = (p^0)^{-1} dp^1 dp^2 dp^3$ is the Lorentz-invariant measure on the hyperboloid of Eq. (1), and integration over the measure $d\mu(r)$ reduces to summation over i and integration over a measure $d\zeta'(r')$ connected with parameter r'.

The distribution function is normalized so that $\int p^0 f(x^\alpha, p^\beta, r) d\nu_r$ is the particle density of a plasma of sort r.

Let $W = W(p^\alpha, r)$ be an arbitrary function of the microscopic variables. Then the following macroscopic field is defined for a known single-particle distribution function:

$$w = w(x^\alpha) = \langle W \rangle = \int W(p^\beta, r) f(x^\alpha, p^\beta, r) d\zeta.$$

Next, for any 4-vector U^α tangent to the hyperboloid of Eq. (1) at some point p^α, the derivative $U^\alpha D_\alpha W(p^\alpha, r)$ is defined. We recall that the tangency condition has the following form:

$$p^\alpha U_\alpha = 0.$$

We define the 4-tensor of the electromagnetic field in the usual way:

$$\Psi^{a0} = E_a, \quad \Psi^{ab} = -\varepsilon_{abc} B_c,$$

where E_a, B_a are the 3-vectors of the electric and magnetic fields, respectively. Maxwell's equations for this case are

$$\partial_\beta \Psi^{\beta\alpha} = 4\pi j_\alpha, \quad \varepsilon^{\alpha\beta\gamma\delta} \partial_\beta \Psi_{\gamma\delta} = 0. \qquad (2)$$

Here the 4-current j^α is the sum of the external current j^α_{ex} and the induced current j^α_{in}:

$$j^\alpha = j^\alpha_{ex} + j^\alpha_{in}.$$

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It is assumed that the total current satisfies the law of conservation of electric charge,

$$\partial_\alpha j^\alpha = 0. \tag{3}$$

We shall interpret Eq. (3) as a limitation on the external 4-current j^α_{ex} For example, it can be assumed that the external charge is determined from the equation

$$\partial_0 j^0_{ex} = -\partial_\alpha j^\alpha_{in} - \partial_a j^a_{ex}.$$

We assume further that the indices i,j run through the values 1, ... K, while the indices I,J run through the values $(3+i)$, $i=1\ldots,K$. If the indices i,j and the indices I,J are used in the same formula, their values are related by $I=i+3$, $J=j+3$. Let us define a set of functions of the microscopic variables p^β, r:

$$J_\alpha(p^\beta,r) = p_\alpha, \quad J_J(p^\beta,r) = \delta_{ji},$$

$$\Phi_\alpha(p^\beta,r) = e_i p_\alpha, \quad r=(i,r'),$$

where e_i is the electric charge of the ith component of the plasma.

The induced 4-current is computed from

$$j_{in\ \alpha} = \langle \Phi_\alpha \rangle. \tag{4}$$

With the the Lorentz forces, accounted for, the distribution function satisfies the kinetic equation

$$p^\alpha \partial_\alpha f + U^\alpha D_\alpha f = \mathrm{St}[f] + S, \quad U^\alpha = \Psi^{\alpha\beta}\Phi_\beta. \tag{5}$$

Here $\mathrm{St}[f]$ in general is a nonlinear operator on the distribution function, called the collision integral, and $S=S(x^\alpha,p^\beta,r)$ is a function of the sources, describing the exchange of matter, momentum, and energy between the plasma and the external medium. We note that Eq. (5) is traditionally used in kinetic theory without sources, which presupposes the solution of a Cauchy problem. The study of the reaction of the system on the sources is a component of the method of Refs. 4–7. In purely mathematical terms, such a formulation preserves the Cauchy problem.

When plasma particles collide, the energy–momentum vector and the number of particles of the components are conserved:

$$\int J_A(p^\alpha,r)\mathrm{St}[f](x^\beta,p^\alpha,r)d\zeta = 0. \tag{6}$$

We shall assume that the set of conserved quantities J_A is complete in the sense that, in the absence of sources and of an electromagnetic field, the equilibrium state is completely characterized by the values of $\langle J_A \rangle$. The distribution function of the equilibrium state has the form

$$f_e = f_e(p^\alpha,r) = (2\pi)^{-3}\exp(F^A J_A). \tag{7}$$

The free parameters in Eq. (7) can be represented as follows:

$$F^\alpha = -\beta u^\alpha, \quad F^I = \beta\mu_i, \tag{8}$$

where β is the inverse temperature, u^α is the 4-velocity of the medium, and μ_i is the chemical potential of the ith component. The equilibrium state of Eq. (7) causes the collision integral to go to zero:

$$\mathrm{St}[f_e] = 0. \tag{9}$$

For this state to satisfy the complete system of equations for a plasma, Eqs. (2), (4), and (5), with zero sources and external currents, the additional condition of electrical neutrality must be satisfied:

$$\langle \Phi_\alpha \rangle_e = 0. \tag{10}$$

The equations of hydrodynamics follow from Eqs (5) and (6):

$$\partial_\alpha Q^\alpha_A = s_A + \Psi^{\alpha\beta}L_{\alpha\beta A},$$

$$Q^\alpha_A = Q^\alpha_A(x^\beta) = \langle J_A p^\alpha \rangle,$$

$$s_A = s_A(x^\alpha) = \int J_A(p^\beta,r)S(x^\alpha,p^\beta,r)d\zeta,$$

$$L_{\alpha\beta A} = \langle \Phi_\beta D_\alpha J_A \rangle.$$

We now investigate the dynamics of linear perturbations about some equilibrium distribution f_0 having the form Eq. (7), caused by weak sources and external currents. We assume the usual representation of the distribution function with linearization: $f=f_0(1+\varphi)$. Then Eq. (5) transforms to

$$(p^\alpha\partial_\alpha - L)\varphi = \beta_0 U^0 + s, \quad s = f_0^{-1}S, \tag{11}$$

where L is a linear operator, defined in terms of the functional derivative of the collision integral:

$$L\varphi = f_0^{-1}D\ \mathrm{St}[f_0](f_0\varphi).$$

The functions φ, considered from the viewpoint of the dependence on the arguments p^α, r, belong to the space H, which it is convenient to consider as a Hilbert space with the scalar product

$$(\varphi_1,\varphi_2) = \int f_0\varphi_1^*\varphi_2 d\zeta.$$

We denote by H_h the subspace spanned by the family of vectors J_A and by H_a the orthogonal complement to H_h:

$$H = H_h \oplus H_a.$$

The metric tensor $\gamma_{AB} = (J_A, J_B)$ is defined in the subspace H_h, and can be used to lower and raise the indices A, B, and C.

The operator L satisfies a number of conditions:

$$LJ_A = 0 \tag{12}$$

[the result of differentiating Eq. (9) with respect to the free parameters];

$$(L\varphi)^* = L\varphi^* \tag{13}$$

(reality);

$$L + L^+ \leqslant 0 \tag{14}$$

(dissipativity);

$$L^+ J_A = 0 \tag{15}$$

[a consequence of the conservation laws given by Eq. (6)].

On the subspace H_a, inequality (14) transforms into a rigorous inequality. Equations (12) and (13) imply $LH_a \subset H_a$ and $L^+ H_a \subset H_a$.

We assume that the interaction of the gas particles is reversible in time at the microscopic level (T-invariance in the quantum theory of Ref. 9). We define in the function space H the time-reversal operator I, which satisfies the conditions

$$I=I^{+}, \quad I^{2}=1, \quad Ip^{a}I=-p^{a}, \quad Ip^{0}I=p^{0}.$$

The integrals J_A are eigenfunctions of the operator I with eigenfunctions ± 1:

$$IJ_A=\varepsilon_A J_A, \quad \varepsilon_A=\pm 1.$$

We also assume that the perturbed equilibrium distribution is invariant with respect to time inversion:

$$If_0=f_0. \tag{16}$$

Then the reversibility at the microscopic level imposes on L the limitation

$$L^{+}=ILI. \tag{17}$$

In a wide class of cases, the operators L and I commute, and, therefore, instead of Eq. (17), the stronger condition $L^{+}=L$ is valid.

For what follows, it is essential that the following equation be satisfied:

$$(J_A,\Phi_a)=0, \tag{18}$$

and, therefore, the functions Φ_a lie within H_a. The proof of these equalities is given in Appendix A (see section A1).

We compute the linear perturbations of the 4-currents $g_A^{\alpha}=\Delta Q_A^{\alpha}$:

$$g_A^{\alpha}=(J_A p^{\alpha},\varphi). \tag{19}$$

As a consequence of Eqs. (11) and (18), the perturbations of the currents satisfy the equations

$$\partial_{\alpha} g_A^{a}=s_A. \tag{20}$$

To close the problem of Eqs. (2) and (20), it is necessary to give constitutive relations, i.e., for example, expressions for the induced 4-currents j_{in}^{α} and the spatial hydrodynamic currents g_A^{α}. To do this, let us study in more detail the kinetic equation, Eq. (11).

We assume that, for an arbitrary function $g=g(x^{\alpha})$, its Fourier transform $g_F(k_{\alpha})$ is defined by

$$g_F=g_F(k_{\alpha})=\int \exp(-ik_{\alpha}x^{\alpha})g(x^{\alpha})dx^{\alpha}.$$

Applying a Fourier transform to Eq. (11), we derive

$$G\varphi_F=s_A-\beta_0 E_{aF}\Phi_a, \quad G=ik_{\alpha}p^{\alpha}-L. \tag{21}$$

We introduce the auxiliary operators: $P_h:H\rightarrow H_h$ and $P_a:H\rightarrow H_a$ are projectors; $I_h:H_h\rightarrow H$ and $I_a:H_a\rightarrow H$ are embedding operators; $G_{hh}=P_hGI_h$, $G_{ah}=P_aGI_h$, $G_{ha}=P_hGI_a$, and $G_{aa}=P_aGI_a$. It is convenient to break up the desired function φ into a hydrodynamic part $h=P_h\varphi$ and a nonhydrodynamic part $a=P_a\varphi$. When this is expanded in the basis in H_h, we get the components $h_A=(J_A,h)$.

We now assume that the sources s in Eq. (11) as functions of the parameters p^{α}, r belong to the space H_h and, consequently, that they are completely characterized by the components s_A. This constitutes the key assumption of the method of Refs. 4–7. The following system of equations is then obtained from Eqs. (21) and (18):

$$G_{hh}h_F+G_{ha}a_F=s_F,$$

$$G_{ah}h_F+G_{aa}a_F=-\beta_0 E_{aF}\Phi_a.$$

Using the last equation, we can express a_F as a function of h and the electric field:

$$a_F=-\beta_0 E_{aF}G_{aa}^{-1}\Phi_a-G_{aa}^{-1}G_{ah}h_F. \tag{22}$$

Substituting φ into Eqs. (4) and (19) and using Eq. (22), we can obtain a representation for the hydrodynamic 4-currents g_A^{α} and the induced currents j_{in}^{α} in terms of the components h_A and the vector E_a. This corresponds to the constitutive relations in form A of Ref. 5. In order to compute these relations in compact form, it is convenient to introduce the coefficients

$$Z_{AB}^{a}=(p^{a}J_A,J_B), \quad R_{AB}^{\alpha\beta}=(P_a p^{\alpha}J_A,G_{aa}^{-1}P_a p^{\beta}J_B).$$

From Eqs. (19), (4), and (22), we derive the desired expressions:

$$g_{AF}^{\alpha}=\beta_0\sum_i e_i R_{AI}^{ab}E_{bF}+(Z_{AB}^{\alpha}-ik_{\beta}R_{AB}^{\alpha\beta})h_F^{B}, \tag{23}$$

$$j_{in\,F}^{\alpha}=\sum_i e_i\left(Z_{IA}^{\alpha}h_F^{A}+\beta_0\sum_j e_j R_{IJ}^{ab}E_{bF}-ik_{\beta}R_{IB}^{\alpha\beta}h_F^{B}\right). \tag{24}$$

Note that the expression for the hydrodynamic currents contains a term with the electric field, while the expression for the induced electric current depends on the hydrodynamic variables. This is characteristic of the electrodynamics of continuous media.[10] Because the dependences of the quantities $R_{AB}^{\alpha\beta}$ on the 4-dimensional wave vector k_{α} do not involve polynomials, the model of Eqs. (23) and (24) is nonlocal in space and in time. The constant coefficients Z_{AB}^{a} determine the reaction of the system to static actions. Therefore, dissipativity effects are described by the coefficient matrix $R_{AB}^{\alpha\beta}$.

The matrix $R_{AB}^{\alpha\beta}$ satisfies a number of conditions. Thus, because of Eq. (13), for a real 4-vector k_{α},

$$R_{AB}^{\alpha\beta}(k_{\gamma})^{*}=R_{AB}^{\alpha\beta}(-k_{\gamma}).$$

We further assume that the theory is invariant under the rotation group $O(3)$, which means that the medium is isotropic and that parity is conserved during collisions. We then have

$$(\rho(g)R)_{AB}^{\alpha\beta}(k_0,gk_b)=R_{AB}^{\alpha\beta}(k_0,k_b) \tag{25}$$

for any $g\in O(3)$. Here ρ is a representation of the $O(3)$ group in the linear space Φ of quantities of the form $R_{AB}^{\alpha\beta}$. Let us identify in Φ the maximal set of linearly independent invariants with respect to the subgroup that conserves the 3-vector k_b:

$$I^{n}=I_{AB}^{n\alpha\beta}(k_c).$$

We choose these invariants so that

$$(I_{AB}^{n\alpha\beta}(k_c))^* = I_{AB}^{n\alpha\beta}(-k_c).$$

Then the most general representation consistent with Eq. (25) has the form

$$R_{AB}^{ab} = I_{AB}^{nab} X_n. \tag{26}$$

Here $X_n = X_n(k_\alpha)$ are scalar functions that satisfy the condition

$$(X_n(k_\alpha))^* = X_n(-k_\alpha).$$

Because of this, they are the Fourier transforms of certain real kernels $Y_n = Y_n(x^\alpha)$:

$$Y_{nF} = X_n.$$

It will be proven below that the functions Y_n vanish outside the light cone of the future (causality).

A specific form of the expansion in Eq. (26) is given in Appendix A (see section A2). Next, substituting Eq. (26) into Eq. (23) and considering the limit $k_\alpha \to 0$, we can identify the terms that correspond to bulk and shear viscosity, thermal conductivity, and diffusion in the expressions for the spatial components of the hydrodynamic currents. The classical transfer coefficients can thus be expressed in terms of scalar series of X_n. These results are given in Appendix A (see section A3).

It is easy to verify that

$$R_{AB}^{\alpha\beta} + R_{BA}^{\beta\alpha *} = -(P_a p^\alpha J_A, G_{aa}^{-1+}(L+L^+)G_{aa}^{-1} P_a p^\beta J_B).$$

From this and from the condition given by the inequality (14), the conditions for dissipativity in nonlocal hydrodynamics follow:

$$(R_{AB}^{\alpha\beta} + R_{BA}^{\beta\alpha *}) C_\alpha^A C_\beta^{B*} \geqslant 0, \tag{27}$$

where the C_α are arbitrary complex quantities. Next, the reciprocity relationships (an analog of the Onsager relations) follow from the condition in Eq. (17):

$$R_{AB}^{\alpha\beta}(k_0, k_c) = \varepsilon_A \varepsilon_B \varepsilon_\alpha \varepsilon_\beta R_{BA}^{\beta\alpha}(k_0, -k_c). \tag{28}$$

By substituting the expansion in Eq. (26) into conditions (27) and (28), we can obtain the corresponding limitations on scalar fields.

There are several other *a priori* conditions on the coefficients $R_{AB}^{\alpha\beta}$ associated with the possible degeneracy of the model. Thus, although the functions J_A are linearly independent by definition, the set of functions J_A and $p^\alpha J_A$ can be linearly dependent. From each equality of the form

$$\lambda^A J_A + \Lambda_\alpha^A p^\alpha J_A = 0, \tag{29}$$

where λ^A and Λ_α^A are constant coefficients, it follows that

$$\Lambda_\alpha^A R_{AB}^{\alpha\beta} = 0, \quad \Lambda_\beta^B R_{AB}^{\alpha\beta} = 0. \tag{30}$$

It is easy to indicate one identity of the form of Eq. (29):

$$\eta_{\alpha\beta} p^\beta \sum_i J_I - J_\alpha = 0,$$

from which follow relationships of the form of Eq. (30):

$$\sum_i R_{IB}^{\alpha\beta} = 0, \quad \sum_i R_{AI}^{\alpha\beta} = 0. \tag{31}$$

If there are no internal degrees of freedom r', there is one more identity of the form of Eq. (29):

$$p^\alpha J_\alpha - \sum_i m^2(i) J_I = 0,$$

which leads to

$$R_{\alpha B}^{\alpha\beta} = 0, \quad R_{A\beta}^{\alpha\beta} = 0. \tag{32}$$

There is interest in considering the Fourier transform of the conductivity tensor σ_F^{ab} the expression for which is obtained from Eq. (24),

$$\sigma_F^{ab} = \beta_0 \sum_i e_i \sum_j e_j R_{IJ}^{ab}. \tag{33}$$

In accordance with the dissipativity and reciprocity conditions of Eqs. (27) and (28), σ_F^{ab} is a symmetric, nonnegative complex definite matrix. The permittivity of the plasma can be calculated from the conductivity tensor in the usual way.[12] In the limit $k_\alpha \to 0$, the expression makes it possible to obtain the static conductivity σ, which is related to the matrix of diffusion coefficients in the usual way (section A4):

$$\sigma = \sum_i e_i \sum_j e_j D_{IJ}.$$

Let us discuss the questions associated with the dissipativity and causality of the hydrodynamic model.

There are two aspects of causality: the causality of the constitutive relations given by Eqs. (23) and (24) and the causality of the dynamic model of Eqs. (2) and (20). To analyze causality, it is convenient to use the method of Ref. 13, which is associated with analytic continuation in the variables k_α into a complex tube:

$$\operatorname{Im} k_0 < 0, \quad \operatorname{Im} k_\alpha \operatorname{Im} k^\alpha > 0. \tag{34}$$

It is easy to see that the coefficients $R_{AB}^{\alpha\beta}$ are analytical functions in the tube (34). In fact, if the wave 4-vector is decomposed into real and imaginary parts $k_\alpha = \alpha_\alpha + i\beta_\alpha$, then

$$G_{aa} + G_{aa}^+ = -I_a(2\beta_\gamma p^\gamma + L + L^+) P_a,$$

and the analyticity of $R_{AB}^{\alpha\beta}$ in tube (34) follows from Eq. (14) and the fact that $\beta_\gamma p^\gamma < 0$. Moreover, from the same reason, for any set of complex quantities C_α^A the inequality

$$(R_{AB}^{\alpha\beta} + R_{BA}^{\beta\alpha *}) C_\alpha^A C_\beta^{B*} \geqslant 0 \tag{35}$$

is satisfied in the tube (34).

Thus, the coefficient functions $R_{AB}^{\alpha\beta}$ are analytical in the tube (34) and are continuous on its boundary. It immediately follows from this[13] that these functions are the Fourier transforms of certain inheritance–nonlocality kernels that go to zero outside the light cone:

$$x^0 \geqslant 0, \quad x_\alpha x^\alpha \geqslant 0. \tag{36}$$

This means that the constitutive relations given by Eqs. (23) and (24) satisfy relativistic causality.

We now consider the propagation of the signal from an event that occurred at the space–time point $x^\alpha=0$ when external electric currents are absent. Then, from Eqs. (20) and (2), we have the system of equations

$$ik_\alpha g^\alpha_{AF}=s_A, \tag{37}$$

$$\varepsilon_{abc}ik_bE_{cF}=-ik_0B_{aF}, \quad ik_aB_{aF}=0, \tag{38}$$

$$\varepsilon_{abc}ik_bB_{cF}=4\pi j^a_{in\ F}+ik_0E_{aF}, \quad ik_aE_{aF}=4\pi j^0_{in\ F}. \tag{39}$$

Here s_A is a set of numbers that characterizes the past event. We assume that the source does not disturb the electrical neutrality:

$$\sum_i e_is_I=0. \tag{40}$$

In consequence of

$$j^\alpha_{in}=\sum_i e_ig^\alpha_I,$$

the consistency condition, Eq. (3), of Maxwell's equations, Eqs. (38) and (39), automatically follows from Eqs. (37) and (40). Since there is no magnetic field in the constitutive relations, Eqs. (23) and (24), it is sufficient to study the system of equations (37) and the equation for the electric field following from Eqs. (38) and (39):

$$i(4\pi k_0)^{-1}(k_0^2E_{aF}-k_bk_bE_{aF}+k_ak_bE_{bF})+j^a_{in\ F}=0. \tag{41}$$

We shall interpret Eqs. (37) and (41) as a system of linear equations for the set of unknown values h^A_F, E_{aF} of the form

$$Ay=z, \tag{42}$$

where $A=(A_{nm})$ is the matrix of the system. It can be seen that, because of the inequality (35) in tube (34), the inequality

$$\mathrm{Re}(A_{nm}C^{n*}C^m)>0 \tag{43}$$

is satisfied for any nonzero set of numbers C^n.

Thus, the solution $y=A^{-1}z$ of the system of Eqs. (42) is determined and analytical in the tube (34). Then, according to Ref. 13, in real space–time, the perturbations of the hydrodynamic quantities and the electromagnetic field go to zero outside the light cone given by the inequalities (36). The relativistic causality of the model has been proven.

Now let us consider the dynamics of the free plasma oscillations. As a condition for the existence of a nontrivial solution of the homogeneous Eqs. (42), we obtain the dispersion relation

$$\det A=0. \tag{44}$$

We shall assume that a real wave 3-vector k_a is given. Then Eq. (44) gives the set of frequencies k_0 of the normal modes of the plasma as a function of the wave number. As a consequence of the inequality (43), solutions with $\mathrm{Im}k_0<0$, corresponding to exponentially increasing modes, are *a fortiori* absent. Since the eigenfrequencies k_0 are associated with the wave number by an analytical dependence, it is possible for the imaginary part, $\mathrm{Im}k_0$, to go to zero only for discrete values of the wave number. In general, the normal vibrations of the plasma damp out exponentially. This implies that the hydrodynamic model is dissipative.

A nonlocal dissipative hydrodynamic model of a relativistic plasma has thus been constructed. The transition from the kinetic theory to hydrodynamics in the class of sources under consideration is exact, since, by solving the self-consistent problem for the electromagnetic field and the hydrodynamic variables given by Eqs. (2), (20), (23), and (24), it is possible to reconstruct the distribution function [see Eq. (22)]. Note that, at a qualitative level, the causality of the hydrodynamic model proven above is a direct consequence of the possibility of the inverse transition to kinetic theory. In fact, if hydrodynamics allowed a process with superluminal signal velocity, it would be possible to find a corresponding process in kinetic theory. The latter contradicts the causality of relativistic kinetic theory.[5]

The method explained here of going to nonlocal hydrodynamics can also be used in the case of a nonrelativistic plasma. The procedure of such a transition is briefly explained in Ref. 14.

Of course, the proposed hydrodynamic model can be used to solve specific problems only if the nonlocality kernels are known explicitly. Two approaches are possible here.

First, it is possible to attempt to compute the resolvent operator G^{-1}_{aa} in explicit form from a given collision integral, and then to determine the coefficients $R^{\alpha\beta}_{AB}$. This is the direct method. It is hard to implement because, for all the collision integrals that are of interest in practice, the operator G^{-1}_{aa} cannot be represented by an analytical expression. However, there is usually a numerical method of solving the problem by means of which the matrix elements of the resolvent operator are calculated numerically to any required accuracy and for any wave 4-vectors k_α. Certain features of the corresponding algorithms are explained in Ref. 15.

Second, a phenomenological approach is possible, in which model expressions are used for the kernels, with a set of free parameters. The values of the latter are chosen from comparison with experiment. Such a method of solving the problem is widely used in all areas of physics and mechanics where, on one hand, it is impossible to do without the kernels, while, on the other hand, the procedure of computing the kernels from first principles (plasticity, turbulence, nonperturbative nonlinear quantum field theory) is unknown. In this case, the model kernels must exactly satisfy the *a priori* relationships obtained in the rigorous theory.

In each specific problem of plasma physics, the range of frequencies and of wave numbers of interest is usually known. Using a hierarchy of interaction processes between the plasma components (corresponding to the hierarchy of the relaxation times), in several cases, it is possible in a given region of values of 4-vectors k_α to consider certain nonlocality kernels trivial, i.e., constant in momentum space or a delta-function in space–time. The resulting hydrodynamic models are less general than is the exact model of Eqs. (2), (20), (23), and (24).

The situation in which the distribution function differs only slightly from the equilibrium function was considered above. The nonlocal hydrodynamic description of a plasma can also be extended to the strongly nonequilibrium case. The detailed theory lies outside the limits of this paper, and only the main ideas will be indicated here. An arbitrary distribution function can be represented as

$$f=f_h+f_a,$$

where f_h is the local-equilibrium distribution function, unambiguously determined by the condition

$$0=\int J_A(p^\beta,r)f_a(x^\alpha,p^\beta,r)d\zeta.$$

We assume that the sources in Eq. (5) belong to the class of variations of the local-equilibrium distribution f_h. In complete analogy with the linear case, it is possible to transform exactly from a kinetic to a hydrodynamic description. The constitutive relations will have the form of complex nonlinear functionals of the electromagnetic field and of the quantities Q_A^0. The procedure for computing these functionals has not been fully developed.

APPENDIX A:

1. Proof of Eq. (18)

Because of the conditions given by Eq. (16) for the equilibrium state f_0, the parameters in Eqs. (8) satisfy the supplementary condition $u^a=0$; i.e., f_0 is a rest state. It is easy to see that Eq. (18) is satisfied for the case in which $A\neq a$ by making the replacement $p^a\rightarrow -p^a$ in the integral when computing the scalar product. Now let $A=a$. In the subsequent formulas, the summation over a is not carried out. Using integration by parts, we get

$$(J_a,\Phi_a)=-\beta_0^{-1}\sum_i e_i(2\pi)^{-3}$$
$$\times\exp(\beta_0\mu_i)\int d\zeta'(r')\int p^aD_a$$
$$\times\exp(-\beta_0p^0)dp^1dp^2dp^3=\beta_0^{-1}\langle\Phi^0\rangle_0.$$

The last expression equals zero because of Eq. (10).

2. Finding an explicit representation of Eq. (26)

It is technically more convenient to analyze the coefficient matrix

$$R_{\alpha A|\beta B}=\eta_{\alpha\gamma}\eta_{\beta\delta}R_{AB}^{\gamma\delta}.$$

We note the obvious symmetry conditions,

$$R_{\alpha\gamma|\beta B}=R_{\gamma\alpha|\beta B},\quad R_{\alpha A|\beta\gamma}=R_{\alpha A|\gamma\beta}. \tag{A1}$$

Because of Eqs. (28) and (A1), it is sufficient to consider the following quantities: $R_{00|00}$, $R_{00|0a}$, $R_{00|ab}$, $R_{00|0I}$, $R_{00|aI}$, $R_{0a|0b}$, $R_{0a|bc}$, $R_{0a|0I}$, $R_{0a|bI}$, $R_{ab|cd}$, $R_{ab|0I}$, $R_{ab|cI}$, $R_{0I|0J}$, $R_{0I|aJ}$, and $R_{aI|bJ}$. We have, successively,

$$R_{00|00}=X_0,\quad R_{00|0a}=ik_aX_1,$$

$$R_{00|ab}=\delta_{ab}X_2+ik_aik_bX_3,\quad R_{00|0I}=X_{4\,I},$$

$$R_{00|aI}=ik_aX_{5\,I},\quad R_{0a|0b}=\delta_{ab}X_6+ik_aik_bX_7,$$

$$R_{0a|bc}=ik_a\delta_{bc}X_8+ik_aik_bik_cX_9+(\delta_{ab}ik_c+\delta_{ac}ik_b)X_{10},$$

$$R_{0a|0I}=ik_aX_{11\,I},\quad R_{0a|bI}=\delta_{ab}X_{12\,I}+ik_aik_bX_{13\,I},$$

$$\begin{aligned}R_{ab|cd}=&\ \delta_{ab}\delta_{cd}X_{14}+(ik_aik_b\delta_{cd}+\delta_{ab}ik_cik_d)X_{15}\\&+(\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc})X_{16}+(\delta_{ac}ik_bik_d\\&+\delta_{ad}ik_bik_c+\delta_{bc}ik_aik_d+\delta_{bd}ik_aik_c)X_{17}\\&+ik_aik_bik_cik_dX_{18},\end{aligned}$$

$$R_{ab|0I}=\delta_{ab}X_{19\,I}+ik_aik_bX_{20\,I},$$

$$\begin{aligned}R_{ab|cI}=&\ ik_c\delta_{ab}X_{21}+ik_aik_bik_cX_{22},\\&+(\delta_{ac}ik_b+\delta_{bc}ik_a)X_{23},\end{aligned}$$

$$R_{0I|0J}=X_{24\,I\,J},\quad R_{0I|aJ}=ik_aX_{25\,I\,J},$$

$$R_{aI|bJ}=\delta_{ab}X_{26\,I\,J}+ik_aik_bX_{27\,I\,J}.$$

We should point out certain consequences of the conditions given by inequalities (27), where $\lambda=k_ak_a$:

$$\mathrm{Re}\,X_6\geqslant 0,\quad \mathrm{Re}\,X_{16}\geqslant 0,\quad \Gamma+\Gamma^+\geqslant 0,\quad \Gamma_{IJ}=X_{26\,I\,J}, \tag{A2}$$

$$\mathrm{Re}\,(9X_{14}-6\lambda X_{15}+6X_{16}-4\lambda X_{17}+\lambda^2X_{18})\geqslant 0. \tag{A3}$$

3. The classical transport coefficients

The connection of the scalar kernels with the classical transport coefficients was studied in detail in Ref. 5. In this paper, Eckhart's definition[1] (in terms of the total particle flux) was used for the velocity of the medium. If the notation of this paper is compared with that of Ref. 5, the following expressions are obtained for the thermal conductivity κ and the coefficients of shear viscosity η_S and bulk viscosity η_V:

$$\kappa=(\beta_0)^2X_6|_{k_\alpha=0},$$

$$\eta_S=X_{16}|_{k_a=0},\quad \eta_V=\left.\left(\frac{2}{3}X_{16}+X_{14}\right)\right|_{k_\alpha=0}.$$

The nonnegative right-hand sides in these equations follow from Eqs. (A2) and (A3). If there are no internal degrees of freedom and Eqs. (32) are consequently valid, the corresponding conditions on the scalar kernels lead to the well-known result $\eta_V=0$.[11] If it is assumed that the diffusive fluxes are generated by gradients of the chemical potentials, using the results of Ref. 5, we again obtain the matrix of diffusion coefficients:

$$D_{ij}=\beta_0X_{27\,I\,J}|_{k_\alpha=0}. \tag{A4}$$

According to Eqs. (28), (31), and (A2), this matrix is symmetric and positive definite and obeys the equations

$$\sum_i D_{ij}=0,\quad \sum_j D_{ij}=0.$$

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