The phase structure of spherical P-spin glass for large values of P

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The limit $P\to\infty$ in the spherical P-spin glass model is examined. The possibility of storing information in the ferromagnetic vacuum of the model is established, and the limit of the noise strength at which the ferromagnetic phase still exists is calculated. © *1996 American Institute of Physics.* [S1063-7761(96)01912-9]

1. INTRODUCTION

The Derrida model[1] is the simplest example of spin glass. The model can be solved exactly even in the first order of replica symmetry breaking.[2] Therefore, it is possible that the model exhibits a unique property, i.e., it enables writing information in the optimum coding mode[3–6] for discrete coupling constants. The system's ability to store information is related to the ferromagnetic phase. This poses the problem of establishing the phase structure of the model and the conditions under which the ferromagnetic phase can exist. This explains the interest in the phase diagram of the spherical model. The problem, of course, is important by itself, since during recent years the spherical model has been thoroughly studied. First we discuss the known results for discrete spins, and then proceed with the case of continuous spins.

The constants $J^0_{i_1\ldots i_P}$ of the model of N spins σ_i are selected so that the vacuum configuration of the Hamiltonian $H(\sigma)$ is fixed:

$$J^0_{i_1\ldots i_P}=\xi_{i_1}\cdots\xi_{i_P}, \tag{1}$$

$$H=-\sum_{1\leqslant i_1<i_2\ldots<i_P\leqslant N}\sum_{k=1}^{\alpha N} C^k_{i_1\ldots i_P}J^0_{i_1\ldots i_P}\sigma_{i_1}\ldots\sigma_{i_P}, \tag{2}$$

where $C^k_{i_1\ldots i_P}$ is the coupling matrix. It takes a unit value only for a single set $(i_1\ldots i_P)$ for each value of k, and is zero in all other cases. The spins σ_i and ξ_i assume the values ±1. When we put

$$\sigma_i=\xi_i, \tag{3}$$

all products in (2) yield $+1$, so that the given point is the point of global minimum of $H(\sigma)$.

The above is trivial, of course. What is not trivial is that the configuration (3) remains (with a probability close to unity) a regular vacuum when noise is introduced into the constants $J^0_{i_1\ldots i_P}$.

If each of the αN constants J^0 remains regular with a probability $(1+m)/2$ and changes its sign with a probability $(1-m)/2$, then (3) remains a regular vacuum[4] so long as

$$\alpha\left[\ln 2+\frac{1+m}{2}\ln\frac{1+m}{2}+\frac{1-m}{2}\ln\frac{1-m}{2}\right]\geqslant\ln 2. \tag{4}$$

So long as this inequality is met, at low temperature the system is in a ferromagnetic vacuum (3) with a magnetization $\langle\sigma_i\xi_i\rangle\sim1$. The inequality (4) derived as the condition for the existence of the ferromagnetic phase of model (1) and (2), in the limit $P\to\infty$ it coincides with the Shannon inequality in information theory. Only the Derrida Hamiltonian (in the statistical-physics approach) makes it possible to saturate this inequality to an equality. For any other choice of $H(\sigma)$ we need a number of constants that is larger than αN in (3) for the ferromagnetic phase with a total magnetization $\langle\sigma_i\xi_i\rangle\sim1$ to exist.

Can a similar Hamiltonian be written for continuous spins? Suppose that we have N spins σ_i restricted by the condition

$$\sum_{i=1}^{N}(\sigma_i)^2=N. \tag{5}$$

The same condition holds for the ξ_i. We again take the Hamiltonian (1) and (2). If $C^k_{i_1\ldots i_P}$ is chosen symmetric in the indices i_α, for $1\leqslant i_\alpha\leqslant N$ we obtain

$$H(\sigma)=-\left(\sum_i\sigma_i\xi_i\right)^P\frac{\alpha N}{N^P}. \tag{6}$$

This function attains its minimum when condition (3) is met. In the $N\gg P\gg1$ limit this minimum is also a minimum of (1) in which P-plets with coinciding indices are forbidden.

To find the analog of condition (4) for the case of continuous spins we must establish the phase structure of spherical P-glass and determine the condition for the existence of the ferromagnetic phase. This problem has been solved for $P=2$ by Kosterlitz *et al.*,[7] who used the theory of random matrices. Here we are interested in large values of P, since only then does the magnetization tend to unity in case of nonzero noise and, as it occurs, the limit of optimum coding is reached.

2. THE PARAMAGNETIC PHASE OF THE MODEL

Let us examine spherical P-spin glass[8–10] with

$$H=-\sum_{1\leqslant i_1<i_2\ldots<i_P\leqslant N}\left(\frac{J_0N}{C_N^P}+J_{i_1\ldots i_P}\sqrt{\frac{N}{C_N^P}}\right)\sigma_{i_1}\ldots\sigma_{i_P}, \tag{7}$$

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where J_0 is the ferromagnetic constant, and the frozen random constants $J_{i_1 \ldots i_p}$ describing the noise have a zero average and a dispersion

$$\langle (j_{i_1 \ldots i_p})^2 \rangle = \frac{J^2}{2}. \tag{8}$$

The ratio J_0/J is the analog of m in (4). Calculating the partition function Z^n by a replica method similar to that used in Ref. 9, we get

$$Z^n = \int_{-i\infty}^{i\infty} \prod_{\alpha<\beta} \frac{N dq_{\alpha\beta} d\lambda_{\alpha\beta}}{2\pi} \prod_{\alpha} \int_{-i\infty}^{i\infty} \frac{\sqrt{N}}{2\pi} d\lambda_{\alpha\alpha} \times \int_{-i\infty}^{i\infty} \frac{dt_\alpha dm_\alpha}{2\pi} \exp(NG), \tag{9}$$

where

$$G = J_0 B \sum_{\alpha} (m_\alpha)^P + \frac{B^2 J^2}{4} \sum_{\alpha,\beta} (q_{\alpha\beta})^P - \frac{1}{2} \sum_{\alpha,\beta} q_{\alpha\beta}\lambda_{\alpha\beta} \sum_{\alpha} t_\alpha m_\alpha + \frac{1}{2} \ln 2\pi + \ln \int_{-\infty}^{\infty} \prod_{\alpha} dx_\alpha \exp\left\{ \frac{1}{2} \sum_{\alpha,\beta} \lambda_{\alpha\beta} x_\alpha x_\beta + \sum_{\alpha} t_\alpha x_\alpha \right\}. \tag{10}$$

Here B is the reciprocal temperature, x_α is the statistical-physics (spin) variables, and q_α is an order parameter. The last two terms on the right-hand side of Eq. (10) result from Gaussian integrals. In (9), $m_\alpha = \langle x_\alpha \rangle$ and $q_{\alpha\beta} = \langle x_\alpha x_\beta \rangle$, and t_α and $\lambda_{\alpha\beta}$ are the corresponding conjugate factors.

The normalization condition yields $q_{\alpha\alpha} = 1$. Integrating with respect to x_α and t_α, we arrive at the following expression:

$$G = \frac{1}{2} \ln 2\pi + J_0 B (m)^P + \frac{B^2 J^2}{4} \sum_{\alpha,\beta} (q_{\alpha\beta})^P + \frac{1}{2} \sum_{\alpha,\beta} \lambda_{\alpha\beta} (m_\alpha m_\beta - q_{\alpha\beta}) - \frac{1}{2} \ln \det\{-\lambda^{-1}\}_{\alpha\beta}. \tag{11}$$

Taking the derivative of this expression with respect to $\lambda_{\alpha\beta}$, we get

$$q_{\alpha\beta} = m_\alpha m_\beta - \{-\lambda^{-1}\}_{\alpha\beta}. \tag{12}$$

At high temperatures the model is in the paramagnetic phase, where $m_\alpha = 0$ and $q_{\alpha\neq\beta} = 0$. From (11) we easily obtain in a manner similar to Ref. 9

$$G = \frac{1}{2}(1 + \ln 2\pi) + \frac{B^2 J^2}{4}. \tag{13}$$

The first term is related to the integration measure and is repeated in the spin-glass and ferromagnetic phases.

3. THE SPIN-GLASS PHASE

Let us examine the case of single breaking of replica symmetry in the spin-glass phase. When the replica symmetry is broken up to a subgroup of rank m_1, we have $q_{\alpha\alpha} = 1$, $q_{\alpha\beta} = q$ for α and β belonging to one block of rank m_1, and $q_{\alpha\beta} = 0$ otherwise. If we allow for (12), the expression (9) becomes

$$G = \frac{1}{2}(1 + \ln 2\pi) + \frac{B^2 J^2}{4}[1 + (m_1 - 1) q^P] + \frac{m_1 - 1}{2m_1} \ln[1 + (m_1 - 1) q]. \tag{14}$$

Here we have employed the fact that a block of a matrix with $(m_1)^2$ elements has $m_1 - 1$ eigenvalues equal to $1 - q$ and one eigenvalue equal to $1 - q + m_1 q$. Taking the derivatives with respect to m_1 and q yields

$$\begin{cases} \dfrac{P B^2 J^2}{4} q^{P-2} = \dfrac{1}{2(1-q)(1-q+m_1 q)}, \\ \dfrac{B^2 J^2}{4} q^P = \dfrac{1}{2m_1^2} \ln \dfrac{1-q+m_1 q}{1-q}. \end{cases} \tag{15}$$

This system can be reduced to a more convenient form:

$$\begin{cases} \dfrac{q^2}{P} = \dfrac{(1-q)(1-q+m_1 q)}{m_1^2} \ln(1-q+m_1 q)(1-q) - \dfrac{q(1-q)}{m_1}, \\ \dfrac{P B^2 J^2}{4} q^P = \dfrac{1}{2(1-q)(1-q+m_1 q)}. \end{cases} \tag{16}$$

We start with the $B \to \infty$ limit for P finite. By introducing the substitution

$$q = 1 - \epsilon, \quad m_1 = \epsilon x, \quad \epsilon \to 0, \tag{17}$$

we can reduce the system (16) in the leading approximation to

$$\begin{cases} \dfrac{P B^2 J^2}{4} = \dfrac{1}{2\epsilon^2 (1+x)}, \\ \dfrac{1}{P} = \dfrac{1+x}{x^2} \ln(1+x). \end{cases} \tag{18}$$

Finding x from (18), we get

$$\epsilon = \frac{1}{BJ} \frac{\sqrt{2 \ln(1+x)}}{x}. \tag{19}$$

The expression for the free energy in this limit has the form

$$G = \frac{BJ}{4} \sqrt{2 \ln(1+x)} \left(2 + \frac{P}{x} \right) - \frac{\ln JB}{2} + \frac{(1 + \ln 2\pi)}{2}. \tag{20}$$

Now let us examine the limit $P \to \infty$. The system (16) becomes

$$\begin{cases} \dfrac{1}{P} = \dfrac{\epsilon}{m_1} \ln \dfrac{m_1}{\epsilon}, \\ \dfrac{P B^2 J^2}{2} = \dfrac{1}{\epsilon m_1}, \end{cases} \tag{21}$$

and the solution of the system (21) can be written as

$$\epsilon=\sqrt{\frac{2}{\ln P}}\frac{1}{PBJ}, \tag{22}$$

$$m_1=\frac{\sqrt{2\ln P}}{BJ}. \tag{23}$$

For the free energy we have the following expression:

$$G=\frac{BJ\sqrt{2\ln P}}{2}-\frac{1}{2}\ln\frac{PBJ}{\sqrt{2\ln P}}+\frac{1}{2}(1+\ln 2\pi). \tag{24}$$

The temperature of transition from the spin-glass phase to the paramagnetic phase can be obtained by comparing (24) and (13):

$$B_c=\frac{\sqrt{2\ln P}}{J}. \tag{25}$$

This expression was obtained without allowing for J_0. The presence of a nonzero J_0 leads to the appearance of paramagnet–spin-glass and ferromagnet–spin-glass transition lines. This case is examined below.

4. THE FERROMAGNETIC PHASE

In the limit $P\to\infty$ we expect total magnetization in the ferromagnetic phase, i.e.,

$$m_\alpha\approx 1,\quad q_{\alpha\beta}\approx 1, \tag{26}$$

with the result that there can be no breaking of replica symmetry in this case. Then in the $n\to 0$ limit we find that

$$\sum_{\alpha\beta} q_{\alpha\beta}^P\sim n^2\to 0.$$

Equation (12) implies that

$$\frac{1}{2}\sum_{\alpha\beta}\lambda_{\alpha\beta}(m_\alpha m_\beta-q_{\alpha\beta})=1. \tag{27}$$

What remains to be done is to calculate the $\lambda_{\alpha\beta}$, $q_{\alpha\beta}$, and $\ln\det\{-\lambda_{\alpha\beta}\}$. Let

$$m_\alpha=m,\quad q_{\alpha\neq\beta}=q,\quad \lambda_{\alpha\alpha}=\lambda_\alpha,\quad \lambda_{\alpha\neq\beta}=\lambda_1. \tag{28}$$

For the λ matrix we have the following representation:

$$\lambda=\lambda_0 I+\lambda_1(A-I)=(\lambda_0-\lambda_1)I+\lambda_1 A, \tag{29}$$

where I is the identity operator, and A is a matrix all of whose elements are equal to 1. For the matrix that is the inverse of λ we have

$$\lambda_1=\frac{1}{\lambda_1-\lambda_0}\left[I+\frac{\lambda_1}{\lambda_1-\lambda_0}A\right]. \tag{30}$$

Equation (12) implies that

$$1=m^2+\frac{1}{\lambda_1-\lambda_0}+\frac{1}{(\lambda_1-\lambda_0)^2}, \tag{31}$$

$$q=m^2+\frac{\lambda_1}{(\lambda_1-\lambda_0)^2}. \tag{32}$$

Calculation of the determinant of $-\lambda_{\alpha\beta}$ yields

$$\begin{aligned}\ln\det\{-\lambda_{\alpha\beta}\}&=\ln(\lambda_1-\lambda_0)^{(n-1)}[\lambda_0+(n-1)\lambda_1]\\&=n\ln(\lambda_1-\lambda_0)+n\frac{\lambda_1}{\lambda_1-\lambda_0}.\end{aligned} \tag{33}$$

For the free energy we have the following expression:

$$\begin{aligned}G=J_0Bm^P+\frac{B^2J^2(1-q^P)}{4}+\frac{\lambda_0(m^2-1)}{2}\\-\frac{\lambda_1(m^2-q)}{2}-\frac{\ln(\lambda_1-\lambda_0)}{2}-\frac{\lambda_1}{2(\lambda_1-\lambda_0)}.\end{aligned} \tag{34}$$

Calculating the derivatives with respect to λ_0 and λ_1 yields

$$PJ_0Bm^{P-2}=\lambda_1-\lambda_0, \tag{35}$$

$$\frac{PB^2J^2q^{P-2}}{2}=\lambda_1. \tag{36}$$

From Eqs. (31), (32), (35), and (36) we can derive the following pair of equations:

$$\frac{PB^2J^2q^{P-1}}{2}=\frac{q-m^2}{(1-q)^2}, \tag{37}$$

$$PJ_0Bm^{P-2}=\frac{1}{1-q}. \tag{38}$$

Now we let P go to ∞. We introduce the substitution

$$1-q=\epsilon,\quad m^2=1-\epsilon x,\quad \epsilon\to 0. \tag{39}$$

Then the system of equations (37) and (38) becomes

$$\epsilon P\exp[-\epsilon P]=\frac{2(x-1)}{B^2J^2}, \tag{40}$$

$$\epsilon P\exp\left[-\frac{\epsilon Px}{2}\right]=\frac{1}{J_0B}. \tag{41}$$

We are interested in the region $B\sim\ln P$. We obtain

$$\epsilon=\frac{J_0}{B} \tag{42}$$

and

$$x=1+\frac{BJ^2}{2J_0}. \tag{43}$$

This yields the following expression for the free energy:

$$\frac{\ln Z}{N}=J_0B-\frac{\ln P}{2}. \tag{44}$$

Now, comparing (44) and (24), we obtain two regions for the ferromagnetic phase:

$$\begin{cases}J_0>\sqrt{\dfrac{\ln P}{2}}J,\\ B>B_c\equiv\dfrac{\sqrt{2\ln P}}{J},\end{cases} \tag{45}$$

and

$$\begin{cases} \dfrac{J_0}{J} > \dfrac{B_c^2 J}{2B} + \dfrac{BJ}{4}, \\ B < B_c. \end{cases} \tag{46}$$

The solution (45) is sufficient for writing information (to the vacuum of the ferromagnetic phase). The multicritical point has the following coordinates:

$$\begin{cases} B_c = \dfrac{\sqrt{2 \ln P}}{J}, \\ J_0 = \sqrt{\dfrac{\ln P}{2}} J. \end{cases} \tag{47}$$

At this point it would be interesting to study the possibility of a Nishimori line. Apparently, direct use of the ideas and methods of Ref. 11 is impossible. But the idea of a resonance between a Gibbs distribution and a nonequilibrium distribution of the coupling constants is too appealing to reject it from the start.

The bounds (45) are the main result of this study. A similar expression has been obtained in the case of Q-colored spins with Potts interaction in the Derrida model, only with $\ln Q$ instead of $\ln P$. However, the analogy is not complete, since in the Potts model for any Q at low temperatures we have total magnetization, while in our case we are forced to take P to infinity.

What amount of information is contained in the vacuum state of N spins fixed to within $1/P$? This quantity must be equal (to within $1/P$ because of the geometric factor) to $N \ln (1/P)$. Since this coincides (see (45)) with the amount of information contained in the coupling constants, we again arrive at the optimal coding example. Thus, the situation differs from Ref. 7, where only partial magnetization is considered and there is no optimum coding. It would be extremely interesting to solve the "rarefied" variant of the model, where in the case of the Derrida model with continuous coupling constants there can be no optimum coding.

It would also be interesting to study the dynamics of the system in the ferromagnetic phase. At large values of P the dynamical temperature of the paramagnet–spin-glass transition remains finite (in our case it tends to zero). Finally, a Hamiltonian of the form (1) can be written for character recognition. Here, possibly, the concept of distance in the character space is more physical than the Hamming generalization for multicolored spins (while in the attraction region this concept is less physical).

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