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NEGATIVE VISCOSITY FOR ROSSBY WAVE AND DRIFT WAVE TURBULENCE

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The possible occurence of a «negative viscosity effect» is studied for Rossby wave and drift wave turbulence. It is assumed that (i) the space and time scales of the wave field are much smaller than the scales of the mean field, and (ii) the small-scale field is sufficiently weak, stationary, and maintained by an external source. Such a formulation is fruitful for studying the effects (characterized by the effective viscosity) of smaller-scale motions upon larger-scale ones. The criteria of large-scale instability due to the negative effective viscosity are derived for the coherent wave motions as well as for small-scale isotropic wave turbulence.

1. INTRODUCTION

The processes of pattern formation have been extensively studied in various hydrodynamic models. One of the aspects of this problem has been called «negative viscosity». This term was introduced when analyzing large-scale geophysical experiments; see the monographs [1, 2]. In the modern literature this term implies two connected classes of phenomena. The first of these is related to the description of anomalous flows of the turbulent kinetic energy through the spectrum toward the region of small wavenumbers in two-dimensional ($2D$) hydrodynamics and to the formation of stationary turbulent spectra. This problem has been studied in Ref. [3] for $2D$ homogeneous isotropic turbulence in a Navier-Stokes (NS) fluid with zero mean velocity, and for $2D$ magnetohydrodynamics in Ref. [4]. Using the same closure techniques of the direct-interaction family, the authors show that the negative eddy damping rate occurs for both cases. Another class of phenomena, to which our paper is devoted, is related to pattern formation when the turbulent spectrum is assumed known. Here the negative-viscosity effect means the appearance of a negative dissipative factor in the equation for the mean flow. From the theoretical viewpoint, generation of large-scale structures is understood as a manifestation of long-wavelength instability in a system of small-scale vortices or waves, the energy of small-scale motions being constant (it is mathematically convenient to treat the small-scale motions as generated by an external source).

A number of analytical studies of the negative viscosity effect were initiated by two-dimensional flow of a viscous incompressible fluid, which is not damped due to existence of an external force periodic along one of the coordinates. In this paper the instability criterion for a sinusoidal velocity profile and the marginal stability curve were derived. Along with this paper, the problem has been considered in Ref. [6] and generalized to an arbitrary periodic velocity profile in Ref. [7].

When studying linear stability and nonlinear regimes appearing, it is convenient to use the two-scale expansion method. In this method it is assumed that the characteristic space and time scales of the basic initial motions are smaller than the scales of the secondary flows. Therefore, it is possible to introduce a small parameter characterizing the ratio of the characteristic scale of small-scale motions to that of large-scale secondary motions. The solution of the hydrodynamic equations is sought in the form of an expansion in the small parameter, while the equation describing the evolution of the large-scale component is obtained from the solvability condition of the initial equations in the corresponding approximation order. Two-scale expansions are widely used in the other problems, which are connected with generation of large-scale fields and structures by small-scale fields and motions. As examples, we mention the papers on the kinetic α-effect [8] and on generation of large-scale convective patterns by helical turbulence [9].

Using the two-scale formalism, the equations of a weakly nonlinear theory for the large-scale motions have been obtained and studied analytically and numerically for problems with the Kolmogorov flows [10, 11]. In the case when the small-scale motions are describable as homogeneous turbulence, negative viscosity effects have been studied in Ref. [12]. In particular, it has been shown for $2D$ NS flows that a homogeneous isotropic small-scale turbulence does not lead to the negative eddy viscosity. A general multiscale formalism for the study of eddy viscosities for incompressible flows of arbitrary dimensionality has been developed in Ref. [13]. In this paper explicit expressions for eddy viscosity in terms of correlation function of the small-scale basic flow have been derived for the low Reynolds number isotropic case (in accordance with Ref. [12], eddy viscosity enhances molecular viscosity), and for the parallel time-independent flow, of which the Kolmogorov flow is an example. Such parallel flow undergoes a negative viscosity instability with respect to large-scale perturbations transverse to the basic flow. Among the papers close in spirit to this group we also mention [14], where the negative viscosity effect during the excitation of a single drift wave (with wavelength greater than the ion Larmor radius at electron temperature) in a magnetized inhomogeneous plasma has been found, and special solutions of the weakly nonlinear equation for large-scale perturbations have been studied.

In contrast to the papers on liquid hydrodynamics mentioned above, the present paper deals with negative viscosity in Rossby wave turbulence and drift wave turbulence. Rossby turbulence is a widespread type of wave motions in the ocean and atmosphere; see, for example, Refs. [15, 16]. Drift turbulence is widespread in magnetized inhomogeneous plasmas of numerous thermonuclear devices and the ionosphere; see, for example, Refs. [17, 18]. It is well-known that, despite the quite different physical origin of these motions, their formal description is very similar [19]. Moreover, Rossby wave turbulence and drift wave turbulence obey identical nonlinear partial differential equation (in the simplest description). Therefore, it is natural to discuss both them together. In order to clarify the discussion and the results, we use the simplest method of analysis which allows us to elucidate in a uncomplicated way the appearance of nontrivial effects and to find out how they differ from non-wave hydrodynamic problems. We assume that it is possible to divide the fields into a large-scale slowly varying part and a small-scale rapidly evolving part. The small-scale field is a wave field, whose level is kept stationary due to the existence of an external source (external force) in the initial equation. The evolution of the large-scale part is calculated by averaging over the small-scale part. In such a formulation the effective (turbulent) viscosity determining the evolution of the large-scale field is a functional of the given spectrum of waves. We use the simplest model spectra to demonstrate the difference from the hydrodynamic problems mentioned above. In particular, we show that small-scale isotropic Rossby and drift wave turbulence can act as a negative effective viscosity on large-scale perturbations. This points to a more substantial role of a nonlocal energy transfer from small scales to larger ones in the case of the Rossby and drift wave turbulence than in the case of $2D$ NS turbulence.

2. EQUATION FOR THE LARGE-SCALE FIELD EVOLUTION

We start from the well-known model two-dimensional equation, which describes the space-time evolution of the stream function in the Rossby wave theory [20]. In dimensionless units

$$\frac{\partial \psi}{\partial t} - \frac{\partial}{\partial t}\Delta\psi - \frac{\partial \psi}{\partial x} + \nu\Delta^2\psi - [\nabla\psi, \nabla\Delta\psi]_z = F. \tag{2.1}$$

Here $\nabla = \mathbf{e}_x\partial/\partial x + \mathbf{e}_y/\partial/\partial y$, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, and ν is the (dimensionless) molecular viscosity of the gas or liquid. Following the known method, used in the turbulence theory, we introduce a source into the right hand side of Eq. (2.1). The role of this source is to maintain the stationary level of the wave turbulence.

For the Rossby wave the x-axis denotes the latitude direction (from West to East), whereas the y-axis denotes the meridional one (from South to North). We note that usually for drift waves the x-axis denotes the radial direction in a thermonuclear device, or the direction in which plasma density varies, whereas the y-axis denotes the azimuthal direction, and thus, one has to replace $\partial\psi/\partial x$ by $-\partial\psi/\partial y$ in Eq. (2.1) in order to follow the conventional notation used in drift wave theory. Then ψ is the dimensionless potential, and ν is the (dimensionless) ion viscosity for magnetized plasmas. However, in this paper, for definiteness, we use the «Rossby wave coordinate frame», i.e., that in which Eq. (2.1) is written. Obviously, the final results can be easily reproduced in a «drift-wave coordinate frame».

In the linear approximation Eq. (2.1) describes the wave propagation with the frequency

$$\omega_R = -\frac{k_x}{1+k^2}, \tag{2.2}$$

and the damping rate

$$\nu_R = \frac{\nu k^4}{1+k^2}, \tag{2.3}$$

where $\mathbf{k}$ is the wave vector, $k^2 = k_x^2 + k_y^2$.

Now we divide the field ψ into mean and fluctuating (turbulent) components:

$$\psi = \overline{\psi} + \psi^T; \tag{2.4}$$

the bar denotes statistical averaging and «T» means «turbulent». After averaging Eq. (2.1) we get

$$\frac{\partial \overline{\psi}}{\partial t} - \frac{\partial}{\partial t}\Delta\overline{\psi} - \frac{\partial \overline{\psi}}{\partial x} + \nu\Delta^2\overline{\psi} = \left[\nabla\overline{\psi}, \nabla\Delta\overline{\psi}\right]_z + \left[\overline{\nabla\psi^T, \nabla\Delta\psi^T}\right]_z. \tag{2.5}$$

To get a closed equation for $\overline{\psi}$ it is necessary to express

$$Q = \left[\overline{\nabla\psi^T, \nabla\Delta\psi^T}\right]_z \tag{2.6}$$

in terms of $\overline{\psi}$. The equation for ψ^T is obtained from Eqs. (2.1) and (2.5):

$$\frac{\partial \psi^T}{\partial t} - \frac{\partial}{\partial t}\Delta\psi^T - \frac{\partial \psi^T}{\partial x} + \nu\Delta^2\psi - \left[\nabla\overline{\psi}, \nabla\Delta\psi^T\right]_z - \left[\nabla\psi^T, \nabla\Delta\overline{\psi}\right]_z - \\ - \left(\left[\nabla\psi^T, \nabla\Delta\psi^T\right]_z - \left[\overline{\nabla\psi^T, \nabla\Delta\psi^T}\right]_z\right). \qquad (2.7)$$

Therefore, the closed equation for $\overline{\psi}$ implies the application of some closure procedure. Since we are interested here in negative viscosity effects for the large-scale flows, we use the following approach. Let us assume that the mean quantities vary on space and time scales which are larger than the characteristic scales of the fluctuation fields. We introduce the characteristic size l of the small-scale field and the characteristic size L of the large-scale field. Then we estimate the ratio of the quantities $\nu\Delta^2\psi^T$, $[\nabla\overline{\psi}, \nabla\Delta\psi^T]_z$, $[\nabla\psi^T, \nabla\Delta\overline{\psi}]_z$, $[\nabla\psi^T, \nabla\Delta\psi^T]_z$. They stand in the ratios

$$1 : \frac{l}{L}\,\mathrm{Re}_L : \frac{l^3}{L^3}\,\mathrm{Re}_L : \mathrm{Re}_l,$$

where $\mathrm{Re}_L \approx \overline{V}L/\nu$ is the Reynolds number of the large-scale motions and $\mathrm{Re}_l \approx V^T l/\nu$ is the Reynolds number of small-scale ones ($\overline{V}$ and V^T are characteristic velocities of large-scale and small-scale motions, respectively). Therefore, for sufficiently small Re_l we can neglect terms in Eq. (2.7) which are quadratic in ψ^T. Furthermore, in accordance with multiscale expansion schemes, we introduce the «slow» variable $\mathbf{X}$ and the fast variable $\mathbf{x}$. The average quantities depend on the slow variable only, whereas the fluctuating components depend on both the fast and slow variables. The following inequality holds:

$$\left|\frac{\partial}{\partial \mathbf{X}}\right| \approx |\mathbf{K}| \ll \left|\frac{\partial}{\partial \mathbf{x}}\right| \approx |\mathbf{k}|, \qquad (2.8)$$

where $\mathbf{K}$ and $\mathbf{k}$ are large-scale and small-scale wave vectors, respectively.

Thus, we can find the solution for ψ^T as an expansion in powers of K, that is,

$$\psi^T = \psi^{(0)}(\mathbf{x}, t) + \psi^{(1)}(\mathbf{x}, \mathbf{X}, t) + \ldots + \psi^{(4)}(\mathbf{x}, \mathbf{X}, t). \qquad (2.9)$$

The solution to order K^4 is presented in Appendix A. Then, the functional dependence of Q through $\overline{\psi}$ is obtained there. So instead of Eq. (2.5) we get the following equation for the large-scale part of ψ:

$$\hat{L}\overline{\psi} = \left(\frac{\partial\overline{\psi}}{\partial X}\frac{\partial}{\partial Y}\Delta_s\overline{\psi} - \frac{\partial\overline{\psi}}{\partial Y}\frac{\partial}{\partial X}\Delta_s\overline{\psi}\right) + \varepsilon_{mn}\varepsilon_{jk}\left[\eta_{mpk}\frac{\partial^3\overline{\psi}}{\partial X_p\partial X_n\partial X_j} - \right. \\ - \nu^{(1)}_{mk}\frac{\partial^2\Delta_s\overline{\psi}}{\partial X_n\partial X_j} + \nu^{(2)}_{mpk}\frac{\partial}{\partial X}\left(\frac{\partial^3\overline{\psi}}{\partial X_p\partial X_n\partial X_j}\right) + \nu^{(3)}_{mlpk}\frac{\partial^4\overline{\psi}}{\partial X_l\partial X_n\partial X_p\partial X_j} + \\ + \varepsilon_{rq}N^{(1)}_{mlkq}\frac{\partial^2}{\partial X_l\partial X_n}\left(\frac{\partial\overline{\psi}}{\partial X_r}\frac{\partial\overline{\psi}}{\partial X_j}\right) + \varepsilon_{rq}N^{(2)}_{kpmq}\frac{\partial\overline{\psi}}{\partial X_j}\frac{\partial^3\overline{\psi}}{\partial X_p\partial X_n\partial X_r} - \\ \left. - \varepsilon_{rq}N^{(3)}_{kpnq}\frac{\partial^2\overline{\psi}}{\partial X_m\partial X_j}\frac{\partial^2\overline{\psi}}{\partial X_p\partial X_r}\right], \qquad (2.10)$$

where

$$\hat{L} = \frac{\partial}{\partial t} - \frac{\partial}{\partial t}\Delta_s - \frac{\partial}{\partial X} + \nu\Delta_s^2, \qquad (2.11)$$

$$\Delta_s \equiv \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}, \tag{2.12}$$

$(j, l, m, n, p, q, r) = x, y$, ε_{mn} is the unit antisymmetric tensor of the second rank,

$$\begin{aligned}
&\varepsilon_{11} = \varepsilon_{22} = 0, \quad \varepsilon_{12} = -\varepsilon_{21} = 1, \quad (m,n) = 1,2,\\
&\eta_{mpk} = \iint d\mathbf{k}\, d\omega\, \Pi_1(k,\omega) 2k_m k_p k_k k^2 (\omega - \omega_R),\\
&\nu^{(1)}_{mk} = \iint d\mathbf{k}\, d\omega\, \Pi_1(k,\omega) \nu_R k_m k_k k^2,\\
&\nu^{(2)}_{mpk} = \iint d\mathbf{k}\, d\omega\, \Pi_2(k,\omega) 4k_m k_p k_k k^2 \nu_R (\omega - \omega_R),\\
&\nu^{(3)}_{mlpk} = \iint d\mathbf{k}\, d\omega\, \Pi_2(k,\omega) 8k_m k_l k_p k_k k^2 \left[\nu k^2 \left(\nu_R^2 - (\omega-\omega_R)^2\right) + \omega\nu_R(\omega-\omega_R)\right],\\
&N^{(1)}_{mlkq} = \iint d\mathbf{k}\, d\omega\, \Pi_2(k,\omega) 2k_m k_l k_q k_k k^2 \left[(\omega-\omega_R)^2 - \nu_R^2\right],\\
&N^{(2)}_{kpmq} = \iint d\mathbf{k}\, d\omega\, \Pi_2(k,\omega) 2k_k k_p k_m k_q k^4 (k^2+1)^{-1},
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
\Pi_1(\mathbf{k},\omega) &= \frac{\Phi(\mathbf{k},\omega)}{(1+k^2)\left[(\omega-\omega_R)^2 + \nu_R^2\right]},\\
\Pi_2(\mathbf{k},\omega) &= \frac{\Phi(\mathbf{k},\omega)}{(1+k^2)^2\left[(\omega-\omega_R)^2 + \nu_R^2\right]^2},
\end{aligned} \tag{2.14}$$

and $\Phi(\mathbf{k},\omega)$ is the space-time spectral density for the small-scale field.

Let us discuss the meaning of the terms in Eqs. (2.10), (2.13). The first term in Q gives the correction to the frequency in the dispersion equation for the large-scale motions. The next three terms in Q are the type terms which lead either to damping (positive effective viscosity) or to growth (negative effective viscosity) of the large-scale motions. The last three terms describe the nonlinear interaction of the large-scale motions. In this paper we are interested mainly in terms of the viscous type. We consider the effects depending on the properties of small-scale wave turbulence. However, in order to make the discussion simpler and to clarify the differences between our paper and the papers mentioned above, in the next Section we demonstrate for the case of the $2D$ flow of a viscous incompressible fluid what effect can be responsible for the appearance of the negative effective (turbulent) viscosity.

3. THE ORIGIN OF THE NEGATIVE VISCOSITY TERM. QUALITATIVE CONSIDERATION

In order to simplify the discussion as much as possible, we consider here the equation for the stream function of a $2D$ incompressible viscous fluid [21]:

$$\frac{\partial}{\partial t}\Delta\psi - \nu\Delta^2\psi + [\nabla\psi, \nabla\Delta\psi]_z = F. \tag{3.1}$$

Inserting ψ in the form (2.4), we get equations analogous to Eqs. (2.5)–(2.7):

$$\frac{\partial}{\partial t}\Delta\overline{\psi} - \nu\Delta^2\overline{\psi} - \overline{[\nabla\psi^T, \nabla\Delta\psi^T]_z} = 0, \quad (3.2)$$

$$\frac{\partial}{\partial t}\Delta\psi^T - \nu\Delta^2\psi^T + [\nabla\overline{\psi}, \nabla\Delta\psi^T]_z + [\nabla\psi^T, \nabla\Delta\overline{\psi}]_z = F. \quad (3.3)$$

It is noteworthy that the third and fourth terms on the left-hand side of Eq. (3.3) describe the interaction between small- and large-scale fields. The term $[\nabla\overline{\psi}, \nabla\Delta\psi^T]_z$ describes the transport of the fluctuation vorticity $W_z^T = \Delta\psi^T$ by the mean flow $\langle\mathbf{V}\rangle = [\mathbf{e}_z, \nabla\overline{\psi}]$ while the term $[\nabla\psi^T, \nabla\Delta\overline{\psi}]_z$ describes the transport of the mean vorticity by the fluctuation component. As in Sec. 2, we introduce the natural physical assumption that the space and time scales of the average quantities are larger than the scales of the fluctuations. Introducing fast and slow variables (x and X, respectively), we use the Fourier transform over the fast variable:

$$\psi^T(\mathbf{x}, t) = \int \frac{d\mathbf{k}\, d\omega}{(2\pi)^3}\hat{\psi}^T(\mathbf{k}, \omega)\exp(-i\omega t + i\mathbf{k}\mathbf{x}).$$

We note that in the Fourier representation the term $[\nabla\overline{\psi}, \nabla\Delta\psi^T]_z$ describes the Doppler shift of the fluctuation frequency. From Eq. (3.3) we get

$$\hat{\psi}^T(\mathbf{k}, \omega) \approx \frac{\hat{F}(\mathbf{k}, \omega)}{k^2(i\omega - \nu k^2)}\left\{1 + \frac{1}{i\omega - \nu k^2}\left(ik_n\varepsilon_{mn}\frac{\partial\overline{\psi}}{\partial X_m} - \frac{ik_m}{k^2}\varepsilon_{mn}\frac{\partial^3\overline{\psi}}{\partial X_n^3}\right)\right\}, \quad (3.4)$$

where it is assumed for simplicity that the terms containing slow spatial derivatives are small. We note once more that the interaction of the fluctuations ψ^T with the mean flow $\overline{\psi}$ (in Eq. (3.4) the terms with the first and third slow derivatives are due to this interaction) causes ψ^T to depend on the slow coordinate. This fact implies that among the terms entering into $-\langle[\nabla\psi^T, \nabla\Delta\psi^T]_z\rangle$ the following term occures:

$$\overline{\varepsilon_{mn}\frac{\partial\psi^T}{\partial x_m}\frac{\partial}{\partial X_n}\Delta_s\psi^T}, \quad (3.5)$$

where Δ_s is the slow Laplacian, as before, and Δ has to be replaced by Δ_s in the first two terms in Eq. (3.2).

Inserting Eq. (3.4) into Eq. (3.5), we can see that Eq. (3.5) gives rise to the following terms in the left-hand side of Eq. (3.2):

$$-\varepsilon_{mn}\varepsilon_{jk}\nu_{mk}^{(1)}\frac{\partial^2}{\partial X_n\partial X_j}\Delta_s\overline{\psi} + \varepsilon_{mn}\varepsilon_{jk}\mu_{mj}\frac{\partial^4}{\partial X_n\partial X_k^3}\Delta_s\overline{\psi}, \quad (3.6)$$

$$\nu_{mk}^{(1)} \doteq \int d\mathbf{k}\, d\omega\langle\hat{F}^2\rangle_{\mathbf{k},\omega}\frac{\nu k^2}{k^4(\omega^2 + \nu^2k^4)^2}k_m k_k, \quad (3.7)$$

$$\mu_{mj} = \int d\mathbf{k}\, d\omega\langle\hat{F}^2\rangle_{\mathbf{k},\omega}\frac{\nu k^2}{k^6(\omega^2 + \nu^2k^4)^2}k_m k_j. \quad (3.8)$$

The first term in Eq. (3.6) is due to the transport of the fluctuation vorticity by the mean flow, while the second term is caused by the transport of the mean vorticity by the fluctuation velocity. The first term corresponds to $\nu^{(1)}_{mk}$, which is calculated for the Rossby wave turbulence in Sec. 2; see Eqs. (2.13). It can be seen that even for isotropic fluctuations the former effect makes a negative contribution to the turbulent viscosity. Indeed, let $\langle \hat{F}^2 \rangle_{\mathbf{k},\omega}$ be an isotropic function of k. Then,

$$\nu^{(1)}_{mk} = \nu^{(1)}\delta_{mk}, \quad \mu_{mj} = \mu\delta_{mj}, \quad \nu^{(1)}, \mu > 0$$

and Eq. (2.6) is rewritten in the form

$$\nu^{(1)}\Delta_s^2\overline{\psi} + \mu\left(\frac{\partial^4}{\partial X^4} + \frac{\partial^4}{\partial Y^4}\right)\Delta_s\overline{\psi}. \tag{3.9}$$

Therefore, the term with $\nu^{(1)}$ gives a negative contribution to the turbulent viscosity. The term with μ describes the dissipation of the large-scale component and bounds the instability region for small wavenumbers.

Thus, the interaction between the large-scale flow and small-scale velocity fluctuations, which manifests itself mainly in the transport of the small-scale fluctuations by the mean flow, gives rise to the viscous-type term with a negative viscosity coefficient in the equation for the mean component. We stress that this conclusion is valid both for the $2D$ model of viscous incompressible fluid, see Eq. (3.1), and for the wave model, see Eq. (2.1).

The above treatment is obviously incomplete: it only points to the mechanism for the appearance of the viscous terms with negative viscosity coefficient in the equation for the mean flow. A detailed consideration demands more accurate analysis of Eqs. (3.2), (3.3) with the two-scale dependence of fluctuations taken into account. The complementary viscous terms can suppress the negative contribution, which is controlled by the transport of the turbulent fluctuations by the large-scale flow. In order to compare our results with those of other authors studying turbulent viscosity by means of the two-scale expansion, and to call attention to the differences between the turbulent viscosity for the $2D$ NS flows and that for the Rossby and drift waves, in Appendix B we obtain and analyze the equation for the average stream function of $2D$ incompressible viscous fluid, see Eq. (B.1). It is a limiting case of Eq. (2.10), if we neglect dispersion ($1 + k^2 \to k^2$) and the eigenfrequency ($\omega_R = 0$) of the waves. In Appendix B we demonstrate that our results correspond to those of Refs. [12, 13] for $2D$ NS flows. We also demonstrate that isotropic small-scale fluctuations do not give rise to a negative viscosity in the framework of $2D$ NS equations. The negative contribution to the eddy viscosity given by $\nu^{(1)}$ in Eq. (3.9) is compensated by the positive contribution. However, in the next Section we demonstrate that this is not the case for the isotropic wave turbulence.

4. NEGATIVE VISCOSITY FOR MODEL SPECTRA OF THE WAVE TURBULENCE

Let us explore the consequences of the general expressions (2.10)–(2.14). We consider the problems arising here, taking the tensor $\nu^{(1)}_{mk}$ as the way of example; see Eqs. (2.13). This tensor is a functional of the space-time spectral function $\Phi(\mathbf{k}, \omega)$ of a given small-scale field. The spectrum has a peak on ω at $\omega \approx \omega_R$ and some characteristic width γ_k. The spectrum is multiplied by the Lorentzian curve

$$\frac{\nu_R}{(\omega - \omega_R)^2 + \nu_R^2},$$

and integrated over ω and **k**. Obviously, the result of integration over ω depends on the ratio between the characteristic widths of the multiplied functions, that is, γ_k and ν_R, whereas the result of integration over **k** depends on the degree of the spectrum anisotropy in **k**. Under these circumstances, it is natural to consider first the simplest model wave spectra leading to the negative viscosity. Thus, we consider the following examples:

1. A small-scale coherent field, that is, the frequency and wavenumber spectra are narrower than the other characteristic widths of the problem.

2. A small-scale isotropic turbulent field. Such a field can be formed in a small-scale region ($k \geq 1$) due to mode coupling, which isotropizes the spectrum (see, e.g., Ref. [22]).

4.1. Coherent wave spectrum

We take the wave spectrum as follows:

$$\Phi(\mathbf{k},\omega) = \frac{\langle\psi^2\rangle}{2}\left[\delta(\omega-\omega_0)\delta(\mathbf{k}-\mathbf{k}_0) + \delta(\omega+\omega_0)\delta(\mathbf{k}+\mathbf{k}_0)\right], \tag{4.1}$$

where $\langle\psi^2\rangle$ is the variance of the fluctuations,

$$\omega_0(\mathbf{k}_0) = -\frac{k_{0x}}{1+k_0^2}.$$

This case implies that the frequency spectrum is sufficiently narrow:

$$\gamma_{k_0} \ll \nu_R(k_0). \tag{4.2}$$

Inserting Eq. (4.1) into Eqs. (2.10)–(2.14), we get the following equation for $\overline{\psi}$:

$$\begin{aligned}
\hat{L}\overline{\psi} = {} & \left(\frac{\partial\overline{\psi}}{\partial X}\frac{\partial}{\partial Y}\Delta_s\overline{\psi} - \frac{\partial\overline{\psi}}{\partial Y}\frac{\partial}{\partial X}\Delta_s\overline{\psi}\right) + \frac{A^2}{2}\nu^{-1}k_0^{-2}\left[k_{0y}^2\frac{\partial^4\overline{\psi}}{\partial X^4} - 2k_{0x}k_{0y}\frac{\partial^4\overline{\psi}}{\partial X^3\partial Y} - \right.\\
& \left. - 2k_{0x}k_{0y}\frac{\partial^4\overline{\psi}}{\partial Y^3\partial X} + k_0^2\frac{\partial^4\overline{\psi}}{\partial X^2\partial Y^2} + k_{0x}^2\frac{\partial^4\overline{\psi}}{\partial Y^4}\right] + 4A^2\nu^{-1}k_0^{-4}\left[-k_{0x}^2k_{0y}^2\frac{\partial^4\overline{\psi}}{\partial X^4} + \right.\\
& + 2k_{0x}k_{0y}(k_{0x}^2 - k_{0y}^2)\frac{\partial^4\overline{\psi}}{\partial X^3\partial Y} + 2k_{0x}k_{0y}(k_{0y}^2 - k_{0x}^2)\frac{\partial^4\overline{\psi}}{\partial Y^3\partial X} + (4k_{0x}^2k_{0y}^2 - k_{0x}^4 - k_{0y}^4)\times\\
& \left.\times\frac{\partial^4\overline{\psi}}{\partial X^2\partial Y^2} - k_{0x}^2k_{0y}^2\frac{\partial^4\overline{\psi}}{\partial Y^4}\right] - A^2\nu^{-2}k_0^{-6}\left[k_{0x}k_{0y}\left(\frac{\partial^2}{\partial Y^2} - \frac{\partial^2}{\partial X^2}\right) + (k_{0x}^2 - k_{0y}^2)\frac{\partial^2}{\partial X\,\partial Y}\right]\times\\
& \times\left(k_{0x}\frac{\partial\overline{\psi}}{\partial Y} - k_{0y}\frac{\partial\overline{\psi}}{\partial X}\right)^2 + \frac{A^2}{2}\nu^{-2}k_0^{-4}\left[k_{0x}k_{0y}\left(\frac{\partial^2\overline{\psi}}{\partial Y^2} - \frac{\partial^2\overline{\psi}}{\partial X^2}\right) + (k_{0x}^2 - k_{0y}^2)\frac{\partial^2\overline{\psi}}{\partial X\,\partial Y}\right]\times\\
& \times\left(k_{0y}\frac{\partial}{\partial X} - k_{0x}\frac{\partial}{\partial Y}\right)^2\overline{\psi} + A^2\nu^{-2}k_0^{-4}\left(k_{0y}\frac{\partial}{\partial X} - k_{0x}\frac{\partial}{\partial Y}\right)\left[k_{0x}k_{0y}\left(\frac{\partial^2\overline{\psi}}{\partial Y^2} - \frac{\partial^2\overline{\psi}}{\partial X^2}\right) + \right.\\
& \left. + (k_{0x}^2 - k_{0y}^2)\frac{\partial^2\overline{\psi}}{\partial X\,\partial Y}\right]\left(k_{0y}\frac{\partial\overline{\psi}}{\partial X} - k_{0x}\frac{\partial\overline{\psi}}{\partial Y}\right).
\end{aligned} \tag{4.3}$$

For simplicity we consider the case when the small-scale field propagates along the x-axis, $\mathbf{k}_0 = k_0\mathbf{e}_x$. Taking $\overline{\psi}$ in the form

$$\overline{\psi} = \exp(-i\Omega t + i\mathbf{K}\mathbf{X}),$$

we get a linear dispersion equation for the large-scale perturbations:

$$\Omega = \frac{-K_x}{1+K^2} - \frac{i}{1+K^2}\left[\nu K_x^4 + \left(\frac{7\langle\psi^2\rangle}{\nu} + 2\nu\right) K_x^2 K_y^2 - \left(\frac{\langle\psi^2\rangle}{\nu} - \nu\right) K_y^4\right]. \quad (4.4)$$

It is worth noting that this equation describes not only the waves but also the large-scale structures. Indeed, it follows from Eq. (4.4) that the most «dangerous» (that is, rapidly growing) are those large-scale perturbations, which are perpendicular to the direction of the wave propagation, that is, $K_x = 0$ and $\mathrm{Re}\,\Omega = 0$. This implies that the negative viscosity effect leads not only to nonlocal energy transfer from small-scale waves to large-scale waves, but also to the generation of stationary structures highly elongated along one of the coordinates. We return once more to the discussion of these possibilities below. The growth rate of such perturbations is

$$\mathrm{Im}\,\Omega = \left(\frac{\langle\psi^2\rangle}{\nu} - \nu\right)\frac{K_y^4}{1+K_y^2}. \quad (4.5)$$

$\mathrm{Im}\,\Omega > 0$ if $\langle\psi^2\rangle > \nu^2$. We stress that in this case we can not take $\nu \to 0$ because such a limit is in contradiction with the inequality (4.2). For «the drift wave coordinate frame» one has to replace K_y by K_x in Eq. (4.5).

The effect described is analogous to the Kolmogorov flow instability of a $2D$ viscous incompressible fluid [5]. For a single drift wave with wavelength greater than the ion Larmor radius at electron temperature in a magnetized plasma this effect has been studied in Ref. [14]. Nonlinear stationary structures formed due to the negative viscosity effect have been also studied there.

4.2. Isotropic wave spectrum

We consider the case when the wave spectrum is isotropic in **k**. This case is close to that considered in Appendix B, so below we compare the equations of the Appendix with those obtained in this Section. Here the model Lorentzian time spectrum is used:

$$\Phi(\mathbf{k}, \omega) = \frac{1}{\pi}\frac{\gamma_k}{(\omega-\omega_R)^2 + \gamma_k^2}\,\Phi(k). \quad (4.6)$$

The width γ_k is a complicated function of $\Phi(k)$, but its explicit form is not discussed here. It should be stressed that we choose the Lorentzian shape for convenience only. It can be easily verified that the result is not changed qualitatively if one chooses other shapes, for example, the Gaussian or in the form of a step; see Eqs. (B.6) where $\omega \to \omega - \omega_R$.

Introducing Eq. (4.6) into Eqs. (2.13), taking the integrals over ω and using the subsidiary integrals (B.7), we arrive at the Eqs. (B.8) for the viscous terms, where instead of Eqs. (B.9) we get

$$\nu^{(1)} = \pi\int dk\, k^3\Phi(k)\frac{k^2}{1+k^2}\frac{1}{\nu_R+\gamma_k},$$
$$\nu^{(3)} = \pi\int dk\, k^3\Phi(k)\frac{k^4}{(1+k^2)^2}\frac{2\nu k^2+\gamma_k}{(\nu_R+\gamma_k)^2}. \quad (4.7)$$

It is worthwhile to note that if we set $1+k^2 \to k^2$ (no dispersion), then, naturally, we get Eqs. (B.9) from Eqs. (4.7). As before, $\nu^{(1)}$ and $\nu^{(3)}$ give negative and positive contributions, respectively, to the effective viscosity. However, due to the wave character of the small-scale perturbations the balance between the two contributions is changed. Indeed, instead of Eq. (B.10) we get

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t}\Delta_s-\frac{\partial}{\partial X}\right)\overline{\psi}+\nu_{eff}\Delta_s^2\overline{\psi}=(1-P)\left(\frac{\partial\overline{\psi}}{\partial X}\frac{\partial}{\partial Y}\Delta_s\overline{\psi}-\frac{\partial\overline{\psi}}{\partial Y}\frac{\partial}{\partial X}\Delta_s\overline{\psi}\right), \tag{4.8}$$

where

$$\begin{aligned} \nu_{eff}&=\nu-\nu^{(1)}+\nu^{(3)}=\nu+\pi\int dk\,\Phi(k)\frac{k^5}{(1+k^2)^2}\frac{\nu k^4-\gamma_k}{(\nu_R+\gamma_k)^2},\\ P&=\pi\int\limits_0^\infty dk\frac{k^5\Phi(k)\left[(1+k^2)\gamma_k+\nu k^2(k^2-2)\right]}{(1+k^2)^2(\gamma_k+\nu_R)^2}. \end{aligned} \tag{4.9}$$

For the case of $2D$ NS flows $\nu_{eff}>0$ holds for any ratio between the spectral width and νk^2; see Eq. (B.11). In contrast, as we can see, two possibilities exist for the Rossby waves.

The first possibility is characterized by $\gamma_k\ll\nu_R$ (narrow isotropic spectrum). This case corresponds to the one when the wave intensity is sufficiently low that the broadening of the spectral lines γ_k is small in comparison not only with ω_R (a widely used definition of weak turbulence), but also with ν_R, which, in turn, is less than ω_R (the damping rate is less than the eigenfrequency). Then, setting $\gamma_k\to 0$, we get

$$\nu_{eff}=\nu+\frac{\langle\psi^2\rangle}{2\nu}. \tag{4.10}$$

This result coincides with Eq. (B.12) and points to the absence of nonlocal energy transfer to the large-scale region.

The second possibility is realized when $\gamma_k\gg\nu_R$ (broad isotropic spectrum). This inequality implies that the intensity of the waves is greater than in the previous case. However, due to the smallness of the damping rate we may still retain the framework of weak turbulence. In this case from Eq. (4.9) one gets

$$\nu_{eff}=\nu-\pi\int\limits_0^\infty dk\,\Phi(k)\frac{k^5}{(1+k^2)^2\gamma_k}, \tag{4.11}$$

and ν_{eff} can be negative for a sufficiently high Rossby wave level. We may roughly estimate the criterion for the negative viscosity as

$$\langle\psi^2\rangle>\nu\gamma, \tag{4.12}$$

where γ is some effective spectral line broadening. It resembles the analogous criterion for the coherent waves; see Eq. (4.5).

The evolution equation for the large-scale perturbations has the form (4.8), where

$$P=\pi\int\limits_0^\infty dk\,\Phi(k)\frac{k^5}{(1+k^2)\gamma_k}.$$

Looking for $\overline{\psi}$ in the form

$$\overline{\psi} = \exp(-i\Omega t + i\mathbf{KX}), \tag{4.13}$$

and inserting Eq. (4.13) into Eq. (4.8), we get the following dispersion equation for the large-scale perturbations:

$$\Omega = -\frac{K_x}{1+K^2} - i\frac{\nu_{eff}K^4}{1+K^2}. \tag{4.14}$$

Equation (4.14) describes two types of motion. The first one ($K_x \neq 0$) is the large-scale wave, while the second one ($K_x = 0$) is the large-scale stationary structure, $\mathrm{Re}\,\Omega = 0$. Therefore, if $\nu_{eff} < 0$ holds, two possibilities exist for the energy flow from the small-scale region. The first one is nonlocal energy transfer from the small-scale waves to the large-scale ones. The large-scale waves grow, and it can happen, that the spectral gap between the two wave regions disappears, the two-scale approximation is violated, and the turbulence becomes nonstationary. We note, however, that this circumstance does not invalidate our treatment, because we consider the initial stage of instability only. Another possibility is related to energy flow from small-scale waves to the large-scale stationary structures highly elongated along one of the coordinates. We remind that as in Sec. 4.1 for «the drift wave coordinate frame» one has to replace K_x by $-K_y$ in Eq. (4.14).

In this theory there are no limits on the growth rate of the large scale instability as K increases. Such restrictions can be obtained by taking into account the higher-order terms in the expansion in powers of K; see Sec. 3. This procedure has been carried out for a single drift wave in a magnetized plasma in Ref. [14]. It has been shown that the terms of order K^6 in the growth rate of the large-scale instability lead to damping of perturbations in the range $K > K_{\max}$ and to the appearance of a maximum of the growth rate for small K.

Since the large-scale perturbations grow due to the negative viscosity effect, the nonlinear term in Eq. (4.8) becomes important. For the subsequent investigation of large-scale structures it is necessary to analyze the nonlinear equation.

5. RESULTS

Here we have studied new effects of generation of large-scale structures. These appear for Rossby wave turbulence in atmosphere and ocean and for drift wave turbulence in magnetized plasmas. The physical reason for their appearance is related to the change in the sign of the effective (turbulent) viscosity in large-scale motions of the medium (negative viscosity). Therefore, the damping of large-scale motions is replaced by growth, which has to be limited due to nonlinear effects. The small-scale field is stationary and maintained by an external source. Such a formulation is fruitful for studying the effects (characterized by the effective viscosity) of smaller-scale motions upon the larger-scale motion. The results obtained are as follows:

1. With the use of the two-scale expansion, an equation is obtained which describes the evolution of the mean stream function in the presence of the small-scale Rossby-wave field or the evolution of the mean potential in the presence of the small-scale drift-wave field. General expressions are obtained for the terms describing the influence of small-scale motion, namely, the viscous terms, the dispersion term and the terms nonlinear in the large-scale field. These

expressions allow one to study the evolution of large-scale motions, with the assumption that the spectrum of the stationary small-scale field is known.

2. The results obtained admit a transition to the hydrodynamics of a viscous incompressible fluid. The previously known results on the eddy viscosity of the small-scale fluid motions are recovered.

3. The qualitative reason for the negative viscosity effect to appear is the transport of small-scale vorticity by the mean flow. This effect leads to the negative viscosity contribution to the effective viscosity governing the large-scale motions.

4. It is shown that the coherent wave motions lead to the negative effective viscosity. The criterion of large-scale instability due to the negative viscosity effect is derived.

5. Opposite to the case of the small-scale isotropic motions of a viscous incompressible fluid, small-scale isotropic Rossby wave and drift wave motions can lead to the negative effective viscosity. It is demonstrated that the effective viscosity can be negative if the spectral line broadening is greater than the linear damping rate.

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APPENDIX A

Derivation of Q

In this Appendix the term Q in Eq. (2.6) is derived. We introduce «slow» variables together with the «fast» ones. The spatial operators are now written in the form:

$$\frac{\partial}{\partial \mathbf{x}} \to \frac{\partial}{\partial \mathbf{x}} + \frac{\partial}{\partial \mathbf{X}},$$

$$\Delta \to \Delta_\perp + 2\frac{\partial^2}{\partial x_p \partial X_p} + \Delta_s, \tag{A.1}$$

$$\Delta^2 \to \Delta_\perp^2 + 2\Delta_\perp\Delta_s + 4\frac{\partial^2}{\partial x_p \partial X_p}\Delta_\perp + 4\frac{\partial^2}{\partial x_p \partial X_p}\Delta_s + \Delta_s^2 + 4\left(\frac{\partial^2}{\partial x_p \partial X_p}\right)^2.$$

Here

$$\Delta_s \equiv \frac{\partial^2}{\partial X_p \partial X_p}, \quad \Delta_\perp = \frac{\partial^2}{\partial x_p \partial x_p}.$$

Then, according to Eqs. (2.9), the term Q can be written in the form

$$Q(x,X,t) = q^{(01)} + q^{(02)} + q^{(03)} + q^{(04)} + q^{(10)} + q^{(20)} + q^{(30)} + q^{(40)} + \\ + q^{(11)} + q^{(12)} + q^{(13)} + q^{(21)} + q^{(22)} + q^{(31)} + O(K^5, K^6, \ldots), \tag{A.2}$$

where

$$q^{(00)} = \varepsilon_{mn}\left\langle \frac{\partial\psi^{(0)}}{\partial x_m}\frac{\partial}{\partial x_n}\Delta_\perp\psi^{(0)}\right\rangle = 0$$

due to the homogeneity of the turbulence,

$$q^{(01)} = \varepsilon_{mn} \left\langle \frac{\partial \psi^{(0)}}{\partial x_m} \left(\frac{\partial^3}{\partial x_n \partial x_p^2} + 2\frac{\partial^3}{\partial x_n \partial x_p \partial X_p} + \frac{\partial^3}{\partial x_n \partial X_p^2} + \frac{\partial^3}{\partial x_p^2 \partial X_n} + \right.\right.$$
$$\left.\left. + 2\frac{\partial^3}{\partial x_p \partial X_p \partial X_n} + \frac{\partial^3}{\partial X_p^2 \partial X_n} \right) \psi^{(1)} \right\rangle, \quad (A.3)$$
$$q^{(10)} = \varepsilon_{mn} \left\langle \left(\frac{\partial}{\partial x_m} + \frac{\partial}{\partial X_m} \right) \psi^{(1)} \frac{\partial}{\partial x_n} \Delta_\perp \psi^{(0)} \right\rangle.$$

The remaining terms in Eq. (A.2) have similar structure. We retain in Eq. (A.2) only those terms which are of order $K^1, \ldots, K^4$. As will be seen below it is just these terms which give rise to the negative viscosity effect.

To calculate the terms in Eq. (A.2) it is necessary to get expressions for $\psi^{(0)}, \psi^{(1)}, \ldots \psi^{(4)}$. Using Eqs. (A.1), (2.7) one finds the the following equations:

$$O(K^0): \frac{\partial \psi^{(0)}}{\partial t} - \frac{\partial}{\partial t}\Delta_\perp \psi^{(0)} - \frac{\partial \psi^{(0)}}{\partial x} + \nu \Delta_\perp^2 \psi^{(0)} = F, \quad (A.4)$$

$$O(K^1): \frac{\partial \psi^{(1)}}{\partial t} - \frac{\partial}{\partial t}\Delta_\perp \psi^{(1)} - \frac{\partial \psi^{(1)}}{\partial x} + \nu \Delta_\perp^2 \psi^{(1)} - \varepsilon_{jk} \frac{\partial \overline{\psi}}{\partial X_j} \frac{\partial}{\partial x_K} \Delta_\perp \psi^{(0)} = 0, \quad (A.5)$$

$$O(K^2): \frac{\partial \psi^{(2)}}{\partial t} - \frac{\partial}{\partial t}\Delta_\perp \psi^{(2)} - \frac{\partial \psi^{(2)}}{\partial x} + \nu \Delta^2 \psi^{(2)} - \frac{\partial \psi^{(1)}}{\partial X} -$$
$$-2\frac{\partial^3 \psi^{(1)}}{\partial t\, \partial x_p \partial X_p} + 4\nu \frac{\partial^2}{\partial x_p \partial X_p} \Delta_\perp \psi^{(1)} - \varepsilon_{jk} \frac{\partial \overline{\psi}}{\partial X_j} \frac{\partial}{\partial x_k} \Delta_\perp \psi^{(1)} = 0. \quad (A.6)$$

We calculate only the terms $\psi^{(0)}$, $\psi^{(1)}$, and $\psi^{(2)}$, since the terms $\psi^{(3)}$, $\psi^{(4)}$ do not contribute to Q, as it will be seen below. Eqs. (A.4), (A.5) and (A.6) can be solved using the Fourier transform over the fast variable x, that is, e.g.,

$$\psi^{(0)}(x,t) = \int d\mathbf{k}\, \hat{\psi}^{(0)}(k,t) e^{ikx}. \quad (A.7)$$

Then, Eq. (A.4) can be transformed to

$$\frac{\partial \hat{\psi}^{(0)}}{\partial t} + (i\omega_R + \nu_R)\hat{\psi}^{(0)} = \frac{F(\mathbf{k},t)}{1+k^2}, \quad (A.8)$$

where

$$\omega_R = -\frac{k_x}{1+k^2}, \quad \nu_R = \frac{\nu k^4}{1+k^2}.$$

The solution of Eq. (A.8) is

$$\hat{\psi}^{(0)}(\mathbf{k},t) = \int_{-\infty}^{t} dt' \frac{F(\mathbf{k},t')}{1+k^2} \exp\left(i\Omega_k(t'-t)\right). \quad (A.9)$$

Here $\Omega_k = \omega_R - i\nu_R$. The solutions of Eqs. (A.5), (A.6) can be obtained in a similar way.

$$\hat{\psi}^{(1)}(\mathbf{X},\mathbf{k},t) = -\varepsilon_{jk} \frac{ik_k k^2}{1+k^2} \frac{\partial \overline{\psi}}{\partial X_j} \int_{-\infty}^{t} dt' \hat{\psi}^{(0)}(\mathbf{k},t') \exp\left(i\Omega_k(t'-t)\right), \quad (A.10)$$

$$\hat{\psi}^{(2)}(\mathbf{X},\mathbf{k},t) = -\varepsilon_{jk}\frac{ik_k k^2}{(1+k^2)^2}\frac{\partial^2\overline{\psi}}{\partial X\,\partial X_j}\int\limits_{-\infty}^{t} dt' e^{i\Omega_k(t'-t)}\int\limits_{-\infty}^{t'} dt''\hat{\psi}^{(0)}(\mathbf{k},t'')\exp\left(i\Omega_k(t''-t')\right) +$$

$$+2\varepsilon_{jk}\frac{k_k k_p k^2}{(1+k^2)^2}\frac{\partial^2\overline{\psi}}{\partial X_p\partial X_j}\int\limits_{-\infty}^{t} dt'\hat{\psi}^{(0)}(\mathbf{k},t')\exp\left(i\Omega_k(t'-t)\right) -$$

$$-2\varepsilon_{jk}\frac{k_p k_k k^2}{(1+k^2)^2}i\Omega_k\frac{\partial^2\overline{\psi}}{\partial X_p\partial X_j}\int\limits_{-\infty}^{t} dt'\exp\left(i\Omega_k(t'-t)\right)\int\limits_{-\infty}^{t'} dt''\hat{\psi}^{(0)}(\mathbf{k},t'')\exp\left(i\Omega_k(t''-t')\right) +$$

$$+4\nu\varepsilon_{jk}\frac{k_p k_k k^4}{(1+k^2)^2}\frac{\partial^2\overline{\psi}}{\partial X_p\partial X_j}\int\limits_{-\infty}^{t} dt'\exp\left(i\Omega_k(t'-t)\right)\int\limits_{-\infty}^{t'} dt''\hat{\psi}^{(0)}(\mathbf{k},t'')\exp\left(i\Omega_k(t''-t')\right) -$$

$$-\varepsilon_{jk}\varepsilon_{lq}\frac{k_k k_q k^2}{(1+k^2)^2}\frac{\partial\overline{\psi}}{\partial X_l}\frac{\partial\overline{\psi}}{\partial X_j}\int\limits_{-\infty}^{t} dt'\exp\left(i\Omega_k(t'-t)\right)\int\limits_{-\infty}^{t'} dt''\hat{\psi}^{(0)}(\mathbf{k},t'')\exp\left(i\Omega_k(t''-t')\right). \quad \text{(A.11)}$$

Now calculate the sum $Q^{(1)} = q^{(01)} + q^{(10)}$. Using the Fourier transform over the fast variables we get

$$q^{(01)}(\mathbf{X},t) = \varepsilon_{mn}\iint d\mathbf{k}\,d\mathbf{k}' ik_m\left[-ik'_n k'^2 - 2k'_n k'_p\frac{\partial}{\partial X_p} + ik'_n\frac{\partial^2}{\partial X_p^2} - k'^2\frac{\partial}{\partial X_n} + \right.$$
$$\left. + 2ik'_p\frac{\partial^2}{\partial X_p\partial X_n} + \frac{\partial^3}{\partial X_p^2\partial X_n}\right]\left\langle\hat{\psi}^{(0)}(\mathbf{k},t)\hat{\psi}^{(1)}(\mathbf{X},\mathbf{k}',t)\right\rangle\exp\left(i(\mathbf{k}+\mathbf{k}')\mathbf{x}\right). \quad \text{(A.12)}$$

It can be easily seen that some terms cancel due to

$$\varepsilon_{mn}k_m k_n = 0.$$

Using Eq. (A.10) we get

$$\left\langle\hat{\psi}^{(0)}(\mathbf{k},t)\hat{\psi}^{(1)}(\mathbf{k}',\mathbf{X},t')\right\rangle = -\varepsilon_{jk}\frac{ik'_k k'^2}{1+k'^2}\frac{\partial\overline{\psi}}{\partial X_j}\times$$
$$\times\int\limits_{-\infty}^{t'} dt_1\left\langle\hat{\psi}^{(0)}(\mathbf{k},t)\hat{\psi}^{(0)}(\mathbf{k}',t_1)\right\rangle\exp\left(i\Omega_{k'}(t'-t_1)\right). \quad \text{(A.13)}$$

Here the assumption about the slow time evolution of $\overline{\psi}$ in comparison with the turbulent term $\hat{\psi}^{(0)}$ is used. This allows us to take the term $\partial\overline{\psi}/\partial X_j$ out of the integral.

For homogeneous turbulence we have

$$\left\langle\hat{\psi}^{(0)}(\mathbf{k},t)\hat{\psi}^{(0)}(\mathbf{k}',t_1)\right\rangle = \Phi(\mathbf{k},t-t_1)\delta(\mathbf{k}+\mathbf{k}'). \quad \text{(A.14)}$$

Then we use the Fourier transform over time,

$$\Phi(\mathbf{k},t-t_1) = \int d\omega\,\Phi(\mathbf{k},\omega)\exp\left(-i\omega(t-t_1)\right). \quad \text{(A.15)}$$

Inserting Eqs. (A.15), (A.14) into Eq. (A.13) and then into Eq. (A.12), we can calculate $q^{(01)}(\mathbf{X},t)$.

Analogous calculations are employed for $q^{(10)}(\mathbf{X},t)$ which is equal to

$$q^{(10)}(\mathbf{X},t) = \varepsilon_{mn} \iint d\mathbf{k}\, d\mathbf{k}' \left(ik_m + \frac{\partial}{\partial X_m} \right) \left(-ik'_n k'^2 \right) \times \times \left\langle \hat{\psi}^{(1)}(\mathbf{X},\mathbf{k},t) \hat{\psi}^{(0)}(\mathbf{k}',t) \right\rangle \exp\left(i(\mathbf{k}+\mathbf{k}')\mathbf{x} \right). \tag{A.16}$$

Then, $Q^{(1)}$ has the form

$$Q^{(1)} = q^{(01)} + q^{(10)} = \varepsilon_{mn}\varepsilon_{jk} \left(\eta_{mpk} \frac{\partial^3 \overline{\psi}}{\partial X_p \partial X_n \partial X_j} - \nu^{(1)}_{mk} \frac{\partial^2 \Delta_s \overline{\psi}}{\partial X_n \partial X_j} \right), \tag{A.17}$$

where η_{mpk}, $\nu^{(1)}_{mk}$ are determined according to Eqs. (2.13). For $Q^{(2)}$ one gets

$$Q^{(2)} = q^{(02)} + q^{(20)},$$

$$q^{(02)}(\mathbf{X},t) = \varepsilon_{mn} \times \times \left\langle \frac{\partial \psi^{(0)}}{\partial x_m} \left(\frac{\partial}{\partial x_n}\Delta_\perp + 2\frac{\partial^3}{\partial x_p \partial x_n \partial X_p} + \frac{\partial}{\partial x_n}\Delta_s + \frac{\partial}{\partial X_n}\Delta_\perp + 2\frac{\partial^3}{\partial x_p \partial X_p \partial X_n} \right) \psi^{(2)} \right\rangle, \tag{A.18}$$

$$q^{(20)}(\mathbf{X},t) = \varepsilon_{mn} \left\langle \left(\frac{\partial}{\partial x_m} + \frac{\partial}{\partial X_m} \right) \psi^{(2)} \frac{\partial}{\partial x_n} \Delta_\perp \psi^{(0)} \right\rangle. \tag{A.19}$$

Carrying out a Fourier transform over the fast variable and inserting the solution for $\hat{\psi}^{(2)}$ into Eqs. (A.18), (A.19) we get

$$Q^{(2)}(\mathbf{X},t) = \varepsilon_{mn}\varepsilon_{jk} \left\{ \nu^{(2)}_{mpk} \frac{\partial}{\partial X} \left(\frac{\partial^3 \overline{\psi}}{\partial X_p \partial X_n \partial X_j} \right) - \nu^{(3)}_{mlpk} \frac{\partial^4 \overline{\psi}}{\partial X_l \partial X_n \partial X_p \partial X_j} + \right. \left. + \varepsilon_{rq} N^{(1)}_{mlkq} \frac{\partial^2}{\partial X_l \partial X_n} \left(\frac{\partial \overline{\psi}}{\partial X_r} \frac{\partial \overline{\psi}}{\partial X_j} \right) \right\}. \tag{A.20}$$

It is easily seen that the sums $q^{(03)} + q^{(30)}$, $q^{(12)} + q^{(21)}$ and $q^{(04)} + q^{(40)}$ vanish by symmetry. Then the sum $q^{(13)} + q^{(31)}$ is equal to zero to fourth order in K. Therefore, only an expression for $q^{(11)}$ should be obtained:

$$q^{(11)}(X,t) = \varepsilon_{mn} \left\langle \left(\frac{\partial \psi^{(1)}}{\partial x_n} \frac{\partial}{\partial X_n} \Delta_\perp \psi^{(1)} + 2\frac{\partial \psi^{(1)}}{\partial x_m} \frac{\partial^3 \psi^{(1)}}{\partial x_m \partial X_p \partial X_n} + \frac{\partial \psi^{(1)}}{\partial X_m} \frac{\partial}{\partial x_n} \Delta_\perp \psi^{(1)} + \right.\right. \left.\left. + 2\frac{\partial \psi^{(1)}}{\partial X_m} \frac{\partial^3 \psi^{(1)}}{\partial x_p \partial x_n \partial X_p} + \frac{\partial \psi^{(1)}}{\partial X_m} \frac{\partial}{\partial X_n} \Delta_\perp \psi^{(1)} \right) \right\rangle. \tag{A.21}$$

Inserting $\hat{\psi}^{(1)}$ (see Eq. (A.10)) into Eq. (A.21) we get

$$q^{(11)}(\mathbf{X},t) = \varepsilon_{mn}\varepsilon_{jk}\varepsilon_{rq} \left(N^{(2)}_{kpmq} \frac{\partial \overline{\psi}}{\partial X_j} \frac{\partial^3 \overline{\psi}}{\partial X_p \partial X_n \partial X_r} + N^{(3)}_{kpnq} \frac{\partial^2 \overline{\psi}}{\partial X_m \partial X_j} \frac{\partial^2 \overline{\psi}}{\partial X_p \partial X_r} \right). \tag{A.22}$$

Now, summing up expressions for $Q^{(1)}$, $Q^{(2)}$ and $q^{(11)}$ we get the final expression for Q to order K^4; see Eq. (2.10).

APPENDIX B

Eddy viscosity of 2D NS flows: a comparison with previous results

In this Appendix we demonstrate a transition from the formulae of Sec. 2 and Appendix A to those describing the turbulent viscosity of a 2D viscous incompressible fluid. It allows us, firstly, to compare our results with those of other authors studying turbulent viscosity in hydrodynamics in the framework of two-scale expansion, and, secondly, to point out the differences arising between calculations of the turbulent viscosity in 2D NS flows and in Rossby waves; see Sec. 4.

The coefficients in the equation for the mean stream function of viscous incompressible fluid are obtained from Eqs. (2.13) by setting $1 + k^2 \to k^2$ (no dispersion) and $\omega_R = 0$ (zero eigenfrequency). We do not discuss effects nonlinear in $\overline{\psi}$ and, therefore, neglect all nonlinear terms. Further, we have $\nu^{(2)}_{mpk} = 0$ because this term is determined by that with $\partial/\partial x$ in Eq. (2.1) Therefore, we have the following equation for $\overline{\psi}$:

$$\frac{\partial}{\partial t}\Delta\overline{\psi} - \nu\Delta^2\overline{\psi} + \varepsilon_{mn}\varepsilon_{jk} \times \\ \times \left(\eta_{mpk}\frac{\partial^3\overline{\psi}}{\partial X_p \partial X_n \partial X_j} - \nu^{(1)}_{mk}\frac{\partial^2}{\partial X_n \partial X_j}\Delta_s\overline{\psi} + \nu^{(3)}_{mlpk}\frac{\partial^4\psi}{\partial X_l \partial X_n \partial X_p \partial X_j}\right) = 0, \quad \text{(B.1)}$$

where

$$\begin{aligned} \eta_{mpk} &= \int d\mathbf{k}\, d\omega \frac{\Phi(\mathbf{k},\omega)}{\omega^2 + \nu^2 k^4}\, 2\omega k_m k_p k_k, \\ \nu^{(1)}_{mk} &= \int d\mathbf{k}\, d\omega \frac{\Phi(\mathbf{k},\omega)}{\omega^2 + \nu^2 k^4}\, \nu k^2 k_m k_k, \\ \nu^{(3)}_{mk} &= \int d\mathbf{k}\, d\omega \frac{\Phi(\mathbf{k},\omega)}{(\omega^2 + \nu^2 k^4)^2}\, 8\nu^3 k^4 k_m k_l k_p k_k. \end{aligned} \quad \text{(B.2)}$$

Naturally, these results can be obtained if we start from Eq. (3.1) and use the method described in Appendix A.

Thus, the sign and the value of the turbulent viscosity are determined by the terms with $\nu^{(1)}_{mk}$, $\nu^{(3)}_{mlpk}$. As is shown in Sec. 3, the term with $\nu^{(1)}_{mk}$ is due to the transport of turbulent vorticity fluctuations by the mean flow. Even for the isotropic small-scale fluctuations this term makes a negative contribution to the turbulent viscosity. However, the term with $\nu^{(3)}_{mlpk}$ can, of course, compensate the negative contribution.

At first, we note that the results which stem from Eqs. (B.1), (B.2) are in accord with the results of [12]. Indeed, let an external source in Eq. (3.1) be homogeneous in space and delta-correlated in time. Then, instead of Eqs. (B.2) we have

$$\begin{aligned} \nu^{(1)}_{mk} &= \int d\mathbf{k}\frac{k_m k_k}{k^4}\,\frac{\langle \hat{W}_0^2\rangle_k}{2\nu k^2}, \\ \nu^{(3)}_{mlpk} &= \int d\mathbf{k}\frac{k_m k_l k_p k_k}{k^6}\,\frac{3\langle \hat{W}_0^2\rangle_k}{\nu k^2}, \end{aligned} \quad \text{(B.3)}$$

where $\langle W_0^2\rangle_k$ is the spatial spectrum of the zeroth-order aproximation of the small-scale field vorticity. Equation (B.1) leads to the equation for the Fourier transform of the large-scale vorticity $\hat{\overline{W}}(\mathbf{K},t)$:

$$\frac{\partial}{\partial t}\hat{\overline{W}}(\mathbf{K},t) = \gamma(\mathbf{K})\hat{\overline{W}}(\mathbf{K},t) - \nu K^2\hat{\overline{W}}(\mathbf{K},t), \tag{B.4}$$

where

$$\gamma(\mathbf{K}) = \int d\mathbf{k}\langle\hat{W}_0^2\rangle_k \frac{[\mathbf{K},\mathbf{k}]_z^2}{2\nu k^6}\left(1 - \frac{6(\mathbf{Kk})^2}{k^2K^2}\right). \tag{B.5}$$

Equation (B.5) coincides with the main term of the expansion in powers of K/k of Eq. (2.15) of Ref. [12].

The case of isotropic small-scale turbulence is the simplest, so we start just from this one. We also use the Lorentzian shape for the spectral line broadening:

$$\Phi(\mathbf{k},\omega) = \frac{1}{\pi}\frac{\gamma_k}{\omega^2+\gamma_k^2}\Phi(k).$$

The Lorentzian shape is generally used; however, one may convince oneself that the result is not changed qualitatively by using another shape instead of Lorentzian one, for example, Gaussian,

$$\Phi(\mathbf{k},\omega) = \frac{1}{\sqrt{2\pi}\,\gamma_k}\exp\left(-\frac{\omega^2}{2\gamma_k^2}\right)\Phi(k), \tag{B.6}$$

or the «step-like» shape,

$$\Phi(\mathbf{k},\omega) = \begin{cases} \dfrac{1}{2\gamma_k}\Phi(k), & |\omega| \le \gamma_k \\ 0, & |\omega| \ge \gamma_k \end{cases}.$$

For the isotropic spectrum we have $\eta_{mpk} = 0$. When calculating $\nu_{mk}^{(1)}$, $\nu_{mlpk}^{(3)}$ the following subsidiary integrals over the asimuthal angle φ of the wavenumber $\mathbf{k}$ are used:

$$\begin{aligned} &\int_0^{2\pi} d\varphi\, k_m k_n = \pi k^2\delta_{mn}, \\ &\int_0^{2\pi} d\varphi\, k_m k_l k_p k_k = \frac{\pi}{4}k^4(\delta_{ml}\delta_{pk} + \delta_{mp}\delta_{lk} + \delta_{mk}\delta_{lp}). \end{aligned} \tag{B.7}$$

Then,

$$\begin{aligned} \nu_{mk}^{(1)} &= \nu^{(1)}\delta_{mk}, \\ \nu_{mk}^{(3)} &= \nu^{(3)}(\delta_{ml}\delta_{pk} + \delta_{mp}\delta_{lk} + \delta_{mk}\delta_{lp}), \end{aligned} \tag{B.8}$$

where

$$\begin{aligned} \nu^{(1)} &= \pi\int dk\, k^3\frac{\Phi(k)}{\nu k^2+\gamma_k}, \\ \nu^{(3)} &= \pi\int dk\, k^3\Phi(k)\frac{2\nu k^2+\gamma_k}{(\nu k^2+\gamma_k)^2}. \end{aligned} \tag{B.9}$$

Equation (B.1) has the form

$$\frac{\partial}{\partial t}\Delta_s\overline{\psi} = \nu_{eff}\Delta_s^2\overline{\psi}, \tag{B.10}$$

where

$$\nu_{eff} = \nu - \nu^{(1)} + \nu^{(3)} = \nu + \pi\int dk\, k^3\Phi(k)\frac{\nu k^2}{(\nu k^2 + \gamma_k)^2}. \tag{B.11}$$

It follows from the expressions obtained that the isotropic small-scale fluctuations do not give rise to a negative viscosity in the framework of the $2D$ NS equations for viscous incompressible fluid. The negative contribution $-\nu^{(1)}$ is compensated by the positive contribution $\nu^{(3)}$ arising in an accurate calculation of all the viscous terms in the framework of our scheme.

If we set $\gamma_k \to 0$ in Eq. (B.11), then instead of Eq. (B.10) we get

$$\frac{\partial}{\partial t}\Delta_s\overline{\psi} = \left(\frac{\langle\psi^2\rangle}{2\nu} + \nu\right)\Delta_s^2\overline{\psi}, \tag{B.12}$$

where $\langle\psi^2\rangle$ is the variance of small-scale fluctuations. This result has been obtained by another method in Ref. [13].

References

1. E. N. Lorenz, *The Nature and the Theory of the General Circulation of the Atmosphere*, WMO (1967).
2. V. P. Starr, *Physics of Negative Viscosity Phenomena*, McGraw-Hill, New York (1968).
3. R. H. Kraichnan, J. Atmos. Sci. **33**, 1521 (1976)
4. A. Pouquet, J. Fluid Mech. **88**, 1 (1978).
5. L. D. Meshalkin and Ya. B. Sinai, Prikladnaya Matematika i Mekhanika **25**, 1700 (1961).
6. J. S. A. Green, J. Fluid Mech. **62**, 273 (1974).
7. V. I. Yudovich, Prikladnaya Matematika i Mekhanika **29**, 453 (1965).
8. U. Frisch, Z. S. She, and P. L. Sulem, Physica D **28**, 382 (1987).
9. S. S. Moiseev, P. B. Rutkevich, A. V. Tur, and V. V. Yanovsky, Zhurn. Eksper. Teor. Fiz. **94**, 144 (1988).
10. A. A. Nepomnyashchii, Prikladnaya Matematika i Mekhanika **40**, 886 (1976).
11. G. I. Sivashinsky, Physica D **17**, 243 (1985).
12. D. Montgomery and T. Hatori, Plasma Phys. Contr. Fus. **26**, 717 (1984).
13. B. Dubrulle and U. Frisch, Phys. Rev. A **43**, 5355 (1991).
14. A. V. Tur, A. V. Chechkin, and V. V. Yanovsky, Phys. Fluids B **4**, 3513 (1992).
15. R. A. Madden, Rev. Geophys. and Space Phys. **17**, 1935 (1979).
16. A. R. Robinson, *Eddies in Marine Sciences*, Springer-Verlag, Berlin, Heidelberg, New York (1983).
17. P. C. Liewer, Nuclear Fusion **25**, 543 (1985).
18. J. La Belle, M. C. Kelley, and C. T. Surko, J. Geophys. Res. **91**, A5, 5513 (1986).
19. A. Hasegawa and C. Maclennan, Phys. Fluids **22**, 2212 (1979).
20. J. Pedlosky, *Geophysical Fluid Dynamics*, Springer-Verlag, New York (1979).
21. L. D. Landau and E. M. Lifshits, *Hydrodynamics*, Nauka, Moscow (1986).
22. A. Hasegawa and K. Mima, Phys. Fluids **21**, 87 (1978).