ЖЭТФ, 1997, том 112, вып. 2(8), стр. 400–428 ©1997

SMALL ANGLE BHABHA SCATTERING AT LEP1. WIDE-NARROW ANGULAR ACCEPTANCE

N. P. Merenkov

National Science Center «Kharkov Institute of Physics and Technology»
310108, Kharkov, Ukraine

Submitted 19 November 1996

Analytical method is applied for the description of small angle Bhabha scattering at LEP1. Inclusive event selection for asymmetrical wide-narrow circular detectors is considered. The QED correction to the Born cross section is calculated with leading and next-to-leading accuracy in second order of perturbation theory and with leading accuracy in third order. All contributions in the second order due to the photonic radiative corrections and pair production are calculated starting from essential Feynman diagrams. Third-order correction is computed by means of the electron structure function method. Second and third-order leading corrections suitable for calorimeter event selection are investigated. Numerical results illustrate the analytical calculations.

1. INTRODUCTION

The small-angle Bhabha scattering (SABS) process is used to measure the luminosity of electron-positron colliders. At LEP1 an experimental accuracy on the luminosity of $\delta\sigma/\sigma < 0.1\%$ has been reached [1]. However, to obtain the total accuracy, a systematic theoretical error must also be added. Accurate determination of the SABS cross section therefore directly affects some physical values measured at LEP1 experiments [2, 3]. Considerable attention has therefore been recently devoted to the Bhabha process [3–11]. The accuracy that has been attained however, is still inadequate. According to these evaluations, the theoretical estimates are still incomplete and their accuracy is far from that which is required.

The theoretical calculation of SABS cross section at LEP1 involves two slightly different problems. The first one is the description of experimental restrictions used for event selection in terms of final particles phase space. The second consists in the writing of matrix element squared with the required accuracy. There are two methods for theoretical investigation of SABS at LEP1: a method based on Monte Carlo calculation [3–5, 7] and analytical method [6, 9–11].

The advantage of the Monte Carlo method is that it can model different types of detectors and event selection [3]. This method, however, cannot use the exact matrix element squared based on essential Feynman diagrams because of the infrared divergence. Therefore, some additional procedures (YFS factor exponentiation [12], utilization of electron structure functions [13]) must be used in order to eliminate this problem and to take into account the leading contribution in the higher orders. Careful attention must be given at this point because of the possibility of double counting. In any case, the next-to-leading second-order correction remains uncertain, and this is a transparent defect of the Monte Carlo method.

The advantage of the analytical method is that it can use the exact matrix element squared. The infrared problem in the context of this approach can be solved in the usual way by taking into account the virtual, real soft, and hard photon emission and pair production in every order of perturbation theory. Any questions about double counting do not arise in analytical calculations. The defect of this method is its low mobility relative to the change in the experimental conditions for event selection. Nevertheless, the analytical calculations are very important because they allow one to check many Monte Carlo calculations for different «ideal» detectors.

Analytical formula for SABS cross sections at LEP1 until now were published for inclusive event selection (IES) when circular symmetrical detectors record only final electron and positron energies [10, 11]. They define first- and second-order corrections to the Born cross section with leading (of the order of $(\alpha L)^n$) and next-to-leading (of the order of $\alpha^n L^{n-1}$) accuracy, as well as third-order correction with leading accuracy only. These contributions will have to be computed in order to reach the required per mille accuracy for SABS cross section at LEP1. Note that such an accuracy selects only collinear (like two-jets, final-state configuration) and semi-collinear (like three-jets one) kinematics.

The case of calorimeter event selection (CES) called in Ref.3 CALO1 and CALO2 for symmetrical and wide-narrow angular acceptance, was considered by the author. The results are being preparing for publication. The latters include the first-order correction with leading and next-to-leading accuracy, as well as second- and third-order corrections with leading accuracy only. Thus, the CES problem of the analytical method is the calculation of next-to-leading, second-order correction.

In this paper we perform full analytical calculation for IES with the wide-narrow angular acceptance. The first- and second-order corrections are derived with next-to-leading accuracy starting from the Feynman diagrams for two-loop elastic electron-positron scattering, one-loop single-photon emission, two-photon emission and pair production. The third-order correction is obtained with leading accuracy with the help of the electron structure function method. The results for leading second- and third-order corrections in the case of CES are also given.

The contents of this paper can be outlined as follows. In Section 2 we introduce the «observable» cross section σ_{exp}, with allowance for the cuts at angles and energies, and obtain the first-order correction. In Section 3 we investigate the second-order corrections. They include the contributions of the processes of pair production (real and virtual) considered in Section 3.1 and two-photon (real and virtual) emission. In Section 3.2 we consider the correction due to the one-side two-photon emission and in Section 3.3 we consider the correction due to opposite side two-photon emission. The expression for second-order photonic correction is given in leading approximation only, while next-to-leading conribution to it is written in Appendix A for symmetrical and wide-narrow detectors. The latter does not contain an auxiliary infrared parameter. In Section 4 we derive the full, leading, third-order correction using the expansion of electron structure functions. In Section 5 we present the numerical results suitable for IES. In Appendix B we give some relations which were used in the analytical calculations and which are very useful for numerical calculation.

2. FIRST-ORDER CORRECTION

We introduce the dimensionless quantity

$$\Sigma = \frac{1}{4\pi\alpha^2} Q_1^2 \sigma_{exp}, \tag{1}$$

where $Q_1^2 = \epsilon^2\theta_1^2$ (ϵ is the beam energy and θ_1 is the minimal angle of the wide detector). The «experimentally» measurable cross section σ_{exp} is defined as follows:

$$\sigma_{exp} = \int dx_1 dx_2 \Theta d^2 q_1^\perp d^2 q_2^\perp \Theta_1^c \Theta_2^c \frac{d\sigma(e^+ + e^- \to e^+ + e^- + X)}{dx_1 dx_2 d^2 q_1^\perp d^2 q_2^\perp}, \tag{2}$$

where X is undetected final particles, and $x_1(x_2)$, and $q_1^\perp$ $(q_2^\perp)$ are the energy fraction and the transverse component of the momentum of the electron (positron) in the final state. The functions Θ_i^c take into account the angular cuts and the function Θ takes into account cutoff on the invariant mass of the detected electron and positron:

$$\Theta_1^c = \theta(\theta_3 - \theta_-)\theta(\theta_- - \theta_1), \quad \Theta_2^c = \theta(\theta_4 - \theta_+)\theta(\theta_+ - \theta_2), \Theta = \theta(x_1 x_2 - x_c),$$

$$\theta_- = \frac{|\mathbf{q}_1^\perp|}{x_1 \epsilon}, \quad \theta_+ = \frac{|\mathbf{q}_2^\perp|}{x_2 \epsilon}. \tag{3}$$

In the case of symmetrical angular acceptance

$$\theta_2 = \theta_1, \quad \theta_3 = \theta_4, \quad \rho = \frac{\theta_3}{\theta_1} > 1$$

but for wide-narrow acceptance

$$\theta_3 > \theta_4 > \theta_2 > \theta_1, \quad \rho_i = \frac{\theta_i}{\theta_1} > 1.$$

For numerical calculation one usually takes

$$\theta_1 = 0.024, \quad \theta_3 = 0.058, \quad \theta_2 = 0.024 + \frac{0.034}{16}, \quad \theta_4 = 0.058 - \frac{0.034}{16}.$$

The first-order correction Σ_1 includes the contributions of the virtual and real soft and hard photon-emission processes

$$\Sigma_1 = \Sigma_{V+S} + \Sigma^H + \Sigma_H. \tag{4}$$

The contribution due to the virtual and real soft photon (with energy less than $\Delta\epsilon, \Delta \ll 1$) can be written as follows (in this case $x_1 = x_2 = 1$, $\mathbf{q}_1^\perp + \mathbf{q}_2^\perp = 0$):

$$\Sigma_{V+S} = 2\frac{\alpha}{\pi} \int\limits_{\rho_2^2}^{\rho_4^2} \frac{dz}{z^2} \left[2(L-1)\ln\Delta + \frac{3}{2}L - 2 \right], \quad L = \ln\frac{\epsilon^2 \theta_1^2 z}{m^2}, \tag{5}$$

where $z = \mathbf{q}_2^{\perp 2}/Q_1^2$, and m is the electron mass.

The second term on the right side of Eq. (4) represents the correction due to the hard photon emission by the electron. In this case we have

$$X = \gamma(1 - x_1, \mathbf{k}^\perp), \quad x_2 = 1, \quad \mathbf{k}^\perp + \mathbf{q}_1^\perp + \mathbf{q}_2^\perp = 0, \quad x_c < x_1 < 1 - \Delta. \tag{6}$$

This expression can be derived by integration of the differential cross section of single-photon emission over the region

$$\rho_2^2 < z < \rho_4^2, \quad x^2 < z_1 = \frac{\mathbf{q}_1^{\perp 2}}{Q_1^2} < x^2 \rho_3^2, \quad -1 < \cos\varphi < 1, \tag{7}$$

where φ is the angle between the vectors $\mathbf{q}_1^\perp$ and $\mathbf{q}_2^\perp$, in the same way as it was done in Ref. 10 for the symmetrical angular acceptance. But at this point we would like to indicate the main features of the method which is used largely in the Section 3 and which is based on the separate calculation of the contributions due to collinear kinematics and semi-collinear kinematics [14].

In collinear kinematics an emitted photon moves inside the cone within polar angles $\theta_\gamma < < \theta_0 \ll 1$ centered along the electron momentum direction (initial: $\mathbf{k}||\mathbf{p}_1$ or final: $\mathbf{k}||\mathbf{q}_1$). In semi-collinear region a photon moves outside this cone. Because such a distinction no longer has physical meaning, the dependence on the auxiliary parameter θ_0 disappears in the total contribution. This is valid for IES and for CES.

Inside collinear kinematics it is necessary to keep the electron mass in the differential cross section

$$d\sigma = \frac{2\alpha^3 s}{\pi^2 q^2}\left[\frac{1+x^2}{s_1 t_1} - \frac{2m^2}{q^2}\left(\frac{1}{s_1^2}+\frac{x^2}{t_1^2}\right)\right] d\Gamma,$$

$$d\Gamma = \frac{d^3q_1 d^3q_2 d^3k}{\epsilon_1 \omega 2\epsilon}\delta^{(4)}(p_1+p_2-k-q_1-q_2)\ , \tag{8}$$

where $q = p_1 - k - q_1$, $s_1 = 2(kq_1)$, $t_1 = 2(kp_1)$, $s = (2p_1p_2)$, and $p_1(p_2)$ is the 4-momentum of the initial electron (positron). If the photon moves inside the initial electron cone

$$s_1 = x(1-x)\epsilon^2\theta_-^2, \quad t_1 = -m^2(1-x)(1+\eta), \quad q^2 = -x^2\epsilon^2\theta_-^2 = -\epsilon^2\theta_+^2,$$

$$d\Gamma = \frac{m^2}{s}\epsilon^2\pi^2 x(1-x)dx\, d\eta\, d\theta_-^2, \quad 0 < \eta = \frac{\theta_\gamma^2\epsilon^2}{m^2} < \frac{\theta_0^2\epsilon^2}{m^2}, \tag{9}$$

and one can derive the following expression after integration over η:

$$\sigma_{\mathbf{k}||\mathbf{p}_1} = \frac{2\alpha^3}{Q_1^2}\int\limits_{\rho_2^2}^{\rho_4^2}\frac{dz}{z^2}\int\limits_{x_c}^{1-\Delta} dx\left[\frac{1+x^2}{1-x}\ln\frac{\theta_0^2\epsilon^2}{m^2} - \frac{2x}{1-x}\right]\theta(x^2\rho_3^2 - z)\ . \tag{10}$$

The right side of Eq. (10) corresponds to the contribution of the narrow strip with the width $2\sqrt{z}\lambda(1-x)$ centered around the line $z = z_1$ in (z, z_1) plane, where $\lambda = \theta_0/\theta_1$. In fact, the condition $\theta_\gamma < \theta_0$ for the initial electron cone can be formulated as follows:

$$|\sqrt{z} - \sqrt{z_1}| < \lambda(1-x), \quad -1 < \cos\varphi < -1 + \frac{\lambda^2(1-x)^2 - (\sqrt{z_1}-\sqrt{z})^2}{2\sqrt{z_1 z}}\ . \tag{11}$$

If photon moves inside the final electron cone

$$s_1 = \frac{1-x}{x}m^2(1+\zeta), \quad t_1 = -(1-x)\epsilon^2\theta_-^2, \quad q^2 = -\epsilon^2\theta_-^2 = -\epsilon^2\theta_+^2,$$

$$d\Gamma = \frac{m^2}{s}\epsilon^2\pi^2 x(1-x)dx d\zeta\frac{d\theta_-^2}{x^2}, \quad 0 < \zeta < \frac{\theta_0^2\epsilon^2 x^2}{m^2}, \tag{12}$$

and integration over ζ leads to

$$\sigma_{\mathbf{k}\|\mathbf{q}_1} = \frac{2\alpha^3}{Q_1^2}\int\limits_{\rho_2^2}^{\rho_4^2}\frac{dz}{z^2}\int\limits_{x_c}^{1-\Delta}dx\left[\frac{1+x^2}{1-x}\ln\frac{\theta_0^2\epsilon^2x^2}{m^2} - \frac{2x}{1-x}\right]. \quad (13)$$

The right side of Eq. (13) corresponds to the contribution of the strip with the width $2\sqrt{z}\times \times x^2(1-x)\lambda$ around the line $z_1 = x^2 z$ in plane (z_1, z). The condition $\theta_\gamma < \theta_0$ for the final electron cone can be formulated as $|\mathbf{r}| < \theta_0$, where $\mathbf{r} = \mathbf{k}/\omega - \mathbf{q}_1/\epsilon_1$, and the latter reads as

$$|\sqrt{z_1} - x\sqrt{z}| < x(1-x)\lambda, \quad -1 < \cos\varphi < -1 + \frac{\lambda^2x^2(1-x)^2 - (\sqrt{z_1} - x\sqrt{z})^2}{2x\sqrt{zz_1}}. \quad (14)$$

Having contributions due to the collinear regions, we now must find the contribution due to the semi-collinear regions. If $m = 0$ on the right side of Eq. (8) then the differential cross section suitable for this case can be written as follows:

$$d\sigma = \frac{\alpha^3 d\varphi\, dz\, dz_1(1+x^2)}{\pi Q_1^2 z(z_1 - xz)}\left[\frac{1}{z_1 + z + 2\sqrt{z_1 z}\cos\varphi} - \frac{x}{z_1 + x^2 z + 2x\sqrt{z_1 z}\cos\varphi}\right]dx. \quad (15)$$

When integrating the first term in the brackets on the right side of Eq. (15) one must use the restriction $\theta_\gamma > \theta_0$ or

$$|\sqrt{z_1} - \sqrt{z}| > (1-x)\lambda, \quad -1 < \cos\varphi < 1,$$

$$|\sqrt{z_1} - \sqrt{z}| < (1-x)\lambda, \quad 1 > \cos\varphi > -1 + \frac{\lambda^2(1-x)^2 - (\sqrt{z_1} - \sqrt{z})^2}{2\sqrt{zz_1}}, \quad (16)$$

and for integration of the second term one must use the restriction $|\mathbf{r}| > \theta_0$ or

$$|\sqrt{z_1} - x\sqrt{z}| > x(1-x)\lambda, \quad -1 < \cos\varphi < 1,$$

$$|\sqrt{z_1} - x\sqrt{z}| < x(1-x)\lambda, \quad 1 > \cos\varphi > -1 + \frac{\lambda^2x^2(1-x)^2 - (\sqrt{z_1} - x\sqrt{z})^2}{2x\sqrt{zz_1}}. \quad (17)$$

The integration (15) over the region (16) gives

$$\sigma_a = \frac{2\alpha^3}{Q_1^2}\int\limits_{\rho_2^2}^{\rho_4^2}\frac{dz}{z^2}\int\limits_{x_c}^{1-\Delta}\frac{1+x^2}{1-x}dx\left[\left(\ln\frac{z}{\lambda^2} + L_2\right)\theta_3^{(x)} + L_3\overline{\theta}_3^{(x)}\right]. \quad (18)$$

Analogously, the integration of right side of Eq. (15) over the region (17) gives

$$\sigma_b = \frac{2\alpha^3}{Q_1^2}\int\limits_{\rho_2^2}^{\rho_4^2}\frac{dz}{z^2}\int\limits_{x_c}^{1-\Delta}\frac{1+x^2}{1-x}dx\left(\ln\frac{z}{x^2\lambda^2} + L_1\right). \quad (19)$$

The values L_i which enter into Eqs. (18) and (19) are defined as follows:

$$L_1 = \ln\left|\frac{x^2(z-1)(\rho_3^2 - z)}{(x-z)(x\rho_3^2 - z)}\right|, \quad L_2 = \ln\left|\frac{(z-x^2)(x^2\rho_3^2 - z)}{x^2(x-z)(x\rho_3^2 - z)}\right|, \quad L_3 = \ln\left|\frac{(z-x^2)(x\rho_3^2 - z)}{(x-z)(x^2\rho_3^2 - z)}\right|.$$

In addition, the following notation for the θ-functions is used:

$$\theta_3^{(x)} = \theta(x^2\rho_3^2 - z), \quad \overline{\theta}_3^{(x)} = 1 - \theta_3^{(x)} = \theta(z - x^2\rho_3^2).$$

Thus, the Σ^H may be represented as the sum of (10), (13), (18), and (19) divided by the factor $4\pi\alpha^2/Q_1^2$ or

$$\Sigma^H = \frac{\alpha}{2\pi} \int\limits_{\rho_2^2}^{\rho_4^2} \frac{dz}{z^2} \int\limits_{x_c}^{1-\Delta} \frac{1+x^2}{1-x} \left[(1+\theta_3^{(x)})(L-1) + K(x,z;\rho_3,1)\right] dx, \tag{20}$$

$$K(x,z;\rho_3,1) = \frac{(1-x)^2}{1+x^2}(1+\theta_3^{(x)}) + L_1 + \theta_3^{(x)}L_2 + \overline{\theta}_3^{(x)}L_3.$$

Here the short notation for the θ-functions is used:

$$\theta_i^{(x)} = \theta(x^2\rho_i^2 - z), \quad \theta_i = \theta(\rho_i^2 - z), \quad \overline{\theta}_i^{(x)} = 1 - \theta_i^{(x)}, \quad \overline{\theta}_i = 1 - \theta_i.$$

It is easy to see that Σ^H for the wide-narrow detectors can be derived from Σ^H for symmetrical detectors (see Ref. 10) by changing the z-integrations limits

$$\int\limits_1^{\rho^2} dz \to \int\limits_{\rho_2^2}^{\rho_4^2} dz \tag{21}$$

and by substituting ρ_3 for ρ under integral sign.

The third term on the right side of Eq. (4) describes the photon emission by a positron. It can be derived by full analogy with Σ^H except for the restrictions on the variables z and z_1:

$$1 < z < \rho_3^2, \quad x^2\rho_2^2 < z_1 < x^2\rho_4^2. \tag{22}$$

The contribution of collinear kinematics ($\mathbf{k}\|\mathbf{p}_2$ and $\mathbf{k}\|\mathbf{q}_2$) to the single hard photon emission cross section corresponds to the integration over the regions inside the strips with a width $2\sqrt{z}(1-x)\lambda$ and $2\sqrt{z}x^2(1-x)\lambda$, respectively. It can be written as follows:

$$\sigma_{\mathbf{k}\|\mathbf{p}_2,\mathbf{k}\|\mathbf{q}_2} = \frac{2\alpha^3}{Q_1^2} \int\limits_1^{\rho_3^2} \frac{dz}{z^2} \int\limits_{x_c}^{1-\Delta} \frac{1+x^2}{1-x} dx \left\{ \left(\ln\frac{\epsilon^2\theta_0^2}{m^2} - \frac{2x}{1-x} \right) \Delta_{42}^{(x)} \times \right.$$
$$\left. \times \left(\ln\frac{\epsilon^2\theta_0^2 x^2}{m^2} - \frac{2x}{1-x} \right) \Delta_{42} \right\}, \tag{23}$$

where

$$\Delta_{42}^{(x)} = \theta_4^{(x)} - \theta_2^{(x)}, \quad \Delta_{42} = \theta_4 - \theta_2. \tag{24}$$

The contribution of the semi-collinear kinematics can be derived by integration (15), taking into account the restrictions (16), (17), and (22). The latter corresponds to regions outside the narrow strips near $z_1 = z$ and $z_1 = x^2 z$, respectively. The result is

$$\sigma_{a+b} = \frac{2\alpha^3}{Q_1^2}\int\limits_1^{\rho_3^2}\frac{dz}{z^2}\int\limits_{x_c}^{1-\Delta}\frac{1+x^2}{1-x}dx\left[\ln\frac{z}{\lambda^2}(\Delta_{42}+\Delta_{42}^{(x)}) + \overline{L}_2\Delta_{42}^{(x)} + (\overline{L}_1 - 2\ln x)\Delta_{42} + \right.$$

$$\left. + \overline{L}_3(\overline{\theta}_4^{(x)} - \theta_2^{(x)}) + \overline{L}_4(\overline{\theta}_4 - \theta_2)\right], \qquad (25)$$

where

$$\overline{L}_1 = \ln\left|\frac{(z-\rho_2^2)(\rho_4^2 - z)x^2}{(x\rho_4^2 - z)(x\rho_2^2 - z)}\right|, \quad \overline{L}_2 = \ln\left|\frac{(z - x^2\rho_2^2)(x^2\rho_4^2 - z)}{x^2(x\rho_4^2 - z)(x\rho_2^2 - z)}\right|,$$
$$\overline{L}_3 = \ln\left|\frac{(z - x^2\rho_2^2)(x\rho_4^2 - z)}{(x^2\rho_4^2 - z)(x\rho_2^2 - z)}\right|, \quad \overline{L}_4 = \ln\left|\frac{(z-\rho_2^2)(x\rho_4^2 - z)}{(\rho_4^2 - z)(x\rho_2^2 - z)}\right|. \qquad (26)$$

The Σ_H is the sum of (23) and (25) divided by $4\pi\alpha^2/Q_1^2$:

$$\Sigma_H = \frac{\alpha}{2\pi}\int\limits_1^{\rho_3^2}\frac{dz}{z^2}\int\limits_{x_c}^{1-\Delta}\frac{1+x^2}{1-x}dx\left[(L-1)(\Delta_{42}+\Delta_{42}^{(x)}) + \widetilde{K}(x,z;\rho_4,\rho_2)\right], \qquad (27)$$

$$\widetilde{K} = \frac{(1-x)^2}{1+x^2}(\Delta_{42}+\Delta_{42}^{(x)}) + \Delta_{42}\overline{L}_1 + \Delta_{42}^{(x)}\overline{L}_2 + (\overline{\theta}_4^{(x)} - \theta_2^{(x)})\overline{L}_3 + (\overline{\theta}_4 - \theta_2)\overline{L}_4.$$

As one can see, the auxiliary parameter θ_0 disappears in the expressions for Σ^H and Σ_H, and the large logarithm acquires the correct appearence. Thus, the separate investigation of contributions due to collinear and semi-collinear kinematics simplifies the calculations and gives a dipper understanding of the underlying physics. This approach is very important for the study of CES when it needs to describe events that belong to the electron (or positron) cluster in a different way compared with events that do not belong to it.

The different parts on the right side of Eq. (4) depend on the auxiliary infrared parameter Δ but the sum does not. It has the form

$$\Sigma_1 = \frac{\alpha}{2\pi}\left\{\int\limits_1^{\rho_3^2}\frac{dz}{z^2}\left[-\Delta_{42} + \int\limits_{x_c}^{1}\left((L-1)P_1(x)(\Delta_{42}+\Delta_{42}^{(x)}) + \frac{1+x^2}{1-x}\widetilde{K}\right)dx\right] + \right.$$

$$\left. + \int\limits_{\rho_2^2}^{\rho_4^2}\frac{dz}{z^2}\left[-1 + \int\limits_{x_c}^{1}\left((L-1)P_1(x)(1+\theta_3^{(x)}) + \frac{1+x^2}{1-x}K\right)dx\right]\right\}, \qquad (28)$$

where

$$P_1(x) = \frac{1+x^2}{1-x}\theta(1-x-\Delta) + (2\ln\Delta + \frac{3}{2})\delta(1-x), \quad \Delta \to 0.$$

In order to eliminate the Δ-dependence, one can use the relations

$$\int\limits_{x_c}^{1}P_1(x)dx = -\int\limits_0^{x_c}\frac{1+x^2}{1-x}dx, \quad \int\limits_{x_c}^{1}P_1(x)\overline{\theta}_3^{(x)}dx = \overline{\theta}_3^{(x_c)}\int\limits_{x_c}^{\sqrt{z}/\rho_3}\frac{1+x^2}{1-x}dx,$$

$$\int\limits_{x_c}^{1}P_1(x)\overline{\Delta}_{42}^{(x)}dx = \theta_4\overline{\theta}_4^{(x_c)}\int\limits_{x_c}^{\sqrt{z}/\rho_4}\frac{1+x^2}{1-x}dx - \theta_2\overline{\theta}_2^{(x_c)}\int\limits_{x_c}^{\sqrt{z}/\rho_2}\frac{1+x^2}{1-x}dx, \qquad (29)$$

where $\overline{\Delta}_{42}^{(x)} = \Delta_{42} - \Delta_{42}^{(x)}$.

The right side of Eq. (28) is the full first-order QED correction to the Born SABS cross section at LEP1 for IES with switched-off vacuum polarization effect. The latter can be taken into account by inserting the quantity $[1 - \Pi(-zQ_1^2)]^{-2}$ under the sign of the z-integration. (For Π see Ref. 3 and the bibliography cited there).

3. SECOND-ORDER CORRECTION

The second-order correction contains the contributions due to two-photon (real and virtual) emission and pair production. As in the symmetrical case, one needs to distinguish between the situations in which additional photons attach only one fermion line (one-side emission) and two fermion lines (opposite-sides emission) in the corresponding Feynman diagrams.

3.1. The contribution of pair production

Consider at first the contribution of the electron-positron pair production Σ^{pair} to the second-order correction:

$$\Sigma^{pair} = \Sigma^{e^+e^-} + \Sigma_{e^+e^-}. \tag{30}$$

To avoid writing some formulas which have the same structure for symmetrical and wide-narrow angular acceptance we will often refer the reader to Ref. 11 in which the details of computation are given for the symmetrical case.

From Section 2 we can write the expression for $\Sigma^{e^+e^-}$ when the created electron-positron pair moves in the electron momentum direction, using the result of Ref.11 for $\Sigma^{e^+e^-}$. It needs only to change the z-integration limits: $(\rho^2, 1) \to (\rho_4^2, \rho_2^2)$ and substitute ρ_3 for ρ everywhere under the integral sign. We can write the result as follows:

$$\Sigma^{e^+e^-} = \frac{\alpha^2}{4\pi^2}\int\limits_{\rho_2^2}^{\rho_4^2}\frac{dz}{z^2}L\left\{L\left(1+\frac{4}{3}\ln(1-x_c)-\frac{2}{3}\int\limits_{x_c}^{1}\frac{dx}{1-x}\overline{\theta}_3^{(x)}\right)-\frac{17}{3}-\frac{8}{3}\zeta_2-\right.$$
$$-\frac{40}{9}\ln(1-x_c)+\frac{8}{3}\ln^2(1-x_c)+\int\limits_{x_c}^{1}\frac{dx}{1-x}\overline{\theta}_3^{(x)}\left(\frac{20}{9}-\frac{8}{3}\ln(1-x)\right)+$$
$$\left.+\int\limits_{x_c}^{1}\left[L\overline{R}(x)(1+\theta_3^{(x)})+\theta_3^{(x)}C_1(x,z;\rho_3)+C_2(x)+d_2(x,z;\rho_3)\right]dx\right\}, \tag{31}$$

$$\overline{R}(x) = (1+x)\left(\ln x - \frac{1}{3}\right) + \frac{1-x}{6x}(4+7x+4x^2),$$

$$C_1(x,z;\rho_3) = -\frac{113}{9}+\frac{142}{9}x-\frac{2}{3}x^2-\frac{4}{3x}-\frac{4}{3}(1+x)\ln(1-x)+\frac{2(1+x^2)}{3(1-x)}\times$$
$$\times\left[2\ln\left|\frac{x^2\rho_3^2-z}{x\rho_3^2-z}\right|-3L_{i2}(1-x)\right]+\left(8x^2+3x-9-\frac{8}{x}-\frac{7}{1-x}\right)\ln x+$$

$$+\frac{2(5x^2-6)}{1-x}\ln^2 x + R(x)\ln\frac{(x^2\rho_3^2-z)^2}{\rho_3^4},$$

$$C_2(x) = -\frac{122}{9} + \frac{133}{9}x + \frac{4}{3}x^2 + \frac{2}{3x} - \frac{4}{3}(1+x)\ln(1-x) + \frac{2(1+x^2)}{(1-x)}L_{i2}(1-x) + \tag{32}$$

$$+\frac{1}{3}\left(-8x^2-32x-20+\frac{8}{x}+\frac{13}{1-x}\right)\ln x + 3(1+x)\ln^2 x, \quad R(x) = 2\overline{R}(x) + \frac{2}{3}(1+x),$$

$$d_2(x,z;\rho_3) = \frac{2(1+x^2)}{3(1-x)}\ln\left|\frac{(z-x^2)(\rho_3^2-z)(z-1)}{(z-x)^2(x^2\rho_3^2-z)}\right| + R(x)\ln\left|\frac{(z-x^2)(\rho_3^2-z)(z-1)}{x^2\rho_3^2-z}\right|.$$

The right side of Eq. (31) does not contain infrared auxiliary parameters because it includes the contributions due to the real and virtual pair production. The contribution of the hard pair takes into account the collinear and semi-collinear kinematics.

If the created electron-positron pair is emitted in the positron momentum direction, the corresponding expression requires more modifications. The source of such modifications is the semi-collinear kinematics, as we saw in Section 2 for the single-photon emission.

The straightforward calculation shows that for the contribution of the semi-collinear region $\mathbf{p}_+\|\mathbf{p}_-$ (we use here notation $\mathbf{p}_\pm$ for the 3-momentum of the positron (electron)) we must include in Eq. (28) of Ref. 11 the expression

$$(\Delta_{42}+\Delta_{42}^{(x)})\ln\frac{z}{\lambda^2} + \Delta_{42}\ln\left|\frac{(z-\rho_2^2)(\rho_4^2-z)}{(z-x\rho_2^2)(x\rho_4^2-z)}\right| + \Delta_{42}^{(x)}\ln\left|\frac{(z-x^2\rho_2^2)(x^2\rho_4^2-z)}{x^2(z-x\rho_2^2)(x\rho_4^2-z)}\right| +$$
$$+(\overline{\theta}_4-\theta_2)\ln\left|\frac{(z-\rho_2^2)(x\rho_4^2-z)}{(z-x\rho_2^2)(\rho_4^2-z)}\right| + (\overline{\theta}_4^{(x)}-\theta_2^{(x)})\ln\left|\frac{(z-x^2\rho_2^2)(x\rho_4^2-z)}{(z-x\rho_2^2)(z-x^2\rho_4^2)}\right|, \tag{33}$$

instead of the expression in curved brackets and change the upper limit of the z-integration: $\rho\to\rho_3$.

For the contribution of the semi-collinear region $\mathbf{p}_+\|\mathbf{q}_1$ the corresponding expression is (see Eq. (33) in Ref. 11)

$$\Delta_{42}\left(\ln\frac{z}{\lambda^2} + \ln\left|\frac{(z-\rho_2^2)(\rho_4^2-z)}{x_2^2\rho_2^2\rho_4^2}\right|\right) + (\overline{\theta}_4-\theta_2)\ln\left|\frac{\rho_4^2(z-\rho_2^2)}{\rho_2^2(z-\rho_4^2)}\right|, \tag{34}$$

and for the semi-collinear region $\mathbf{p}_-\|\mathbf{p}_1$ (see Eq. (38) in Ref. 11) the corresponding expression is

$$\Delta_{42}^{(x)}\left(\ln\frac{z}{\lambda^2} + \ln\left|\frac{(z-x^2\rho_2^2)(x^2\rho_4^2-z)}{x_1^2x^4\rho_2^2\rho_4^2}\right|\right) + (\overline{\theta}_4^{(x)}-\theta_2^{(x)})\ln\left|\frac{\rho_4^2(z-x^2\rho_2^2)}{\rho_2^2(z-x^2\rho_4^2)}\right|. \tag{35}$$

In the symmetrical limit we have $\rho_3=\rho_4=\rho$, $\rho_2=1$, and

$$\Delta_{42}\to\theta(\rho^2-z)\theta(z-1), \quad \Delta_{42}^{(x)}\to\theta(x^2\rho^2-z), \quad \overline{\theta}_4^{(x)}\to\theta(z-x^2\rho^2), \quad \overline{\theta}_4,\theta_2,\theta_2^{(x)}\to 0, \tag{36}$$

and Eqs. (33)–(35) reduce to the corresponding expressions derived in Ref. 11.

The modification of the contributions due to virtual, real, soft and hard collinear pair production includes the change of the z-integral upper limit: $\rho\to\rho_3$ and a trivial change of the θ-functions under the integral sign: $\theta(x^2\rho^2-z)\to\Delta_{42}^{(x)}$, $1\to\Delta_{42}$. The sum of all the contributions has the form

$$\Sigma_{e^+e^-} = \frac{\alpha^2}{4\pi^2}\int\limits_1^{\rho_3^2}\frac{dz}{z^2}L\left\{L\left[\Delta_{42}(1+\frac{4}{3}\ln(1-x_c)) - \frac{2}{3}\int\limits_{x_c}^{1}\frac{dx}{1-x}\overline{\Delta}_{42}^{(x)}\right] + \Delta_{42}\left(-\frac{17}{3}-\frac{8}{3}\zeta_2 - \right.\right.$$

$$\left. - \frac{40}{9}\ln(1-x_c) + \frac{8}{3}\ln^2(1-x_c)\right) + \int\limits_{x_c}^{1}\frac{dx}{1-x}\overline{\Delta}_{42}^{(x)}\left(\frac{20}{9}-\frac{8}{3}\ln(1-x)\right) +$$

$$+\int\limits_{x_c}^{1}\left[L\overline{R}(x)(\Delta_{42}+\Delta_{42}^{(x)}) + \Delta_{42}^{(x)}C_1(x,z;\rho_2) + \Delta_{42}\left(C_2(x)+\overline{d}_2(x,z;\rho_2)\right) + (\overline{\theta}_4^{(x)} - \theta_4^{(x)})\times\right.$$

$$\times\left(\frac{2(1+x^2)}{3(1-x)}\ln\left|\frac{(x^2\rho_2^2-z)(x\rho_4^2-z)}{(x^2\rho_4^2-z)(x\rho_2^2-z)}\right| + R(x)\ln\left|\frac{(x^2\rho_2^2-z)\rho_4^2}{(x^2\rho_4^2-z)\rho_2^2}\right|\right) + (\overline{\theta}_4-\theta_4)\times$$

$$\left.\left.\times\left(\frac{2(1+x^2)}{3(1-x)}\ln\left|\frac{(x\rho_4^2-z)(z-\rho_2^2)}{(x\rho_2^2-z)(z-\rho_4^2)}\right| + R(x)\ln\left|\frac{(\rho_2^2-z)\rho_4^2}{(\rho_4^2-z)\rho_2^2}\right|\right)\right]\right\}, \tag{37}$$

$$\overline{d}_2(x,z;\rho_2) = \frac{2(1+x^2)}{3(1-x)}\ln\frac{(z-\rho_2^2)^2}{(z-x\rho_2^2)^2} + 2R(x)\ln\frac{z-\rho_2^2}{\rho_2^2}.$$

Using Eq. (36) we can verify that the right side of Eq. (30) goes over to the corresponding expression for symmetrical angular acceptance.

3.2. The contribution of the one-side two-photon emission

In this section we give the analytical expressions for all contributions to the second-order correction which appear due to the one-side two-photon (real and virtual) emission. The master formula, which does not contain the infrared auxiliary parameter Δ, is written only for the leading approximation, and next-to-leading contribution to it is given in Appendix A.

As before we differentiate between the radiation along the electron and positron momentum directions:

$$\Sigma_2 = \Sigma^{\gamma\gamma} + \Sigma_{\gamma\gamma}, \quad \Sigma^{\gamma\gamma} = \Sigma^{(S+V)^2} + \Sigma^{(S+V)H} + \Sigma^{HH},$$

$$\Sigma_{\gamma\gamma} = \Sigma_{(S+V)^2} + \Sigma_{(S+V)H} + \Sigma_{HH}. \tag{38}$$

The contribution of the virtual and real soft photons is the same for electron and positron emission:

$$\Sigma_{(S+V)^2} = \Sigma^{(S+V)^2} = \frac{\alpha^2}{\pi^2}\int\limits_{\rho_2^2}^{\rho_4^2}\frac{dz}{z^2}L\times$$

$$\times\left[L\left(2\ln^2\Delta + 3\ln\Delta + \frac{9}{8}\right) - 4\ln^2\Delta - 7\ln\Delta + 3\zeta_3 - \frac{3}{2}\zeta_2 - \frac{45}{16}\right]. \tag{39}$$

The virtual and real soft photon correction to the single hard photon emission differs for a photon that moves along the electron momentum direction and positron direction. In the first case the corresponding contribution can be derived with the help of the result for a symmetrical detector (see Eq. (50) in Ref. 10) using the substitutions $(\rho_4^2,\ \rho_2^2)$ instead of $(\rho^2,\ 1)$ for the z-integration limits and ρ_3 instead of ρ under the integral sign. Therefore,

$$\Sigma^{(S+V)H} = \frac{\alpha^2}{2\pi^2}\int\limits_{\rho_2^2}^{\rho_4^2}\frac{dz}{z^2}L\int\limits_{x_c}^{1-\Delta}\frac{1+x^2}{1-x}dx\left\{\left(2\ln\Delta-\ln x+\frac{3}{2}\right)\left[K(x,z;\rho_3,1)+(L-1)(1+\theta_3^{(x)})\right]+\right.$$
$$+\frac{1}{2}\ln^2 x-\frac{(1-x)^2}{2(1+x^2)}+(1+\theta_3^{(x)})(-2+\ln x-2\ln\Delta)+\overline{\theta}_3^{(x)}\left[\frac{1}{2}L\ln x+\right.$$
$$\left.\left.+\ 2\ln\Delta\ln x-\ln x\ln(1-x)-\ln^2 x-L_{i2}(1-x)-\frac{x(1-x)+4x\ln x}{2(1+x^2)}\right]\right\}. \quad (40)$$

In order to obtain the expression for $\Sigma_{(S+V)H}$ we must change in the right side of Eq. (39):

i) the limits of z-integration: $(\rho_4^2,\ \rho_2^2)\to(\rho_3^2,\ 1)$,

$$\text{ii) } K(x,z:\rho_3,1)\to\widetilde{K}(x,z:\rho_4,\rho_2),\quad \theta_3^{(x)}\to\Delta_{42}^{(x)},\quad \overline{\theta}_3^{(x)}\to\overline{\Delta}_{42}^{(x)},\quad 1\to\Delta_{42}. \quad (41)$$

The contribution of two hard photons emitted in the electron momentum directon may be obtained in the same way as $\Sigma^{(S+V)H}$, using the known result for symmetrical detectors (see Eq. (54) in Ref. 10):

$$\Sigma^{HH} = \frac{\alpha^2}{4\pi^2}\int\limits_{\rho_2^2}^{\rho_4^2}\frac{dz}{z^2}L\int\limits_{x_c}^{1-2\Delta}dx\int\limits_{\Delta}^{1-x-\Delta}dx_1\frac{I^{HH}}{x_1(1-x-x_1)(1-x_1)^2}, \quad (42)$$

$$I^{HH} = \overline{A}\theta_3^{(x)}+\overline{B}+\overline{C}\theta_3^{(1-x_1)},$$

$$\overline{A} = \gamma\beta\left(\frac{L}{2}+\ln\frac{(x^2\rho_3^2-z)^2}{x^2(x(1-x_1)\rho_3^2-z)^2}\right)+\zeta\ln\frac{(1-x_1)^2(1-x-x_1)}{xx_1}+\gamma_A,$$
$$\overline{B} = \gamma\beta\left(\frac{L}{2}+\ln\left|\frac{x^2(z-1)(\rho_3^2-z)(z-x^2)(z-(1-x_1)^2)^2(\rho_3^2x(1-x_1)-z)^2}{(\rho_3^2(1-x_1)^2-z)^2(z-(1-x_1))^2(z-x(1-x_1))^2(\rho_3^2x^2-z)}\right|\right)+$$
$$+\zeta\ln\frac{(1-x_1)^2x_1}{x(1-x-x_1)}+\delta_B, \quad (43)$$
$$\overline{C} = \gamma\beta\left(L+2\ln\left|\frac{x(\rho_3^2(1-x_1)^2-z)^2}{(1-x_1)^2(\rho_3^2x(1-x_1)-z)(\rho_3^2(1-x_1)-z)}\right|\right)-$$
$$-2(1-x_1)\beta-2x(1-x_1)\gamma,$$

where

$$\gamma = 1+(1-x_1)^2,\quad \beta = x^2+(1-x_1)^2,\quad \zeta = x^2+(1-x_1)^4,$$

$$\gamma_A = xx_1(1-x-x_1)-x_1^2(1-x-x_1)^2-2(1-x_1)\beta,$$

$$\delta_B = xx_1(1-x-x_1) - x_1^2(1-x-x_1)^2 - 2x(1-x_1)\gamma.$$

Unfortunately, it is impossible to give such a simple prescription as (41) in order to obtain Σ_{HH} from Eqs. (42) and (43). In the case of radiation of two hard photons along the positron momentum direction additional detailed analysis of semi-collinear kinematics is required. All essential points of such an analysis are given in Section 2, and the reader can make all calculations with the help of the formulas given in Appendix B of Ref. 10. The final result is

$$\Sigma_{HH} = \frac{\alpha^2}{4\pi^2} \int\limits_1^{\rho_3^2} \frac{dz}{z^2} L \int\limits_{x_c}^{1-2\Delta} dx \int\limits_{\Delta}^{1-x-\Delta} dx_1 \frac{I_{HH}}{x_1(1-x-x_1)(1-x_1)^2}, \tag{44}$$

$$I_{HH} = \widetilde{A}\Delta_{42}^{(x)} + \widetilde{C}\Delta_{42}^{(1-x_1)} + \widetilde{B}\Delta_{42} + (\overline{\theta}_4^{(x)} - \theta_2^{(x)})a + (\overline{\theta}_4^{(1-x_1)} - \theta_2^{(1-x_1)})c + (\overline{\theta}_4 - \theta_2)b,$$

$$a = \gamma\beta \ln\left|\frac{(\rho_4^2 x(1-x_1) - z)(\rho_2^2 x^2 - z)}{(\rho_2^2 x(1-x_1) - z)(\rho_4^2 x^2 - z)}\right|, \quad b = \gamma\beta \ln\left|\frac{(\rho_4^2(1-x_1) - z)(\rho_2^2 - z)}{(\rho_2^2(1-x_1) - z)(\rho_4^2 - z)}\right|,$$

$$c = \gamma\beta \ln\left|\frac{(\rho_4^2 x(1-x_1) - z)(\rho_2^2(1-x_1)^2 - z)^2(\rho_4^2(1-x_1) - z)}{(\rho_2^2 x(1-x_1) - z)(\rho_4^2(1-x_1)^2 - z)^2(\rho_2^2(1-x_1) - z)}\right|,$$

$$\widetilde{A} = \gamma\beta\left(\frac{L}{2} + \ln\left|\frac{(\rho_4^2 x^2 - z)(\rho_2^2 x^2 - z)}{x^2(\rho_4^2 x(1-x_1) - z)(\rho_2^2 x(1-x_1) - z)}\right|\right) + \zeta \ln\frac{(1-x_1)^2(1-x-x_1)}{xx_1} + \gamma_A,$$

$$\widetilde{B} = \gamma\beta\left(\frac{L}{2} + \ln\left|\frac{x^2(\rho_4^2 - z)(\rho_2^2 - z)}{(\rho_4^2(1-x_1) - z)(\rho_2^2(1-x_1) - z)}\right|\right) + \zeta \ln\frac{(1-x_1)^2 x_1}{x(1-x-x_1)} + \delta_B,$$

$$\widetilde{C} = \gamma\beta\left(L + \ln\left|\frac{x^2(\rho_4^2(1-x_1)^2 - z)^2(\rho_2^2(1-x_1)^2 - z)^2}{(1-x_1)^4(\rho_4^2 x(1-x_1) - z)(\rho_2^2 x(1-x_1) - z)(\rho_4^2(1-x_1) - z)(\rho_2^2(1-x_1) - z)}\right|\right) - $$
$$- 2(1-x_1)(\beta + x\gamma).$$

As one can see, the separate contributions to the right side of Eq. (38) depend on the infrared auxiliary parameter Δ but $\Sigma^{\gamma\gamma}$ and $\Sigma_{\gamma\gamma}$ do not. The elimination of Δ-dependence analytically required considerable effort. The leading terms are given below (for the next-to-leading terms see Appendix A):

$$\Sigma^{\gamma\gamma L} = \frac{\alpha^2}{4\pi^2} \int\limits_{\rho_2^2}^{\rho_4^2} \frac{dz}{z^2} L^2 \int\limits_{x_c}^{1} dx \left[\frac{1}{2}(1 + \theta_3^{(x)}) P_2(x) + \int\limits_x^1 \frac{dt}{t} P_1(t) P_1\left(\frac{x}{t}\right) \theta_3^{(t)}\right], \tag{45}$$

$$\Sigma_{\gamma\gamma}^{L} = \frac{\alpha^2}{4\pi^2} \int\limits_1^{\rho_3^2} \frac{dz}{z^2} L^2 \int\limits_{x_c}^{1} dx \left[\frac{1}{2}(\Delta_{42} + \Delta_{42}^{(x)}) P_2(x) + \int\limits_x^1 \frac{dt}{t} P_1(t) P_1\left(\frac{x}{t}\right) \Delta_{42}^{(t)}\right], \tag{46}$$

where

$$P_2(x) = P_1 \otimes P_1 = \int_x^1 \frac{dt}{t} P_1(t) P_1\left(\frac{x}{t}\right) = \lim_{\Delta\to 0}\left\{\left[\left(2\ln\Delta + \frac{3}{2}\right)^2 - 4\zeta_2\right]\delta(1-x) + \right.$$

$$\left. + 2\left[\frac{1+x^2}{1-x}\left(2\ln(1-x) - \ln x + \frac{3}{2}\right) + \frac{1}{2}(1+x)\ln x - 1 + x\right]\theta(1-x-\Delta)\right\}, \quad (47)$$

$$\int_0^1 P_2(x)dx = 0.$$

The expressions (45) and (46) are not convenient for numerical calculations. The suitable expressions can be written as follows:

$$\Sigma^{\gamma\gamma L} = \frac{\alpha^2}{4\pi^2}\left\{-2\int_{\rho_2^2}^{\rho_4^2}\frac{dz}{z^2}L^2\int_0^{x_c}P_2(x)dx - \int_{m_{23}}^{\rho_4^2}\frac{dz}{z^2}L^2\int_{x_c}^{\sqrt{z}/\rho_3}\left[P_1(x)g\left(\frac{x_c}{x}\right) + \frac{1}{2}P_2(x)\right]dx\right\}, \quad (48)$$

$$\Sigma^L_{\gamma\gamma} = \frac{\alpha^2}{4\pi^2}\left\{-2\int_{\rho_2^2}^{\rho_4^2}\frac{dz}{z^2}L^2\int_0^{x_c}P_2(x)dx - \int_{m_{14}}^{\rho_4^2}\frac{dz}{z^2}L^2\int_{x_c}^{\sqrt{z}/\rho_4}\left[P_1(x)g\left(\frac{x_c}{x}\right) + \frac{1}{2}P_2(x)\right]dx\right\} +$$

$$\left. + \int_{m_{12}}^{\rho_2^2}\frac{dz}{z^2}L^2\int_{x_c}^{\sqrt{z}/\rho_2}\left[P_1(x)g\left(\frac{x_c}{x}\right) + \frac{1}{2}P_2(x)\right]dx\right\}, \quad (49)$$

where

$$g(y) = y + \frac{y^2}{2} + 2\ln(1-y), \quad m_{23} = \max(\rho_2^2,\ x_c^2\rho_3^2),$$

$$m_{14} = \max(1,\ x_c^2\rho_4^2), \quad m_{12} = \max(1,\ x_c^2\rho_2^2).$$

The last two formulas can be derived with the help of the relations given in Appendix B. The integration over the x-variables in Eqs. (45) and (46) can be performed with the help of the formulas

$$\int^x P_2(y)dy = F_2(x), \quad \int^x P_1(y)g\left(\frac{x_c}{y}\right)dy = F_g(x), \quad \int^x P_1(y)dy = -g(x), \quad x < 1, \quad (50)$$

$$F_2(x) = -2x - \frac{x^2}{4} + (x + \frac{x^2}{2})\ln\frac{x^3}{(1-x)^4} + 4\ln(1-x)\ln\frac{x}{1-x} + 4L_{i2}(x), \quad (51)$$

$$F_g(x) = -\frac{x_c^2}{2x} + (2x + x^2)\ln x + \left(x_c + \frac{x_c^2}{2}\right)\ln\frac{x}{(1-x)^2} + \left(2x_c + \frac{x_c^2}{2} - 2x - \frac{x^2}{2}\right) \times$$

$$\times \ln(x - x_c) + 4L_{i2}(x) + 4L_{i2}\left(\frac{1-x}{1-x_c}\right), \quad x_c < x < 1. \quad (52)$$

Therefore, the second-order leading contribution to the SABS cross section at LEP1 can be expressed in terms of one integral over the z-variable.

It is useful to note that for CES the leading contributions in all orders of perturbation theory take into account the emission of photons in initial state only. Thus, the corresponding correction due to the one-side two photon (real and virtual) emission is

$$\Sigma^{\gamma\gamma L}_{CES} = -\frac{1}{8}\left(\frac{\alpha}{\pi}\right)^2 \int\limits_{\rho_2^2}^{\rho_4^2} \frac{dz}{z^2} L^2 \left\{ F_2(x_c) + \left[F_2\left(\frac{\sqrt{z}}{\rho_3}\right) - F_2(x_c) \right] \overline{\theta}_3^{(x_c)} \right\}, \tag{53}$$

$$\Sigma^{L}_{\gamma\gamma\ CES} = -\frac{1}{8}\left(\frac{\alpha}{\pi}\right)^2 \left\{ \int\limits_{\rho_2^2}^{\rho_4^2} \frac{dz}{z^2} L^2 F_2(x_c) + \int\limits_{1}^{\rho_4^2} \frac{dz}{z^2} L^2 \left[F_2\left(\frac{\sqrt{z}}{\rho_4}\right) - F_2(x_c) \right] \overline{\theta}_4^{(x_c)} - \right.$$

$$\left. - \int\limits_{1}^{\rho_2^2} \frac{dz}{z^2} L^2 \left[F_2\left(\frac{\sqrt{z}}{\rho_2}\right) - F_2(x_c) \right] \overline{\theta}_2^{(x_c)} \right\}. \tag{54}$$

3.3. Second order correction due to the opposite-side photon emission

In this section we calculate analytically the expression for

$$\Sigma^{\gamma}_{\gamma} = \Sigma^{S+V}_{S+V} + \Sigma^{H}_{S+V} + \Sigma^{S+V}_{H} + \Sigma^{H}_{H}. \tag{55}$$

The quantity Σ^{γ}_{γ} does not depend on the infrared auxiliary parameter Δ because it contains all contributions due to the virtual, real, soft and hard photon emission.

The first term on the right side of Eq. (55) takes into account only the «opposite-side» virtual and real soft photon corrections

$$\Sigma^{S+V}_{S+V} = \frac{\alpha^2}{\pi^2} \int\limits_{\rho_2^2}^{\rho_4^2} \frac{dz}{z^2} L \left[L(4\ln^2\Delta + 6\ln\Delta + \frac{9}{4}) - 6 - 14\ln\Delta - 8\ln^2\Delta \right]. \tag{56}$$

The contribution of one-loop virtual and real soft photon corrections to the hard single-photon emission can be written as follows:

$$\Sigma^{H}_{S+V} = \frac{\alpha^2}{2\pi^2} \int\limits_{\rho_2^2}^{\rho_4^2} \frac{dz}{z^2} \left[2(L-1)\ln\Delta + \frac{3}{2}L - 2 \right] \times$$

$$\times \int\limits_{x_c}^{1-\Delta} \frac{1+x^2}{1-x} \left[(1 + \theta_3^{(x)})(L-1) + K(x, z; \rho_3, 1) \right], \tag{57}$$

$$\Sigma^{S+V}_{H} = \frac{\alpha^2}{2\pi^2} \int\limits_{1}^{\rho_3^2} \frac{dz}{z^2} \left[2(L-1)\ln\Delta + \frac{3}{2}L - 2 \right] \times$$

$$\times \int\limits_{x_c}^{1-\Delta} \frac{1+x^2}{1-x} \left[(\Delta_{42} + \Delta_{42}^{(x)})(L-1) + \widetilde{K}(x, z; \rho_4, \rho_2) \right] dx. \tag{58}$$

In order to find the contribution of the two opposite-side hard photon emission to Σ_γ^γ, it is convenient to use the factorization theorem for the differential cross sections of two-jets processes in QED [16]:

$$\Sigma_H^H = \frac{\alpha^2}{4\pi^2}\int\limits_0^\infty \frac{dz}{z^2}\int\limits_{x_c}^{1-\Delta} dx_1 \int\limits_{\frac{x_c}{x_1}}^{1-\Delta} dx_2 \frac{1+x_1^2}{1-x_1}\,\frac{1+x_2^2}{1-x_2}\Phi(x_1,z;\rho_3,1)\Phi(x_2,z;\rho_4,\rho_2), \tag{59}$$

$$\Phi(x,z;\rho_3,1) = (\Delta_{31}+\Delta_{31}^{(x)})(L-1) + \frac{(1-x)^2}{1+x^2}(\Delta_{31}+\Delta_{31}^{(x)}) + \Delta_{31}L_1 + \Delta_{31}^{(x)}L_2 +$$
$$+ (\overline{\theta}_3^{(x)} - \theta_1^{(x)})L_3 + (\overline{\theta}_3 - \theta_1)\ln\left|\frac{(x\rho_3^2 - z)(z-1)}{(z-x)(\rho_3^2 - z)}\right|, \tag{60}$$

$$\Phi(x,z;\rho_4,\rho_2) = (\Delta_{42}+\Delta_{42}^{(x)})(L-1) + \widetilde{K}(x,z;\rho_4,\rho_2), \tag{61}$$

$$\Delta_{31} = \theta_3 - \theta_1,\quad \Delta_{31}^{(x)} = \theta_3^{(x)} - \theta_1^{(x)},\quad \theta_1 = \theta(1-z),\quad \theta_1^{(x)} = \theta(x^2 - z).$$

The Δ-dependence of the separate terms on the right side of Eq. (55) can be eliminated analytically in the whole sum. The leading contribution is expressed in terms of the electron structure functions as follows:

$$\Sigma_\gamma^{\gamma L} = \frac{\alpha^2}{4\pi^2}\int\limits_0^\infty \frac{dz}{z^2}L^2 \int\limits_{x_c}^{1} dx_1 \int\limits_{\frac{x_c}{x_1}}^{1} dx_2 P_1(x_1)P_1(x_2)(\Delta_{31}+\Delta_{31}^{(x_1)})(\Delta_{42}+\Delta_{42}^{(x_2)})\ . \tag{62}$$

The next-to-leading contribution to Σ_γ^γ is given in Appendix A.

The form of Σ_γ^γ suitable for numerical counting can be written in terms of the functions $F_2(x)$ and $F_g(x)$ in the same manner as it was done at the end of Section 3.2:

$$\Sigma_\gamma^{\gamma L} = \frac{\alpha^2}{4\pi^2}\left\{ -\int\limits_{\rho_2^2}^{\rho_4^2}\frac{dz}{z^2}L^2\left[4(1)F_2(x_c) + 2(1)\left(F_g\left(\frac{\sqrt{z}}{\rho_3}\right) - F_g(x_c)\right)\overline{\theta}_3^{(x_c)} - \right.\right.$$
$$-\int\limits_1^{\rho_4^2}\frac{dz}{z^2}L^2 2(1)\left(F_g\left(\frac{\sqrt{z}}{\rho_4}\right) - F_g(x_c)\right)\overline{\theta}_4^{(x_c)} + \int\limits_1^{\rho_2^2}\frac{dz}{z^2}L^2 2(1)\left(F_g\left(\frac{\sqrt{z}}{\rho_2}\right) - F_g(x_c)\right)\overline{\theta}_2^{(x_c)} +$$
$$+\int\limits_{x_c\rho_3\rho_4}^{\rho_4^2}\frac{dz}{z^2}L^2\left[F_g\left(\frac{\sqrt{z}}{\rho_4}\right) - F_g\left(\frac{x_c\rho_3}{\sqrt{z}}\right) + g\left(\frac{\sqrt{z}}{\rho_3}\right)\left(g\left(\frac{\sqrt{z}}{\rho_4}\right) - g\left(\frac{x_c\rho_3}{\sqrt{z}}\right)\right)\right] +$$
$$+\int\limits_{x_c\rho_2}^{1}\frac{dz}{z^2}L^2\left[F_g(\sqrt{z}) - F_g\left(\frac{x_c\rho_2}{\sqrt{z}}\right) + g\left(\frac{\sqrt{z}}{\rho_2}\right)\left(g(\sqrt{z}) - g\left(\frac{x_c\rho_2}{\sqrt{z}}\right)\right)\right] -$$
$$-\int\limits_{x_c\rho_4}^{1}\frac{dz}{z^2}L^2\left[F_g\left(\frac{\sqrt{z}}{\rho_4}\right) - F_g\left(\frac{x_c}{\sqrt{z}}\right) + g(\sqrt{z})\left(g\left(\frac{\sqrt{z}}{\rho_4}\right) - g\left(\frac{x_c}{\sqrt{z}}\right)\right)\right] -$$
$$\left.-\int\limits_{x_c\rho_3\rho_2}^{\rho_2^2}\frac{dz}{z^2}L^2\left[F_g\left(\frac{\sqrt{z}}{\rho_3}\right) - F_g\left(\frac{x_c\rho_2}{\sqrt{z}}\right) + g\left(\frac{\sqrt{z}}{\rho_2}\right)\left(g\left(\frac{\sqrt{z}}{\rho_3}\right) - g\left(\frac{x_c\rho_2}{\sqrt{z}}\right)\right)\right]\right\}. \tag{63}$$

On the right side of Eq. (63) the quantities in brackets are suitable for CES, when only the initial state radiation is taken into account.

4. THIRD-ORDER CORRECTION

Within the required accuracy only the leading contribution to the third-order correction must be kept. The latter becomes more important than the next-to-leading one for LEP2 because of the increase in the energy. In order to evalulate it, one can use the iteration up to third order of the master equation for the electron structure function [13]:

$$D(x, \alpha_{eff}) = D^{NS}(x, \alpha_{eff}) + D^S(x, \alpha_{eff}). \tag{64}$$

The iterative form of non-singlet component of Eq. (64) is

$$D^{NS}(x, \alpha_{eff}) = \delta(1-x) + \sum_{k=1}^{\infty} \frac{1}{k!}\left(\frac{\alpha_{eff}}{2\pi}\right)^k P_1(x)^{\otimes k},$$
$$\underbrace{P_1(x)\otimes\cdots\otimes P_1(x)}_{k} = P_1(x)^{\otimes k}, \quad P_1(x)\otimes P_1(x) = \int_x^1 P_1(t) P_1\left(\frac{x}{t}\right)\frac{dt}{t}. \tag{65}$$

Up to the third-order the singlet component of Eq. (64) is [13]

$$D^S(x, \alpha_{eff}) = \frac{1}{2!}\left(\frac{\alpha_{eff}}{2\pi}\right)^2 R(x) + \frac{1}{3!}\left(\frac{\alpha_{eff}}{2\pi}\right)^3 \left[2P_1\otimes R(x) - \frac{2}{3}R(x)\right], \tag{66}$$

where $R(x)$ is defined by Eq. (31). The effective coupling α_{eff} in Eqs. (64)–(66) represents the integral of the running QED constant

$$\frac{\alpha_{eff}}{2\pi} = \int_0^L \frac{\alpha dt}{2\pi(1-\alpha t/3\pi)} = \frac{3}{2}\ln\left(1 - \frac{\alpha L}{3\pi}\right)^{-1}. \tag{67}$$

The nonsinglet structure function describes the photon emission and pair production without allowance for the identity of final fermions, while the singlet structure function is responsible just for the identity effects.

Up to the third order the electron structure function has the form

$$D(x, L) = \delta(1-x) + \frac{\alpha L}{2\pi}P_1(x) + \frac{1}{2}\left(\frac{\alpha L}{2\pi}\right)^2\left(P_2(x) + \frac{2}{3}P_1(x) + R(x)\right) + \frac{1}{3}\left(\frac{\alpha L}{2\pi}\right)^3 \times$$
$$\times\left[\frac{1}{2}P_3(x) + P_2(x) + \frac{4}{9}P_1(x) + \frac{2}{3}R(x) + R^{''}(x)\right], \quad R^{''}(x) = P_1\otimes R(x). \tag{68}$$

For the functions $P_3(x)$ and $R^{''}(x)$ see Ref. 13.

The factorization form of the differential cross section [16] leads to

$$\Sigma^L = \int_0^{\infty}\frac{dz}{z^2}\int_{x_c}^1 dx_1 \int_{\frac{x_c}{x_1}}^1 dx_2 C(x_1, L)C(x_2, L), \tag{69}$$

$$C(x_1,L)=\int\limits_{x_1}^{1}\frac{dt}{t}D(t)D\left(\frac{x_1}{t}\right)\Delta_{31}^{(t)},\quad C(x_2,L)=\int\limits_{x_2}^{1}\frac{dt}{t}D(t)D\left(\frac{x_2}{t}\right)\Delta_{42}^{(t)}.$$

The expansion of $C(x_1,L)$ is

$$C(x_1,L)=\delta(1-x_1)\Delta_{31}^{(x_1)}+\frac{\alpha L}{2\pi}P_1(x_1)(\Delta_{31}^{(x_1)}+\Delta_{31})+$$
$$+\left(\frac{\alpha L}{2\pi}\right)^2\left[C_2(x_1)(\Delta_{31}^{(x_1)}+\Delta_{31})+\int\limits_{x_1}^{1}\frac{dt}{t}\Delta_{31}^{(t)}\overline{C}_2(x_1,t)\right]+$$
$$+\left(\frac{\alpha L}{2\pi}\right)^3\left[C_3(x_1)(\Delta_{31}^{(x_1)}+\Delta_{31})+\int\limits_{x_1}^{1}\frac{dt}{t}\Delta_{31}^{(t)}\overline{C}_3(x_1,t)\right],\tag{70}$$

$$C_2(x)=\frac{1}{2}P_2(x)+\frac{1}{3}P_1(x)+\frac{1}{2}R(x),\quad \overline{C}_2(x,t)=P_1(t)P_1\left(\frac{x}{t}\right),$$

$$C_3(x)=\frac{1}{6}P_3(x)+\frac{1}{3}P_2(x)+\frac{4}{27}P_1(x)+\frac{2}{9}R(x)+\frac{1}{3}R^{p}(x),\tag{71}$$

$$\overline{C}_3(x,t)=P_1(t)C_2\left(\frac{x}{t}\right)+C_2(t)P_1\left(\frac{x}{t}\right),$$

and the same for $C(x_2,L)$ with the substitution x_2 instead of x_1 and $\Delta_{42}^{(x_2)}$ (Δ_{42}) instead of $\Delta_{31}^{(x_1)}$ (Δ_{31}).

Because of the θ-functions under integral sign one has to distinguish between $\int\limits_x^1 dt\,t^{-1}A(t)B(x/t)\Delta_{31}^{(t)}$ and $\int\limits_x^1 dt\,t^{-1}B(t)A(x/t)\Delta_{31}^{(t)}$.

In the case of CES one must take into account the initial-state radiation only. Therefore, instead of (70) we can write

$$C_{CES}(x_1,L)=\Delta_{31}^{(x_1)}\left[\delta(1-x_1)+\frac{\alpha L}{2\pi}P_1(x_1)+\left(\frac{\alpha L}{2\pi}\right)^2C_2(x_1)+\left(\frac{\alpha L}{2\pi}\right)^3C_3(x_1)\right]\tag{72}$$

and likewise for $C(x_2,L)$.

The last step is to write the third-order contribution on the right side of Eq. (69):

$$\Sigma_3^L=\left(\frac{\alpha}{2\pi}\right)^3\int\limits_0^\infty\frac{dz}{z^2}L^3\int\limits_{x_c}^{1}dx\left(Z_1+\int\limits_{\frac{x_c}{x}}^{1}dx_1Z_2\right),\tag{73}$$

$$Z_1=(2\Delta_{42}+\Delta_{42}^{(x)}\Delta_{31}+\Delta_{31}^{(x)}\Delta_{42})C_3(x)+\int\limits_x^1\frac{dt}{t}(\Delta_{42}^{(t)}\Delta_{31}+\Delta_{31}^{(t)}\Delta_{42})\overline{C}_3(x,t),$$

$$Z_2 = \left[(\Delta_{31} + \Delta_{31}^{(x)})(\Delta_{42} + \Delta_{42}^{(x_1)}) + (\Delta_{31} + \Delta_{31}^{(x_1)})(\Delta_{42} + \Delta_{42}^{(x)})\right] P_1(x) C_2(x_1) +$$
$$+ P_1(x) \int_{x_1}^{1} \left[\Delta_{31}^{(t)}\Delta_{42} + \Delta_{42}^{(t)}\Delta_{31} + \Delta_{31}^{(x)}\Delta_{42}^{(t)} + \Delta_{42}^{(x)}\Delta_{31}^{(t)}\right] \frac{dt}{t} \overline{C}_2(x_1, t).$$

When writing the expressions for Z_1 and Z_2 it is assumed that $\Delta_{31}\Delta_{42} = \Delta_{42}$. In the case of CES the expressions for Z_1 and Z_2 can be written as follows:

$$Z_1 = (\Delta_{42}^{(x)}\Delta_{31} + \Delta_{31}^{(x)}\Delta_{42}) C_3(x), \quad Z_2 = (\Delta_{42}^{(x)}\Delta_{31}^{(x_1)} + \Delta_{42}^{(x_1)}\Delta_{31}^{(x)}) P_1(x) C_2(x_1). \tag{74}$$

Using the relations given in Appendix B we can represent the right side of Eq. (73) in the form suitable for numerical calculations as double integral over the z- and x-variables. It can be written as follows:

$$\Sigma_3^L = \Sigma_3^0 + \Sigma_0^3 + \Sigma_2^1 + \Sigma_1^2, \tag{75}$$

where the superscript (subscript) shows the number of additional particles (real and virtual) emitted by the electron (positron). The one-side emission contributes to the right side of Eq. (75) as

$$\Sigma_3^0 + \Sigma_0^3 = \left(\frac{\alpha}{2\pi}\right)^3 \left\{ \int_{\rho_2^2}^{\rho_4^2} \frac{dz}{z^2} L^3 \left[-2\int_0^{x_c} F_p(x)dx + 2\int_{x_c}^{1} F_r(x)dx - \right.\right.$$
$$\left. - \overline{\theta}_3^{(x_c)} \int_{x_c}^{\sqrt{z}/\rho_3} F_{pr}(x, x_c)dx \right] - \int_1^{\rho_4^2} \frac{dz}{z^2} L^3 \overline{\theta}_4^{(x_c)} \int_{x_c}^{\sqrt{z}/\rho_4} F_{pr}(x, x_c)dx +$$
$$\left. + \int_1^{\rho_2^2} \frac{dz}{z^2} L^3 \overline{\theta}_2^{(x_c)} \int_{x_c}^{\sqrt{z}/\rho_2} F_{pr}(x, x_c)dx \right\}, \tag{76}$$

where

$$F_p(x) = \frac{4}{3}P_3(x) + \frac{4}{3}P_2(x) + \frac{8}{27}P_1(x), \quad F_r(x) = \frac{4}{9}R(x) + \frac{5}{3}R^p(x),$$

$$F_{pr}(x, x_c) = \frac{1}{6}P_3(x) + \frac{1}{2}P_2(x)\left[\frac{2}{3} + g\left(\frac{x_c}{x}\right)\right] + P_1(x)\left[\frac{4}{27} + \frac{1}{2}f\left(\frac{x_c}{x}\right) + \right.$$
$$\left. + \frac{2}{3}g\left(\frac{x_c}{x}\right) + \frac{1}{2}r\left(\frac{x_c}{x}; 1\right)\right] + R(x)\left[\frac{2}{9} + \frac{1}{2}g\left(\frac{x_c}{x}\right)\right] + \frac{1}{3}R^p(x),$$

$$r(z, 1) = \int_z^1 R(x)dx = -\frac{22}{9} + z + z^2 + \frac{4}{9}z^3 - \left(\frac{4}{3} + 2z + z^2\right)\ln z,$$

$$f(z) = -F_2(z).$$

In the case of CES the corresponding contribution can be derived by inserting the functions F_p^c, F_r^c, and F_{pr}^c on the right side of Eq. (76) instead of functions F_p, F_r, and F_{pr}, respectively, where

$$F_{pr}^c(x) = C_3(x), \quad F_p^c(x) = \frac{1}{6}P_3(x) + \frac{1}{3}P_2(x) + \frac{4}{27}P_1(x),$$

$$F_r^c(x) = \frac{2}{9}R(x) + \frac{1}{3}R^p(x).$$

The contribution of the opposite-side emission to the right side of Eq. (75) is

$$\Sigma_2^1 + \Sigma_1^2 = \left(\frac{\alpha}{2\pi}\right)^3 \left\{ \int\limits_{\rho_2^2}^{\rho_4^2} \frac{dz}{z^2} L^3 \left[\int\limits_0^{x_c} \left(-8P_3(x) - \frac{8}{3}P_2(x) \right) dx + \right.\right.$$

$$\left. + 4\int\limits_{x_c}^{1} R^p(x)dx - \overline{\theta}_3^{(x_c)} \int\limits_{x_c}^{\sqrt{z}/\rho_3} \left(H(x,x_c) + 2g(\frac{x_c}{x})h(x;\sqrt{z}/\rho_3)\right) dx \right] -$$

$$- \int\limits_{1}^{\rho_4^2} \frac{dz}{z^2} L^3 \overline{\theta}_4^{(x_c)} \int\limits_{x_c}^{\sqrt{z}/\rho_4} \left(H(x,x_c) + 2g\left(\frac{x_c}{x}\right) h(x;\sqrt{z}/\rho_4)\right) dx +$$

$$+ \int\limits_{1}^{\rho_2^2} \frac{dz}{z^2} L^3 \overline{\theta}_2^{(x_c)} \int\limits_{x_c}^{\sqrt{z}/\rho_2} \left(H(x,x_c) + 2g\left(\frac{x_c}{x}\right) h(x;\sqrt{z}/\rho_2)\right) dx +$$

$$+ \int\limits_{x_c\rho_3\rho_4}^{\rho_4^2} \frac{dz}{z^2} L^3 \left[\int\limits_{\frac{x_c\rho_4}{\sqrt{z}}}^{\sqrt{z}/\rho_3} \left(P_1(x) G\left(\frac{x_c}{x}; \frac{\sqrt{z}}{\rho_4}\right) + g\left(\frac{x_c}{x}; \frac{\sqrt{z}}{\rho_4}\right) h\left(x; \frac{\sqrt{z}}{\rho_3}\right)\right) dx + (\rho_3 \leftrightarrow \rho_4) \right] +$$

$$+ \int\limits_{x_c\rho_2}^{1} \frac{dz}{z^2} L^3 \left[\int\limits_{\frac{x_c\rho_2}{\sqrt{z}}}^{\sqrt{z}/1} \left(P_1(x) G\left(\frac{x_c}{x}; \frac{\sqrt{z}}{\rho_2}\right) + g\left(\frac{x_c}{x}; \frac{\sqrt{z}}{\rho_2}\right) h\left(x; \frac{\sqrt{z}}{1}\right)\right) dx + (\rho_2 \leftrightarrow 1) \right] -$$

$$- \int\limits_{x_c\rho_3\rho_2}^{\rho_2^2} \frac{dz}{z^2} L^3 \left[\int\limits_{\frac{x_c\rho_2}{\sqrt{z}}}^{\sqrt{z}/\rho_3} \left(P_1(x) G\left(\frac{x_c}{x}; \frac{\sqrt{z}}{\rho_2}\right) + g\left(\frac{x_c}{x}; \frac{\sqrt{z}}{\rho_2}\right) h\left(x; \frac{\sqrt{z}}{\rho_3}\right)\right) dx + (\rho_3 \leftrightarrow \rho_2) \right] -$$

$$\left. - \int\limits_{x_c\rho_4}^{1} \frac{dz}{z^2} L^3 \left[\int\limits_{\frac{x_c\rho_4}{\sqrt{z}}}^{\sqrt{z}/1} \left(P_1(x) G\left(\frac{x_c}{x}; \frac{\sqrt{z}}{\rho_4}\right) + g\left(\frac{x_c}{x}; \frac{\sqrt{z}}{\rho_4}\right) h\left(x; \frac{\sqrt{z}}{1}\right)\right) dx + (\rho_4 \leftrightarrow 1) \right] \right\}, \quad (77)$$

where

$$g(a;b) = g(a) - g(b), \quad G(a;b) = G(a) - G(b), \quad G(z) = \frac{1}{2}f(z) + \frac{1}{3}g(z) + \frac{1}{2}r(z),$$

$$H(x; x_c) = P_1(x)\left[2f\left(\frac{x_c}{x}\right) + \frac{4}{3}g\left(\frac{x_c}{x}\right) + r\left(\frac{x_c}{x};1\right)\right] + g\left(\frac{x_c}{x}\right)[P_2(x) + R(x)],$$

$$h(x; \sqrt{z}/\rho) = \int\limits_x^{\sqrt{z}/\rho} \frac{dt}{t} P_1(t) P_1\left(\frac{x}{t}\right) =$$

$$= \frac{1+x^2}{1-x}\left(\frac{3}{2} + 2\ln\frac{(\sqrt{z}/\rho - x)(1-x)}{(1-\sqrt{z}/\rho)x}\right) - 1 + x - \frac{\sqrt{z}}{\rho} + \frac{x\rho}{\sqrt{z}} - (1+x)\ln\frac{\sqrt{z}}{x\rho}.$$

Note that the substitutions inside the straight brackets concern either the limits of x-integration or the expressions under the x-integral sign.

In the case of CES the right side of Eq. (77) requires the following modifications: i) the coefficient at $P_3(x)$ must be reduced eight times, the coefficients at $P_2(x)$ and $R^{\gamma}(x)$ must be reduced four times; ii) it must be assumed that $h = 0$ and $H^c(x, x_c)$ must be inserted instead of $H(x, x_c)$, where

$$H^c(x, x_c) = P_1(x)\left[\frac{1}{2}f\left(\frac{x_c}{x}\right) + \frac{2}{3}g\left(\frac{x_c}{x}\right) + \frac{1}{2}r\left(\frac{x_c}{x};1\right)\right] + \frac{1}{2}g\left(\frac{x_c}{x}\right)[P_2(x) + R(x)].$$

5. THE NUMERICAL RESULTS

The numerical calculations were carried out for the beam energy $\epsilon = 46.15$ GeV, and the limited angles of the circular detectors were taken from Eq. (3). The Born cross section

$$\Sigma_B = \frac{4\pi\alpha^2}{Q_1^2}\int\limits_{\rho_2^2}^{\rho_4^2} \frac{dz}{z^2}\left(1 - \theta_1^2\frac{z}{2}\right)$$

(in the symmetrical wide-wide case the limits of integration are 1 and ρ_3^2) is 175.922 nb for the ww angular acceptance and 139.971 nb for the nn and wn angular acceptances.

The results of our calculations of the QED correction with the switched-off vacuum polarization are shown in Tables 1–3. For comparsion we give also the corresponding numbers derived by the help of Monte Carlo program BHLUMI [3].

As one can see from Table 1 there is an approximately constant difference on the level of 0.3‰ between our analytical and Monte Carlo results within first-order correction. The possible reason for this effect is as follows. In the analytical calculation we systematically ignore the terms with $\theta^2 \simeq |t|/s$ as compared with unity. It is well known, however, that such terms have double logarithmic asymptotic [17] and parametrically equal $(\alpha|t|/\pi s)\ln^2(|t|/s)$ which is just 0.1‰ for the LEP1 conditions. Monte Carlo BHLUMI program, to the best of our knowledge, takes into account all first-order contributions [18].

Table 2 gives the absolute values of the second-order correction to the SABS cross section, with allowance for the leading and next-to-leading contributions. The correction due to the pair production is small, in agreement with the results of the Ref. 6. The second-order photonic correction is represented as a sum of the leading contribution and next-to-leading one. As one can see, the next-to-leading part is not negligible.

Table 1

The SABS cross section (in nb) with first- and second-order photonic correction

	First-order correction				Second-order correction			
x_c	BHLUMI ww	ww	nn	wn	BHLUMI ww	ww	nn	wn
0.1	166.046	166.008	130.813	134.504	166.892	166.958	131.674	134.808
0.3	164.740	164.702	129.797	133.416	165.374	165.447	130.524	133.583
0.5	162.241	162.203	128.001	131.428	162.530	162.574	128.474	131.127
0.7	155.431	155.390	122.922	125.809	155.668	155.597	123.206	125.225
0.9	134.390	134.334	106.478	107.945	137.342	137.153	108.820	109.667

Table 2

The second-order absolute correction to the SABS cross section (in nb)

	Pair production			Two-photon emission		
x_c	ww	nn	wn	ww	nn	wn
0.1	0.007	−0.004	0.015	0.742 + 0.208	0.679 + 0.182	0.249 + 0.091
0.3	−0.033	−0.033	−0.020	0.546 + 0.199	0.556 + 0.171	0.069 + 0.098
0.5	−0.058	−0.050	−0.041	0.140 + 0.231	0.291 + 0.182	−0.314 + 0.134
0.7	−0.090	−0.074	−0.069	−0.027 + 0.234	0.117 + 0.187	−0.571 + 0.170
0.9	−0.142	−0.115	−0.115	2.961 − 0.142	2.458 − 0.116	1.822 − 0.090

Table 3

Leading third-order correction and SABS cross section as obtained in this work

	Third-order correction			SABS cross section at LEP1		
x_c	ww	nn	wn	ww	nn	wn
0.1	−0.055	−0.047	−0.006	166.910	131.623	134.817
0.3	−0.065	−0.053	−0.018	165.349	10.438	133.545
0.5	−0.036	−0.040	0.004	162.472	128.384	131.090
0.7	0.089	0.058	0.124	155.596	123.190	125.310
0.9	0.291	0.220	0.331	137.307	108.927	109.893

Table 3 gives the absolute value of the leading third-order correction and the SABS cross section with all corrections obtained in this work. The third-order correction takes into account the three-photon emission and pair production which is accompanied by single-photon radiation. At large values of x_c this correction is comparable with the second-order next-to-leading correction. This effect increases under the conditions of LEP2.

6. CONCLUSION

In this paper we give the analytical calculation of the QED correction to SABS cross section at LEP1 for the case of inclusive event selection and wide-narrow angular acceptance. They include the leading and next-to-leading contributions in first and second orders of perturbation theory and leading contribution in third order. The leading contributions in the case of calorimeter event selection are obtained for any form of the final electron and positron cluster. The result is represented in the form of a manifold integral with definite limits, and the functions under integral sign have no physical singularities. No problem arises with infrared divergence and double counting.

The selection of essential Feynman diagrams, utilization of Sudakov's variables, relevant to this problem, impact factor representation of the differential cross section due to the t-channel photon exchange, and the electron structure function method and investigation of underlying kinematics were very useful in this work. We emphasize separately the simplifications connected with the impact factor representation, which allows us to represent the differential cross sections of two-jet processes in QED in factorized form. The latter allows us to use the cutoff θ-functions for the final electron and positron independently at the level of differential cross section. The calculation does not require to go to c.m.s. of the underlying subprocess (as in Ref. 6) and avoids the corresponding complications.

At this point, we wish to comment on the analytical calculation of the leading contribution due to the photon emission and pair production carried out in Ref. 6. Authors of those articles used as the master formula for description of the QED corrections to the SABS cross section due to the initial-state radiation the representation valid for the cross sections of Drell-Yan process [19], electron-positron annihilation into muons (or hadrons) [20], and large-angle Bhabha scattering [21]. In this set, however, the SABS process has a very particular feature: only for it two different scales exist. The first one is the momentum transfer squared t; this scale defines the cross section. The second scale is the full c.m.s. energy squared $s = 4\epsilon^2$ and $\theta^2 \sim |t|/s \ll 1$ has the status of a small correction.

The t-scale physics is very simple and is defined by the peripheral interaction of the electron and positron due to the one-photon exchange, provided that the momentum transfer is strictly perpendicular: $t = -\mathbf{q}^2$. The s-scale physics is more complicated. At the Born level it is seen as a contribution of the annihilation diagram and also permits the energy and longitudinal momentum exchange for the contribution of the scattering diagram. The first-order QED correction for the s-scale cross section includes the contributions of the box diagrams, the large-angle photon emission and the up-down interference because both, the eikonal representation for the scattering amplitude and the factorization form of the differential cross section break down. In the second order large-angle pair production and two-photon emission appear.

The structure function used in Ref. 6 controls the t-scale cross section only and is not related to s-scale cross section because physics of different scales evolves by its own laws. This is well known from the analysis of such different problems of physics as, for example, higher twist corrections in QCD [22] and turbulence phenomenon in hydrodynamics [23].

On the other hand, only the scattered diagram contributes to the Born cross section used in Ref. 6. But every time when the annihilation diagram as compared with the scattering diagram is neglected, one must automatically neglect θ^2 as compared with unity everywhere including the Born cross section and the experimental cuts in order to be consistent. Taking into account these arguments, we must simplify the master formula in Ref. 6 by eliminating the terms proportional to $\xi \sim |t|/s \ll 1$ and ξ^2 in the numerator of Eq. (5) and in the cutoff restrictions. It then becames adequate to the one obtained in Ref. 10 and the one used in this work.

Numerical evaluations show a good agreement with Monte Carlo calculations within first-order correction, but an agreement for higher-order corrections will require additional efforts.

I thank E. Kuraev and L. Trentadue for fruitful discussions and critical remarks. I also thank A. Arbuzov and G. Gach for assistance with the numerical calculations. This work was supported by INTAS grant №93-1867.

APPENDIX A

Let us first consider the next-to-leading second order Δ-independent contribution due to the one-side, two-photon emission. We first give the analytical expression for the symmetrical case, because it was not published until now. (I do not give special notation for the next-to-leading contribution to Σ, keeping in mind that only such terms are considered in this Appendix):

$$\Sigma^{\gamma\gamma} = \Sigma_{\gamma\gamma} = \frac{1}{4}\left(\frac{\alpha}{\pi}\right)^2 \int_1^{\rho^2} \frac{dz}{z^2} L\, Y, \tag{A.1}$$

$$Y = y + \int_{x_c}^{1} dx \left\{ A + \int_0^{1-x} dx_1 \left[\frac{1}{x_1}\, 4\, \frac{1+x^2}{1-x}(\theta_\rho^{(x)} l_1 + l_2) + \left(-1 - \frac{1+x}{1-x_1} - \right.\right.\right.$$
$$\left.\left. - \frac{x}{(1-x_1)^2}\right)(l_4 + \theta_\rho^{(x)} l_3 + 2\theta_\rho^{(1-x_1)} l_5) + \frac{2(1+x)}{1-x_1}\theta_\rho^{(1-x_1)}\right] -$$
$$\left. - 4\,\frac{1+x^2}{1-x}\,\overline{\theta}_\rho^{(x)} \left[\int_{1-\sqrt{z}/\rho}^{1-x} dx_1 \left(\frac{1}{x_1} l_5 + \frac{2}{x_2}\ln\frac{x}{1-x_1}\right) + \int_0^{\sqrt{z}/\rho - x} \frac{dx_1}{x_1}\, l_6\right]\right\},$$

$$y = 12\zeta_3 + 10\zeta_2 - \frac{45}{4} - 16\ln^2(1-x_c) - 28\ln(1-x_c),$$

$$A = (1+\theta_\rho^{(x)})\left[2(5+2x) + 4(x+3)\ln(1-x) + 4\,\frac{1+x^2}{1-x}\ln x\right] +$$
$$+ 2\,\frac{1+x^2}{1-x}\left[\left(\frac{3}{2} - \ln x\right) K(x,z;\rho,1) - \frac{1}{2}\ln^2 x - \frac{(1-x)^2}{2(1+x^2)} + \right.$$
$$\left. + 2\ln(1-x)\left(\theta_\rho^{(x)} \ln\left|\frac{x^2\rho^2 - z}{x\rho^2 - z}\right| + \ln\left|\frac{(z-1)(z-x^2)(\rho^2 - z)}{(z-x)^2(x\rho^2 - z)}\right|\right)\right] +$$
$$+ \overline{\theta}_\rho^{(x)}\left[\frac{16}{1-x}\ln(1-x) + \frac{14}{1-x} - (1-x)\ln x + \right.$$
$$+ 2\,\frac{1+x^2}{1-x}\left(-\frac{3}{2}\ln^2 x + 3\ln x\ln(1-x) - L_{i2}(1-x) - \frac{x(1-x) + 4x\ln x}{2(1+x^2)} + \right.$$
$$\left.\left. + \frac{(1+x)^2}{1+x^2}\ln\left|\frac{(\sqrt{z} - x\rho)}{\rho - \sqrt{z}}\right| + 2\ln\left|\frac{\sqrt{z} - x\rho}{\rho}\right| \ln\left|\frac{x(x\rho^2 - z)}{x^2\rho^2 - z}\right|\right)\right],$$

$$l_1 = \ln\left|\frac{(x^2\rho^2 - z)(x\rho^2 - z)}{(x(1-x_1)\rho^2 - z)(x(x+x_1)\rho^2 - z)}\right|, \quad l_3 = \ln\left|\frac{(1-x_1)^2(1-x-x_1)(x^2\rho^2 - z)^2}{x^3 x_1 (x(1-x_1)\rho^2 - z)^2}\right|,$$

$$l_2 = \ln\left|\frac{(z-x)^2(z-(1-x_1)^2)(z-(x+x_1)^2)}{(z-(1-x_1))(z-x(1-x_1))((x+x_1)-z)(x(x+x_1)-z)}\right| + \\ + \ln\left|\frac{((1-x_1)^2\rho^2-z)((x+x_1)^2\rho^2-z)(x\rho^2-z)}{((x+x_1)\rho^2-z)((1-x_1)\rho^2-z)(x^2\rho^2-z)}\right|,$$

$$l_4 = \ln\left|\frac{(1-x_1)^2 x x_1(z-1)(z-x^2)(z-(1-x_1)^2)^2}{x_2(z-(1-x_1))^2(z-x(1-x_1))^2}\right| + \ln\left|\frac{(\rho^2-z)(x(1-x_1)\rho^2-z)^2}{(x^2\rho^2-z)((1-x_1)^2\rho^2-z)^2}\right|,$$

$$l_5 = \ln\left|\frac{x((1-x_1)^2\rho^2-z)^2}{(1-x_1)^2(x(1-x_1)\rho^2-z)((1-x_1)\rho^2-z)^2}\right|,$$

$$l_6 = \ln\left|\frac{(x\rho^2-z)((x+x_1)^2\rho^2-z)^2}{(x^2\rho^2-z)(x(x+x_1)\rho^2-z)((x+x_1)\rho^2-z)}\right|.$$

For the wide-narrow angular acceptance we need to consider only the case of the positron emission $\Sigma_{\gamma\gamma}$, because the corresponding expression for the electron emission $\Sigma^{\gamma\gamma}$ is Eq. (A.1) with (ρ_4^2, ρ_2^2) as the limits of z–integration and ρ_3 instead ρ under the integral sign.

The analytical expression for $\Sigma_{\gamma\gamma}$ has the form

$$\Sigma_{\gamma\gamma} = \frac{1}{4}\left(\frac{\alpha}{\pi}\right)^2 \int\limits_1^{\rho_3^2} \frac{dz}{z^2} L\, A_N^W, \tag{A.2}$$

$$A_N^W = y\Delta_{42} + \int\limits_{x_c}^1 dx \Bigg\{ \Delta_{42}\left[4(4+3x) + 6(x+3)\ln(1-x) + \left(x-1+4\,\frac{1+x^2}{1-x}\right)\ln x\right] + \\ +\Delta_{42}^{(x)}\left[(1-x)(3+\ln x)+2(x+3)\ln(1-x)+4\frac{1+x^2}{1-x}\ln x\right] + \overline{\Delta}_{42}^{(x)}\frac{2}{1-x}\times \\ \times\left(4+(1+x)^2\right)\ln(1-x) + 2\,\frac{(1+x)^2}{1-x}\left(\theta_4\overline{\theta}_4^{(x)}\ln\left|\frac{\sqrt{z}-x\rho_4}{\rho_4-\sqrt{z}}\right| - \theta_2\overline{\theta}_2^{(x)}\ln\left|\frac{\sqrt{z}-x\rho_2}{\rho_2-\sqrt{z}}\right|\right) + \\ + \frac{1+x^2}{1-x}B + \int\limits_0^{1-x} dx_1\left[2\,\frac{1+x^2}{(1-x)x_1}\left(\Delta_{42}^{(x)}l_{1+} + \Delta_{42}l_{2+} + (\overline{\theta}_4^{(x)} - \theta_2^{(x)})l_{1-} + (\overline{\theta}_4 - \theta_2)l_{2-}\right) + \right. \\ + \left(-1-\frac{1+x}{1-x_1}-\frac{x}{(1-x_1)^2}\right)\left(\Delta_{42}^{(x)}\left(\ln\frac{(1-x_1)^2x_2}{x^3x_1}+l_{3+}\right) + \Delta_{42}\left(\ln\frac{(1-x_1)^2xx_1}{x_2}+l_{4+}\right) + \right. \\ + \Delta_{42}^{(1-x_1)}\left(2\ln\frac{x}{(1-x_1)^2}+l_{5+}\right) + (\overline{\theta}_4^{(x)} - \theta_2^{(x)})l_{3-} + (\overline{\theta}_4-\theta_2)l_{4-} + \left.(\overline{\theta}_4^{(1-x_1)} - \theta_2^{(1-x_1)})l_{5-}\right) + \\ + \left. 2\,\frac{1+x}{1-x_1}\Delta_{42}^{(1-x_1)}\right] + 2\,\frac{1+x^2}{1-x}\theta_4\overline{\theta}_4^{(x)}\left[\int\limits_{1-\sqrt{z}/\rho_4}^{1-x} dx_1\left(\frac{1}{x_1}\bar{l}_6 - \frac{4}{x_2}\ln\frac{x}{1-x_1}\right) + \right. \\ + \left.\int\limits_0^{\sqrt{z}/\rho_4-x}\frac{dx_1}{x_1}\bar{l}_7\right] + 2\frac{1+x^2}{1-x}\theta_2\overline{\theta}_2^{(x)}\left[\int\limits_{1-\sqrt{z}/\rho_2}^{1-x} dx_1\left(\frac{1}{x_1}\tilde{l}_6 + \frac{4}{x_2}\ln\frac{x}{1-x_1}\right) + \int\limits_0^{\sqrt{z}/\rho_2-x}\frac{dx_1}{x_1}\tilde{l}_7\right]\Bigg\},$$

$$B = \Delta_{42}\left(-2\ln^2 x + 2\ln(1-x)\ln\left|\frac{x^4(z-\rho_2^2)^2(z-x^2\rho_2^2)(x^2\rho_4^2-z)(\rho_4^2-z)^2}{(z-x\rho_2^2)^3(x\rho_4^2-z)^3}\right|\right) +$$
$$+ \Delta_{42}^{(x)}\left(\ln^2 x + 2\ln(1-x)\ln\left|\frac{(z-x^2\rho_2^2)(x^2\rho_4^2-z)}{x^4(z-x\rho_2^2)(x\rho_4^2-z)}\right|\right) + (3-2\ln x)\widetilde{K}(x,z;\rho_4,\rho_2) +$$
$$+ \overline{\Delta}_{42}^{(x)}\left(7 - 2\ln x\ln(1-x) - 2\ln^2 x - 2L_{i2}(1-x) - \frac{x(1-x)+4x\ln x}{1+x^2}\right) +$$
$$+ 2(\overline{\theta}_4 - \theta_2)\ln(1-x)\ln\left|\frac{(x\rho_4^2-z)^3(z-\rho_2^2)^2(z-x^2\rho_2^2)}{(\rho_4^2-z)^2(x^2\rho_4^2-z)(z-x\rho_2^2)^3}\right| +$$
$$+ 2(\overline{\theta}_4^{(x)} - \theta_2^{(x)})\ln(1-x)\ln\left|\frac{(z-x^2\rho_2^2)(x\rho_4^2-z)}{(x^2\rho_4^2-z)(x\rho_2^2-z)}\right| +$$
$$+ 4\theta_4\overline{\theta}_4^{(x)}\ln\left|\frac{x\rho_4-\sqrt{z}}{\rho_4}\right|\ln\left|\frac{x(x\rho_4^2-z)}{x^2\rho_4^2-z}\right| + 4\theta_2\overline{\theta}_2^{(x)}\ln\left|\frac{\sqrt{z}-x\rho_2}{\rho_2}\right|\ln\left|\frac{z-x^2\rho_2^2}{x(z-x\rho_2^2)}\right|,$$

$$l_{1\pm} = (1\pm\hat{c})\ln\left|\frac{(z-x^2\rho_2^2)(z-x\rho_2^2)}{(z-x(1-x_1)\rho_2^2)(z-x(x+x_1)\rho_2^2)}\right|,$$

$$l_{2\pm} = (1\pm\hat{c})\left[\ln\left|\frac{(z-x\rho_2^2)^3(z-(1-x_1)^2\rho_2^2)^2(z-(x+x_1)^2\rho_2^2)^2}{(z-x^2\rho_2^2)(z-x(1-x_1)\rho_2^2)(z-x(x+x_1)\rho_2^2)(z-(1-x_1)\rho_2^2)^2(z-(x+x_1)\rho_2^2)^2}\right|\right],$$

$$l_{3\pm} = (1\pm\hat{c})\ln\left|\frac{z-x^2\rho_2^2}{z-x(1-x_1)\rho_2^2}\right|, \quad l_{4\pm} = (1\pm\hat{c})\ln\left|\frac{z-\rho_2^2}{z-(1-x_1)\rho_2^2}\right|,$$

$$l_{5\pm} = (1\pm\hat{c})\ln\left|\frac{(z-(1-x_1)^2\rho_2^2)^2}{(z-x(1-x_1)\rho_2^2)(z-(1-x_1)\rho_2^2)}\right|,$$

$$\tilde{l}_6 = \ln\left|\frac{x^2(z-(1-x_1)^2\rho_2^2)^4}{(1-x_1)^4(z-x(1-x_1)\rho_2^2)^2(z-(1-x_1)\rho_2^2)^2}\right|,$$

$$\tilde{l}_7 = \ln\left|\frac{(z-x\rho_2^2)^2(z-(x+x_1)^2\rho_2^2)^4}{(z-x^2\rho_2^2)^2(z-x(x+x_1)\rho_2^2)^2(z-(x+x_1)\rho_2^2)^2}\right|, \quad \bar{l}_6 = -\hat{c}\tilde{l}_6, \quad \bar{l}_7 = -\hat{c}\tilde{l}_7,$$

where $x_2 = 1 - x - x_1$, and $\hat{c}$ is the operator of the substitution:

$$\hat{c}f(\rho_2) = f(\rho_4). \tag{A.3}$$

It can be verified that in the symmetrical limit Eq. (A.2) coincides with Eq. (A.1).

For the opposite-side emission the next-to-leading contribution to Σ in the symmetrical case is

$$\Sigma_\gamma^\gamma = \left(\frac{\alpha}{\pi}\right)^2 L\int_0^\infty \frac{dz}{z^2}\,T, \tag{A.4}$$

$$T = A\theta_\rho\overline{\theta}_1 - \int_{x_c}^1 dx\left[\frac{1+x^2}{2(1-x)}N(x,z;\rho,1) + \Xi(x) + \frac{\overline{\Xi}(x)}{1-x}\right]\times$$
$$\times\int_{x_c/x_1}^1 dx_1\left[(1+x_1)\Xi(x_1) + \frac{2\overline{\Xi}(x_1)}{1-x_1}\right], \tag{A.5}$$

where

$$A = -6 - 14\ln(1-x_c) - 8\ln^2(1-x_c) + \int\limits_{x_c}^{1} dx \Big\{ 7(1+x) +$$
$$+ \frac{1+x^2}{2(1-x)}\left[3K(x,z;\rho,1) + 7\overline{\theta}_\rho^{(x)}\right] + 2\ln\frac{x-x_c}{x}\left[(3+x)(1+\theta_\rho^{(x)}) +\right.$$
$$\left. + \frac{4}{1-x}\overline{\theta}_\rho^{(x)} + \frac{1+x^2}{1-x}N(x,z;\rho,1)\right] + \frac{8}{1-x}\ln\frac{x(1-x_c)}{x-x_c}\Big\}. \tag{A.6}$$

We introduce the following reduced notation for θ-functions:

$$\Xi(x) = \theta_\rho\overline{\theta}_1 + \theta_\rho^{(x)}\overline{\theta}_1^{(x)}, \quad \overline{\Xi}(x) = \theta_\rho\overline{\theta}_\rho^{(x)} - \theta_1\overline{\theta}_1^{(x)}. \tag{A.7}$$

The quantity $K(x,z;\rho,1)$ in the expression for A is the K-factor for the single-photon emission, and the quantity $N(x,z;\rho,1)$ can be derived with the help of Eq. (10) in the following way:

$$N(x,z;\rho,1) = \left(\widetilde{K}(x,z;\rho_4,\rho_2) - \frac{(1-x)^2}{1+x^2}(\Delta_{42} + \Delta_{42}^{(x)})\right)\Bigg|_{\rho_4=\rho,\ \rho_2=1}. \tag{A.8}$$

Note that $N(1,z;\rho,1) = 0$.

In the wide-narrow case the corresponding formula for Σ_γ^γ may be written as follows:

$$\Sigma_\gamma^\gamma = \frac{\alpha^2}{\pi^2}L\int\limits_0^\infty \frac{dz}{z^2}T_N^W, \tag{A.9}$$

where

$$T_N^W = \widetilde{A} - \frac{1}{2}\Bigg\{\int\limits_{x_c}^{1} dx\left[\frac{1+x^2}{2(1-x)}N(x,z;\rho_3,1) + \Xi_{31}(x) + \frac{1}{1-x}\overline{\Delta}_{31}^{(x)}\right] \times$$
$$\times \int\limits_{x_c/x}^{1} dx_1\left[(1+x_1)\Xi_{42}(x) + \frac{2}{1-x_1}\overline{\Delta}_{42}^{(x)}\right] +$$
$$+ \int\limits_{x_c}^{1} dx\left[\frac{1+x^2}{2(1-x)}N(x,z;\rho_4,\rho_2) + \Xi_{42}(x) + \frac{1}{1-x}\overline{\Delta}_{42}^{(x)}\right] \times$$
$$\times \int\limits_{x_c/x}^{1} dx_1\left[(1+x_1)\Xi_{31}(x) + \frac{2}{1-x_1}\overline{\Delta}_{31}^{(x)}\right]\Bigg\}, \tag{A.10}$$

where

$$\widetilde{A} = \left(-6 - 14\ln(1-x_c) - 8\ln^2(1-x_c)\right)\Delta_{42} +$$
$$+ \int\limits_{x_c}^{1} dx\Big\{\Delta_{42}\left[7(1+x) + \frac{8}{1-x}\ln\frac{x(1-x_c)}{x-x_c}\right] +$$

$$+\frac{1+x^2}{2(1-x)}\left[\frac{3}{2}\Delta_{42}\overline{K}(x,z;\rho_3,1)+\frac{3}{2}\Delta_{31}\widetilde{K}(x,z;\rho_4,\rho_2)+\right.$$
$$\left.+\frac{7}{2}(\Delta_{42}\overline{\Delta}_{31}^{(x)}+\Delta_{31}\overline{\Delta}_{42}^{(x)})\right]+\ln\frac{x-x_c}{x}\left[(3+x)(\Delta_{31}\Xi_{42}(x)+\Delta_{42}\Xi_{31}(x))+\right.$$
$$+\frac{4}{1-x}(\overline{\Delta}_{42}^{(x)}\Delta_{31}+\overline{\Delta}_{31}^{(x)}\Delta_{42})+\frac{1+x^2}{1-x}(\Delta_{42}N(x,z;\rho_3,1)+$$
$$\left.\left.+\Delta_{31}N(x,z;\rho_4,\rho_2))\right]\right\}, \tag{A.11}$$

and

$$\Xi_{42}(x)=\theta_4\overline{\theta}_2+\theta_4^{(x)}\overline{\theta}_2^{(x)}=\Delta_{42}+\Delta_{42}^{(x)},$$
$$\Xi_{31}(x)=\Delta_{31}+\Delta_{31}^{(x)},\quad \overline{\Delta}_{31}^{(x)}=\Delta_{31}-\Delta_{31}^{(x)}.$$

It is obvious that in the symmetrical limit Eq. (A.9) coincides with (A.4).

APPENDIX B

Here we give some relations which are used in the analytical calculations and which may be useful for the numerical computations.

For the case of the emission along the electron momentum direction these relations are

$$\int\limits_{\rho_2^2}^{\rho_4^2}dz\int\limits_{x_c}^{1}dx\,\overline{\theta}_3^{(x)}=\int\limits_{\rho_2^2}^{\rho_4^2}dz\,\overline{\theta}_3^{(x_c)}\int\limits_{x_c}^{\sqrt{z}/\rho_3}dx,$$
$$\int\limits_{\rho_2^2}^{\rho_4^2}dz\int\limits_{x_c}^{1}dx\int\limits_{0}^{1-x}dx_1\overline{\theta}_3^{(1-x_1)}=\int\limits_{\rho_2^2}^{\rho_4^2}dz\,\overline{\theta}_3^{(x_c)}\int\limits_{x_c}^{\sqrt{z}/\rho_3}dx\int\limits_{1-\sqrt{z}/\rho_3}^{1-x}dx_1. \tag{B.1}$$

For the case of the emission along the positron direction they are

$$\int\limits_{1}^{\rho_3^2}dz\int\limits_{x_c}^{1}dx\,[\overline{\theta}_4^{(x)}-\theta_2^{(x)}]=\int\limits_{1}^{\rho_3^2}dz\int\limits_{x_c}^{1}dx\,[\overline{\theta}_4-\theta_2+\theta_4\overline{\theta}_4^{(x)}+\theta_2\overline{\theta}_2^{(x)}]=$$
$$=\int\limits_{1}^{\rho_3^2}dz\left\{(\overline{\theta}_4-\theta_2)\int\limits_{x_c}^{1}dx+\theta_4\overline{\theta}_4^{(x_c)}\int\limits_{x_c}^{\sqrt{z}/\rho_4}dx+\theta_2\overline{\theta}_2^{(x_c)}\int\limits_{x_c}^{\sqrt{z}/\rho_2}dx\right\},$$
$$\int\limits_{1}^{\rho_3^2}dz\int\limits_{x_c}^{1}dx\int\limits_{0}^{1-x}dx_1\,[\overline{\theta}_4^{(1-x_1)}-\theta_2^{(1-x_1)}]=\int\limits_{1}^{\rho_3^2}dz\int\limits_{x_c}^{1}dx\left\{(\overline{\theta}_4-\theta_2)+\int\limits_{0}^{1-x}dx_1+\right.$$
$$\left.+\overline{\theta}_4^{(x_c)}\theta_4\int\limits_{x_c}^{\sqrt{z}/\rho_4}dx\int\limits_{1-\frac{\sqrt{z}}{\rho_4}}^{1-x}dx_1+\overline{\theta}_2^{(x_c)}\theta_2\int\limits_{x_c}^{\sqrt{z}/\rho_2}dx\int\limits_{1-\sqrt{z}/\rho_2}^{1-x}dx_1\right\}. \tag{B.2}$$

Some additional relations arise for the case of opposite-side emission. Let us consider first the integration-limits restrictions for the product of the θ-functions in the symmetrical case

$$\theta_3\overline{\theta}_3^{(x_1)}\overline{\theta}_3^{(x_2)}, \quad \theta_1\overline{\theta}_3^{(x_1)}\overline{\theta}_1^{(x_2)}, \quad \theta_1\overline{\theta}_1^{(x_1)}\overline{\theta}_1^{(x_2)}. \tag{B.3}$$

At first, we use the formulas (B.1) and eliminate $\overline{\theta}_i^{(x_2)}$ using the following changes: 1) $\overline{\theta}_i^{(x_2)} \to \to \overline{\theta}_i^{(\frac{x_c}{x_1})}$, 2) the upper limit of x_2 integration in the case of $\overline{\theta}_3^{(x_2)}$ must be replaced by $\sqrt{z}/\rho_3$ and in the case of $\overline{\theta}_1^{(x_2)}$ by $\sqrt{z}$.

Thus we have three regions defined by the quantities in (z, x_1) plane:

$$\begin{gathered} \rho^2 = z, \quad z = x_1^2\rho^2, \quad z = \frac{x_c^2\rho^2}{x_1^2}, \\ 1 = z, \quad z = x_1^2\rho^2, \quad z = \frac{x_1}{x_c^2}, \\ 1 = z, \quad z = x_1^2, \quad z = \frac{x_1^2\rho^2}{x_c^2}. \end{gathered} \tag{B.4}$$

The limits of integrations may be transformed as follows:

$$\begin{gathered} \int \theta_3\overline{\theta}_3^{(x_1)}\overline{\theta}_3^{(x_2)} \to \int\limits_{x_c\rho^2}^{\rho^2} dz \int\limits_{x_c\rho/\sqrt{z}}^{\sqrt{z}/\rho} dx_1 \int\limits_{x_c/x_1}^{\sqrt{z}/\rho} dx_2, \\ \int \theta_3\overline{\theta}_1^{(x_1)}\overline{\theta}_1^{(x_2)} \to \int\limits_{x_c\rho}^{1} dz \int\limits_{x_c/\sqrt{z}}^{\sqrt{z}/\rho} dx_1 \int\limits_{x_c/x_1}^{\sqrt{z}} dx_2, \end{gathered} \tag{B.5}$$

and for $\int \theta_1\overline{\theta}_1^{(x_1)}\overline{\theta}_1^{(x_2)}$ the formulas can be derived from the above formulas by setting $\rho = 1$. For the wide-narrow case the prescription is similar:

$$\int \theta_4\overline{\theta}_4^{(x_1)}\overline{\theta}_3^{(x_2)} \to \int\limits_{x_c\rho_3}^{\rho_4^2} dz \int\limits_{x_c\rho_3/\sqrt{z}}^{\sqrt{z}/\rho_4} dx_1 \int\limits_{x_c/x_1}^{\sqrt{z}/\rho_3} dx_2. \tag{B.6}$$

The other variants of the restrictions in the wide-narrow angular acceptance may be written as follows:

$$\begin{gathered} \int \theta_1\overline{\theta}_2^{(x_1)}\overline{\theta}_1^{(x_2)} \to \int\limits_{x_c\rho_2}^{1} dz \int\limits_{\frac{x_c\rho_2}{\sqrt{z}}}^{\sqrt{z}} dx_1 \int\limits_{\frac{x_c}{x_1}}^{\sqrt{z}/\rho_2} dx_2, \\ \int \theta_1\overline{\theta}_4^{(x_1)}\overline{\theta}_1^{(x_2)} \to \int\limits_{x_c\rho_4}^{1} dz \int\limits_{x_c\rho_4/\sqrt{z}}^{\sqrt{z}} dx_1 \int\limits_{x_c/x_1}^{\sqrt{z}/\rho_4} dx_2, \\ \int \theta_2\overline{\theta}_2^{(x_1)}\overline{\theta}_3^{(x_2)} \to \int\limits_{x_c\rho_2\rho_3}^{\rho_2^2} dz \int\limits_{x_c\rho_3/\sqrt{z}}^{\sqrt{z}/\rho_2} dx_1 \int\limits_{x_c/x_1}^{\sqrt{z}/\rho_3} dx_2. \end{gathered} \tag{B.7}$$

References

1. The LEP Collaboration: ALEPH, DELPI, L3 and OPAL and the LEP Electroweak Working Group, CERN–PPE/95; B. Pietrzyk, preprint LAPP-Exp-94.18, *Invited talk at the Conf. Radiative Corrections: Status and Outlook*, Galtinburg, TN, USA, 1994; I. C. Brock et al., Preprint CERN-PPE/96-89, CMU-HEP/96-04, 1996.
2. G. Barbiellini et al., *Neutrino Counting in Z Physics at LEP*, conv. L. Trentadue, ed. by G. Altarelli, R. Kleiss, and C. Verzegnassi, CERN Report 89-08.
3. H. Anlauf et al., *Events Generator for Bhabha Scattering*, conv. S. Jadach and O. Nicrosini, Yell. Rep. CERN 96-01, **2**, 229.
4. S. Jadach, E. Richter-Was, B. F. L. Ward, and Z. Was, Comput. Phys. Commun. **70**, 305 (1992).
5. G. Montagna et al., Comput. Phys. Commun. **76**, 328 (1993); M. Cacciori, G. Montagna, and F. Piccinini, Comput. Phys. Commun. **90**, 301 (1995); CERN-TH/95-169; G. Montanga et al., Nucl. Phys. B **401**, 3 (1993).
6. S. Jadach, M. Skrzypek, and B. F. L. Ward, Phys. Rev. D **47**, 3733 (1993); S. Jadach, E. Richter-Was, B. F. L. Ward, and Z. Was, Phys. Lett. B **260**, 438 (1991).
7. S. Jadach, E. Richter-Was, B. F. L. Ward, and Z. Was, Phys. Lett. B **353**, 349, 362 (1995); S. Jadach, M. Melles, B. F. L. Ward, and S. A. Yost, Phys. Lett. B **377**, 168 (1966).
8. W. Beenakker, F. A. Berends, and S. C. van der Marck, Nucl. Phys. B **355**, 281 (1991); W. Beenakker and B. Pietrzyk, Phys. Lett. B **304**, 366 (1993).
9. M. Gaffo, H. Czyz, E. Remiddi, Nuovo Cim. A **105**, 271 (1992); Int. J. Mod. Phys. **4**, 591 (1993); Phys. Lett. B **327**, 369 (1994); G. Montanga, O. Nicrosini, and F. Piccinini, Preprint FNT/T–96/8.
10. A. B. Arbuzov et al., Yell. Rep. CERN 95-03, p. 369; preprint CERN-TH/95-313, UPRF-95-438; Nucl. Phys. B **485**, 457 (1997).
11. A. B. Arbuzov, E. Kuraev, N. P. Merenkov, and L. Trentadue, JETP **81**, 638 (1995); Preprint CERN-TH/95-241, JINR-E2-95-110.
12. D. R. Yennie, S. C. Frautchi, and H. Suura, Ann. Phys. **13**, 379 (1961).
13. L. N. Lipatov, Yad. Fiz. **20**, 94 (1974); G. Altarelli and G. Parisi, Nucl. Phys. B **126**, 298 (1977); M. Skrzypek, Acta Phys. Pol. B **23**, 135 (1992).
14. N. P. Merenkov, Yad. Fiz. **48**, 1073 (1988); **50**, 469 (1989).
15. T. D. Lee and M. Nauenberg, Phys. Rev. B **133**, 1549 (1964).
16. H. Cheng and T. T. Wu, Phys. Rev. Lett. **23**, 670 (1969); V. G. Zima and N. P. Merenkov, Yad. Fiz. **25**, 998 (1976); V. N. Baier, V. S. Fadin, V. Khoze, and E. Kuraev, Phys. Rep. **78**, 294 (1981).
17. V. G. Gorshkov, Uspcchi Fiz. Nauk **110**, 45 (1973).
18. S. Jadach and B. W. L. Ward, Phys. Rev. D **40**, 3582 (1989).
19. S. Drell and T. M. Yan, Phys. Rev. Lett. **25**, 316 (1970).
20. E. A. Kuraev and V. S. Fadin, Yad. Fiz. **41**, 466 (1985).
21. W. Beenakker, F. A. Berends, and S. C. van der Marck, Nucl. Phys. B **349**, 323 (1991).
22. A. P. Bukhvostov, E. A. Kuraev, and L. N. Lipatov, Yad. Fiz. **38**, 439 (1983); **39**, 194 (1984).
23. R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, and H. C. Morris, *Solitons and Nonlinear Wave Equations*, Academic Press, Inc. (Harcourt Brace Jovanovich, Publishers); S. S. Moiseyev, P. B. Rutkevich, A. V. Tur, and V. V. Yanovsky, Zh. Eksp. Teor. Fiz. **94**, 144 (1988); A. V. Chechkin, A. V. Tur, and V. V. Yanovsky, Phys. Fluid B **4**, 3513 (1992).