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# THE ROLE OF HELICITY IN TURBULENT MHD FLOWS

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We have studied the behavior of a helical homogeneous small-scale MHD turbulent flow under the action of a weak inhomogeneous large-scale disturbance. We have shown that turbulent energy redistribution in the presence of nonzero helicity occurs mainly over large scales. Helicity increases correlation time, leading to the weakening of a direct cascade and to the formation of steep spectra over small scales, with simultaneous turbulent energy growth over large scales. Furthermore, an expression for the effective viscosity of the mean flow is derived. It is shown that the magnetic field, in addition to the helicity, reduces the effective viscosity of the medium. This may be important in the study of MHD flow around obstacles in the presence of an external magnetic field.

## 1. INTRODUCTION

The problem of self-organization of a turbulent MHD flow with magnetic Reynolds number $Re_m \ll 1$ in an external homogeneous magnetic field has long been under discussion (see, for instance, [1–4], with references therein). Obviously, an external magnetic field causes a rearrangement of the topological structure of a turbulent flow. Specifically, if the original turbulence (in absence of a magnetic field) is isotropic, it becomes anisotropic in the presence of a magnetic field. Furthermore, in presence of a magnetic field, the spectral and dynamical properties of turbulence can change.

As demonstrated by numerous experiments (see, e.g., [4–6]), turbulence spectrum varies with the magnetic field. It should be emphasized, however, that turbulence essentially always remains three-dimensional, although there exists a tendency to quasi-two-dimerization. Over small scales, the spectral dependence of the turbulent energy $E_t$ on wave number $k$ is of the form $E_t \sim k^{-\alpha}$, where the exponent $\alpha$ varies with increasing magnetic field from $-5/3$ (at $\mathbf{B} = 0$) to between $-2$ and $-7/3$ (at low $\mathbf{B}$ values) [1, 4]. With growing magnetic field, $\alpha$ ranges from $-11/3$ to $-4$, and the turbulence becomes highly intermittent [4].

It is noteworthy that such magnetic field-dependent behavior of the turbulence spectrum is observed only in those experiments where turbulent flow is generated either by drawing a grid through the medium [1] or in the presence of a honeycomb [4].

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For the entire subsequent analysis, it is important to note that turbulence becomes helical in an external magnetic field. This means that the one-point correlation function $H = \langle \mathbf{v}[\nabla\mathbf{v}]\rangle \neq 0$, where $\mathbf{v}$ is the flow velocity. On the other hand, purely helical turbulence is characterized by the quantity $\alpha = -7/3$ [7], which is in agreement with experimental results [1, 4]. In the general case, helicity, along with energy, is the most important feature of a turbulent flow. Helicity, being the second invariant of Euler's equation, just like energy [8], exerts a significant influence on the evolution and stability of turbulent and laminar flows [9]. Helicity is probably one of the main sources of magnetic field generation and maintenance in astrophysical objects [10]. In the absence of a magnetic field, helical turbulence is unstable against large-scale disturbances [11]. This leads to energy redistribution between large-scale and small-scale fluctuations. On the other hand, helicity leads to an efficient viscosity decrease in the mean flow, i.e., to a decrease in Reynolds stresses [12].

The present paper deals with the behavior of small-scale helical turbulence in an external homogeneous magnetic field and under a weak large-scale disturbance. We also examine the effect of a magnetic field on the viscosity of such turbulence.

## 2. PRINCIPAL EQUATIONS

Let us write the system of MHD equations in dimensionless form for an incompressible fluid:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\nabla)\mathbf{u} = -\nabla P + \frac{1}{Re}\Delta\mathbf{u} + \frac{N}{Re_m}[[\nabla\mathbf{B}]\mathbf{B}] + \mathbf{F}, \tag{1}$$

$$\frac{\partial \mathbf{B}}{\partial t} = [\nabla\,[\mathbf{uB}]] + \frac{1}{Re_m}\nabla^2\mathbf{B}, \tag{2}$$

$$\nabla\mathbf{u} = \nabla\mathbf{B} = 0,$$

where $\mathbf{F}$ is an external non-electromagnetic force and $P$ is pressure. The problem is characterized by three dimensionless numbers: the Reynolds number $Re = UL/\nu$, the magnetic Reynolds number $Re_m$ and the magnetic interaction parameter $N = \sigma B^2 L/\rho U$ (here $\rho, \nu$ are fluid density and viscosity, $U$ and $L$ are characteristic velocity and dimension).

We represent all fields as a sum of averaged and fluctuating values:

$$\mathbf{u} = \langle\mathbf{u}\rangle + \mathbf{u}', \quad \mathbf{B} = \mathbf{B}_0 + \mathbf{h}, \quad P = \langle P\rangle + P', \quad \mathbf{F} = \langle\mathbf{F}\rangle + \mathbf{F}',$$

$$\mathbf{B}_0 \gg \mathbf{h}, \quad \langle\mathbf{u}'\rangle = \langle\mathbf{h}\rangle = \langle P'\rangle = \langle\mathbf{F}'\rangle = 0,$$

where $\langle\ldots\rangle$ denotes averaging over an ensemble.

Assuming first that $\langle\mathbf{u}\rangle = 0$ and $Re_m \ll 1$, one can easily derive an equation for $\mathbf{u}'$ up to the second order of $\mathbf{h}$:

$$\frac{\partial \mathbf{u}'}{\partial t} + (\mathbf{u}'\nabla)\mathbf{u}' - \langle(\mathbf{u}'\nabla)\mathbf{u}'\rangle = -\nabla P' + \frac{1}{Re}\Delta\mathbf{u}' + \frac{N}{Re_m}[[\nabla\mathbf{h}]\mathbf{B}_0] + \mathbf{F}'.$$

Let us examine the stability of a small-scale turbulent flow under the magnetic field with respect to weak large-scale non-uniform disturbances. In this case, we represent the turbulent field as a sum of the initial turbulent field $\mathbf{u}^{(0)}$ and its disturbance $\mathbf{u}^{(1)}$:

$$\mathbf{u}' = \mathbf{u}^{(0)} + \mathbf{u}^{(1)},$$

where $\mathbf{u}^{(0)} \gg \mathbf{u}^{(1)}$.

We introduce the notation for the correlation functions:

$$Q^{00}_{ij}(\xi,\tau) = \left\langle u^{(0)}_i(\mathbf{x},t)\; u^{(0)}_j(\mathbf{x}+\xi,t+\tau)\right\rangle,$$

$$Q^{10}_{ij}(\mathbf{x},\xi,t,\tau) = \left\langle u^{(1)}_i(\mathbf{x},t)\; u^{(0)}_j(\mathbf{x}+\xi,t+\tau)\right\rangle,$$

and, in the same manner, for correlators of higher order.

Assuming that the external force $\mathbf{F}'$ maintains the initial small-scale helical turbulence, we can derive an equation for $Q^{00}_{ij} = Q^{00}_{ij}(\xi,\tau)$:

$$\frac{\partial Q^{00}_{ij}}{\partial t} + \nabla_k Q^{000}_{kij} = -\nabla_i\left\langle P^{(0)}u^{(0)}_j\right\rangle + \frac{1}{Re}\Delta Q^{00}_{ij} + \frac{N}{\Delta}(\nabla_i\nabla_3 Q^{00}_{3j} - \nabla^2_3 Q^{00}_{ij}) + \left\langle F'_i u^{(0)}_j\right\rangle,$$

where the magnetic term is obtained from the Eq.(2) with allowance for the above assumptions, and $\mathbf{B}_0$ is directed along the third coordinate.

In a similar way, we can write an equation for $Q^{10}_{ij} = Q^{10}_{ij}(\mathbf{x},\xi,t,\tau)$:

$$\frac{\partial Q^{10}_{ij}}{\partial t} - \frac{1}{Re}\Delta Q^{10}_{ij} + \nabla_k(Q^{100}_{ikj} + Q^{100}_{kij}) = -\nabla_i\left\langle P^{(1)}u^{(0)}_j\right\rangle + \frac{N}{\Delta}(\nabla_i\nabla_3 Q^{10}_{3j} - \nabla^2_3 Q^{10}_{ij}).$$

Eliminating the pressure from this equation, we obtain the final equation for $Q^{10}_{ij}$ :

$$\frac{\partial Q^{10}_{ij}}{\partial \tau} - \frac{1}{Re}\Delta Q^{10}_{ij} + \Pi_{im}\nabla_k(Q^{100}_{mkj} + Q^{100}_{kmj}) = -N\cos^2\theta Q^{10}_{ij}, \tag{3}$$

where $\Pi_{im} = (\delta_{im} - \nabla_i\nabla_m/\Delta)$ is a projection operator, and the operator $\cos^2\theta = \nabla^2_3/\Delta$.

Similarly to Eq. (3), we obtain an equation for the third moments $Q^{100}_{ijk} = Q^{10}_{ij}(\mathbf{x},\xi,\xi',t,\tau,\tau')$ :

$$\left(\frac{\partial}{\partial\tau} - \frac{1}{Re}\Delta + N\cos^2\theta\right) Q^{100}_{ijn} + \Pi_{im}\nabla_k(Q^{1000}_{mkjn} + Q^{1000}_{kmjn} - Q^{10}_{mk}Q^{00}_{jn} - Q^{10}_{mk}Q^{00}_{jn}) = 0. \tag{4}$$

To complete the system of Eqs. (3) and (4), we use the results from [12], where a finitness of a correlation time is taken into account through two-scale analogue of Orszag eddy damped quasi normal markovian (EDQNM) approximation [13]. This approximation consists in the replacement of the fourth order moment cumulant in equation for third moment on effective damping term, proportional to a square-law combination of pair moments. This approach is analogous to other traditional turbulent second order closures [14]. EDQNM approximation for strong MHD turbulence was considered in detail in [15].

In this approximation, we obtain from Eq. (4)

$$Q^{100}_{ipj} = -\tilde{\tau}\Pi_{im}\nabla_k Q^{1000}_{mkpj},$$

where the three-point correlator

$$Q^{1000}_{mkpj}(x,t,x,t,\xi,\tau,\xi',\tau') = Q^{10}_{mk}(x,t,x,t)Q^{00}_{pj}(\xi-\xi',\tau-\tau')+$$

$$+Q^{10}_{mp}(x,t,\xi,\tau)Q^{00}_{kj}(\xi',\tau') + Q^{10}_{mj}(x,t,\xi',\tau')Q^{00}_{kp}(\xi,\tau) \quad (5)$$

and

$$\tilde{\tau} = \left(\frac{1}{\tau^*} + N\cos^2\theta\right)^{-1},$$

and the correlation time $\tau^* \simeq L_{tur}/E^{1/2}_{tur}$ ($L_{tur}$ and $E_{tur}$ being the characteristic scale and average energy of turbulent flow, respectively).

To substitute the expression (5) for the third moment into Eq. (3), we pass to the limit $\xi'\to\xi, \tau'\to\tau$. Here (see Appendix) we take into account that $A\delta_{ij} = Q^{00}_{ij}(0,0)$ and $C\varepsilon_{ijp} = \partial Q^{00}_{ij}(\xi,\tau)/\partial\xi_p|_{\xi,\tau\to 0}$ depend on $B$ (or $N$), i.e., on the magnetic field. This results in

$$\left(\frac{\partial}{\partial\tau} - \left(\frac{1}{Re} + A\tilde{\tau}\right)\Delta + N\cos^2\theta\right) Q^{10}_{ij}(\mathbf{x},t,\xi,\tau) - \tilde{\tau}CH^{10}_{ij} =$$
$$= \tilde{\tau}\nabla_p\left[\frac{\partial}{\partial x_k}Q^{10}_{ip}(\mathbf{x},t,0,0)Q^{00}_{kj}(\xi,\tau) + \frac{\partial}{\partial x_s}Q^{10}_{kp}(\mathbf{x},t,0,0)\nabla_k\xi_s Q^{00}_{lj}(\xi,\tau) + \right.$$
$$\left. + Q^{10}_{kp}(\mathbf{x},t,0,0)\nabla_k Q^{00}_{ij}(\xi,\tau)\right], \quad (6)$$

where $A = A(N,0,0)$ and $C = C(N,0,0)$ are scalar functions of $N$, $H_{ij} = \varepsilon_{ikl}\nabla_k\times \times Q^{10}_{lj}(\mathbf{x},t,\xi,\tau)$.

## 3. INSTABILITY OF THE SECOND MOMENTS

To study the stability of the system (6), we apply the operator $\varepsilon_{lmi}\nabla_m$, and write resulting system in the homogeneous form:

$$\left(\nabla_\tau - \frac{1}{Re}\Delta - \tilde{\tau}A\Delta + N\cos^2\theta\right) Q^{10}_{ij} - \tilde{\tau}CH^{10}_{ij} = 0,$$
$$\left(\nabla_\tau - \frac{1}{Re}\Delta - \tilde{\tau}A\Delta + N\cos^2\theta\right) H^{10}_{ij} + \tilde{\tau}C\Delta Q^{10}_{ij} = 0. \quad (7)$$

An equation for $Q^{10}_{ij}$ follows from (7):

$$\left((\nabla_\tau - \nu_H\Delta + N\cos^2\theta)^2 + \tilde{\tau}^2C^2\Delta\right) Q^{10}_{ij} = 0, \quad (8)$$

where the effective viscosity $\nu_H = 1/Re + \tilde{\tau}A$. Passing into $k$-space in (8), we derive an expression for the decay factor $\gamma = -i\omega$:

$$\gamma = -\nu_H k^2 - N\cos^2\theta + \tilde{\tau}|Ck|, \quad (9)$$

where $\theta$ denotes the angle between $\mathbf{k}$ and $\mathbf{B}_0$. It follows from the form of Eq. (9) that helicity increases relaxation time. In other words, helicity prolongs vortex life-time. However, helicity influence is practically imperceptible in case of $\nu_H k^2 + N\cos^2\theta \gg \tilde{\tau}|Ck|$. Hence, at sufficiently weak magnetic fields the helicity effect is most essential over large scales, i.e. at low $k$ values. Under the condition $\gamma > 0$, we obtain from Eqn. (9) the instability condition:

$$\frac{1}{2}k_0\left(1-\left(1-\frac{4N\cos^2\theta}{k_0^2\nu_H}\right)^{1/2}\right) < k < \frac{1}{2}k_0\left(1+\left(1-\frac{4N\cos^2\theta}{k_0^2\nu_H}\right)^{1/2}\right), \qquad (10)$$

where $k_0 = \tilde{\tau}|C|/\nu_H$.

Let us study two limiting cases. Let $\cos\theta = 0$, i.e., consider modes for which $\mathbf{k}\perp\mathbf{B}_0$. In this case, (10) acquires the form of the purely hydrodynamic limit [11]:

$$0 < k < k_0. \qquad (11)$$

Here

$$k_0 = \frac{\tau^*|C|}{1/Re + A\tau^*},$$

and the dependence of the scale $k_0$ on the magnetic field is contained only in the coefficients $A$ and $C$. As a rule, $A\tau^* \gg 1/Re$, which results in $k_0 = C/A$. Consequently, $k_0$ dependence on the magnetic field is determined by the ratio of helicity to turbulence intensity.

On the other hand, at $\cos\theta = 1$, i.e., in modes for which $\mathbf{k} \parallel \mathbf{B}_0$, we obtain the condition (10) in the form

$$\frac{1}{2}\left(k_0-\left(k_0^2-\frac{4N}{\nu_H}\right)^{1/2}\right) < k < \frac{1}{2}\left(k_0+\left(k_0^2-\frac{4N}{\nu_H}\right)^{1/2}\right), \qquad (12)$$

where

$$k_0 = \frac{\tilde{\tau}|C|}{1/Re + A\tilde{\tau}}, \quad \tilde{\tau} = \frac{\tau^*}{1+N\tau^*}.$$

It is evident from (12) that at $\mathbf{B}_0 = 0$, this interval coincides with that of the modes with $\mathbf{k}\perp\mathbf{B}_0$. With growing magnetic field, $\tilde{\tau}$ decreases, i.e., the effective correlation time decreases. Simultaneously, the instability interval is reduced and vanishes at the fields described by $4N/\nu_H = k_0^2$.

It follows from the conditions (11) and (12) that at scales

$$0 < k < \frac{1}{2}\left(k_0-\left(k_0^2-\frac{4N}{\nu_H}\right)^{1/2}\right), \qquad (13)$$

$$\frac{1}{2}\left(k_0+\left(k_0^2-\frac{4N}{\nu_H}\right)^{1/2}\right) < k < k_0,$$

modes with $\mathbf{k}\perp\mathbf{B}_0$ are unstable against large-scale disturbances, whereas those with $\mathbf{k} \parallel \mathbf{B}_0$ are attenuated.

Thus, the energy of a large-scale disturbance is redistributed among the scales, so that modes transverse to the magnetic field predominantly grow.

## 4. TURBULENT VISCOSITY

To study the influence of the magnetic field on viscosity, we examine the variations in hydrodynamic viscosity (in the absence of a magnetic field) in an external homogeneous magnetic field.

We consider the case of $\langle \mathbf{u} \rangle \neq 0$ and $\langle \mathbf{u} \rangle \ll \mathbf{u}^{(0)}$. In this case, the equation for $\langle \mathbf{u} \rangle$,

$$\frac{\partial \langle \mathbf{u} \rangle}{\partial t} + \langle \mathbf{u} \rangle (\nabla \langle \mathbf{u} \rangle) - \langle (\mathbf{u}' \nabla) \mathbf{u}' \rangle = -\nabla \langle p \rangle + \frac{1}{Re} \Delta \langle \mathbf{u} \rangle + \frac{N}{Re_m} \langle [\nabla \mathbf{B}] \mathbf{B}] \rangle + \langle \mathbf{F} \rangle , \tag{14}$$

involves Reynolds stresses $\langle (\mathbf{u}' \nabla) \mathbf{u}' \rangle$ that depend only on $Q_{ij}^{10}$ for homogeneous turbulence, as shown in [12, 17]. On the other hand, additional terms appear on the right-hand side of Eq. (6) for $Q_{ij}^{10}(\xi, \tau, \mathbf{x}, t)$ :

$$\left[ \left( \frac{\partial}{\partial \tau} - \frac{1}{Re} \Delta - \tilde{\tau} A \Delta + N \cos^2 \theta \right) \delta_{il} - \tilde{\tau} C \varepsilon_{ikl} \nabla_k \right] Q_{lj}^{10}(\xi, \tau, \mathbf{x}, t) =$$
$$= - \langle u_p \rangle \nabla_p Q_{ij}^{00} - \frac{\partial}{\partial x_p} \langle u_i \rangle Q_{pj}^{00} - \frac{\partial \langle u_p \rangle}{\partial x_s} \xi_s \nabla_p Q_{ij}^{00} +$$
$$+ \tilde{\tau} \nabla_p \left[ \frac{\partial Q_{ip}^{10}(\mathbf{x}, t, 0, 0)}{\partial x_k} Q_{kj}^{00} + \frac{\partial Q_{kp}^{10}(\mathbf{x}, t, 0, 0)}{\partial x_s} \nabla_k (\xi_s Q_{ij}^{00}) + Q_{kp}^{10}(\mathbf{x}, t, 0, 0) \nabla_k Q_{ij}^{00} \right] , \tag{15}$$

where $u_p = u_p(\mathbf{x}, t)$, $Q_{ij}^{00} = Q_{ij}^{00}(\xi, \tau, N)$.

Taking into account that $j$ is a dummy index in Eq. (15), which allows us to write this equation in the vector form

$$\frac{\partial y_i}{\partial \tau} = A_{ij} y_j + f_i,$$

its formal solution being of the form

$$y_i(t) = Y_{ij}(t) y_i(0) + \int_0^t Y_{ik}(Y_{kj}^{-1}(t') f_j(t')) dt',$$

with the matrix $Y_{ij}$ satisfying the homogeneous equation

$$\frac{\partial Y_{ij}}{\partial \tau} = A_{ik} Y_{kj}, \tag{16}$$

the solution of Eq. (14) has the form

$$Y_{ij}(\tau) = \exp\left( -(\nu_H k^2 + N \cos^2 \theta) \right) \times$$
$$\times \left[ \mathrm{ch}(\tilde{C} k \tau) \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) + \frac{k_i k_j}{k^2} + i \varepsilon_{ikj} \frac{k_k}{k} \mathrm{sh}(\tilde{C} k \tau) \right] , \tag{17}$$

where $\tilde{C} = \tilde{\tau} C$.

In this case, the expression for $Q_{ij}^{10}$ takes the form

$$Q_{ij}^{10} = \int\limits_0^\infty \exp(-(\nu_H k^2 + N\overline{\cos^2\theta}\tau)) \left\{ \left[ \left( -\frac{4\pi k^2}{3}\tilde{A}_0 - \frac{4\pi k^3}{15}\frac{\partial \tilde{A}_0}{\partial k} \right) \text{ch}(\tilde{C}k\tau) + \right.\right.$$

$$\left. + \frac{4\pi k^4}{15}\frac{\partial \hat{C}}{\partial k}\text{sh}(\tilde{C}k\tau) \right] \left( \frac{\partial \langle u_i \rangle}{\partial x_j} + \frac{\partial \langle u_j \rangle}{\partial x_i} \right) -$$

$$\left. - \frac{4\pi}{15}\tilde{\tau}\left( k^2\tilde{A}_0 \text{ch}(\tilde{C}k\tau) - k^3\tilde{C}_0 \text{sh}(\tilde{C}k\tau) \right)\left( Q_{ij}^{10} + Q_{ji}^{10} \right) \right\} dk\, d\tau + Z_{ij}, \qquad (18)$$

where it is taken into account that $Q_{ij}^{00}(k,\tau) = \hat{Q}'_{ij}$ (see Appendix), $Z_{ij}$ are terms unrelated to the viscosity, and

$$\tilde{A}_0 = \frac{A_0}{1 + N\dfrac{2k^2 Re\overline{\cos^2\theta} + N\overline{\cos^4\theta}}{k^4 Re^{-2} + \omega^2}},$$

$$\tilde{C}_0 = \frac{C_0}{1 + N\dfrac{2k^2 Re\overline{\cos^2\theta} + N\overline{\cos^4\theta}}{k^4 Re^{-2} + \omega^2}}. \qquad (19)$$

For the sake of clarity, we have applied the mean-value theorem when integrating over $\theta$, resulting in the appearance of $\overline{\cos^2\theta}$. At $N \to 0$, the expression (18) passes to the hydrodynamic limit [12]. In this case, noting that Reynolds stress appears in the equation for $\langle \mathbf{u} \rangle$ in the form

$$-\frac{\partial}{\partial x_j}(Q_{ij}^{10} + Q_{ji}^{10}),$$

we obtain the expression for the viscosity $\nu_t^m$ in a magnetic field:

$$\nu_t^m = \left[ 1 + \frac{8\pi\tilde{\tau}}{15}\int\limits_0^\infty \left( k^4\tilde{A}_0(k,\tau)\text{ch}(\tilde{C}k\tau) - k^5\tilde{C}_0(k,\tau)\text{sh}(\tilde{C}k\tau) \right) \times \right.$$

$$\left. \times \exp\left( -(\nu_H k^2\tau + N\overline{\cos^2\theta})\tau \right) dk\, d\tau \right]^{-1} \int\limits_0^\infty \exp\left( -(\nu_H k^2\tau + N\overline{\cos^2\theta})\tau \right) \times$$

$$\times \left[ \left( \frac{8\pi k^2}{3}\tilde{A}_0(k,\tau) + \frac{8\pi k^3}{15}\frac{\partial \tilde{A}_0(k,\tau)}{\partial k} \right) \text{ch}(\tilde{C}k\tau) - \frac{8\pi k^4}{15}\frac{\partial \tilde{C}_0(k,\tau)}{\partial k}\text{sh}(\tilde{C}k\tau) \right] dk\, d\tau. \qquad (20)$$

It follows from (20) that in MHD flows with $Re_m \ll 1$, the turbulent viscosity decreases with increasing magnetic field.

## 5. DISCUSSION

As demonstrated above, a magnetic field alters the properties of homogeneous turbulence in a most significant manner. The existence of nonzero mean helicity results in instability of turbulent MHD flow with respect to weak large-scale disturbances. However, the instability of

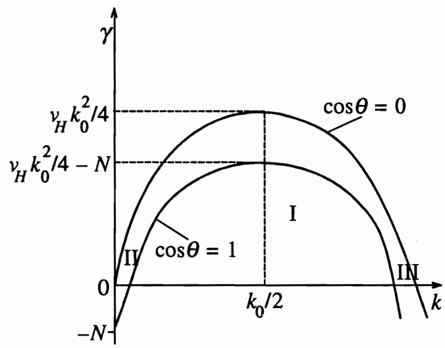


**Fig. 1.** Dependence of $\gamma$ on $k$ for two limiting values, $\cos\theta = 0$ and $\cos\theta = 1$

helical MHD turbulence has distinctive features in comparison with the instability of helical turbulent hydrodynamic flow in the absence of the magnetic field.

Figure 1 schematically shows the bahavior of the damping factor $\gamma$ for two limiting values of $\cos^2\theta$. The regions I, II, and III correspond to the conditions (10) and (13), respectively.

Let us examine the behavior of the components of the correlation tensor in these regions. The incompressibility condition in $k$-space,

$$k_i Q_{ij}^{10} = 0, \tag{21}$$

leads to the following relation between the components:

$$\left|\frac{k_3}{k_\perp}\right| = 2\left|\frac{Q_\perp}{Q_{3j}^{10}}\right|, \tag{22}$$

where for the sake of simplicity and without any loss of generality, we have assumed that $k_1 Q_{1j}^{10} = k_2 Q_{2j}^{10} = k_\perp Q_\perp$. On the other hand,

$$k_i k_i = k^2. \tag{23}$$

Multiplying (23) by $Q_\perp$ and assuming that $Q_1 \approx Q_2$, we find with the help of (22) that

$$Q_{3j}^{10} = \sqrt{2}\,\mathrm{tg}\,\theta \cdot Q_\perp. \tag{24}$$

Taking into consideration the fact that the energy density at fixed $k$ is

$$E(k) = 2Q_\perp + Q_3, \tag{25}$$

where $Q_3 = Q_{33}^{10}$, we finally obtain

$$E(k)\cos\theta = (2\cos\theta + \sqrt{2}\sin\theta)Q_\perp. \tag{26}$$

Thus, $\cos\theta$ is a measure of energy distribution among components along and across the magnetic field at specified $k$.

At $\cos\theta = 0$ we obtain $Q_\perp = 0$, and all the energy of the given mode is concentrated in the component parallel to the magnetic field, i.e., $E(k) = Q_3$. Here $k^2 = 2k_\perp^2$, i.e., in this mode fluctuations are normal to the magnetic field.

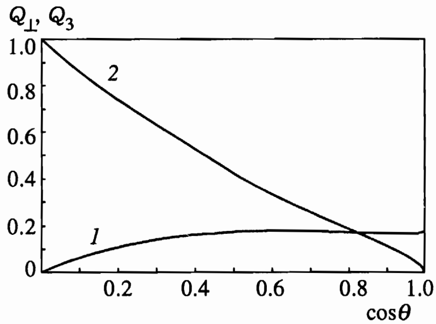


**Fig. 2.** Angle dependence of $Q_\perp$ (*1*) and $Q_3$ (*2*) (normalized by $\tilde{E}$) for $Nt = 1$

At $\cos\theta = 1$ we observe the opposite situation: all the energy of these modes with $k^2 = k_3^2$ is concentrated in fluctuations normal to the field, i.e., $E(k) = 2Q_\perp$, which oscillate along the magnetic field.

If we take into account the form of $E(k) = Q_{ii}^{10}(k)$ and (8) for larger scales,

$$E(k,t) = \tilde{E}(k,t)\exp(-Nt\cos^2\theta)$$

then we obtain at some fixed time $t$

$$Q_\perp = \tilde{E}\exp(-Nt\cos^2\theta)\frac{\cos\theta}{2\cos\theta + \sqrt{2}\sin\theta},$$
$$Q_3 = \tilde{E}\exp(-Nt\cos^2\theta)\frac{\sqrt{2}\sin\theta}{2\cos\theta + \sqrt{2}\sin\theta}. \qquad (27)$$

Figure 2 represents the behavior of $Q_\perp/\tilde{E}$ and $Q_3/\tilde{E}$ for $Nt = 1$ as a function of angle. One can easily see that modes with the same $k$ behave differently, depending on $\cos\theta$.

Returning to the instability of the second moments, we have the following. In regions II and III, corresponding to the conditions (13), modes with $\cos\theta = 1$ are attenuated, whereas modes with $\cos\theta = 0$ grow. Hence, energy must be transferred from modes with $\cos\theta = 0$ to modes with $\cos\theta = 1$. In region I, energy growth is observed in all modes (but at different growth rates). In this case energy transfer between modes at a fixed $k$ probably proceeds in such a way that at $\theta < \pi/4$, energy is transferred from $Q_3$ to $Q_\perp$, and conversely at $\theta > \pi/4$, from $Q_\perp$ to $Q_3$. However, in these cases the fluctuation amplitude will grow in a different manner. This is related to Joule dissipation, which is greatest at $\cos\theta = 1$, and vanishes at $\cos\theta = 0$.

If we analyze the role of helicity, it reduces to the following. By increasing vortices, lifetime at large scales, helicity slows down a direct Obukhov cascade from larger to smaller scales. Thus, it leads to incoming energy redistribution over large scales, i.e., an increase in vortex lifetime increases the probability of vortex mergers. On the other hand, at high $k$ values, helicity play essentially no role, and at these scales turbulence is dissipated. The joint action of these two processes results in the energy growth at large scales, and in an efficient «eating away» of smaller scales that are weakly supplied with energy «from above» due to the presence of nonzero one-point helicity in the system. Since helicity grows with magnetic field [18], at the same time, the connection between large and small scales is disturbed more strongly. Experimentally [4], this leads to the energy spectrum steepening over the small-scale range, up to $\alpha = 11/3$–$16/3$ with increasing magnetic field. Thus, the energy of a weak large-scale disturbance is redistributed among modes with differing $\cos\theta$, the cascade along the spectrum being weak.

In contrast, region I decreases with growing magnetic field, and fluctuation growth at $0 < k < k_0$ is mainly connected with the growth of the energy of longitudinal fluctuations. Here, however, one must bear in mind that turbulence remains three-dimensional, but the process of energy transfer from one component to another at a given fixed $k$ results from rapidly occurring processes (instabilities). This leads to the generation of a quasi-two-dimensional fluctuation pattern (symmetric about the magnetic field).

It should be noted that in the intermediate range of $0 < \cos\theta < 1$, there exist fluctuations along all three components; for instance at $\theta = \pi/4$ the intensity of modes along and across the magnetic field is the same, and they have the same instability growth rate. In the vicinity of this point, namely at $\theta \cong \pi/4$, energy exchange between components with given $k$ is probably absent.

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## APPENDIX

The influence of an external uniform magnetic field on the behavior of correlations in a turbulent medium has been studied in Ref. [2]. In a magnetic field, the second moment of the velocity field acquires the form

$$\hat{Q}_{ij}(\mathbf{k},\omega) = \frac{\hat{Q}'_{ij}(\mathbf{k},\omega)}{1 + \dfrac{(\mathbf{kB}_0)^2}{\mu\rho}\dfrac{2\eta\nu k^4 - 2\omega^2 + (\mathbf{kB}_0)^2/\mu\rho}{(\eta^2k^4+\omega^2)(\nu^2k^4+\omega^2)}}, \tag{A.1}$$

where $\hat{Q}'_{ij}(\mathbf{k},\omega)$ is a correlation function in the absence of the magnetic field, and $\eta, \nu, \rho$ are magnetic and hydrodynamic viscosities and fluid density, respectively. Here we retain, for convenience, the notation of Ref. [2]. Assuming that turbulence is helical and isotropic in the absence of the magnetic field, we can write $\hat{Q}'_{ij}(\mathbf{k},\omega)$ as follows (no matter whether $H^0(0,t) = 0$ or not):

$$\hat{Q}'_{ij}(\mathbf{k},\omega) = \hat{A}_0(\mathbf{k},\omega)\left(\delta_{ij} - \frac{k_ik_j}{k^2}\right) + i\hat{C}_0(\mathbf{k},\omega)\varepsilon_{ijs}k_s. \tag{A.2}$$

When passing into $x$-space, $Q^{00}_{ij} = \int \hat{Q}_{ij}(\mathbf{k},\omega)d^3k d\omega$ in the limit $\xi = x - x' \to 0, \tau = t - t' \to 0$.

In case of $Re_m \ll 1$ we obtain from (A.1)

$$\hat{Q}_{ij}(\mathbf{k},\omega) = \frac{\hat{Q}'_{ij}(\mathbf{k},\omega)}{1 + \dfrac{(\mathbf{kB})^2}{\mu\rho}\dfrac{2\eta\nu k^4 + (\mathbf{kB})^2/\mu\rho}{\eta^2k^4(\nu^2k^4+\omega^2)}} = \frac{\hat{Q}'_{ij}(\mathbf{k},\omega)}{1 + \dfrac{k^2\cos^2\theta\ B_0^2}{\mu\rho\eta^4}\dfrac{2\eta\nu + \cos^2\theta\ B_0^2/\mu\rho k^2}{\nu^2k^4+\omega^2}},$$

or in dimensionless form

$$\hat{Q}_{ij}(\mathbf{k},\omega) = \frac{\hat{Q}'_{ij}(\mathbf{k},\omega)}{1 + N\dfrac{2k^2 Re\cos^2\theta\ + N\cos^4\theta}{k^4Re^{-2}+\omega^2}}. \tag{A.3}$$

Consequently,

$$A(N,0,0) = \frac{1}{2}\int \frac{\delta_{ij}\hat{Q}'_{ij}(\mathbf{k},\omega)}{1+N\dfrac{2k^2 Re\cos^2\theta + N\cos^4\theta}{k^4 Re^{-2}+\omega^2}} d^3k\, d\omega,$$

$$C(N,0,0) = -\frac{i}{3}\varepsilon_{ijk}\int \frac{k_k\hat{Q}'_{ij}(\mathbf{k},\omega)}{1+N\dfrac{2k^2 Re\cos^2\theta + N\cos^4\theta}{k^4 Re^{-2}+\omega^2}} d^3k\, d\omega.$$

Clearly, both $A(N,0,0)$ and $C(N,0,0)$ decrease with increasing $N$ (or magnetic field).

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