ЖЭТФ, 1998, том 113, вып. 4, стр. 1253–1260

STABILITY OF STRONGLY LOCALIZED EXCITATIONS IN DISCRETE MEDIA WITH CUBIC NONLINEARITY

S. Darmanyan, A. Kobyakov

Institute of Spectroscopy, Russian Academy of Sciences
142092, Troitsk, Moscow region, Russia

F. Lederer

Institute of Solid State Theory and Theoretical Optics, Friedrich-Schiller-Universität Jena
07743, Jena, Germany

Submitted 6 August 1997

By using a linear analysis it is analytically shown that the stability of strongly localized modes depends on their symmetry, the sign of nonlinearity, and the degree of localization. The existence of a stable, bright, even mode of the discrete nonlinear Schrödinger equation is demonstrated and confirmed by direct numerical simulations. Possible applications to all-optical switching are discussed.

In the past decade many investigations have been devoted to intrinsic localized modes in discrete nonlinear systems due to their relevance to different branches of science, e.g., solid state physics, nonlinear optics, and biology. The fundamental properties of localized structures were used to explain some thermodynamic effects in solids (e.g., nonexponential energy relaxation), polaron and defect dynamics in anharmonic lattices and quantum crystals, etc. (see Refs. [1–6] and the bibliography cited there). Many physical phenomena such as modulational instability of plane waves [7, 8], formation and stability of temporal solitons [9, 10], and the recurrence effect [11] occur in discrete systems in a quite different way compared to those in extensively studied continuum systems. The discretness of the medium is responsible for new physical effects that could not be forecast in studying the continuum model. Some of the theoretically predicted properties of discrete systems, in particular, modulational instability of plane waves, existence and dynamics of bright and dark localized states, were already verified experimentally [12, 13].

In many cases the evolution of the initial excitation may be described by the discrete nonlinear Schrödinger equation (DNLSE), which is one of the fundamental equations in nonlinear physics. For instance, it governs electron-phonon interaction in a one-dimensional ionic crystal or mediates nonlinear processes in biology, where it is called a discrete self-trapping equation [1]. Another spectacular example is the evolution of the electromagnetic field in an array of linearly coupled waveguides, which have a great potential in applications for performing all-optical switching, steering, and demultiplexing. Exploiting of such waveguide arrays for power and phase controlled all-optical information processing was discussed in many papers (see Refs. [14, 15] and the bibliography cited there). However, from the point of view of obtaining a practical device the number of excited channels in the array should be minimized. Fortunately, discrete systems are able to support the so-called strongly localized modes (SLMs), which contain only a few excited components and hence exactly suit the above-mentioned criterion. In contrast to an inhomogeneous discrete system, this intrinsic localization is a pure nonlinear effect which appears to be very promising in optical information processing. However, to optimize the switching process, the boundaries between stable and unstable propagation of the SLM have to be identified.

As far as the structure of the SLM is concerned, two basic types of SLMs can be distinguished, i.e., odd (centered on-site) and even modes (centered between sites). In both cases the adjacent components may oscillate either in- (unstaggered modes) or out-of-phase (staggered modes) [16, 17] depending on the sign of the nonlinearity. As was already mentioned, the stability of SLMs against perturbations affects substantially the dynamics of the mode and is therefore an important issue to be addressed. The problem can be tackled by using various approaches, e.g., direct numerical calculations or a method based on the so-called Peierls–Nabarro (PN) potential [16, 17]. It is evident that the former method cannot cover the entire problem; i.e., study of the effect of variation of all parameters involved on the stability. The latter method relies on the PN potential (PN barrier) of both types of solutions, providing no information about the instability gain. Moreover, as was demonstrated in Ref. [17], to consistently interpret the results obtained, one must introduce concepts such as the negative mass for staggered modes. Another technique, which is based on a variational approach, was applied to investigate the existence and stability of relatively weak localized modes of the generalized DNLSE [18]. Finally, the onset of chaos, including the so-called microchaos for three coupled oscillators, has been studied by calculating the Lyapunov exponent [19].

As a result of these previous studies, all even SLMs of the DNLSE with the Kerr-like nonlinearity have been assumed to be unstable. In this paper we prove for the first time the existence of a stable even mode in the system described by the DNLSE and give an analytical criterion for its stability. We show that a direct linear analysis can be exploited to straightforwardly investigate the stability of the entire family of SLMs. This technique provides a clear physical picture of the onset of SLM dynamics. The analytical results concerning the regions of instability as well as the respective gain permit us to draw conclusions for all-optical switching in waveguide arrays.

The DNLSE under consideration is

$$i\frac{dE_n}{dt} + c(E_{n+1} + E_{n-1}) + \lambda|E_n|^2 E_n = 0, \tag{1}$$

where t and n stand for the evolution parameter and the site index, respectively, E_n represents the excitation at the nth site, c is the linear coupling coefficient, and λ is the effective nonlinear coefficient. All quantities are dimensionless. This can be achieved by a convenient normalization using characteristic scales for the evolution variable and the amplitude of the excitation. In case of waveguide arrays t denotes the propagation distance along the waveguide.

In order to identify SLMs we take advantage of a method reported in Refs. [1, 4]. Inserting $E_n = e_n \exp(i\omega t)$ into (1), where e_n represent the respective amplitudes of a bright localized mode, we obtain a system of a few algebraic equations. Thus, for the even mode $e_n = A(\ldots, 0, \alpha_3, \alpha_2, 1, s, s\alpha_2, s\alpha_3, 0, \ldots)$, $|n| = 1, 2, 3, \ldots$, $s = \pm 1$ we obtain the following equation with the requirement for strong localization $|\alpha_3| \ll |\alpha_2| \ll 1$, $a_n \approx 0$ for $n > 3$:

$$\omega \equiv \omega_e = \lambda A^2 + sc + \frac{c^2}{\lambda A^2}, \quad \alpha_2 \equiv \alpha = \frac{c}{\lambda A^2} - s\left(\frac{c}{\lambda A^2}\right)^2, \quad \alpha_3 = \left(\frac{c}{\lambda A^2}\right)^2, \tag{2}$$

where for symmetry reasons the subscript $n = 0$ has been dropped.

Analogously for the odd mode the ansatz $e_n = B(\ldots, 0, \beta_2, \beta_1, \beta_0, s\beta_1, s\beta_2, 0, \ldots)$, $|\beta_2| \ll |\beta_1| \ll 1$, gives

$$\omega \equiv \omega_{os} = \lambda B^2 + \frac{2c^2}{\lambda B^2}, \quad \beta_0 = 1, \quad \beta_1 \equiv \beta = \frac{c}{\lambda B^2}, \quad \beta_2 = \beta_1^2, \quad s = 1 \tag{3a}$$

for the symmetric mode and

$$\omega \equiv \omega_{oa} = \lambda B^2 + \frac{2c^2}{\lambda B^2}, \quad \beta_0 = 0, \quad \beta_1 = 1, \quad \beta_2 \equiv \beta = \frac{c}{\lambda B^2}, \quad s = -1 \tag{3b}$$

for the antisymmetric mode. Here the subscripts e and o represent the even and the odd mode, respectively, and the parameter $s = \pm 1$ defines the symmetry of the mode. In deriving (2) and (3) we restricted the analysis to the second-order terms concerning the small parameters α, and β for no more than six excitations. In concentrating on the physical aspect of the problem we restrict the discussion to the first-order approximation. For sufficiently strong localization they provide a reasonable accuracy, which was confirmed by a direct numerical solution of (1). The difference between numerical and approximate analytical solutions merely amounts to a few percents. A detailed study which takes higher-order terms into account represents a separate subject and is beyond the scope of this paper. Hence, in what follows we mainly deal with *strongly* localized modes and assume that for $\alpha, \beta < \nu_{sl} \approx 0.2$ the second-order terms can be ignored.

To study the stability of SLMs we impose complex perturbations $\delta_n(t)$ on each nonzero excitation amplitude [20]. We begin with the even mode and insert the perturbed profile $e_n = A(\ldots, 0, \alpha + \delta_{-2}, 1 + \delta_{-1}, s + \delta_{+1}, \alpha + \delta_{+2}, 0, \ldots)$ into (1). A subsequent linearization yields an eighth-order system of equations for the real-valued variables, which is only numerically solvable. However, a considerable simplification can be achieved by a proper decomposition of the perturbations into symmetric and antisymmetric components as $\delta_j^{\pm} = \delta_{+j} \pm \delta_{-j}$ ($j = 1, 2$) [19], which leads to a decoupling of the system. Separating real and imaginary parts of the perturbations $\delta_j^{\pm} = \delta_{jr}^{\pm} + i\delta_{ji}^{\pm}$ and introducing the scaled time $\tau_e = \omega_e t$, we obtain two independent systems for the column vector $\overline{\delta}^{\pm} = (\delta_{1r}^{\pm}, \delta_{1i}^{\pm}, \delta_{2r}^{\pm}, \delta_{2i}^{\pm})$

$$\frac{d\overline{\delta}^{\pm}}{d\tau_e} = \begin{pmatrix} 0 & (s-p)\alpha & 0 & -\alpha \\ 2-(3s-p)\alpha & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & 1 \\ \alpha & 0 & -1 & 0 \end{pmatrix} \overline{\delta}^{\pm}, \tag{4}$$

where $p = \pm 1$ stands for the symmetric (δ_j^+) and antisymmetric (δ_j^-) perturbation, respectively. If we introduce $\overline{\delta}^{\pm} \propto \exp(g\tau_e)$, then the eigenvalues g of (4) are given by the biquadratic equation

$$g^4 + \left[1 + 2(p-s)\alpha + 2(3-2ps)\alpha^2\right] g^2 + 2(p-s)\alpha + 2(3-2ps)\alpha^2 + 2(p-2s)\alpha^3 + \alpha^4 = 0. \tag{5}$$

If the symmetry of the perturbation coincides with that of the SLM ($s = p$), Eq. (5) does not exhibit real-valued solutions provided that α is small, as required ($\alpha \le \nu_{sl}$). Thus, the SLM is always stable against those perturbations which was numerically verified. In contrast, if the perturbation has the opposite symmetry of the SLM ($p = -s$) the SLM can become unstable [$\mathrm{Re}(g) \neq 0$].

We observed two basically different kinds of SLMs dynamics. Both staggered and unstaggered modes are always unstable with respect to symmetric and antisymmetric perturbations, respectively, whereas SLMs with $\alpha s < 0$ are unstable only if the modulus of the amplitude α exceeds some critical value, i.e., $|\alpha| > \alpha_{cr}$. This particular even SLMs are neither staggered nor unstaggered and can be obtained from those by changing the phase of excitations on sites $n \ge 1$ by π. Hence, we call these modes *twisted staggered* (TS) ($s = 1$,

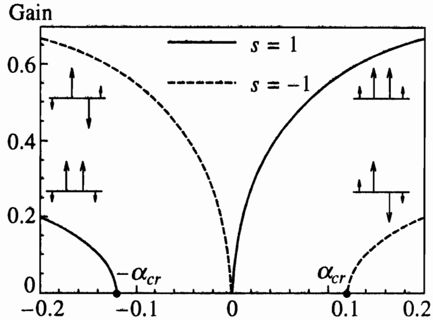

Fig. 1. Instability gain [Re(g)] plotted as a function of the amplitude α for even SLMs. The insets show the shape of the respective SLMs, where the twisted modes are sketched at the bottom

$\alpha < 0$) and *twisted unstaggered* (TU) ($s = -1$, $\alpha > 0$) SLM. It is worth realizing that the continuous NLSE limit does not exhibit a solution of that topology.

In analyzing the solutions of (5) we can ignore higher than quadratic terms in α and thus obtain a compact expression for the instability gain. If the linear coupling (c) and the nonlinearity (λ) have the same sign (i.e. $\alpha > 0$) the unstaggered SLM ($s = 1$) is always unstable against antisymmetric perturbations where the gain of instability

$$g \approx 2\sqrt{s\alpha}\,(1 - 5s\alpha/4), \quad s\alpha > 0 \tag{6}$$

increases with α (see Fig. 1). The instability of unstaggered modes is confirmed by a direct numerical solution of (1). The decay of the antisymmetrically perturbed unstaggered SLM and its subsequent transformation into an odd mode can be clearly recognized in Fig. 2. As can be anticipated from (6) (see also Fig. 1), the transition time decreases with α due to the increase in the instability gain. A change of α from 0.13 (Fig. 2*a*) to 0.15 (Fig. 2*b*) causes a significant reduction of that transition time to the stable odd mode. We mention that for relatively small amplitudes ($\alpha \approx 0.1$) the intermediate, asymmetric, oscillating state is fairly persistent and can be thus considered a quasi-stationary state.

In contrast to the behavior of the unstaggered SLM, which is in agreement with the results previously reported [16–18], the TU mode ($s = -1$) becomes unstable against symmetric perturbations only beyond the critical amplitude and the corresponding gain is

$$g \approx \sqrt{-s\alpha - \alpha_{cr}}, \quad s\alpha + \alpha_{cr} < 0, \quad \alpha_{cr} \approx 0.12. \tag{7}$$

This has the consequence that TU SLMs are stable against *any* perturbation if $\alpha < \alpha_{cr}$. For the case where c and λ have opposite signs (i.e., $\alpha < 0$) the situation is reversed (see the left side of Fig. 1) and the TS mode exhibits stability for that particular region of $|\alpha|$. These predictions were double-checked by numerically solving (1), imposing an *asymmetric* perturbation on the TU SLM. If α does not exceed the critical value α_{cr}, the TU mode is stable, exhibiting only slight oscillations evoked by the perturbation (see Fig. 3*a*). If α grows larger and exceeds the critical value, the TU SLM becomes unstable and decays eventually (see Fig. 3*b*). Thus, the existence of a stable even SLM of the DNLSE has been proven. The stability of the twisted modes might be explained by the fact that neither the TU nor the TS variant have a topological counterpart among odd SLMs. Hence, such a twisted SLM cannot transform to an odd SLM and stability arguments based on the PN barrier do not apply here. Beyond the critical value α_{cr} instability manifests itself in a spreading of the mode and sets in if the localization becomes weaker due to an increasing secondary amplitude $|\alpha|$ (see Fig. 3*b*). We note that accounting for

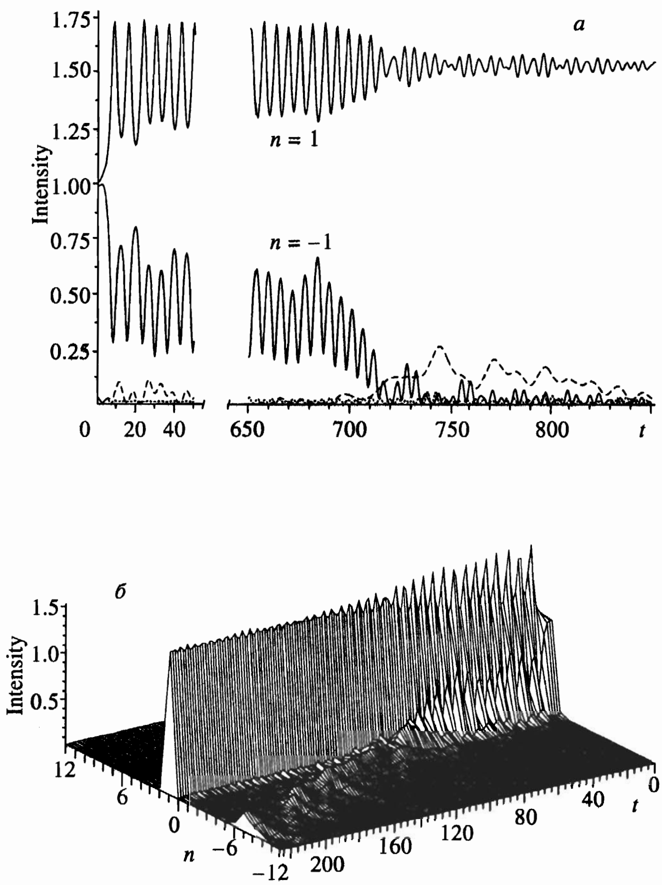

Fig. 2. Evolution of an unstaggered SLM ($s = 1$) antisymmetrically perturbed; $\lambda = 1$, $A = 1$, $\overline{\delta}^{-} = (0.01, 0, 0, 0)$; (*a*) $\alpha = 0.13$, evolution of the four initial excitations (solid lines $n = \pm 1$, dashed line $n = -2$, dotted line $n = 2$); (*b*) $\alpha = 0.15$, evolution of the mode

second-order terms in conjunction with the excitations at the sites $|n| = 3$ does not significantly change the instability regions and the gain. The transition from stability to instability, which is caused by a slight change of α at the input, can be potentially exploited for all-optical switching (e.g., see the drastic change of the output intensity in the waveguide labeled $n = -1$ in Figs. 3*a* and 3*b*, respectively).

Following the same procedure one can likewise study the stability of odd SLMs. For example, if we ignore the second-order corrections for the odd symmetric SLM in (3a) and impose complex perturbations $\varepsilon_n(t)$, we obtain from (1) and the subsequent linearization a six-order system of ordinary differential equations. By decomposing the perturbation into the symmetric and antisymmetric components $\varepsilon_1^{\pm} = \varepsilon_{+1} \pm \varepsilon_{-1}$ one can easily infer that the equation for ε_1^{-} can be separated, and that it yields the solution $\varepsilon_1^{-}(t) = \varepsilon_1^{-} \exp(-i\omega_o t)$. Obviously, this type of perturbation does not provoke any instability of the system. Thus, one needs only to study the stability with respect to symmetric perturbations. Separating real and imaginary parts

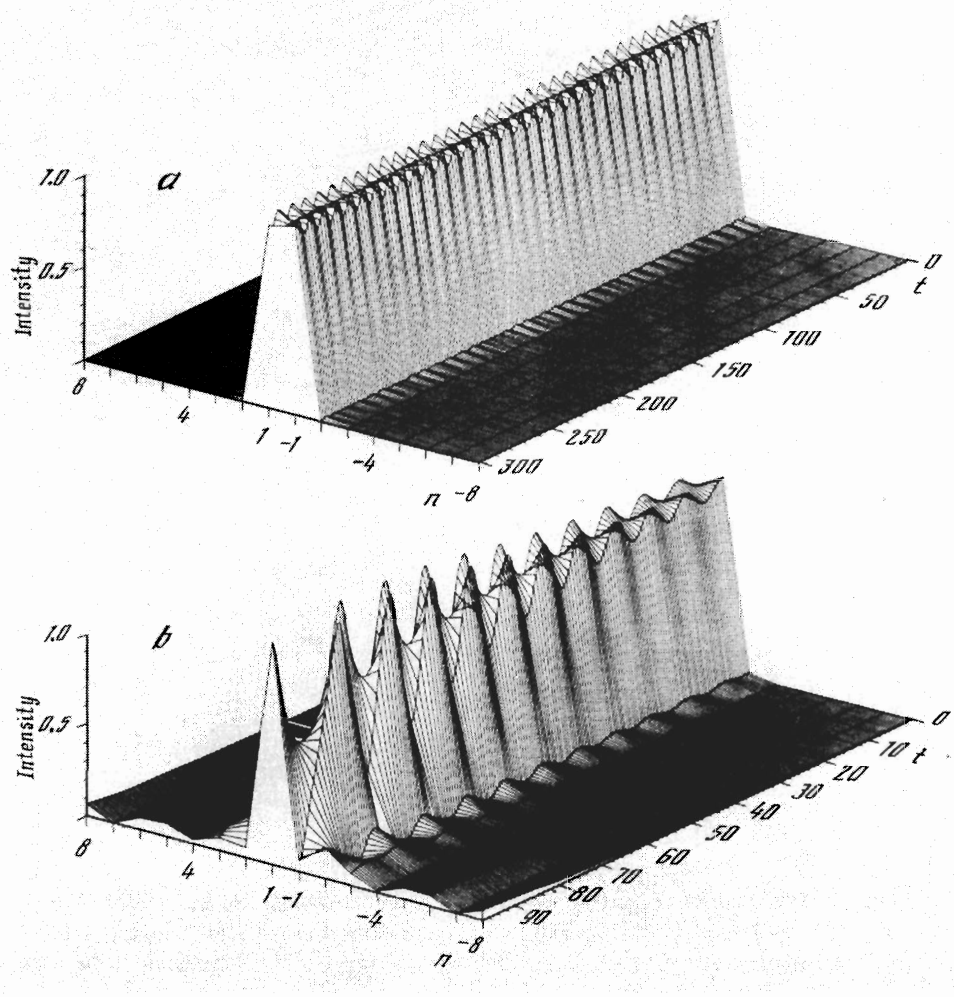

Fig. 3. Propagation of the perturbed twisted unstaggered SLM ($s = -1$), $\lambda = 1$, $A = 1$; (*a*) $\alpha = 0.11 < \alpha_{cr}$, $\overline{\delta}^{+} = (0.04, -0.02, 0.02, -0.02)$, $\overline{\delta}^{-} = (0, -0.04, -0.04, 0.04)$; (*b*) $\alpha = 0.16 > \alpha_{cr}$, $\overline{\delta}^{+} = (0.04, 0, 0, 0)$

of the perturbations $\varepsilon_0 = \varepsilon_{0r} + i\varepsilon_{0i}$, $\varepsilon_1^+ = \varepsilon_{1r}^+ + i\varepsilon_{1i}^+$, we obtain a system of four linear equations

$$\frac{d\overline{\varepsilon}}{d\tau_{os}} = \begin{pmatrix} 0 & 0 & 0 & -\beta \\ 2 & 0 & \beta & 0 \\ 0 & -2\beta & 0 & 1 \\ 2\beta & 0 & -1 & 0 \end{pmatrix} \overline{\varepsilon}, \qquad (8)$$

where $\tau_{os} = \omega_{os} t$ is the scaled time, and $\overline{\varepsilon} = (\varepsilon_{0r}, \varepsilon_{0i}, \varepsilon_{1r}^+, \varepsilon_{1i}^+)$ is the perturbation vector. Again the corresponding eigenvalue problem represents a simple biquadratic equation, which

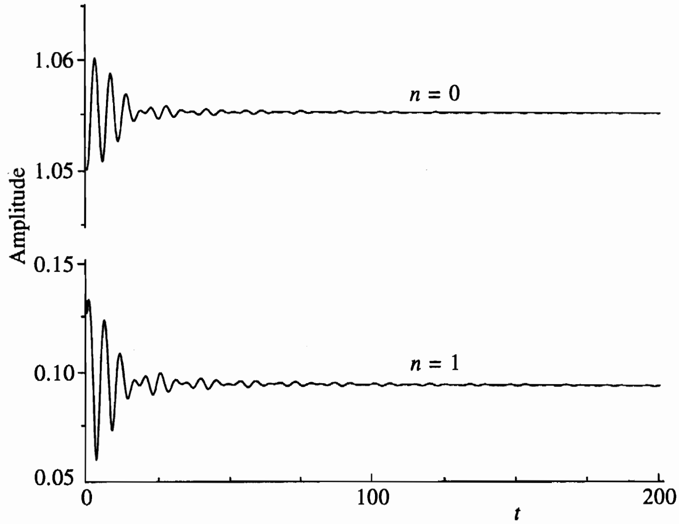

Fig. 4. Evolution of the perturbed odd SLM. The amplitudes of the central (upper curve) and the secondary (lower curve) excitations are shown for $\lambda = 1$, $B = 1$, $\beta = 0.1$, $\overline{\varepsilon} = (0.05, 0, 0.05, 0)$; $\varepsilon^- = i \cdot 0.05$

now reads as

$$g^4 + (1 + 4\beta^2)g^2 + 4\beta^2(1 + \beta^2) = 0, \tag{9}$$

where $\mathrm{Re}(g)$ also represents the instability gain. We straightforwardly obtain a nonzero gain $\mathrm{Re}(g)$ only provided that the secondary excitation $|\beta| > 1/\sqrt{8} \approx 0.35$. Such an instability causes the spreading of the mode in both directions in n. The larger the instability gain, the faster the unstable SLM decays and the excitation is spread over the entire array. However, the above values for the secondary excitation are beyond the required small-parameter limit for β. Thus, we may draw the conclusion that the odd, *strongly* localized mode (3a) is stable against small perturbations. This result was confirmed numerically. Figure 4 shows the evolution of a perturbed odd SLM, where a complex perturbation was superimposed on a solution of (3a). Obviously, the perturbation results only in the quickly damping oscillations near the SLM. Because (1) has a continuum set of SLM solutions, which depend on the amplitude B, the perturbed solution eventually transforms into a stable SLM with a new amplitude determined by the strength of perturbation. Analogously, one can show that the odd antisymmetric SLM is also stable.

In conclusion, we have demonstrated that by using a direct linear analysis the stability behavior of intrinsic, strongly localized modes of the discrete nonlinear Schrödinger equation can be analytically predicted. The regions of instability and the respective gain have been explicitly calculated. The familiar stability of odd modes was confirmed. It was shown for the first time that twisted *even* modes can be also stable provided that the secondary amplitudes are below a certain critical value.

S. D. and F. L. gratefully acknowledge a grant from the Deutsche Forschungsgemeinschaft, Bonn. S. D. is also thankful for the partial support through grant 96-0334049 of Russian

Foundation of Basic Research. The authors are grateful to E. Schmidt for invaluable help with the numerics.

References

1. A. C. Scott and L. Macneil, Phys. Lett. A 98, 87 (1983); J. C. Eilbeck, P. S. Lomdahl, and A. C. Scott, Physica D **16**, 318 (1985).
2. A. S. Dolgov, Fiz. Tverd. Tela **28**, 1641 (1986), [Sov. Phys. Sol. State **28**, 902 (1986)].
3. A. J. Sievers and S. Takeno, Phys. Rev. Lett. **61**, 970 (1988).
4. J. B. Page, Phys. Rev. B **41**, 7835 (1990).
5. V. M. Burlakov and S. A. Kiselev, Zh. Eksp. Teor. Fiz. **99**, 1526 (1991) [Sov. Phys. JETP **72**, 854 (1991)].
6. G. P. Tsironis and S. Aubry, Phys. Rev. Lett. **77**, 5225 (1996).
7. Yu. S. Kivshar and M. Peyrard, Phys. Rev. A **46**, 3198 (1992).
8. V. M. Burlakov, S. A. Darmanyan, and V. N. Pyrkov, Zh. Eksp. Teor. Fiz. **108**, 904 (1995) [Sov. Phys. JETP **81**, 496 (1995)].
9. A. A. Aceves, G. G. Luther, C. De Angelis, A. M. Rubenchik, and S. K. Turitsyn, Phys. Rev. Lett. **75**, 73 (1995).
10. S. A. Darmanyan, I. Relke, and F. Lederer, Phys. Rev. E **55**, 7662 (1996).
11. V. M. Burlakov, S. A. Darmanyan, and V. N. Pyrkov, Phys. Rev. B **54**, 3257 (1996).
12. B. Denardo, B. Galvin, A. Greenfield, A. Larraza, S. Putterman, and W. Wright, Phys. Rev. Lett. **68**, 1730 (1992).
13. P. Marquie, J. M. Bilbaut, and M. Remoissenet, Phys. Rev. E **51**, 6127 (1995).
14. A. Aceves, C. De Angelis, T. Peschel, R. Muschall, F. Lederer, S. Trillo, and S. Wabnitz, Phys. Rev. E **53**, 1172 (1996).
15. W. Krolikowski and Yu. S. Kivshar, JOSA B **13**, 876 (1996).
16. Yu. S. Kivshar and D. K. Campbell, Phys. Rev. E **48**, 3077 (1993).
17. D. Cai, A. R. Bishop, and N. Gronbech-Jensen, Phys. Rev. Lett. **72**, 591 (1994).
18. E. W. Laedke, O. Kluth, and K. H. Spatschek, Phys. Rev. E **54**, 4299 (1996).
19. N. Finlayson, K. J. Blow, L. J. Bernstein, and K. W. DeLong, Phys. Rev. A **48**, 3863 (1993).
20. K. W. Sandusky, J. B. Page, and K. E. Schmidt, Phys. Rev. B **46**, 6161 (1992).