Stationary solution of the Rayleigh–Taylor instability for spatially periodic flows: questions of uniqueness, dimensionality, and universality

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The stationary solutions of the Rayleigh–Taylor instability for spatially periodic flows with general symmetry are investigated here for the first time. The existence of a set of stationary solutions is established. The question of its dimensionality in function space is resolved on the basis of an analysis of the symmetry of the initial perturbation. The interrelationship between the dimensionality of the solution set and the symmetry of the flow is found. The dimensionality of the solution set is established for flows invariant with respect to one of five symmorphic two-dimensional groups. The nonuniversal character of the set of stationary solutions of the Rayleigh–Taylor instability is demonstrated. For flows in a tube, on the contrary, universality of the solution set, along with its independence of the symmetry of the initial perturbation, is assumed. The problem of the free boundary in the Rayleigh–Taylor instability is solved in the first two approximations, and their convergence is investigated. The dependence of the velocity and Fourier harmonics on the parameters of the problem is found. Possible symmetry violations of the flow are analyzed. Limits to previously studied cases are investigated, and their accuracy is established. Questions of the stability of the solutions obtained and the possibility of a physically correct statement of the problem are discussed.
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1. INTRODUCTION

The Rayleigh–Taylor instability plays an important role in hydrodynamics, the physics of high energy densities, astrophysics, etc. Such an instability arises in a compressible medium when the pressure and density gradients are oppositely directed.[1] The simplest case of the Rayleigh–Taylor instability is the instability in a uniform gravity field of the free surface separating a heavy upper liquid from a lighter lower liquid. The density of the lower liquid is frequently taken to be equal to zero, and the upper liquid is assumed to be incompressible. Then (see Ref. 2) the linear stage of the development of this instability with growth rate $\sqrt{gk}$, where g is the acceleration, $k=2\pi/\lambda$, and λ is the wavelength of the perturbation, is quite brief, and even for amplitudes of the perturbation of the order of $\lambda/10$ nonlinear effects begin to act: a periodic system of bubbles and jets forms (see Fig. 1), where the motion of a bubble becomes asymptotically stationary.[2] The problem of finding the stationary solution of the Rayleigh–Taylor instability is considered to be one of the classical problems of fluid dynamics.[3] The question of its uniqueness, first posed by Garabedian,[4] received quantitative treatment only recently by Inogamov and Chekhlov,[5] who established the existence of a one-parameter family of solutions of this problem for a two-dimensional periodic system of bubbles and jets. The bulk of previous studies[2–5] were restricted, however, to planar flows in light of the difficulties that are met in the three-dimensional problem.[6] The first analytical study of spatial flow and proof that it has a one-parameter family of stationary solutions was carried out by Abarzhi and Inogamov,[7] who emphasized the importance of the symmetry of three-dimensional flow in the stationary problem. The present paper is dedicated to a general analysis of the question of the uniqueness of stationary spatial flow in the Rayleigh–Taylor instability.

2. THE SET OF STATIONARY SOLUTIONS OF THE RAYLEIGH–TAYLOR INSTABILITY

Let us consider the asymptotic ($t\to\infty$) stage of development of the Rayleigh–Taylor instability, namely stationary flow of the incompressible liquid with potential $\Phi(x,y,z)$ satisfying

$$\Delta\Phi=0,\quad \nabla\Phi|_{z=+\infty}=0$$

and the conditions on the free surface of the liquid $z=z^*(x,y)$

$$\begin{aligned} -\frac{\partial\Phi}{\partial t}\bigg|_{z=z^*(x,y)} &= \frac{1}{2}(\nabla\Phi(x,y,z))^2+gz|_{z=z^*(x,y)}=0,\\ -\frac{\partial z^*(x,y)}{\partial t} &= \nabla z^*(x,y)\nabla\Phi(x,y,z)|_{z=z^*(x,y)}=0. \end{aligned} \tag{1}$$

Let the perturbation at time $t=0$ be periodic in the xy plane. This translational invariance leads in the limit $t\to\infty$ to the formation of a spatially periodic flow of bubbles and jets[2,3,5–7] (see Fig. 2). The condition of periodicity of the flow is very important for the question of uniqueness of the stationary solution of system of equations (1). Indeed, the problems flow in a tube and spatially periodic flow, despite their equivalence from the point of view of finding the solution, are physically different. For example, the spatially periodic, stationary solutions turn out to be unstable against period doubling,[2,6] which in principle is impossible for flow in a tube. The correct transition from periodic flow to flow in

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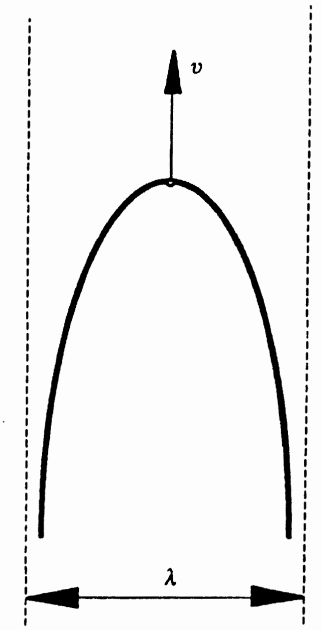

FIG. 1. Stationary motion—the nonlinear stage of the Rayleigh–Taylor instability.

a tube consists in introducing an infinitesimally small viscosity, which at once limits the number of possible solutions.[8]

We further assume that the initial perturbation at the time $t=0$ is invariant with respect to any symmetry group **G**. The total number of such two-dimensional Fedorov groups containing the subgroup of translations in the plane is equal to 17 (Ref. 9). It is clear that the symmetry of the spatial flow at times $t\approx 0$ is determined by the star $\{\mathbf{k}^*\}$ of that one of the wave vectors allowed by **G** that is associated with the largest growth rate. A general study of the further development of the Rayleigh–Taylor instability for all 17 spatial groups, however, goes beyond the scope of the present work.

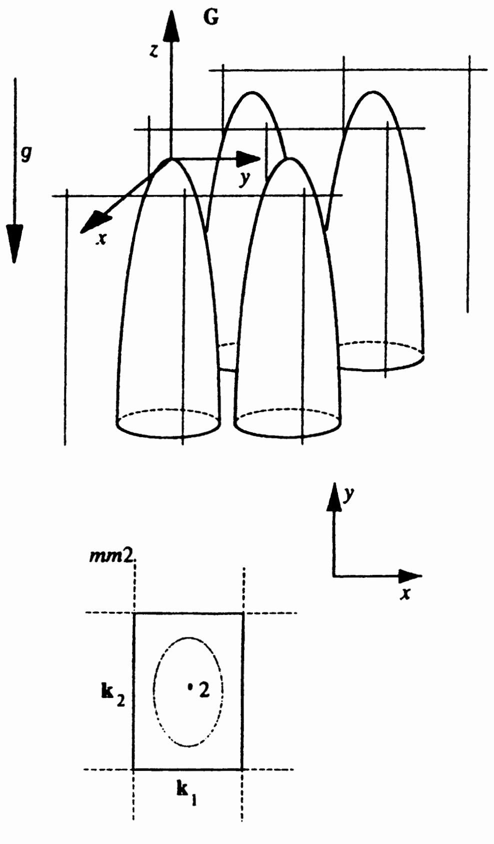

FIG. 2. Spatially period flow. Spatial two-dimensional group **G** of flows $pmm2$. Independent translations $\mathbf{k}_1$, $\mathbf{k}_2$, where $\mathbf{k}_2 \neq \mathbf{G}[\mathbf{k}_1]$.

Let us analyze the changes in the symmetry of a flow that is invariant at $t=0$ with respect to one of the five symmorphic two-dimensional groups

$$p1,\, p2,\, pmm2,\, p4mm,\, p6mm. \tag{2}$$

It is easy to show that for any of these groups the largest growth rate of the initial perturbation will correspond to the star of the wave vector, $\mathbf{k}^*=\mathbf{0}$. If at $t=0$ there is only one point per unit cell, with a maximum extremal value of the velocity, then at times $t\approx 0$ the symmetry of the flow will not change. Moreover, under the natural condition that different representations of the group **G** not mix at times $t\rightarrow\infty$ the stationary three-dimensional flow will be invariant with respect to the same spatial group **G**.

We will now adduce some qualitative arguments in favor of the existence of a continuum of solutions of the problem which are invariant with respect to one of the groups (2), and on behalf of a connection between its dimensionality and its symmetry.

We assume that we have stationary flow, periodic in the xy plane, that this flow is unbroken, and that the free boundary $z^*(x,y)$ is smooth. Since one of the boundary conditions of the problem $(\nabla\Phi|_{z=+\infty}=0)$ is degenerate, such a flow is completely determined by prescribing the spatial period $\hat{\mathbf{k}}$ and the velocity of the stationary flow.[8] Equations (1), however, contain another independent physical parameter—the acceleration g, which has the dimensions v^2k and is therefore free.[4,5,7] This means that the stationary periodic solution is not unique. Each of this set of nonunique solutions has its own Froude number $v/\sqrt{g\lambda}$. From physical considerations it is clear that of these solutions those that are singular and "finite" will be first of all those whose Froude numbers are bounded by values of order unity, and secondly "isolated jet flows."[7,5]

What is the dimensionality of the set of stationary solutions? Note that the problem (the Laplace equation with boundary conditions) allows, generally speaking, the existence of solutions of the most general type. Their number can be restricted only by requiring invariance of the flow with respect to one of the groups (2). Obviously, the flow (or solution) will be symmetric with respect to one of the groups (2) (call it **G**) if both the potential Φ and the free surface $z^*(x,y)$ are invariant with respect to this group. The lower the symmetry the group **G** possesses, the larger, generally speaking, should be the set of possible solutions. Thus, to describe plane-periodic flow that is invariant with respect to either the group containing a fourth-order axis ($p4mm$) or the group containing a sixth-order axis ($p6mm$), requires that we prescribe one independent translation **k**. The conditions for invariance of the potential Φ and the free surface $z^*(x,y)$ with respect to any of these groups lead to the result that the continuum of stationary solutions will be described by three independent physical parameters: the acceleration g, a unique, characteristic period $\mathbf{k}$, and the velocity v. The set of stationary solutions thus turns out to be a one-parameter set.[7] Each of these solutions is prescribed by its free surface $z^*(x,y)$, with which its Froude number $v/\sqrt{g\lambda}$ in Eqs. (1) is uniquely associated. It is possible (see Ref. 7) to parametrize these free surfaces by one of the radii of curvature at the top of a bubble (the second radius is connected with the first by a relation that depends only on the symmetry). Then one of the two end-points of the one-parameter family of stationary solutions will restrict the Froude numbers to values of order unity, and the other will correspond to an isolated jet with close-to-infinite radius of curvature of the bubble.

If the periodic flow is invariant with respect to the groups $p1$, $p2$, $pmm2$, then its description will require the assignment of two independent translations $\mathbf{k}_1$, $\mathbf{k}_2$ ($\mathbf{k}_2 \neq \mathbf{G}[\mathbf{k}_1]$) in the xy plane (see Fig. 2). The set of stationary solutions in this case turns out to be a two-parameter set: among the physical dimensioned parameters of the problem, besides the acceleration g and velocity v, there will be not one, but two independent periods $\mathbf{k}_1$ and $\mathbf{k}_2$. How do the stationary solutions of this two-parameter family differ? As in the previous case of high-symmetry flows, the free surface in Eqs. (1), as before, is not uniquely determined by $\mathbf{k}_1$ and $\mathbf{k}_2$; however, now the symmetry relations do not impose any restrictions on its form. If, for example, we parametrize the family of stationary solutions by the radii of curvature at the top of the bubble, then they will not be connected by a symmetry relation (see Fig. 2). The points in the plane of these two independent variables will completely describe all permitted (by Eqs. (1)) free surfaces. Each of these points will have uniquely associated with it its own potential and, by virtue of Eqs. (1), its own velocity and Froude number. In analogy with the single-parameter family, the two-parameter family of stationary solutions should have two end-regions (not points, as for a one-parameter family). One of them will correspond to flows with close-to-zero isolated-jet Froude numbers. The region of physically permissible parameter values will be restricted, as before, by the terminal solutions with Froude numbers of order unity.

Also note that the previously investigated planar flows,[2–6] where the absence of an initial perturbation along one of the independent translations in the xy plane leads in the asymptotic stage to a periodic chain of bubbles and jets, may be considered as a limiting case of three-dimensional flows with symmetry $p1$, $p2$, $pmm2$. These planar flows are symmetrically distinct (their group is $pm11$) and should form a one-parameter subfamily of the two-parameter set.

Finally, the importance of translational invariance in the stationary problem (1) must be emphasized: in contrast to the periodic problem, flow in a tube, like viscous flow, is uniquely determined by one linear dimension and its geometry, i.e., by the ratio k_2/k_1 (Ref. 8). Therefore the dimensionality of the solution set for flow in a tube with broken symmetry should remain unchanged.

That a stationary periodic flow (1) should not be uniquely determined by the values $\{g,\mathbf{k}\}$ is of course unphysical and shows that the problem has been incorrectly formulated. In our above description of the asymptotic stage we proceeded from general considerations based on the symmetry of the initial perturbation. This, however, does not allow us to draw any conclusions about the connection between any of the stationary solutions and the initial data (roughly speaking, that is to say, the amplitude of the initial perturbation, in analogy with a Stokes wave[9]). In addition, a determination of the stability of each of the stationary solutions with allowance for the surface tension (questions which are not considered in this work) will possibly permit a physically correct statement of the problem.

3. STATIONARY FLOWS

Let the initial perturbation be invariant with respect to the group $pmm2$, see Fig. 2. We seek the stationary solution of (1) in the form of an expansion of the potential in eigenfunctions of the Laplace operator. In a coordinate system comoving with the bubble with constant velocity v

$$\Phi=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Phi_{mn}\times\left(\cos mk_1x\cos nk_2y\,\frac{\exp(-\sqrt{k_1^2m^2+k_2^2n^2}\,z)}{\sqrt{k_1^2m^2+k_2^2n^2}}+z\right), \tag{3}$$

where k_1 and k_2 are the wave vectors along the x and y axes, respectively, and $\hat{\Phi}$ is the matrix of Fourier amplitudes, with $\Phi_{mn}\neq\Phi_{nm}$ and $\Phi_{00}=0$. In the comoving coordinate system, the vertices of the bubble are the braking points, and the rising speed of the bubbles in the laboratory frame is $v=-\Sigma_{m,n}\Phi_{mn}$. The symmetry $pmm2$ leaves arbitrary the value of k_3—the inverse length in the z direction. In Eqs. (1) and (3) it is easy to take limits to arrive at the cases investigated in Refs. 5 and 7, of planar flow ($k_1=k$, $k_2=0$, and $\Phi_{mn}=0$ for all $n\neq0$) and flow with $p4mm$ symmetry ($k_1=k$, $k_2=k$, and $\Phi_{mn}=\Phi_{nm}$).

The stationary flow (1) is assumed to be unbroken, and the free surface continuous. Therefore the potential (3) and boundary conditions (1) can be expanded near a braking point in the unit cell. We note at once that such an approach to a solution does not allow us to consistently introduce surface tension and viscosity into the problem or to obtain (in contrast to Ref. 5, also see footnote in Ref. 7) an analytic solution in higher approximations.

The expansion of the boundary conditions near the vertex (0,0,0) of the bubble has the form

$$\sum_{s,p,q}\overline{(spq)}x^{2s}y^{2p}z^q=0,\quad \sum_{s,p,q}(spq)x^{2s}y^{2p}z^q=0 \tag{4}$$

respectively for the Bernoulli equation and the equation describing the absence of flow through the free boundary. We next expand the free surface in an even power series in x and y. Let $z_d=\Sigma_{i,j}\gamma_{ij}x^{2i}y^{2j}$ correspond to the first, and $z_k=\Sigma_{i,j}\beta_{ij}x^{2i}y^{2j}$ to the second of Eqs. (4). Then the equalities $\gamma_{ij}=\beta_{ij}$ for all i and j determine the values of the matrix elements Φ_{mn} and the shape of the free surface $z^*=z_d=z_k$.

In what follows it will be convenient to transform to dimensionless coordinates $k_1x\to x$, $k_2y\to y$, $k_3z\to z$, $v/\sqrt{g/k_3}\to v$, $\Phi_{mn}/\sqrt{g/k_3}\to\Phi_{mn}$.

As in Ref. 7, we introduce the moments $\{M\}$ as functions of the Fourier amplitudes:

$$M(\alpha,\beta,\gamma)=\sum_{m,n}\Phi_{mn}m^{\alpha}n^{\beta}\left(\sqrt{\left(\frac{k_1}{k_3}\right)^2m^2+\left(\frac{k_2}{k_3}\right)^2n^2}\right)^{\gamma}, \tag{5}$$

$$M(0,0,0)=-v. \tag{5a}$$

We then have

$$\left(\frac{k_1}{k_3}\right)^2M(\alpha+2,\beta,\gamma-1)+\left(\frac{k_2}{k_3}\right)^2M(\alpha,\beta+2,\gamma-1)$$
$$=M(\alpha,\beta,\gamma+1). \tag{6}$$

As can be easily seen from Eqs. (1), the coefficients $(\overline{spq})$ are quadratic in the moments, and the coefficients (spq) depend on β_{ij} and linearly on the moments (see Appendix A). We choose from among the moments $\{M\}$ the linearly independent moments $M(2,0,-1)\equiv M_{1x}$, $M(0,2,-1)\equiv M_{1y}$, $M(2,0,0)\equiv M_{2x}$, $M(0,2,0)\equiv M_{2y}$ (first-order moments) and $M(4,0,0)$, $M(0,4,0)$, $M(2,2,0)$, $M(2,2,-1)$, $M(4,0,-1)$, $M(0,4,-1)$ (these moments appear in the second order of the expansions (4)). Expressing the functions $(\overline{spq})$ and (spq) entering into expansions (4) in terms of them, and then β_{ij} and γ_{ij} (see Appendix B), we easily obtain expressions for the equations $\gamma_{ij}=\beta_{ij}$.

Note that the dimensional expressions for the moments $\{M\}$, the equations $\gamma_{ij}=\beta_{ij}$ for all i and j, and their solutions—the radii of curvature, Fourier amplitudes, and velocity—do not contain any dependence on the free (for $pmm2$ symmetry) inverse length in the z direction k_3.

Thus,

$$\gamma_{10}=\beta_{10}:\to3\left(\frac{k_1}{k_3}\right)^4M_{1x}^3+\left(\frac{k_1}{k_3}\right)^2\left(\frac{k_2}{k_3}\right)^2M_{1x}^2M_{2y}-M_{2x}=0,$$
$$\gamma_{01}=\beta_{01}:\to3\left(\frac{k_2}{k_3}\right)^4M_{1y}^3+\left(\frac{k_2}{k_3}\right)^2\left(\frac{k_1}{k_3}\right)^2M_{1y}^2M_{1x}-M_{2y}=0. \tag{7}$$

The equations $\gamma_{ij}=\beta_{ij}$ for $i+j=2$ are derived in Appendix C. In the system $\gamma_{ij}=\beta_{ij}$, $i+j=n=1, 2, \dots,\infty$, Eqs. (7) for $i+j=1$ are inhomogeneous in the moments. As in Ref. 7, it is easy to show that the equations of the higher approximations for $i+j\geqslant2$ are homogeneous in the moments. Setting $k_1=k_2=k_3$ and $M(\alpha,\beta,\gamma)=M(\beta,\alpha,\gamma)$ in the system $\gamma_{ij}=\beta_{ij}$, $i+j=N=1, 2, \dots,\infty$ (see Eqs. (7) and Appendix C), we arrive at the flow with symmetry $p4mm$ (Ref. 7). For $M(\alpha,\beta,\gamma)=0$ for arbitrary $\beta\neq0$, $\gamma\neq0,-1$ and $k_1=k_3$, $k_2=0$, the equations $\gamma_{ij}=\beta_{ij}$ describe planar flow.[5]

Thus, we will seek the solutions of problem (1) by successive approximations in the function space $\hat{\Phi}$. The equations $\gamma_{ij}=\beta_{ij}$, $i+j\leqslant N$, derived in terms of the moments $\{M\}$, do not contain any additional parameters. Free parameters can be introduced in them only through the choice of variables—the Fourier amplitudes for approximation of the boundary conditions to Nth order. The number of these parameters (0, 1, 2, . . .) should be such that, first, the solutions of the systems of equations $\gamma_{ij}=\beta_{ij}$, $i+j\leqslant N$ converge with increasing N and, second, the limits of planar flow and spatial flow with symmetry $p4mm$ exist. The choice itself of additional variables is, of course, arbitrary. The solutions, therefore, corresponding in each approximation to a different choice of variables should not differ too much.

For fixed values of g, k_1, and k_2, the velocity and radii of curvature of the free surface at the braking point (R_x,R_y) are the measurable physical quantities that describe the $pmm2$-invariant stationary flow. Their connection with the Fourier amplitudes is easy to determine and is given by the identities $R_x=-1/2\beta_{10}$, $R_y=-1/2\beta_{01}$ (see Appendix B) for the radii of curvature and relation (5a) for the velocity v (the dimensional values of the radii of curvature are related to their dimensionless values by the transformations $R_xk_3/k_1^2\to R_x$ and $R_yk_3/k_2^2\to R_y$). As was mentioned above, the physical values of the parameters R_x and R_y in the two-parameter set of solutions of (1) are $R_{x\,\mathrm{cr}}\leqslant R_x\leqslant\infty$, $R_{y\,\mathrm{cr}}\leqslant R_y\leqslant\infty$, where $R_{x\,\mathrm{cr}}$ and $R_{y\,\mathrm{cr}}$ restrict the Froude numbers allowed by (1) to values of order unity.

Let the number of equations N_e at each order of approximation N be linked with the number of variables N_t (the number of Fourier amplitudes) by the relation $N_t=N_e+2$. We solve any N_e-1 equations of such a system for N_e-1 of the N_t variables. Transforming in the remaining equations from the Fourier amplitudes to the velocity v and radii of curvature R_x and R_y, we obtain an expression $f(v,R_x,R_y)=0$ from which we find the velocity v and thereupon the harmonics Φ_{mn} as functions of the parameters R_x and R_y, i.e., a two-parameter solution set in the Nth approximation. If the radii of curvature R_x and R_y are related in any way, e.g., by a symmetry relation or a geometrical relation, then the solution set will be a one-parameter set (the number of independent variables in this case $N_t=N_e+1$). For a zero-parameter solution, obviously, we have $N_t=N_e$, and the values of the radii of curvature R_x and R_y are fixed.

4. TWO-PARAMETER FAMILY OF SOLUTIONS

4.1. First approximation; analysis

We will now solve the problem of finding the stationary solution of the Rayleigh–Taylor instability in the first order of the expansion of Eq. (1) in powers $x^{2i}y^{2j}$ near a braking point: $\gamma_{ij}=\beta_{ij}$, $i+j=N=1$, $N_e=2$ [see Eqs. (7)].

We introduce the velocity v as a "length": $m(\alpha,\beta,\gamma)=M(\alpha,\beta,\gamma)/v$, $\varphi_{mn}=\Phi_{mn}/v$.

Then we have identically ($[v]=\sqrt{g/k_3}$)

$$v=\frac{1}{(k_1/k_3)m_{1x}}\sqrt{\frac{m_{2x}}{3(k_1/k_3)^2m_{1x}+(k_2/k_3)^2m_{1y}}}$$
$$=\frac{1}{(k_2/k_3)m_{1y}}\sqrt{\frac{m_{2y}}{3(k_2/k_3)^2m_{1y}+(k_1/k_3)^2m_{1x}}},$$
$$R_x=\frac{3\left(\dfrac{k_1}{k_3}\right)^2m_{1x}+\left(\dfrac{k_2}{k_3}\right)^2m_{1y}}{m_{2x}}, \tag{8}$$

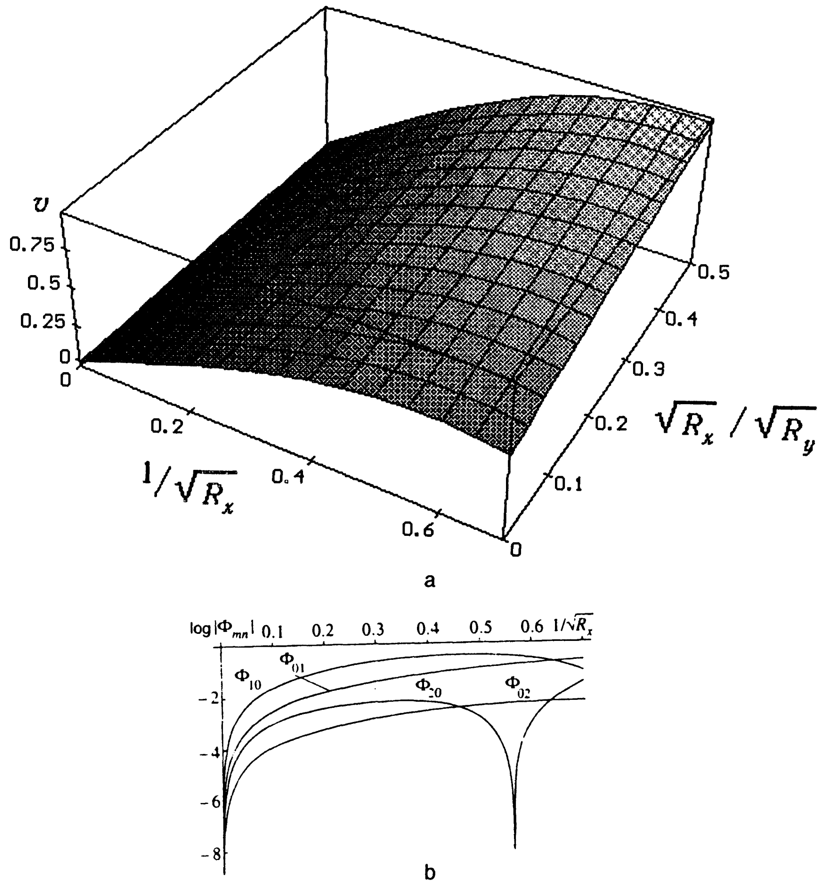

FIG. 3. Solutions of the problem of stationary flow in the first approximation in the physical region; $k_2=k_1/2$; a) the velocity $v(R_x,R_y)$ as a function of the parameters; b) decrease of the absolute value of the harmonics $\Phi_{mn}(R_x,R_y)$ with growth of $m+n$ in the physical region; $\sqrt{R_x/R_y}=0.8$ for dimensional values of R_x and R_y; $|\Phi_{0n}(R_x,R_y)|\to 0$ as $R_y\to\infty$. Units are as follows: $[v]=\sqrt{g/k_1}$, $[\Phi_{mn}]=\sqrt{g/k_1}$, $[R_x]=1/k_1$, $[R_y]=k_1/k_2^2$.

$$R_y=\frac{\left(\frac{k_1}{k_3}\right)^2 m_{1x}+3\left(\frac{k_1}{k_3}\right)^2 m_{1y}}{m_{2y}}.$$

Relations (8) allow us to transform from the third-order system of inhomogeneous equations $\gamma_{ij}=\beta_{ij}$ with $i+j=1$ to the linear equation (5a)

$$m(0,0,0)=-1 \tag{9}$$

and, in line with relations (8), to the relation

$$k_1^2 m_{1x}^2 R_x=k_2^2 m_{1y}^2 R_y. \tag{9a}$$

Let us now construct the two-parameter solution set. We choose the Fourier amplitudes Φ_{10}, Φ_{01} and Φ_{20}, Φ_{02}, as our variables. Then

$$m(0,0,0)=\varphi_{10}+\varphi_{01}+\varphi_{20}+\varphi_{02}$$
$$=\frac{1}{2}(3m_{1x}-m_{2x}+3m_{1y}-m_{2y}).$$

From relations (8) and (9) it is easy to find the velocity and Fourier amplitudes:

$$v=\frac{1}{2}\left(\frac{3(R_x-k_1/k_3)}{R_x^{3/2}}+\frac{3(R_y-k_2/k_3)}{R_y^{3/2}}-\frac{(k_1/k_3)R_xR_y^{1/2}+(k_2/k_3)R_x^{1/2}R_y}{R_x^{3/2}R_y^{3/2}}\right),$$

$$\Phi_{10}=-\frac{1}{R_x^{1/2}}\left(2-3\frac{k_1}{k_3}\frac{1}{R_x}-\frac{k_2}{k_3}\frac{1}{R_x^{1/2}R_y^{1/2}}\right),$$

$$\Phi_{01}=-\frac{1}{R_y^{1/2}}\left(2-\frac{k_1}{k_3}\frac{1}{R_x^{1/2}R_y^{1/2}}-3\frac{k_2}{k_3}\frac{1}{R_y}\right), \tag{10}$$

$$\Phi_{20}=\frac{1}{2R_x^{1/2}}\left(1-3\frac{k_1}{k_3}\frac{1}{R_x}-\frac{k_2}{k_3}\frac{1}{R_x^{1/2}R_y^{1/2}}\right),$$

$$\Phi_{02}=\frac{1}{2R_y^{1/2}}\left(1-\frac{k_1}{k_3}\frac{1}{R_x^{1/2}R_y^{1/2}}-3\frac{k_2}{k_3}\frac{1}{R_y}\right).$$

As could be expected, expressed in terms of dimensional quantities, the solution (10) is seen not to depend on k_3. As is easy to see from Eqs. (10), the limiting values $R_x\to\infty$ and $R_y\to\infty$ are associated with planar flows (in the x and y directions, respectively) whereas for $k_1=k_2$ and $R_x=R_y$ the solution (10) describes flows with $p4mm$ symmetry. Expressions (10) give the solution of the problem to the first approximation in the entire physical region of parameter values $R_{x\,\mathrm{cr}}\leq R_x\leq\infty$, $R_{y\,\mathrm{cr}}\leq R_y\leq\infty$ (Fig. 3) even though, strictly speaking, a valid description of the free surface in the limit $R_x\to\infty$, $R_y\to\infty$ requires the assignment of a large number of harmonics in expansion (3) (see Ref. 7). Indeed, it is easy to see from the solution (10) that the absolute values of the amplitudes $\Phi_{mn}(R_x,R_y)$ decrease with the order of the harmonic for arbitrary values of R_x and R_y in the physical region:

$$|\Phi_{mn}(R_x,R_y)| \gg |\Phi_{(m+1)n}(R_x,R_y)|, \tag{11}$$

where

$$\Phi_{mn}(R_x,R_y) \geqslant \Phi_{nm}(R_x,R_y) \quad \text{for } m \geqslant n.$$

The falloff shown in relation (11) is exponential (see Fig. 3b):

$$|\Phi_{(m+1)n}(R_x,R_y)| \approx |\Phi_{mn}(R_x,R_y)| \exp(-al), \quad a>0,$$

Let us analyze expressions (10).

In the limit $R_x \to \infty$, $R_y \to \infty$ for all k_1 and k_2 for isolated jet flows we find

$$v(R_x,R_y) \to \frac{3}{2}\left(\frac{1}{R_x^{1/2}} + \frac{1}{R_y^{1/2}}\right). \tag{12}$$

For definiteness, from here on we will only consider flows with natural conditions (Fig. 2): for $k_1 \geqslant k_2$ in Eqs. (1) and (3), first $R_x \leqslant R_y$ and, second, $\Phi_{mn} \geqslant \Phi_{nm}$ for $m \geqslant n$. We now expand expressions (10) in the limit $k_2 \to 0$, $1/R_y \to 0$ in powers of the small parameters $1/R_y$ and k_2 (see Ref. 5):

$$v(R_x,R_y) \to \frac{3}{2R^{3/2}}\left(R_x - \frac{k_1}{k_3}\right) + \frac{3}{2\sqrt{R_y}} - \frac{k_2}{k_3}\frac{1}{2\sqrt{R_y R_x}} - \frac{k_1}{k_3}\frac{1}{2R_y\sqrt{R_x}}. \tag{13}$$

With the help of the latter expansion it is easy to determine the accuracy of the passage to the limit of planar flow[5] with $k_2 \equiv 0$ from spatial flow. A flow will be planar if the dimensionless radius of curvature at its braking point diverges, $1/R_y \to 0$ (see Fig. 4a). In other words, a flow with dimensional radius of curvature tending as $k_2 \to 0$ to infinity faster than $1/k_2^2$.

For $k_1 \approx k_2 \approx k$ and $R_x \approx R_y \approx R$, Eqs. (10) easily yield (see Fig. 4b)

$$v(R_x,R_y) \to \frac{3R - 4(k/k_3)}{R^{3/2}} + \frac{3(R-4(k/k_3))}{4R^{5/2}}(\Delta R_x + \Delta R_y) + \frac{3R - 8(k/k_3)}{2R^{3/2}}\frac{\Delta k_1 + \Delta k_2}{k_3}, \tag{14}$$

where

$$\Delta R_x = R - R_x, \quad \Delta R_y = R - R_y, \quad \Delta k_1 = k - k_1,$$

$$\Delta k_2 = k - k_2.$$

In the first approximation the critical parameter values $R_{x\,\mathrm{cr}}$, $R_{y\,\mathrm{cr}}$ will be those for which

$$|\Phi_{mn}(R_{x\,\mathrm{cr}},R_{y\,\mathrm{cr}})| \approx |\Phi_{(m+1)n}(R_{x\,\mathrm{cr}},R_{y\,\mathrm{cr}})|.$$

As can be easily estimated from Eqs. (10), for $k_1 \geqslant k_2$ the conditions $R_x \leqslant R_y$ and $\Phi_{m0} \geqslant \Phi_{0n}$ are necessarily fulfilled for $\sqrt{R_x/R_y} \leqslant k_2/k_1$. For such flows, it follows from Eqs. (10) that $R_{x\,\mathrm{cr}} \approx (2.5 \pm 0.5)(k_1/k_3)$ with $R_{y\,\mathrm{cr}} \geqslant (2.5 \pm 0.5) \times (k_2/k_3)(k_1/k_2)^3$. It should be emphasized that for such an estimate the accuracy of the asymptotic limits (13) and (14) of planar flow and flow having symmetry $p4mm$ is necessarily fulfilled in the critical region:

$$R_{y\,\mathrm{cr}} \to \infty, \quad R_{y\,\mathrm{cr}} \approx \frac{1}{k_2^2}, \quad R_{x\,\mathrm{cr}} \approx (2.5 \pm 0.5)\frac{k_1}{k_3} \quad \text{for } k_2 \approx 0,$$

$$R_{y\,\mathrm{cr}} \approx R_{x\,\mathrm{cr}} \approx (2.5 \pm 0.5)\frac{k_1}{k_3} \quad \text{for } k_2 \approx k_1.$$

We will now show that for arbitrary values of k_1 and k_2 in the first of Eqs. (10) (the equation for the velocity v as a function of the parameters R_x and R_y) there are no singular, geometrically distinguished values of k_2/k_1 in the given region.

Indeed, let the velocity be extremal on some curve $u(R_x,R_y)=0$. Then

$$\frac{\partial u(R_x,R_y)/\partial(1/\sqrt{R_x})}{\partial u(R_x,R_y)/\partial(/1\sqrt{R_y})} = \frac{\partial v(R_x,R_y)/\partial(1/\sqrt{R_x})}{\partial v(R_x,R_y)/\partial(1/\sqrt{R_y})}.$$

From Eq. (10) we easily find that

$$\frac{\partial v}{\partial(1/\sqrt{R_x})} - \frac{\partial v}{\partial(1/\sqrt{R_y})} = \frac{k_1-k_2}{k_3}\left(\frac{1}{\sqrt{R_x}} - \frac{1}{R_y}\right)^2 + 8\frac{k_1-k_2}{k_3}\frac{1}{R_y} + 8\frac{k_1}{k_3}\left(\frac{1}{R_x} - \frac{1}{R_y}\right),$$

i.e., for $R_x < R_y$ and $k_1 > k_2$

$$\frac{\partial v}{\partial(1/\sqrt{R_x})} > \frac{\partial v}{\partial(1/\sqrt{R_y})}. \tag{15}$$

It is also easy to show that for any $k_1 \neq k_2$ the points $(R_{x\,\mathrm{ext}}, R_{y\,\mathrm{ext}})$ at which the velocity attains an extreme value [see Eq. (10)] lie outside the limits of the region in question.

Thus, $pmm2$ symmetry does not distinguish between parameter values of any singular (dependent on the ratio k_2/k_1) points or curves. It can be easily seen from Eqs. (10) and (15) that for flows with $p4mm$ symmetry and $k_2 = k_1 = k$ for given k the extreme (maximum) values of the velocity $v(R_x,R_y)$ lie on the line $R_x - R_y = 0$ [in (14) $\Delta R_x = -\Delta R_y$, see Fig. 4b]. The lowest velocity in the rectangular geometry for $k_1 > k_2$ always belongs to the flow with the boundary condition $1/R_y = 0$ (Fig. 3a).

Thus, even in the first approximation (4) the accuracy of the limits (12)–(14) and the absence of singularities in the velocity as a function of the parameters (15) serve to confirm that the solution set with symmetry $pmm2$ is a two-parameter set. Indeed, we will now show that one- and zero-parameter sets cannot be solution sets of this problem.

We assume that the dimensionality of the solution set

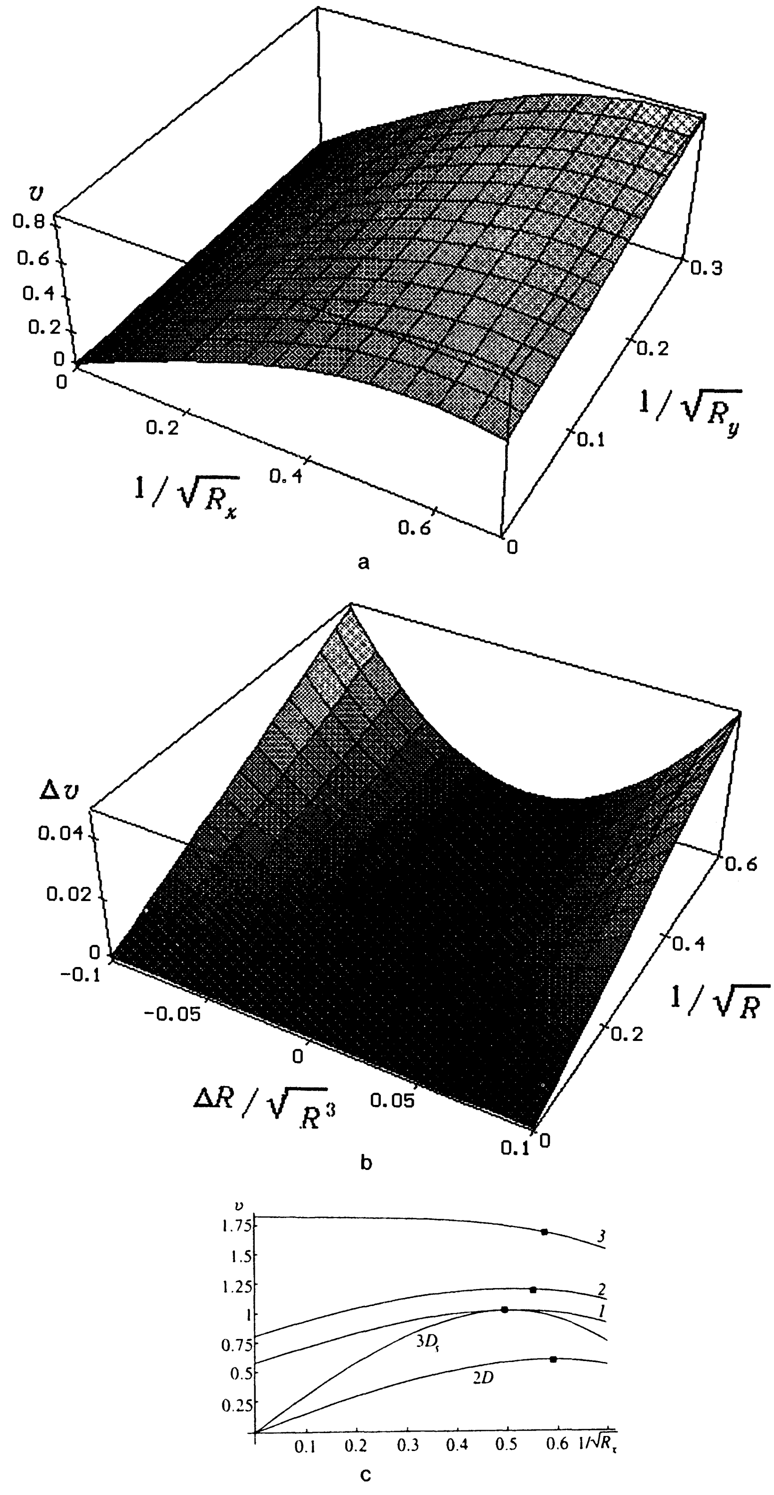

FIG. 4. The velocity $v(R_x, R_y)$ in the first approximation: a) in the limit of planar ($pm11$) flow: $k_2 \equiv 0$, $1/R_y \to 0$; b) in the limit of spatial flow with symmetry $p4mm$: $k_1 \equiv k_2$, $R^{-1/2} = R_x^{-1/2} + R_y^{-1/2}/2$, $\Delta R^{-1/2} = R_x^{-1/2} - R_y^{-1/2}/2$, $\Delta v = v(\Delta R = 0) - v(\Delta R \approx 0)$, $[R] = 1/k_1$, $[\Delta R] = 1/k_1$; c) as a function of one parameter for $k_2 = k_1$ (*1*), $k_2 = k_1/2$ (*2*), $k_2 = k_1/10$ (*3*). The zero-parameter solutions are indicated by small filled squares. Curves $2D$ and $3D_s$ correspond to one-parameter families of solutions for periodic planar flow with symmetry $pm11$ and spatial flow with symmetry $p4mm$, respectively.

does not depend on the symmetry, that the flow is not governed by the geometry k_2/k_1, and that by varying this ratio it is possible to go from the spatial ($p4mm$) to the planar ($pm11$) periodic solution. Thus, as can be easily seen, in the limit $k_1 \approx k_2 \approx k$ the corrections to the absolute values of the velocity and Fourier amplitudes should not depend on the sign of $k_1 - k_2$.

In the first approximation a relation between R_x and R_y that depends on k_1 and k_2 can be obtained from the condition that one of the amplitudes (10) equals zero for the one-parameter family of solutions, as well as the solution of the system of equations $\{\Phi_{20} = 0, \Phi_{02} = 0\}$ for the zero-parameter family (see Ref. 7). We immediately throw out the solutions $1/R_x = 0$, $1/R_y = 0$ of the equations $\Phi_{20} = 0$ and $\Phi_{02} = 0$ as not depending on k_1 and k_2; as before, we normalize the dimensionless velocity by k_3. The expressions thus obtained for the velocity and Fourier harmonics are quite lengthy and are not given here. For the zero-parameter solution in the limit $k_1 \approx k_2 \approx k$ we find ($\Delta k_1 = k - k_1$, $\Delta k_2 = k - k_2$)

$$R_x=\frac{4k}{k_3}-\frac{10\Delta k_1+2\Delta k_2}{3k_3},\quad R_y=\frac{4k}{k_3}-\frac{2\Delta k_1+10\Delta k_2}{3k_3},$$

$$v(R_x,R_y)=\frac{1}{\sqrt{k/k_3}}\left(1+\frac{\Delta k_1+\Delta k_2}{4}\right),$$

$$\Phi_{10}(R_x,R_y)=-\frac{1}{2\sqrt{k/k_3}}\left(1+\frac{5\Delta k_1+\Delta k_2}{12}\right),$$

$$\Phi_{01}(R_x,R_y)=-\frac{1}{2\sqrt{k/k_3}}\left(1+\frac{\Delta k_1+5\Delta k_2}{12}\right). \tag{16}$$

In the second limit, $k_2\rightarrow\infty$, this solution goes over to

$$R_x=3\frac{k_1}{k_3}\left(1+\frac{k_2}{k_1}\right),\quad R_y=\frac{k_1}{3k_3}\left(1+17\frac{k_2}{k_1}\right),$$

$$v(R_x,R_y)=\frac{4}{\sqrt{3}\sqrt{k_1/k_3}}\left(1-\frac{13}{2}\frac{k_2}{k_1}\right),$$

$$\Phi_{10}(R_x,R_y)=-\frac{1}{\sqrt{3}\sqrt{k_1/k_3}}\left(1-\frac{k_2}{2k_1}\right),$$

$$\Phi_{01}(R_x,R_y)=-\frac{\sqrt{3}}{\sqrt{k_1/k_3}}\left(1-\frac{17}{2}\frac{k_2}{2k_1}\right). \tag{17}$$

In analogy with Eqs. (16) and (17), irrespective of the choice of variable, that is, Fourier amplitude, the functions describing the one-parameter family of solutions have the form (Fig. 4c)

$$v(R),\Phi_{mn}(R)\approx A\left(R,\frac{k}{k_3}\right)\left(1+C\left(R,\frac{k}{k_3}\right)\frac{\Delta k_1+\Delta k_2}{k_3}\right) \tag{18}$$

in the limit $k_1\approx k_2\approx k$ and the form

$$v(R),\Phi_{mn}(R)\approx B\left(R,\frac{k_1}{k_3}\right)\left(1-D\left(R,\frac{k_1}{k_3}\right)\frac{k_2}{k_1}\right) \tag{19}$$

in the limit of planar flow we have $k_2\rightarrow\infty$. In expressions (18) and (19) A, B, C, and D are functions of the indicated arguments.

Thus, when $p4mm$ symmetry is broken, corrections to the velocity and Fourier amplitudes for the zero- and one-parameter solution families are linear in the small quantity k_1-k_2, and as k_1-k_2 grows so does the velocity (see Ref. 7). This means that when the symmetry is lowered from $p4mm$ to $pmm2$ both the zero- and the one-parameter solutions of (1), even in the first approximation, move into the unphysical region (Fig. 4c).

As was already mentioned, the choice of additional variables—the Fourier amplitudes—to approximate the boundary conditions in each order is, of course, arbitrary. If, when solving problem (1) in the first order of the expansion of (1) in the terms $x^{2i}y^{2j}$ near a braking point, we choose as the harmonic variables Φ_{10}, Φ_{01}, and Φ_{20}, Φ_{11}, then the velocity as a function of R_x and R_y will have the form (see Ref. 7)

$$v=\frac{1}{2}\left(\frac{3(R_x-k_1/k_3)}{R_x^{3/2}}+\frac{3(a(k_1,k_2)R_y-(k_2/k_3)b(k_1,k_2))}{R_y^{3/2}}-\frac{((k_1/k_3)b(k_1,k_2)R_xR_y^{1/2}+(k_2/k_3)R_x^{1/2}R_y)}{R_x^{3/2}R_y^{3/2}}\right), \tag{20}$$

where

$$a(k_1,k_2)=\frac{1}{3}\frac{3k_1-\sqrt{k_1^2+k_2^2}}{k_2-\sqrt{k_1^2+k_2^2}},$$

$$b(k_1,k_2)=\frac{3k_1+2k_2-\sqrt{k_1^2+k_2^2}}{k_2-\sqrt{k_1^2+k_2^2}}.$$

The first-order approximations (19) and (20), as expected, do not differ much. For physical values of the parameters R_x and R_y, the difference between the magnitudes of the velocities for a different choice of variables is quite insignificant and amounts to around 0.1–1%. We will not adduce expressions for the amplitudes $\Phi_{mn}(R_x,R_y)$, but note only that the difference between the corresponding values of the Fourier harmonics in (10) and (20) reaches levels ranging from 1 to 10%. Finally, cases (11)–(19) easily carry over to the choice of variables Φ_{10}, Φ_{01}, and Φ_{20}, Φ_{11} in (20).

4.2. Second approximation; analysis

We will now construct a two-parameter set of solutions of (1) in the second order of the expansion in terms $x^{2i}y^{2j}$ near a braking point $\gamma_{ij}=\beta_{ij}$, $i+j\leqslant N=2$, wherefore $N_e=5$ (see Eqs. (7) and Appendix C) and the number of variables $N_t=N_e+2=7$. We choose the following Fourier amplitudes: Φ_{10}, Φ_{01}, Φ_{11}, Φ_{20}, Φ_{02}, and Φ_{30}, Φ_{03}. As in the first approximation, invoking relations (8) and (9) we lower the order of the inhomogeneous equations of system (7). After a number of lengthy transformations, transforming from the variables Φ_{10}, Φ_{01}, Φ_{11}, Φ_{20}, Φ_{02}, Φ_{30}, and Φ_{03}, with the help of Eqs. (5) and (8), to the variables m_{1x}, m_{2x}, φ_{11}, m_{1y}, m_{2y}, φ_{30}, and φ_{03}, we solve any three equations of the resulting system for the variables φ_{11}, φ_{30}, and φ_{03}, and transform with the help of Eqs. (8) and (9a) in the remaining equation from the variables m_{1x}, m_{2x}, m_{1y}, and m_{2y} to the variables v, R_x, and R_y:

$$v(R_x,R_y)=-\frac{P(\sqrt{R_x},\sqrt{R_y})}{Q(\sqrt{R_x},\sqrt{R_y})}, \tag{21}$$

where $P(R_x,R_y)$ and $Q(R_x,R_y)$ are polynomials in $\sqrt{R_x}$ and $\sqrt{R_y}$ whose coefficients depend on k_1/k_3 and k_2/k_3 (Fig. 5). Exact expressions for the velocity and amplitudes are extraordinarily involved and are not given here.

As in the previously investigated[5,7] cases of planar and highly symmetric flows the second approximation (21), in contrast to the first (10), does not give solutions of (1) over the entire physical region of parameter values.[1)] Thus, ac-

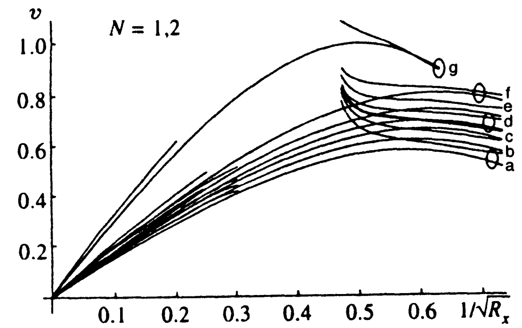

FIG. 5. Two-parameter set of solutions of the Rayleigh–Taylor instability in the approximations $N=1,\ 2$. The velocity $v(R_x,R_y)$ for a) for all $k_2<k_1$, $1/R_y\equiv 0$, b) $k_2=k_1/4$, $C=0.05$, c) $k_2=k_1/4$, $C=0.10$ and $k_2=k_1/2$, $C=0.10$, d) $k_2=3k_1/4$, $C=0.15$, e) $k_2=k_1/2$, $C=0.20$, f) $k_2=3k_1/4$, $C=0.30$, g) $k_2=k_1$, $R_y=R_x$, $C=(k_1/k_2)\sqrt{R_x/R_y}$ with dimensional values of R_x and R_y; ovals denote critical values of R_x.

cording to Eq. (21), for arbitrary values of k_1 and k_2 the absolute values of the amplitudes will fall off with growth of the harmonic index for

$$R_{x,y\ \mathrm{cr}}\leqslant R_{x,y}\leqslant R^*_{x,y} \quad \text{and} \quad R^{**}_{x,y}\leqslant R_{x,y}\leqslant\infty, \tag{22}$$

where

$$R_{x\ \mathrm{cr}}\approx(2.5\pm0.5)\,\frac{k_1}{k_3},\quad R^*_x\approx(5.5\pm0.5)\,\frac{k_1}{k_3},$$

$$R^{**}_x\approx(70\pm25)\,\frac{k_1}{k_3},$$

and, in analogy with the first approximation (see Fig. 5)

$$R_{y\ \mathrm{cr}}\approx(2.5\pm0.5)\,\frac{k_2}{k_3}\left(\frac{k_1}{k_2}\right)^3,$$

$$R^*_y\approx(5.5\pm0.5)\,\frac{k_2}{k_3}\left(\frac{k_1}{k_2}\right)^3,$$

$$R^{**}_y\approx(70\pm25)\,\frac{k_2}{k_3}\left(\frac{k_1}{k_2}\right)^3.$$

For isolated jet flows in the limit $R_x\rightarrow\infty$, $R_y\rightarrow\infty$ for all k_1 and k_2, with the help of Eq. (21) we find that

$$v(R_x,R_y)\rightarrow\frac{5}{3}\left(\frac{1}{R_x^{1/2}}+\frac{1}{R_y^{1/2}}\right)+O\left(\frac{1}{\sqrt{R_x}},\frac{1}{\sqrt{R_y}}\right)^3. \tag{23}$$

Let us now expand the second-order approximation of the velocity (21) in the small parameters $1/R_y$ and k_2 in the limit of planar flow $k_2\rightarrow0$, $1/R_y\rightarrow0$. Since as in (13) for $1/R_y\equiv0$ the velocity does not depend on k_2, this expansion, generally speaking, does not contain terms of the form k_2^i. For small $1/R_y$, k_2 and $R_y(k_2/k_3)^2\gg1$ we have

$$v\left(R_x,\frac{k_1}{k_3},R_y\,\frac{k_2}{k_3}\right)$$

$$\rightarrow\frac{4R_x^3-27(k_1/k_3)R_x^2+36(k_1/k_3)^2R_x-19(k_1/k_3)^3}{12(R_x-5k_1/k_3)R_x^{3/2}}$$

$$+\frac{5}{3\sqrt{R_y}}-\frac{k_2}{k_3}\frac{1}{\sqrt{R_y}}f\left(R_x,\frac{k_1}{k_3}\right). \tag{24}$$

Let us now consider the limit of highly symmetric flow $k_1\approx k_2\approx k$, $R_x\approx R_y\approx R$ (the expression for $v(R,k/k_3,R,k/k_3)$ is given in Ref. 7):

$$v\left(R_x,\frac{k_1}{k_3},R_y\,\frac{k_2}{k_3}\right)-v\left(R,\frac{k}{k_3},R\,\frac{k}{k_3}\right)$$

$$\rightarrow F^{|R|}\left(R,\frac{k}{k_3}\right)(\Delta R_x+\Delta R_y)+F^{|k|}\left(R,\frac{k}{k_3}\right)$$

$$\times\left(\frac{\Delta k_1+\Delta k_2}{k_3}\right). \tag{25}$$

In analogy with the inequality (15), for arbitrary values of k_1 and k_2 there are no geometry-dependent distinct values in the allowed region (22) for the velocity $v(R_x,R_y)$ in the second approximation (21). For $k_2=k_1$ the correction to the velocity (21), (25) will be quadratic in the deviations of R_x and R_y from the line $R_x-R_y=0$ and for values of $R\geqslant R^*$ and $R\approx R_{\mathrm{cr}}$ are numerically small and negative. Planar flow ($pm11$) with $1/R_x\equiv0$ for arbitrary $k_1>k_2$ is the slowest. The zero- and one-parameter families of solutions of the second approximation of (1) for $k_1\neq k_2$ are found in the unphysical region of phase space.

Like the solution (10), after transforming back to dimensioned variables the solution (21) does not depend on k_3.

4.3 Questions of convergence

Thus, we have found solutions of the problem (1) in successive approximations $N=1,\ 2$: the velocity and Fourier amplitudes (10) and (21) are functions of two parameters, and in each approximation in the allowed region of parameter values the amplitudes fall off in absolute value with growth of the harmonic index.

If the solutions of the problem (1) form a two-parameter set, then as the number of approximations grows there should exist a functional limit over the parameters for all the quantities describing the flow. So let us analyze expressions (10) and (21) from the standpoint of the convergence.

For any nonzero k_1 and k_2 in the allowed region of variation of the parameters the surfaces $v(R_x,R_y)$ are densely grouped together, and their convergence with growth of the approximation is shown in Fig. 6. It is easy to compare the limiting expansions of the velocity (12), (23), (13), (24), and (14), (25). In each of them the expressions corresponding to different approximations have the same functional dependence on the expansion parameters, and the difference between corresponding coefficients in the allowed region is not great (Fig. 7).

In the symmetrically distinct cases $k_2\equiv0$ and $k_1\equiv k_2$ the nature of the convergence of the solutions obtained in different approximations varies, see Figs. 8a and b. Let us first consider flows with $k_2\equiv0$. From Eqs. (13) and (24) we easily find that for small $1/R_y\approx0$ for arbitrary values of R_x in the allowed region the difference between the approximations grows with $1/R_y$ as $1/6R_y$. This leads to an abrupt deterioration of the convergence even for moderate deviations of $1/R_y$ from zero (note that the dimensional radius of curvature, equal to R_yk_3/k_2^2, remains infinite). Therefore in

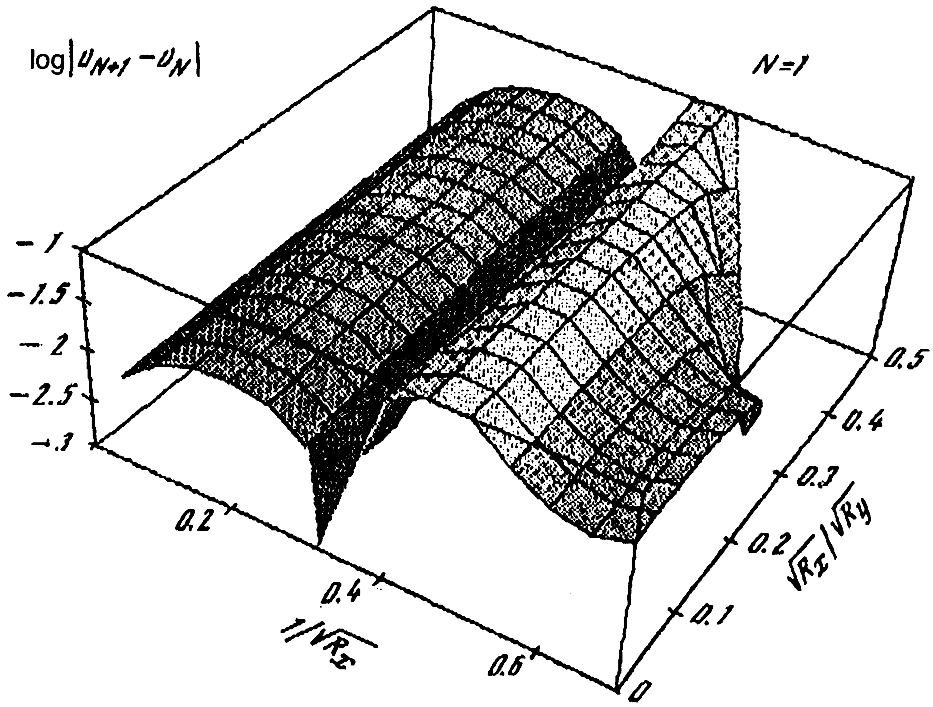

FIG. 6. Convergence of the surfaces $v_N(R_x, R_y)$ with increase in the order of the approximation N, $k_2 = k_1/2$.

the case $k_2 \equiv 0$ the only possible value of R_y at which convergence of the solutions of the successive approximations of (1) exists is $1/R_y \equiv 0$ (Fig. 8a).

Also, $p4mm$ symmetry with $k_1 \equiv k_2$ lowers the dimensionality of the solution set of (1), (3). In this case the velocity $v(R_x, R_y)$ is extremal on the line $R_x - R_y = 0$, where also the successive approximations achieve the best convergence (Fig. 8b).

Note that the behavior of any of the amplitudes is similar to that of the velocity.

It should be stressed that for arbitrary values of k_1 and k_2 the zero- and one-parameter solutions of problem (1), obtained in the first two orders of the expansion near a braking point, $N = i + j = 1$, 2, obviously diverge (Fig. 8c, inequality (15), Eqs. (16)).

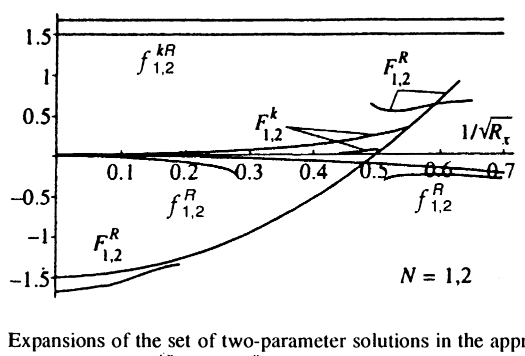

FIG. 7. Expansions of the set of two-parameter solutions in the approximations $N = 1$, 2. Here f^{kR}_N and f^R_N are the coefficients of the terms $\approx 1/\sqrt{k_2^2 R_y}$ and $\approx 1/\sqrt{R_y}$ (dimensioned coordinates) of the expansion of the velocity as functions of the parameters in the limit $k_2 \to 0$, $1/R_y \to 0$; F^k and F^R are the coefficients of the terms $\approx \Delta(k_1 - k_2)$ and $\approx \Delta(R_x - R_y)$ of the expansion of the two-parameter solutions in the limit $k_1 \approx k_2$, $R_x \approx R_y$.

5. DISCUSSION AND CONCLUSIONS

As has been shown above, stationary, spatially periodic solutions of problem (1) in the Rayleigh–Taylor instability form a set whose dimensionality is determined by the symmetry of the initial perturbation.

Thus, flow with symmetry $pmm2$ is described by the independent parameters $\{g, v, k_1, k_2\}$, and the solution set forms a double continuum (surface) in phase space $\hat{\Phi}$. Successive approximations to this ideal ($N \to \infty$) family of solutions have been obtained here for the first time (unfortunately, in contrast to Refs. 5 and 7, we were not able to advance beyond the second-order approximation of (1) due to technical difficulties). These approximations converge: first of all, in each approximation the harmonics $\{\Phi_{mn}\}$ fall off in absolute value as the combined index $m+n$ increases, and second, for all of the quantities describing the flow (1) there exist functional limits in the parameters as the order of approximation increases:

$$\hat{\Phi} = \lim_{N \to \infty} \hat{\Phi}_N, \quad v = \lim_{N \to \infty} v_N.$$

The exponential nature of the convergence in all these cases (Figs. 3 and 6) testifies to the smoothness of the ideal solution.[4,5,7]

Each of the solutions in the ideal two-parameter set corresponds to an exact solution of problem (1) with its own free surface and uniquely associated Froude number. For $pmm2$ symmetry in (1) the inverse length in the z direction (3) remains indeterminate.[8] However, the solution is smooth:

$$|\Phi_{mn}|_{m+n=l+1} \ll |\Phi_{mn}|_{m+n=l},$$

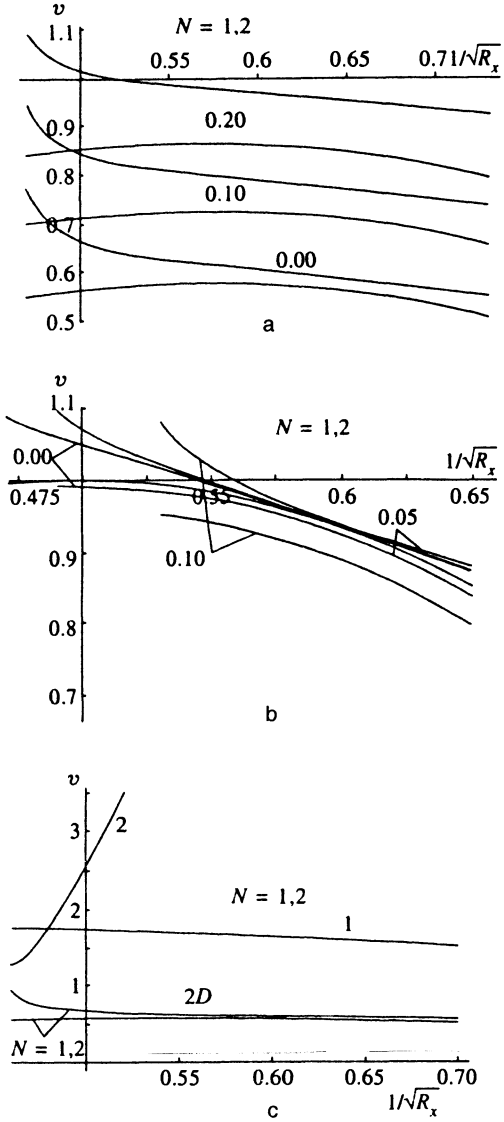

FIG. 8. Properties of the convergence of the solutions with increase in the order of the approximation N: a) in the limit of planar flow $pm11$, $k_2\equiv 0$, $1/R_y\to 0$, the numbers alongside the curves are the values of $1/R_y$; b) in the limit of spatial flow with symmetry $p4mm$, $k_1\equiv k_2$, $R^{-1/2}=R_x^{-1/2}+R_y^{-1/2}$, $\Delta R^{-1/2}=R_x^{-1/2}-R_y^{-1/2}$, the numbers alongside the curves are the values of ΔR, $[\Delta R]=1/k_1$, $[R]=1/k_1$; c) one-parameter solutions in the limit $k_2\to 0$, the curves labeled $2D$ correspond to planar periodic flow.

and the main contribution comes from the first terms in the expansion (3):

$$\Phi_{10}\cos k_1 x\,\frac{\exp(-k_1 z)}{k_1}+\Phi_{01}\cos k_2 y\,\frac{\exp(-k_2 z)}{k_2},$$

so the characteristic length in the z direction is of order $1/k_1$, $1/k_2$. Without loss of generality, for $k_1\geqslant k_2$, we will set $k_3=k_1$ and, in the dimensional quantities, we will set $\mathrm{Fr}=v/\sqrt{g\lambda_1}$. However, it must be emphasized again that the dimensional solution of problem (1) (the velocity, Fourier harmonics, radii of curvature R_x, R_y, and the free surface) does not depend on k_3 [see Eqs. (10), (20), and (21)].

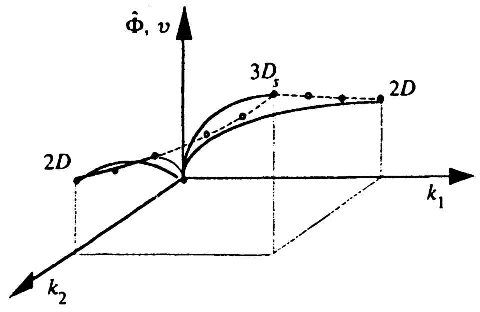

FIG. 9. Two-parameter set of stationary solutions and its end region for periodic flow that is invariant with respect to the space group $pmm2$, in the phase space $\hat{\Phi}$. The filled and empty circles denote points from the end regions of the solution set.

In one of the boundary regions of the ideal two-parameter solution set with isolated jet flows (see Fig. 9) in the dimensional coordinates

$$R_x\to\infty,\quad R_y\to\infty,\quad \hat{\Phi}\approx 1/\sqrt{R},$$

the desired dependence of the velocity has the form

$$v\to\sqrt{g}\left(\frac{2}{\sqrt{k_1^2R_x}}+\frac{2}{\sqrt{k_2^2R_y}}\right). \tag{26}$$

Note that in this boundary region

$$v=\lim_{N\to\infty} v_N=\left(X_N\sqrt{\frac{g}{k_1^2R_x}}+Y_N\sqrt{\frac{g}{k_2^2R_y}}\right),$$

where N is the order of the approximation (see Ref. 7).

Near the other end-region of the ideal solution set, limiting the physical values of R_x, R_y, v, Φ_{mn}, and Fr the convergence of the successive approximations deteriorates abruptly. In this critical region, roughly defined in the first two approximations (see Fig. 9), the velocity and radii of curvature of the free surface range from

$$R_{x\,\mathrm{cr}}^-,R_{y\,\mathrm{cr}}^-,v_{\mathrm{cr}}^-(R_{x\,\mathrm{cr}}^-,R_{y\,\mathrm{cr}}^-),\ \Phi_{mn\,\mathrm{cr}}^-(R_{x\,\mathrm{cr}}^-,R_{y\,\mathrm{cr}}^-) \tag{27a}$$

to

$$R_{x\,\mathrm{cr}}^+,R_{y\,\mathrm{cr}}^+,v_{\mathrm{cr}}^+(R_{x\,\mathrm{cr}}^+,R_{y\,\mathrm{cr}}^+),$$

$$\Phi_{mn\,\mathrm{cr}}^+(R_{x\,\mathrm{cr}}^+,R_{y\,\mathrm{cr}}^+). \tag{27b}$$

For $k_1 > k_2$ in dimensional coordinates

$$R^-_{x\,\mathrm{cr}} = 2.0/k_1, \quad R^-_{y\,\mathrm{cr}} = c/k_2, \quad c \to \infty,$$

$$v^-_{\mathrm{cr}}(R^-_{x\,\mathrm{cr}}, R^-_{y\,\mathrm{cr}}) = 0.56\sqrt{g/k_1}, \quad \mathrm{Fr} = 0.22.$$

We emphasize that for $R^-_{y\,\mathrm{cr}} \equiv \infty$ (for all $n \neq 0$, $\Phi_{mn} \equiv 0$) the solution becomes invariant with respect to the group $pm11$, not $pmm2$. Flows with this value of R_y are planar and form, as expected, a symmetrically distinct one-parameter subfamily of the two-parameter set (Fig. 5).

The location of the other boundary point of the end-region depends on the ratio k_2/k_1 in such a way that large values of k_2/k_1 correspond to large values of $v^+_{\mathrm{cr}}(R^+_{x\,\mathrm{cr}}, R^+_{y\,\mathrm{cr}})$ (see Refs. 5 and 7). Let $R^+_{x\,\mathrm{cr}} = R^+_{x\,\mathrm{cr}}(k_2, k_1)$ and $R^+_{y\,\mathrm{cr}} = R^+_{y\,\mathrm{cr}}(k_2, k_1)$. From the estimate outlined above for the physical region of parameter values it follows that the dimensional quantities $R^+_{x\,\mathrm{cr}}$ and $R^+_{y\,\mathrm{cr}}$ are linked by the relation $R^+_{x\,\mathrm{cr}}/R^+_{y\,\mathrm{cr}} \approx (k_2/k_1)^4$, with $\partial R^+_{x\,\mathrm{cr}}/\partial k_2 > 0$ and $\partial R^+_{y\,\mathrm{cr}}/\partial k_2 < 0$ for $k_1 > k_2$.

In the limit $k_2 \to k_1$, for $R^+_{x\,\mathrm{cr}} = 3.0/k_1$, $R^+_{y\,\mathrm{cr}} = (3.0/k_2)(k_1/k_2)^3$, the Froude number satisfies $\mathrm{Fr} \to 0.38$ (see Ref. 7) and the expansion of the critical values of the velocity in the small parameter $(k_1 - k_2)/k_1$ has the form

$$\frac{v^+_{\mathrm{cr}}(R^+_{x\,\mathrm{cr}}, R^+_{y\,\mathrm{cr}})}{\sqrt{g/k_1}} = 0.964 - \left(-0.316 + 0.112 \left. \frac{\partial R^+_{x\,\mathrm{cr}}}{\partial k_2} \right|_{R^+_{x\,\mathrm{cr}} = 3.0/k_1} \right) \frac{k_1 - k_2}{k_1}. \tag{28}$$

But in the limit $k_2 \to 0$, for $R^+_{x\,\mathrm{cr}} = 2.0/k_1$, $R^+_{y\,\mathrm{cr}} \to c(k_2)/k_2$, where $c(k_2) \approx 1/k_2^3 \to \infty$, the Froude number satisfies $\mathrm{Fr} \to 0.22$ (Ref. 5), and

$$\frac{v^+_{\mathrm{cr}}(R^+_{x\,\mathrm{cr}}, R^+_{y\,\mathrm{cr}})}{\sqrt{g/k_1}} = 0.565 + \left(1.296 + 0.072 \left. \frac{\partial R^+_{x\,\mathrm{cr}}}{\partial k_2} \right|_{R^+_{x\,\mathrm{cr}} = 2.0/k_1} \right) \frac{k_2}{k_1}. \tag{29}$$

Corresponding expressions for $\Phi^+_{mn\,\mathrm{cr}}(R^+_{x\,\mathrm{cr}}, R^+_{y\,\mathrm{cr}})$ and the free surface are given in Appendix D.

From Eqs. (28) and (29) it can be easily seen that the critical values $R^+_{x\,\mathrm{cr}}$ and $R^+_{y\,\mathrm{cr}}$ weakly depend on the ratio k_2/k_1 in the limit $k_2 \to 0$, whereas in the limit $k_2 \to k_1$ the value of the derivative $\partial R^+_{x\,\mathrm{cr}}/\partial k_2$ should be quite large (28), which finds itself in total agreement with the results obtained (see Fig. 5).

If the symmetry of the flow is lowered from $pmm2$ to $pm11$, then in the limit $k_2 \to 0$ the surface of solutions in phase space degenerates into a curve.[5] Such flows are described by only three independent physical quantities $\{g, v, k_1\}$ and the solution set is a one-parameter set, in complete agreement with Ref. 5. Its end-points are $R_y \equiv \infty$, $R_{x\,\mathrm{cr}} = 2.0/k_1$ with $v_{\mathrm{cr}}(R_{x\,\mathrm{cr}}) = 0.56\sqrt{g/k_1}$ and $R_x \to \infty$ with $v_{\mathrm{cr}}(R_x) \to 2\sqrt{gk_1}/\sqrt{k_1 R_x}$.

In the limit $k_2 \to k_1$, $p4mm$ symmetry distinguishes in the two-parameter set solutions forming a simple continuum:[7] the one-parameter family prescribed by the values $\{g, v, k_1\}$ $R_x = R_y = R$ with end-points $R_{cr} = 3.0/k_1$, $v_{\mathrm{cr}}(R_{\mathrm{cr}}) = 0.96\sqrt{g/k_1}$ and $R \to \infty$, $v_{\mathrm{cr}}(R) \to 4\sqrt{gk_1}/\sqrt{k_1 R}$ (see Ref. 7).

Thus, families of stationary periodic solutions for planar flow with $pm11$ symmetry and spatial flow with symmetry $p4mm$ are isolated curves in phase space $\hat{\Phi}$ (Ref. 7). In order to make the transition in (1) via symmetry breaking from spatial flow to planar flow, it is necessary to enlarge the dimensionality of the solution set (Fig. 9). For a prescribed value of one of the characteristic translations, planar flows possess the lowest velocity (Froude number) and spatial flows with $p4mm$ symmetry the highest allowed by (1) (Fig. 9).

The main results obtained here may be carried over to the case of $p6mm$-invariant periodic flow. Note that in this case the transition from spatial flow to planar flow will be accompanied by a lowering of the symmetry from $p6mm$ to $mm2$ and, consequently, the transition in (1) from spatial to planar solutions should also take place in the two-parameter set in phase space.

Thus, translationally invariant stationary solutions of the Rayleigh–Taylor instability are not universal in the sense of the dimensionality of their set. Stationary solutions of the Rayleigh–Taylor instability for flows in tubes, on the contrary, should be universal: as was noted above, symmetry violations cannot affect the dimensionality of their set. Allowing for viscosity near the wall will have the obvious result that these solutions will be a point (points) in phase space.[3]

Garabedian, who was the first to enunciate the hypothesis that the stationary solution of the Rayleigh–Taylor inhomogeneity is nonunique,[4] assumed that only the fastest flow is realized in (1) and that all the others are unstable. It is interesting that in the investigated cases[2,6] only solutions with critical values of the Froude number were observed. Thus, for planar flow the experimental Froude number Fr_{2D} was found to be equal to 0.22 (Ref. 2), and for spatial flow with a square lattice the value $\mathrm{Fr}_{3D_s} = 0.38$ was obtained by numerical experiment.[6] The largest velocity in the one-parameter families of solutions[5,7] corresponded to the Froude numbers $\mathrm{Fr}_{2D} = 0.26$ and $\mathrm{Fr}_{3D_s} = 0.48$ for planar flow and "square" spatial flow, respectively. In the case of $pmm2$-invariant flow, it may be expected that some solution $\{v_{\mathrm{cr}}(R_{x\,\mathrm{cr}}, R_{y\,\mathrm{cr}}), R_{x\,\mathrm{cr}}, R_{y\,\mathrm{cr}}\}$ in the critical region will be realized experimentally. However, the stability of stationary flows (1) has so far not been examined. The selection of solutions, their evolution and stability criteria remain as subjects for future study.

Let us comment now on the solution method employed in this work. It should be emphasized that the nonuniqueness of the solution of the problem of stationary flow is dictated, primarily, by physical considerations. Let the dimensionality of the solution set be equal to P. If the number of additional parameters introduced in (3) is less than P, then the solutions obtained in successive approximations will not converge in phase space to any limit. If, on the other hand, the number of additional variables in (3) is greater than P, then functional convergence of successive approximations will be realized on a P-dimensional surface of phase space. Note that in this case we may be talking not simply about the convergence of $(P+s)$-dimensional approximations of (1) with $s\geqslant 0$, but also about their best convergence. Indeed, the solution of the problem (1) is assumed to be continuous, and the physical values of $\hat{\Phi}$ are such that $|\Phi_{(m+1)n}|\ll|\Phi_{mn}|$. Therefore, in any finite approximation, allowing for $P+s$ additional variables can give only small corrections in the physical region to the P-dimensional solution, and the $(P+s)$-dimensional solution set should be extremal on a P-dimensional surface of phase space. Will convergence of $(P+s)$-dimensional successive approximations be achieved on a P-dimensional surface of phase space or will their best convergence depend on the specific form of the approximation of the boundary conditions and its symmetry? Thus, if the dimensionality of the set of stationary solutions, determined by the symmetry of the initial perturbation, is equal to P, then the P-parameter solutions found in the successive approximations will always be distinct in the physical region from the convergence point of view, irrespective of the particular solution approach, for arbitrary choice of the variables used to approximate the boundary conditions.

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APPENDIX A

After Eqs. (1) and (3) are transformed to dimensionless coordinates $k_1x\rightarrow x$, $k_2y\rightarrow y$, $k_3z\rightarrow z$, $v/\sqrt{g/k}\rightarrow v$, $\Phi_{mn}/\sqrt{g/k_3}\rightarrow\Phi_{mn}$, the relations for the corresponding Bernoulli functions $(\overline{spq})$ take the form

$$(\overline{000})=0,\quad(\overline{001})=2/\sqrt{g/k_3},$$

$$(\overline{spq})=\frac{(-1)^{s+p+q}}{(2s)!(2p)!q!}$$

$$\times\sum_{i=0}^{s}\sum_{j=0}^{p}\sum_{r=0}^{q}(C_q^r[C_{2s}^{2i}C_{2p}^{2j}M(2i,2j,r)M(2s-2i,2p-2j,q-r)-(k_1/k_3)^2C_{2s}^{2i+1}C_{2p}^{2j}M(2i+2,2j,r-1)M(2s-2i,2p-2j,q-r-1)-(k_2/k_3)^2C_{2s}^{2i}C_{2p}^{2j+1}M(2i,2j+2,r-1)M(2s-2i,2p-2j,q-r-1)]-2M(2s,2p,q)M(0,0,0)).$$

where (spq) are functions that depend on β_{ij} and linearly on the moments and are defined by the condition that there be no flow through the free boundary. In the Nth-order expansion in the dimensionless coordinates this condition has the form

$$-\sum_{s+p+q=N}(spq)_0x^{2s}y^{2p}z_k^q+R_N(000)_1+Q_N(000)_2+\sum_{l=1}^{N-1}\left\{R_l\sum_{s+p+q=N-l}(spq)_1x^{2s}y^{2p}z_k^q+Q_l\times\sum_{s+p+q=N-l}(spq)_2x^{2s}y^{2p}z_k^q\right\}=\sum_{s+p+q=N}(spq)_2x^{2s}y^{2p}z_k^q,$$

where

$$(000)_0=0,\quad(spq)_0=\frac{(-1)^{s+p+q}}{(2s)!(2p)!q!}M(2s,2p,q),$$

$$(spq)_1=\frac{(-1)^{s+p+q}}{(2s+1)!(2p)!q!}\left(\frac{k_1}{k_3}\right)^2M(2s+2,2p,q-1),$$

$$(spq)_2=\frac{(-1)^{s+p+q}}{(2s)!(2p+1)!q!}\left(\frac{k_2}{k_3}\right)^2M(2s,2p+2,q-1),$$

$$R_l=\sum_{i=0}^{l}2i\beta_{i,l-i}x^{2i}y^{2(l-i)},\quad Q_l=\sum_{i=0}^{l}2(l-i)\beta_{i,l-j}x^{2i}y^{2(l-i)},$$

$$R_l+Q_l=2l\sum_{i=0}^{l}\beta_{i,l-j}x^{2i}y^{2(l-i)}.$$

APPENDIX B

By virtue of symmetry, it is easy to obtain relations for γ_{ij} and β_{ij} by making the substitutions $(spq)_0\rightarrow(psq)_0$ and $(spq)_1\rightarrow(psq)_2$ in the expressions for γ_{ij} and β_{ij}, respectively. For γ_{ij} and β_{ij} we obtain

$$(\overline{001})\gamma_{10}=-(\overline{100}),$$

$$(\overline{001})^3\gamma_{20}=-((\overline{002})(\overline{100})^2-(\overline{001})(\overline{100})(\overline{101})+(\overline{001})^2(\overline{200})),$$

$$(\overline{001})^3\gamma_{11}=-(2(\overline{002})(\overline{100})(\overline{010})+(\overline{001})^2(\overline{110})-(\overline{001})(\overline{100})(\overline{011})-(\overline{001})(\overline{010})(\overline{101})),$$

$$\beta_{10}=\frac{(100)_0}{-(001)_0+2(000)_1}.$$

$$\beta_{20}=-((002)_0(100)_0^2-(001)_0(100)_0(101)_0+(001)_0^2(200)_0+2(100)_0(101)_0(000)_1-4(001)_0(200)_0(000)_1+4(200)_0(000)_1^2-2(100)_0^2(001)_1+2(001)_0(100)_0(100)_1-4(100)_0(000)_1(100)_1)/((001)_0-4(000)_1)\times((001)_0-2(000)_1)^2,$$

$$\beta_{11}=-(2(002)_0(100)_0(010)_0-(001)_0(100)_0(011)_0$$

$$-(001)_0(010)_0(101)_0+2(010)_0(101)_0(000)_1$$
$$-2(001)_0(110)_0(000)_1-2(010)_0(100)_0(001)_1$$
$$+2(001)_0(100)_0(010)_1+2(011)_0(100)_0(000)_1$$
$$-2(001)_0(110)_0(000)_1+\ldots)/((001)_0-2(000)_1)$$
$$\times((001)_0-2(000)_2)((001)_1-2(000)_1$$
$$-2(000)_2).$$

APPENDIX C

The equations of the system $\gamma_{ij}=\beta_{ij}$ with $i+j=N\geq 2$ are homogeneous in the moments. Invoking relations (8), we can transform from the moments M_{1x}, M_{1y}, M_{2x}, and M_{2y} to the variables v, R_x, and R_y and velocity-normalized moments $m(\alpha\beta\gamma)=M(\alpha\beta\gamma)/v$. The equations of the system $\gamma_{ij}=\beta_{ij}$ then take the form

$\gamma_{20}-\beta_{20}=0\rightarrow$:

$$m(4,0,0)-6m(2,2,-1)\left(\frac{k_2}{k_3}\right)^2-\frac{1}{R_x}2m(4,0,-1)$$
$$\times\left[\frac{5}{R_x}\left(2+\left(\frac{k_1}{k_3}\right)^2\right)+2\frac{k_2}{k_1}\frac{1}{R_x^{1/2}R_y^{1/2}}\right]$$
$$-\frac{1}{v}\left[\frac{9}{R_x^2R_y^{3/2}}\left(\frac{k_2}{k_3}\right)^3+15\frac{k_1}{k_3}\left(\frac{k_2}{k_3}\right)^2\frac{1}{R_x^{5/2}R_y}\right.$$
$$\left.+123\left(\frac{k_1}{k_3}\right)^2\frac{k_2}{k_3}\frac{1}{R_x^3R_y^{1/2}}+285\left(\frac{k_1}{k_3}\right)^3\frac{1}{R_x^{7/2}}\right]=0,$$

$\gamma_{11}-\beta_{11}=0\rightarrow$:

$$m(2,2,0)-3m(2,2,-1)\left(3\left(\frac{k_1}{k_3}\right)^2\frac{1}{R_x}+4\frac{k_1}{k_3}\frac{k_2}{k_3}\frac{1}{R_x^{1/2}}\frac{1}{R_y^{1/2}}\right.$$
$$\left.+3\left(\frac{k_2}{k_3}\right)^2\frac{1}{R_y}\right)-m(4,0,-1)\left(\frac{k_1}{k_3}\right)^2\frac{1}{R_y}-m(0,4,$$
$$-1)\left(\frac{k_2}{k_3}\right)^2\frac{1}{R_x}-\frac{1}{v}\left[27\left(\frac{k_1}{k_3}\right)^3\frac{1}{R_x^{5/2}R_y}\right.$$
$$+27\left(\frac{k_2}{k_3}\right)^3\frac{1}{R_xR_y^{5/2}}+45\frac{k_1}{k_3}\left(\frac{k_2}{k_3}\right)^2\frac{1}{R_x^{3/2}R_y^2}$$
$$\left.+45\left(\frac{k_1}{k_3}\right)^2\frac{k_2}{k_3}\frac{1}{R_x^2R_y^{3/2}}\right]=0.$$

The equation $\gamma_{02}-\beta_{02}=0$ is easily obtained from the equation $\gamma_{20}-\beta_{20}=0$ by interchanging the subscripts, thus: $k_1\leftrightarrow k_2$ and $R_x\leftrightarrow R_y$.

APPENDIX D

For $k_2\rightarrow k_1$, $\Delta k=(k_1-k_2)$, $R'_{x\,\mathrm{cr}}=(\partial R^+_{x\,\mathrm{cr}}/\partial k_2)_{R^+_{x\,\mathrm{cr}}=3.0/k_1}$, the expressions for the Fourier harmonics take the form

$$\Phi^+_{10\,\mathrm{cr}}(R^+_{x\,\mathrm{cr}},R^+_{y\,\mathrm{cr}})/\sqrt{g/k_1}=-0.294+(-0.035+0.291R'_{x\,\mathrm{cr}})\Delta k/k_1,$$

$$\Phi^+_{10\,\mathrm{cr}}(R^+_{x\,\mathrm{cr}},R^+_{y\,\mathrm{cr}})/\sqrt{g/k_1}=-0.294+(-1.364+0.291R'_{x\,\mathrm{cr}})\Delta k/k_1,$$

$$\Phi^+_{11\,\mathrm{cr}}(R^+_{x\,\mathrm{cr}},R^+_{y\,\mathrm{cr}})/\sqrt{g/k_1}=-0.403+(1.181-0.318R'_{x\,\mathrm{cr}})\Delta k/k_1,$$

$$\Phi^+_{20\,\mathrm{cr}}(R^+_{x\,\mathrm{cr}},R^+_{y\,\mathrm{cr}})/\sqrt{g/k_1}=0.039-(0.606+0.064R'_{x\,\mathrm{cr}})\Delta k/k_1,$$

$$\Phi^+_{02\,\mathrm{cr}}(R^+_{x\,\mathrm{cr}},R^+_{y\,\mathrm{cr}})/\sqrt{g/k_1}=0.039+(0.008-0.064R'_{x\,\mathrm{cr}})\Delta k/k_1,$$

$$\Phi^+_{30\,\mathrm{cr}}(R^+_{x\,\mathrm{cr}},R^+_{y\,\mathrm{cr}})/\sqrt{g/k_1}=-0.025+(0.185-0.011R'_{x\,\mathrm{cr}})\Delta k/k_1,$$

$$\Phi^+_{03\,\mathrm{cr}}(R^+_{x\,\mathrm{cr}},R^+_{y\,\mathrm{cr}})/\sqrt{g/k_1}=-0.025+(0.317-0.011R'_{x\,\mathrm{cr}})\Delta k/k_1,$$

and the free surface is given by

$$z^*/k_1=(-0.167-0.056R'_{x\,\mathrm{cr}}\Delta k/k_1)x^2+(-0.167+(0.333-0.056R'_{x\,\mathrm{cr}})\Delta k/k_1)y^2+(-0.021+(0.177-0.028R'_{x\,\mathrm{cr}})\Delta k/k_1)x^2y^2+(-0.001-(0.040+0.005R'_{x\,\mathrm{cr}})\Delta k/k_1)x^4+(-0.001+(0.005-0.005R'_{x\,\mathrm{cr}})\Delta k/k_1)y^4.$$

In the limit $k_2\rightarrow 0$ for $R'_{x\,\mathrm{cr}}=(\partial R^+_{x\,\mathrm{cr}}/\partial k_2)_{R^+_{x\,\mathrm{cr}}=2.0/k_1}$ the Fourier amplitudes take the form

$$\Phi^+_{10\,\mathrm{cr}}(R^+_{x\,\mathrm{cr}},R^+_{y\,\mathrm{cr}})/\sqrt{g/k_1}=-0.457-0.259R'_{x\,\mathrm{cr}}k_2/k_1,$$

$$\Phi^+_{20\,\mathrm{cr}}(R^+_{x\,\mathrm{cr}},R^+_{y\,\mathrm{cr}})/\sqrt{g/k_1}=-0.074+0.126R'_{x\,\mathrm{cr}}k_2/k_1,$$

$$\Phi^+_{30\,\mathrm{cr}}(R^+_{x\,\mathrm{cr}},R^+_{y\,\mathrm{cr}})/\sqrt{g/k_1}=-0.034+0.061R'_{x\,\mathrm{cr}}k_2/k_1,$$

$$\Phi^+_{01\,\mathrm{cr}}(R^+_{x\,\mathrm{cr}},R^+_{y\,\mathrm{cr}})/\sqrt{g/k_1}=-2.121k_2/k_1,$$

$$\Phi^+_{02\,\mathrm{cr}}(R^+_{x\,\mathrm{cr}},R^+_{y\,\mathrm{cr}})/\sqrt{g/k_1}=1.061k_2/k_1,$$

$$\Phi^+_{03\,\mathrm{cr}}(R^+_{x\,\mathrm{cr}},R^+_{y\,\mathrm{cr}})/\sqrt{g/k_1}=-0.236k_2/k_1,$$

$$\Phi^+_{11\,\mathrm{cr}}(R^+_{x\,\mathrm{cr}},R^+_{y\,\mathrm{cr}})/\sqrt{g/k_1}=0.000,$$

and the free surface is given by

$$z^*/k_1=(-0.25+0.125R'_{x\,\mathrm{cr}}k_2/k_1)x^2+(-0.017+0.034R'_{x\,\mathrm{cr}}k_2/k_1)x^4+(0.250k_2/k_1)x^4.$$

[1)]This is possibly due to the presence of extrema on the one-parameter curves[5,7] and local extrema on the two-parameter solution surfaces.

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