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STABILITY PROBLEM IN NONLINEAR WAVE PROPAGATION

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An explicit expression for the excitation spectrum of the stationary solutions of a nonlinear wave equation is obtained. It is found that all branches of many-valued solutions of a nonlinear wave equation between the $(2K+1, 2K+2)$ turning points (branch points in the complex plane of the nonlinearity parameter) are unstable. Some parts of branches between the $(2K, 2K+1)$ turning points are also unstable. The instability of the latter is related to the possibility that pairs of complex conjugate eigenvalues cross the real axis in the κ plane.

1. INTRODUCTION

In the Ref. [1–5] it was found that for a transverse electromagnetic wave (Fig. 1) propagating in a nonlinear medium, many states are possible for a given amplitude of the incident wave. The nonlinear medium was taken in the form of a slab. The reflection and transmission coefficients in this case are functionals of the state. In a linear medium, there exists only one state for a given incident wave, and this state is stable against small perturbations. In a nonlinear medium, some of the states are stable and some are unstable against small perturbations. This property is very important for practical purposes. In this paper we study the problem of stability for all states. The main result is as follows: all solutions of the nonlinear problem can be parametrized by one parameter ρ_1, which is equal to the transparency of the nonlinear medium. This parameter ρ_1 is a multivalued function of the effective nonlinearity μ. The graph of $\rho_1 = \rho_1(\mu)$ has turning points (see Fig. 2). All branches between the $2K+1$ and $2K+2$ ($K = 0, 1, 2, \ldots$) turning points are unstable, and some parts of branches between the $2K$ and $2K+1$ turning points are unstable against small perturbations. In thermodynamics it is also possible to find many solutions for given external conditions. Some of them are stable, some are not. But there always exists a state that yields the absolute minimum of the free energy. All other stable states can be considered metastable. Only quantum or thermal fluctuations can lead to transitions between different metastable states. In a dynamical problem, on the other hand, there is no

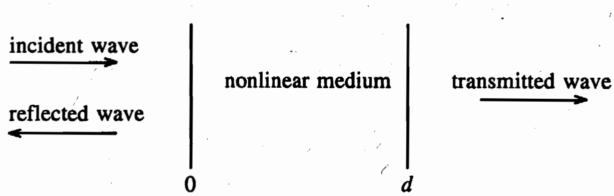

Fig. 1. Geometry of wave propagation

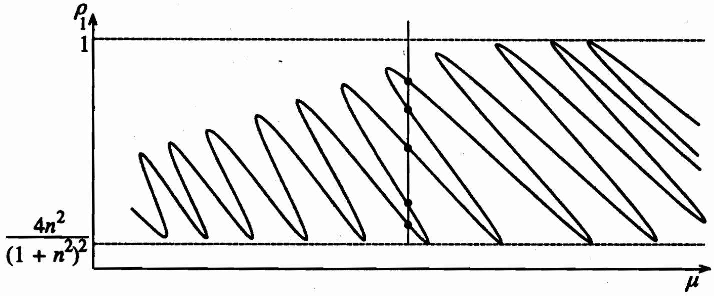

Fig. 2. Dependence of ρ_1 on μ; the number of solutions for given μ

general principle that distinquishes one solution from all other local stable states. Which state will be realized after a transition from an unstable state is still an unsolved problem.

2. FORMULATION OF STABILITY PROBLEM

We investigate the stability of solutions of the wave equation

$$\frac{\partial^2 E}{\partial x^2} - \frac{1}{2c^2}\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial E}(\epsilon E^2) = 0, \qquad \epsilon = n^2(1+\mu_0 E^2) \tag{1}$$

in a slab of length d. In Eq. (1), n is the refractive index and c the speed of light. In the general case, μ_0 is a function of position x. In real materials $\mu_0 \sim E_0^{-2}$, where E_0 is the electric field on the atomic scale. Hence, in real materials $\mu_0|E|^2 \ll 1$. Only such a case will be considered below. We investigate the stability of solutions of Eq. (1), that take the form

$$E_0 = \mathrm{Re}(e^{-i\omega t}\psi_0(x)) \tag{2}$$

for an incident wave given by

$$A\exp\left(\frac{i\omega}{c}x\right). \tag{3}$$

It is convenient to use the dimensionless variables

$$y = \frac{\omega n}{c}x, \quad \mu = \frac{3}{4}\mu_0|A|^2, \quad \psi_0 \to A\psi, \quad b = \frac{\omega n}{c}d. \tag{4}$$

Then Eq. (1) takes the form [5] (inside the slab)

$$(1+2\mu|\psi|^2)\psi + \frac{\partial^2\psi}{\partial y^2} = 0, \tag{5}$$

with boundary conditions

$$\psi_{(0)} = 1+R, \quad \psi'_{(0)} = \frac{i}{n}(1-R),$$

$$\psi_{(b)} = Te^{ib/n}, \quad \psi'_{(b)} = \frac{i}{n}Te^{ib/n}. \tag{6}$$

The unknown coefficients R and T can be eliminated from Eq. (5), and we obtain

$$\psi_{(0)} - in\psi'_{(0)} = 2, \quad \psi_{(b)} + in\psi'_{(b)} = 0. \tag{7}$$

We represent ψ in the form [5]

$$\psi = \sqrt{\rho}e^{i\alpha}. \tag{8}$$

The value of the function $\rho_{(y)}$ at point b ($\rho_1 = \rho(b)$) completely determines the modulus of the transmission and reflection coefficients:

$$|T|^2 = \rho_1, \quad |R|^2 = 1 - \rho_1. \tag{9}$$

Equation (1) for the stationary solutions of type (7) can be reduced to a function ρ only [5], and its solutions in the general case are elliptic functions. The condition $|\mu E^2| \ll 1$ drastically simplifies the investigation of solutions of type (8) [5].

The qualitative dependence of ρ_1 on the effective nonlinearity is shown in Fig. 2 (see also Eq. (48)).

We are able now to formulate the stability problem for solutions of type (8). We seek a solution of Eq. (1) in the form

$$E = E_0 + \tilde{E}, \tag{10}$$

where E_0 is given by Eqs. (2), (8), and

$$E_1 = \mathrm{Re}\left[e^{-i\omega t}\left(E_1 e^{\kappa\omega t} + E_2 e^{\kappa^*\omega t}\right)\right]. \tag{11}$$

The boundary conditions for the function $\tilde{E}$ correspond to the outgoing wave:

$$\left(\frac{E_1'}{E_1}\right)_0 = -\frac{i}{n}(1+i\kappa), \quad \left(\frac{E_1'}{E_1}\right)_b = \frac{i}{n}(1+i\kappa),$$
$$\left(\frac{E_2'}{E_2}\right)_0 = -\frac{i}{n}(1+i\kappa^*), \quad \left(\frac{E_2'}{E_2}\right)_b = \frac{i}{n}(1+i\kappa^*). \tag{12}$$

Inserting Eq. (11) into Eq. (1), we obtain

$$\frac{\partial^2 E_1}{\partial y^2} + (1+i\kappa)^2\left\{(1+2\mu|\psi|^2)E_1 + 2\mu(|\psi|^2 E_1 + \psi^2 E_2^*)\right\} = 0,$$
$$\frac{\partial^2 E_2}{\partial y^2} + (1+i\kappa^*)^2\left\{(1+2\mu|\psi|^2)E_2 + 2\mu(|\psi|^2 E_2 + \psi^2 E_1^*)\right\} = 0. \tag{13}$$

Note that ψ is the solution of Eq. (5) with boundary conditions given by (6). The system of equations (13) with boundary conditions (12) can be considered an eigenvalue problem for the symmetric operator $\hat{L}$. The explicit form of operator $\hat{L}$ is given by Eq. (13). We easily obtain for the first line

$$\hat{L}_{11} = \frac{\partial^2}{\partial y^2} + (1+4\mu|\psi|^2), \qquad \hat{L}_{12} = 0,$$
$$\hat{L}_{13} = 2\mu\cos(2\alpha)|\psi|^2, \quad \hat{L}_{14} = 2\mu\sin(2\alpha)|\psi|^2. \tag{14}$$

As a trivial fact, we note that if κ is an eigenvalue, then so is κ^*.

3. STABILITY IN LINEAR MEDIUM

Consider as a starting point the linear case ($\mu = 0$). In the linear case, the system of equations (13) decouples into two independent subsystems for the quantities E_1 and E_2. Solution of the first of Eqs. (13) yields

$$E_1 = A_1 \cos((b-y)(1+i\kappa)) + A_2 \sin((b-y)(1+i\kappa)). \tag{15}$$

From the boundary conditions (12), we obtain

$$A_2 = -\frac{i}{n}A_1, \qquad \mathrm{tg}(b(1+i\kappa)) = -\frac{2in}{n^2+1}. \tag{16}$$

Solution of the second of Eq. (16) yields

$$\kappa = -\frac{1}{b}\ln\frac{n+1}{n-1} + i\left(1 - \frac{\pi N}{b}\right); \quad N = 0, \pm1, \pm2... \tag{17}$$

The second of Eqs. (13) yields

$$\kappa = -\frac{1}{b}\ln\frac{n+1}{n-1} - i\left(1 - \frac{\pi N}{b}\right); \quad N = 0, \pm1, \pm2... \tag{18}$$

Hence the full spectrum of the operator $\hat{L}$ in the linear case is given by Eqs. (17) and (18). The real part of all eigenvalues is negative, hence the unique solution of (5) is stable in the linear case.

If the length b of the slab is given by

$$b = \pi N_0 + \varepsilon, \qquad |\varepsilon| \ll 1, \tag{19}$$

then for $n \gg 1$, two eigenvalues of the linear problem are closer to zero then all others:

$$\kappa_{1,2} = -\frac{1}{b}\ln\left(\frac{n+1}{n-1}\right) \pm \frac{i\varepsilon}{b}. \tag{20}$$

As we will see below, these two eigenvalues play a special role in the nonlinear problem.

4. STABILITY PROBLEM IN NONLINEAR MEDIUM

Points where $\partial\mu/\partial\rho_1 = 0$ are turning points. It is easy to show from Eqs. (5) and (7) that the function $\partial\psi/\partial\rho_1$ is a solution of the system of equations (13) for $\kappa = 0$. This means that of the two branches entering at the turning point, one is unstable, because in the vicinity of the turning point $\kappa \sim \delta\rho_1$. We prove this statement below.

We use the following simple expression [5] for ρ and α:

$$\rho = \rho_1\left\{\frac{1}{n^2} + \left(1 - \frac{1}{n^2}\right)\cos^2\left(\frac{(y-b)}{1 - 3\mu\rho_1\left(1 + 1/n^2\right)/4}\right)\right\}, \tag{21}$$

$$\alpha = \alpha_{(0)} + \frac{1}{n}\int_0^y \frac{dy_1}{1/n^2 + (1-1/n^2)\cos^2\left(\dfrac{b-y_1}{1-3\mu\left(1+1/n^2\right)\rho_1/4}\right)} =$$

$$= \alpha_{(0)} + \left(1 - \frac{3\mu}{4}\left(1+\frac{1}{n^2}\right)\rho_1\right)\left[\operatorname{arctg}\left\{\frac{1}{n}\operatorname{tg}\left(\frac{y-b}{1-3\mu\left(1+1/n^2\right)\rho_1/4}\right)\right\} + \right.$$

$$\left. + \operatorname{arctg}\left\{\frac{1}{n}\operatorname{tg}\left(\frac{b}{1-3\mu\left(1+1/n^2\right)\rho_1/4}\right)\right\}\right].$$

In Eq. (21), $\alpha_{(0)}$ is the phase of ψ at the point $y = 0$:

$$\alpha_{(0)} = \operatorname{arctg}\left[\frac{n\partial\rho/\partial y}{2(\rho_1+\rho)}\right]_{y=0}. \qquad (22)$$

Equations (5) and (21) enable us to represent the function ψ in a form that yields an explicit physical picture of nonlinear wave propagation. Inserting the expression (21) into Eq. (5), we find that ψ can be represented in the form

$$\psi = e^{i\lambda_1(b-y)}\left(Ae^{i\gamma(b-y)} + Be^{-i\gamma(b-y)}\right) + e^{i\lambda_2(b-y)}\left(Ce^{i\gamma(b-y)} + De^{-i(\gamma-\lambda)(b-y)}\right), \qquad (23)$$

where

$$\gamma = \frac{1}{1-3\mu\rho_1(1+1/n^2)/4}.$$

Equation (5) leads to the following expression for the quantities $\lambda_{1,2}$:

$$\lambda_{1,2} = \pm\lambda, \qquad (24)$$

where $\lambda = \mu\rho_1/2n$. From the same Eq. (5), we also obtain two equations for the quantities A, B and C, D. As a result, we have

$$\psi = B\left[\left(1+\frac{1}{n}\right)\exp\left(-i(\gamma-\lambda)(b-y)\right) + \left(1-\frac{1}{n}\right)\exp\left(i(\gamma+\lambda)(b-y)\right)\right] +$$

$$+ C\left[\left(1+\frac{1}{n}\right)\exp\left(i(\gamma-\lambda)(b-y)\right) + \left(1-\frac{1}{n}\right)\exp\left(-i(\gamma+\lambda)(b-y)\right)\right]. \qquad (25)$$

Now from the boundary condition (7) at $y = b$, we obtain one equation for B and C

$$C = B\frac{\mu\rho_1}{4}\left(\frac{1}{2}+\frac{3}{2n^2}\right). \qquad (26)$$

Finally, recalling that $|\psi_{(b)}| = \sqrt{\rho_1}$, we have

$$\psi = \frac{\sqrt{\rho_1}}{2}\left[\left(1+\frac{1}{n}\right)\exp\left(-i\gamma(b-y)\right) + \left(1-\frac{1}{n}\right)\exp\left(i\gamma(b-y)\right)\right]\exp\left(i\tilde{\varphi}_c + i\lambda(b-y)\right), \qquad (27)$$

where $\tilde{\varphi}_c$ is some constant, that is simply related to the phase α_0 given by Eq. (22).

Equation (27) means that the speed of light is slightly different for waves moving in opposite directions; and this is one of the main effects of nonlinearity.

Below, we use expressions (21) and (27) to solve the stability problem.

5. DERIVATION OF THE EQUATION OF THE SPECTRUM κ

The spectral points κ, defined by Eqs. (13) with boundary conditions (12), are the roots of some equation that is an analytic function of κ and the associated parameter ρ_1. Near the turning points, there exists a region where the two eigenvalues are real. This enables us to consider only real-valued κ. The equations for κ obtained under the assumption that κ is real can be analytically continued into the complex κ plane. The roots of this equation also yield the complex values of κ of the initial eigenvalue problem, given by Eqs. (12) and (13).

For real values of κ, we obtain the following eigenvalue problem:

$$\frac{\partial^2 E_1}{\partial y^2} + (1+i\kappa)^2\left\{(1+2\mu|\psi|^2)E_1 + 2\mu(|\psi|^2 E_1 + \psi^2 E_1^*)\right\} = 0, \tag{28}$$

$$\frac{E'_{1(0)}}{E_{1(0)}} = -\frac{i}{n}(1+i\kappa), \qquad \frac{E'_{1(b)}}{E_{1(b)}} = \frac{i}{n}(1+i\kappa), \tag{29}$$

where $|\psi|$ is given by Eq. (21) and ψ is given by Eq. (27). Equation (28) has four linearly independent solutions. We seek them in the form

$$\tilde{E}_1 = Ae^{i(\gamma+\beta)(b-y)} + Be^{i(\gamma-\beta^*+\mu\rho_1/n)(b-y)} + Ce^{-i(\gamma-\beta)(b-y)} + De^{-i(\gamma+\beta^*-\mu\rho_1/n)(b-y)}, \tag{30}$$

where A, B, C, D and β are complex numbers. We omitted in expression (30) higher harmonics with small amplitudes of order $O(A\mu)$. We also put

$$E_1 = e^{i\tilde{\varphi}_c}\tilde{E}_1. \tag{31}$$

Inserting expression (30) for quantity $\tilde{E}_1$ in to Eq. (28), we obtain the following system of equations for the A, B, C, D:

$$\hat{A}\begin{pmatrix} A \\ B^* \\ C \\ D^* \end{pmatrix} = 0, \tag{32}$$

where the matrix $\hat{A}$ is

$$\hat{A} = \begin{pmatrix} \frac{1}{4}\left(1+\frac{1}{n^2}\right)+\frac{i\kappa-\beta}{\mu\rho_1} & \frac{1}{4}\left(1-\frac{1}{n}\right)^2 & \frac{1}{2}\left(1-\frac{1}{n^2}\right) & \frac{1}{2}\left(1-\frac{1}{n^2}\right) \\ \frac{1}{2}\left(1-\frac{1}{n^2}\right) & \frac{1}{2}\left(1-\frac{1}{n^2}\right) & \frac{1}{4}\left(1+\frac{1}{n^2}\right)+\frac{i\kappa+\beta}{\mu\rho_1} & \frac{1}{4}\left(1+\frac{1}{n}\right)^2 \\ \frac{1}{2n}+\frac{i\kappa-\beta}{\mu\rho_1} & \frac{1}{2n}+\frac{i\kappa-\beta}{\mu\rho_1} & 0 & 0 \\ 0 & 0 & -\frac{1}{2n}+\frac{i\kappa+\beta}{\mu\rho_1} & -\frac{1}{2n}+\frac{i\kappa+\beta}{\mu\rho_1} \end{pmatrix}. \tag{33}$$

The values of β are solutions

$$\det\hat{A} = 0. \tag{34}$$

A simple calculation of det $\hat{A}$ yields

$$\left(\frac{1}{2n}+\frac{i\kappa-\beta}{\mu\rho_1}\right)^2\left(-\frac{1}{2n}+\frac{i\kappa+\beta}{\mu\rho_1}\right)^2=0. \tag{35}$$

We see that each eigenvalue β is doubly degenerate. As a result, we obtain only two linearly independent solutions of Eq. (28) of the form (30):

$$\tilde{E}_1=i\left[A\exp\left(i(\gamma+\mu\rho_1/2n+i\kappa)(b-y)\right)+C\exp\left(-i(\gamma-\mu\rho_1/2n-i\kappa)(b-y)\right)\right], \tag{36}$$

where A and C are real numbers.

Two other solutions of the system of equations (28) can be found in the form

$$\tilde{E}_1=(iy+A_1)\exp\left(i(\gamma+\mu\rho_1/2n+i\kappa)(b-y)\right)+C_1\exp\left(-i(\gamma-\mu\rho_1/2n-i\kappa)(b-y)\right) \tag{37}$$

for $\beta=\mu\rho_1/2n+i\kappa$, and

$$\tilde{E}_1=(iy+C_2)\exp\left(-i(\gamma-\mu\rho_1/2n+i\kappa)(b-y)\right)+A_2\exp\left(i(\gamma+\mu\rho_1/2n-i\kappa)(b-y)\right) \tag{38}$$

for $\beta=\mu\rho_1/2n-i\kappa$.

Inserting expressions (37) and (38) into Eq. (28), we obtain the following system of equations for the coefficients $A_{1,2}$ and $C_{1,2}$:

$$\begin{gathered}(A_1+A_1^*)\frac{\mu\rho_1}{4}\left(1-\frac{1}{n}\right)^2+\frac{\mu\rho_1}{2}\left(1-\frac{1}{n^2}\right)(C_1+C_1^*)+1=0,\\ (C_1+C_1^*)\frac{\mu\rho_1}{4}\left(1+\frac{1}{n^2}\right)+(A_1+A_1^*)\frac{\mu\rho_1}{2}\left(1-\frac{1}{n^2}\right)+2i\kappa C_1=0,\end{gathered} \tag{39}$$

and

$$\begin{gathered}(C_2+C_2^*)\frac{\mu\rho_1}{4}\left(1+\frac{1}{n^2}\right)+\frac{\mu\rho_1}{2}\left(1-\frac{1}{n^2}\right)(A_2+A_2^*)-1=0,\\ (A_2+A_2^*)\frac{\mu\rho_1}{4}\left(1-\frac{1}{n}\right)^2+\frac{\mu\rho_1}{2}\left(1-\frac{1}{n^2}\right)(C_2+C_2^*)+2i\kappa A_2=0.\end{gathered} \tag{40}$$

The solutions of the system of equations (39), (40) are

$$A_1=-\frac{2}{\mu\rho_1(1-1/n)^2},\quad C_1=-\frac{i}{\kappa}\frac{1+1/n}{1-1/n}, \tag{41}$$

$$A_2=\frac{i}{\kappa}\frac{1-1/n}{1+1/n},\quad C_2=\frac{2}{\mu\rho_1(1+1/n)^2}. \tag{42}$$

Hence, the general solution of the system of Eqs. (28) is

$$\begin{aligned}\tilde{E}_1&=i\left(A\exp\left(i(\gamma+\frac{\mu\rho_1}{2n}+i\kappa)(b-y)\right)+B\exp\left(-i\left(\gamma-\frac{\mu\rho_1}{2n}+i\kappa\right)(b-y)\right)\right)+\\ &+Ce^{-\kappa(b-y)}\left[\left(-iy+\frac{2}{\mu\rho_1(1-1/n)^2}\right)\exp\left(i\left(\gamma+\frac{\mu\rho_1}{2n}\right)(b-y)\right)+\frac{i}{\kappa}\frac{1+1/n}{1-1/n}\times\right.\\ &\left.\times\exp\left(-i\left(\gamma-\frac{\mu\rho_1}{2n}\right)(b-y)\right)\right]+De^{\kappa(b-y)}\left[\left(iy+\frac{2}{\mu\rho_1(1+1/n)^2}\right)\times\right.\\ &\left.\times\exp\left(-i\left(\gamma-\frac{\mu\rho_1}{2n}\right)(b-y)\right)+\frac{i}{\kappa}\frac{1-1/n}{1+1/n}\exp\left(i\left(\gamma+\frac{\mu\rho_1}{2n}\right)(b-y)\right)\right].\end{aligned} \tag{43}$$

In Eq. (41), A, B, C and D are real numbers.

The boundary conditions (29) at $y = b$ yield the first pair of equations for the coefficients A, B, C, D. In the leading approximation for the parameter μ, we have

$$C = D\left(\frac{1-1/n}{1+1/n}\right)^3,$$

$$A\left(\frac{1+1/n}{1-1/n}\right) - B - C\left(\frac{1+1/n}{1-1/n}\right)\left(b+\frac{1}{\kappa}\right) + D\left(-b+\frac{1}{\kappa}\right) = 0. \tag{44}$$

The second pair of equations for the coefficients A, B, C, D we obtain from the boundary conditions (29) at $y = 0$:

$$\begin{aligned}
&A\left(1-\frac{1}{n^2}\right)e^{-\kappa b}\cos(\gamma_+ b) - B\left(1+\frac{1}{n}\right)^2 e^{\kappa b}\cos(\gamma_- b) + \\
&+Ce^{-\kappa b}\left(\frac{1+1/n}{1-1/n}\right)\left(\frac{2\sin(\gamma_+ b)}{\mu\rho_1} - \frac{1}{\kappa}\left(1+\frac{1}{n}\right)^2\cos(\gamma_- b)\right) + \\
&+De^{\kappa b}\left(\frac{2\sin(\gamma_- b)}{\mu\rho_1} + \frac{1}{\kappa}\left(1-\frac{1}{n}\right)^2\cos(\gamma_+ b)\right) = 0, \\
&A\left(1-\frac{1}{n^2}\right)e^{-\kappa b}\sin(\gamma_+ b) + Be^{\kappa b}\left(1+\frac{1}{n}\right)^2\sin(\gamma_- b) + \\
&+Ce^{-\kappa b}\left(\frac{1+1/n}{1-1/n}\right)\left(-\frac{2\cos(\gamma_+ b)}{\mu\rho_1} + \frac{1}{\kappa}\left(1+\frac{1}{n}\right)^2\sin(\gamma_- b)\right) + \\
&+De^{\kappa b}\left(\frac{2\cos(\gamma_- b)}{\mu\rho_1} + \frac{1}{\kappa}\left(1-\frac{1}{n}\right)^2\sin(\gamma_+ b)\right) = 0,
\end{aligned} \tag{45}$$

where

$$\gamma_\pm = \gamma \pm \frac{\mu\rho_1}{2n}. \tag{46}$$

The condition that the system of Eqs. (44) and (45) has nontrivial solutions leads to an algebraic equation for the spectrum κ of the eigenvalue problem, given by Eq. (28) with boundary conditions (29). To obtain this equation, it is convenient to eliminate the coefficient C from the system of equations (44), (45), and to calculate the determinant of third order. The result of this calculation is

$$\begin{aligned}
&\frac{\sin(2\gamma b)}{\kappa}\left(e^{2\kappa b}\left(1+\frac{1}{n}\right)^2 - e^{-2\kappa b}\left(1-\frac{1}{n}\right)^2\right) - \frac{4}{\mu\rho_1}\cos(2\gamma b) + \\
&+\frac{2}{\mu\rho_1}\left(e^{2\kappa b}\left(\frac{1+1/n}{1-1/n}\right)^2 + e^{-2\kappa b}\left(\frac{1-1/n}{1+1/n}\right)^2\right) + \\
&+\left[b\left(\left(1+\frac{1}{n}\right)^2 + \left(1-\frac{1}{n}\right)^2\right) - \frac{1}{\kappa}\left(\left(1+\frac{1}{n}\right)^2 - \left(1-\frac{1}{n}\right)^2\right)\right]\sin(2\gamma b) = 0.
\end{aligned} \tag{47}$$

To simplify Eq. (47), we use the equation [5] for ρ_1

$$1-\cos(2\gamma b)=\frac{8(1-\rho_1)}{\rho_1}\frac{n^2}{(n^2-1)^2}. \tag{48}$$

Inserting the expression for $\cos(2\gamma b)$ from Eq. (48) into Eq. (47), we obtain the following equation for the spectrum points κ:

$$\frac{32n^2}{\mu\rho_1^2 b(n^2-1)^2}+\frac{\sin(2\gamma b)}{\kappa b}\left[\left(e^{2\kappa b}-1\right)\left(1+\frac{1}{n}\right)^2-\left(e^{-2\kappa b}-1\right)\left(1-\frac{1}{n}\right)^2\right]+$$
$$+\sin(2\gamma b)\left(\left(1+\frac{1}{n}\right)^2+\left(1-\frac{1}{n}\right)^2\right)+$$
$$+\frac{2}{b\mu\rho_1}\left[\left(e^{2\kappa b}-1\right)\left(\frac{1+1/n}{1-1/n}\right)^2+\left(e^{-2\kappa b}-1\right)\left(\frac{1-1/n}{1+1/n}\right)^2\right]=0. \tag{49}$$

As noted above, Eq. (49) solves the general eigenvalue problem given by Eqs. (12) and (13).

Near the turning points, Eq. (49) has two real solutions. If $n\gg 1$, then both are in the range $|\kappa b|\ll 1$. In the range $|\kappa b|\ll 1$, we obtain from Eq. (49) the quadratic equation

$$\left[\frac{16n^2}{\mu\rho_1^2 b(n^2-1)^2}+3\frac{n^2+1}{n^2}\sin(2\gamma b)\right]+\kappa b\left[\frac{16n(n^2+1)}{\mu\rho_1 b(n^2-1)^2}+\frac{4}{n}\sin(2\gamma b)\right]+$$
$$+4\kappa^2 b^2\left[\frac{1}{\mu\rho_1 b}\frac{n^4+6n^2+1}{(n^2-1)^2}+\frac{n^2+1}{3n^2}\sin(2\gamma b)\right]=0. \tag{50}$$

The first term in Eq. (50) vanishes at the turning points, because the equation for the turning points is precisely the free term in Eq. (50):

$$\frac{16n^2}{\mu\rho_1^2 b(n^2-1)^2}+3\frac{n^2+1}{n^2}\sin(2\gamma b)=0. \tag{51}$$

The last statement immediately follows from Eq. (48). The coefficients of the terms κb and $(\kappa b)^2$ in Eq. (50) are both positive at the turning points, because ρ_1 is bounded from above and below:

$$\frac{4n^2}{(n^2+1)^2}<\rho_1\leq 1. \tag{52}$$

The inequality (52) is a consequence of Eq. (48). Hence, near the turning points the two eigenvalues are real. One of them can be found from Eq. (50) for any value of the refractive index n; it changes sign at the turning points. The second eigenvalue is also small ($|\kappa b|\ll 1$) near the turning points only if $n\gg 1$. For all other eigenvalues $|\kappa b|\gtrsim 1$.

We are able now to give a qualitative picture of the movement of these two «lowest» eigenvalues. It is presented in Fig. 3. At some values of μ, the two conjugate eigenvalues reach the real axis in the κ plane (point (1,1) in Fig. 2). After collision, they become real. One of them moves along the real axis towards the origin, and reaches it at point 2 (turning point, branch point in the μ-plane). After that, one branch is unstable (see Fig. 1). At point 3, both eigenvalues reach their extreme values and start to decrease in absolute value. At point 4

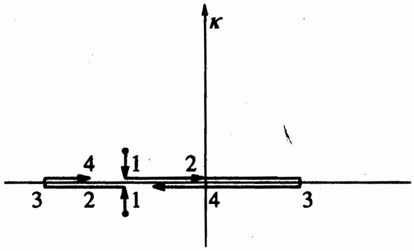

Fig. 3. Trajectory of two «conjugate» eigenvalues, one of them passing through zero

we go to the next turning point. Note that in the limit $\mu \to 0$, Eq. (49) yields for κ the values of the linear problem (Eqs. (17), (18)).

To complete our investigation, we need to prove the possibility of the complex eigenvalues (with $\operatorname{Im}\kappa \neq 0$) crossing the real axis $\operatorname{Re}\kappa = 0$ when the nonlinearity parameter μ changes from zero to some finite value.

Suppose that for some value of the parameters $(b\mu\rho_1, \gamma b)$ a purely imaginary solution of Eq. (47) exists with $\kappa = i\tilde{\kappa}$ ($\tilde{\kappa}$ is real). Then from Eq. (47), we obtain two equations:

$$\frac{4}{\mu\rho_1 b} = -\sin(2\gamma b)\frac{\operatorname{tg}(\tilde{\kappa}b)}{\tilde{\kappa}b}\,\frac{(n^2-1)^2}{n^2(n^2+1)}, \tag{53}$$

and

$$\sin(2\gamma b)\frac{2(n^2+1)}{n^2}\left(1+\frac{\sin(2\tilde{\kappa}b)}{\tilde{\kappa}b}\right) - \frac{4}{\mu\rho_1 b}\left[\cos(2\gamma b) - \cos(2\tilde{\kappa}b)\frac{n^4+6n^2+1}{(n^2-1)^2}\right] = 0. \tag{54}$$

Inserting the expression for $\mu\rho_1 b$ from Eq. (53) into Eq. (54), we find one equation for γb and $\tilde{\kappa}b$

$$\frac{2(n^2+1)}{n^2}\left(1+\frac{\sin(2\tilde{\kappa}b)}{\tilde{\kappa}b}\right) + \\ + \frac{\operatorname{tg}(\tilde{\kappa}b)}{\tilde{\kappa}bn^2(n^2+1)}\left[(n^2-1)^2\cos(2\gamma b) - \cos(2\tilde{\kappa}b)(n^4+6n^2+1)\right] = 0. \tag{55}$$

The function on the left-hand side of Eq. (55) is an even function of the parameter $\tilde{\kappa}b$. Hence, we can investigate just the region $\tilde{\kappa}b \geq 0$. Consider first $\tilde{\kappa}b$ in the range

$$\tilde{\kappa}b \in \frac{\pi}{2}\,[2K_1, 2K+1], \qquad K = 0, 1, \ldots \tag{56}$$

That is,

$$\tilde{\kappa}b = \pi K + X, \qquad 0 < X < \pi/2. \tag{57}$$

In the range given by Eq. (57), we have

$$f(X) = \frac{2(n^2+1)}{n^2}\left(1+\frac{\sin(2X)}{\tilde{\kappa}b}\right) + \frac{\operatorname{tg}X}{\tilde{\kappa}bn^2(n^2+1)}\times \\ \times\left[(n^2-1)^2\cos(2\gamma b) - \cos(2X)(n^4+6n^2+1)\right] = 0, \tag{58}$$

$$0 < X < \pi/2.$$

It is easy to prove that the function on the left-hand side of Eq. (58) is positive for $0 \le X < \pi/2$. To check this, we change the $\cos(2\gamma b)$ to -1 and find

$$f(X) \ge \frac{2(n^2+1)}{n^2}\left(1+\frac{\sin(2X)}{2\tilde{\kappa}b}\right) + \frac{8\,\mathrm{tg}\,X}{\tilde{\kappa}b(n^2+1)}\sin^2 X, \tag{59}$$

$$0 < X < \pi/2.$$

In the range $0 < X < \pi/2$, the expression on the right-hand side of the inequality (59) is positive, and hence in the range

$$\tilde{\kappa}b \in \frac{\pi}{2}\,[2K, 2K+1]\,,$$

Eq. (55) does not have a solution.

Consider now the range

$$\tilde{\kappa}b \in \frac{\pi}{2}\,[2K+1, 2K+2]\,, \tag{60}$$

that is,

$$\tilde{\kappa}b = \pi K + \pi/2 + X; \qquad 0 < X < \pi/2. \tag{61}$$

In the range (61), we obtain from Eq. (55)

$$\tilde{f}(X) = \frac{2(n^2+1)}{n^2}\left(1-\frac{\sin(2X)}{\tilde{\kappa}b}\right) - \\ - \frac{\mathrm{ctg}\,X}{\tilde{\kappa}bn^2(n^2+1)}\left[(n^2-1)^2\cos(2\gamma b) + \cos(2X)(n^4+6n^2+1)\right] = 0. \tag{62}$$

For any value of the parameter γb, the function $\tilde{f}(x)$ given by Eq. (62) varies from $-\infty$ to $2(n^2+1)/n^2$ when x goes from zero to $\pi/2$. Hence, there exists a minimum of one solution of Eq. (55) in the range

$$\tilde{\kappa}b \in \pm\frac{\pi}{2}\,[2K+1, 2K+2]\,, \qquad K = 0, 1, 2. \tag{63}$$

Taking Eq. (53) into account, we find, that new unstable modes (with $\mathrm{Im}\,\kappa \neq 0$) always appear outside some neighborhood of the turning points, because the quantity $\mu\sin(2\gamma b)$ is negative at all turning points ($\mu\sin(2\gamma b) < 0$). The system of Eqs. (48), (53), (62) can have a solution only if $|\mu b|$ is sufficiently large, so we find that all branches between the turning points ($2K+1, 2K+2$; $K = 0, 1, 2...$) are unstable. For $|\mu b|$ greater than some critical value $b\mu_{cr}(n)$, some parts of branches between the turning points ($2K, 2K+1$, $K = 1, 2...$) become unstable. The instability of these branches is related to pairs of conjugate eigenvalues crossing the real axis $\mathrm{Re}\,\kappa = 0$. The crossing points are given by Eqs. (53) and (55).

6. LIMITING CASE OF WEAK NONLINEARITY $|\mu|B \ll 1$

In the case $|\mu|b \ll 1$, the turning point can exist only if the refractive index $n \gg 1$, so $|\mu|bn \gtrsim 1$. In the vicinity of the turning points, γb is close to πN, where N is an integer:

$$\gamma b = \pi N + \epsilon + \frac{3\mu\rho_1}{4}b. \tag{64}$$

In the range of parameters considered here, Eq. (48) can be reduced to the cubic equation

$$Y^3 + Y\left(4 - \frac{\epsilon^2 n^2}{3}\right) - \left(\frac{2\epsilon^2 n^3}{27} + \frac{8\epsilon n}{3} + 3\mu bn\right) = 0, \tag{65}$$

where

$$Y = \frac{2\epsilon n}{3} + \frac{3\mu bn\rho_1}{4}. \tag{66}$$

From Eq. (65) we find that at the turning points, Y is given by

$$Y = \pm(\epsilon^2 n^2 - 12)^{1/2}/3. \tag{67}$$

From Eq. (51), it follows that the turning points exist only if $\mu\epsilon < 0$.

Between the turning points, all three solutions of Eq. (65) are given by

$$Y_k = \frac{2}{3}(n^2\epsilon^2 - 12)^{1/2}\cos\varphi_k, \tag{68}$$

where

$$\varphi_k = \frac{2\pi K}{3} + \frac{1}{3}\arccos\left(\frac{\epsilon^3 n^3 + 36\epsilon n + 81\mu bn/2}{(n^2\epsilon^2 - 12)^{1/2}}\right), \quad K = 0, 1, 2. \tag{69}$$

Equation (50) for κ can be substantially simplified in this case

$$\kappa^2 b^2 + \frac{4\kappa b}{n} + \frac{1}{n^2}\left[4 + 3Y^2 - \frac{n^2\epsilon^2}{3}\right] = 0. \tag{70}$$

The solutions of this equation are

$$(\kappa b)_{1,2} = -\frac{2}{n} \pm \frac{1}{n}\sqrt{\frac{n^2\epsilon^2}{3} - 3Y^2}. \tag{71}$$

It is easy to see that between the turning points, one mode has positive values of κb, hence the branch between the turning points is unstable.

7. CONCLUSIONS

In this paper we formulate an algebraic equation for the excitation spectrum κ that solves the problem of the stability of the solutions of a nonlinear wave equation in a slab. It is found that all branches between the $(2K+1, 2K+2)$ turning points are always unstable. Some parts of branches between the $(2K, 2K + 1)$ turning points are also unstable. The instability of the latter is associated with the possibility that pairs of complex conjugate eigenvalues cross the real axis in the κ-plane. Such a phenomenon can take place only if the effective nonlinearity is sufficiently strong ($|\mu| > \mu_{cr}(n)$). In that event, the temporal behavior of transitions between stationary states, when the amplitude of the incident wave varies, can be very complicated.

It was possible to obtain an explicit expression for the excitation spectrum, but only by virtue of the weak nonlinearity of the coefficient in the wave equation. Strong nonlinear effects result from the large length of the nonlinear medium compared to the wavelength scale.

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References

1. H. M. Gibbs, *Optical Bistability: Controlling Light with Light*, Academic Press, New York (1985).
2. Wei Chen and D. L. Mills, Phys. Rev. B **35**, 524 (1987); Phys. Rev. B **36**, 6269 (1987).
3. H. M. Gibbs, G. Khitrova, and N. Peyghambarian, *Nonlinear Photonics*, Springer-Verlag Berlin-Heidelberg (1990).
4. A. C. Newell and J. V. Moloney, *Nonlinear Optics*, Addison-Wesley, Reading, MA (1992).
5. Yu. N. Ovchinnikov and I. M. Sigal, Preprint, University of Toronto (1997).