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PASSIVE SCALAR IN A LARGE-SCALE VELOCITY FIELD

I. Kolokolov[a,e], *V. Lebedev*[b,c*], *M. Stepanov*[d,e]

[a] *Budker Institute of Nuclear Physics*
630090, Novosibirsk, Russia

[b] *Landau Institute for Theoretical Physics, Russian Academy of Sciences*
117940, Moscow, Russia

[c] *Department of Physics of Complex Systems, Weizmann Institute of Science*
76100, Rehovot, Israel

[d] *Institute of Automation and Electrometry, Russian Academy of Sciences*
630090, Novosibirsk, Russia

[e] *Novosibirsk State University*
630090, Novosibirsk, Russia

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We consider advection of a passive scalar $\theta(t, \mathbf{r})$ by an incompressible large-scale turbulent flow. In the framework of the Kraichnan model all PDF's (probability distribution functions) for the single-point statistics of θ and for the passive scalar difference $\theta(\mathbf{r}_1) - \theta(\mathbf{r}_2)$ (for separations $\mathbf{r}_1 - \mathbf{r}_2$ lying in the convective interval) are found.

INTRODUCTION

We treat advection of a passive scalar field $\theta(t, \mathbf{r})$ by an incompressible turbulent flow; the role of the scalar can be played by temperature or by pollutant density. The velocity field is assumed to contain motions from some interval of scales restricted from below by L_v. A steady situation with a permanent random supply of the passive scalar is considered. We wish to establish statistics of the passive scalar θ for scales that are less than both the scale L_v and the pumping scale L, and larger than the diffusion scale r_{dif} (for definiteness we assume that $L < L_v$). Such a convective interval of scales exists if the Peclet number $\mathrm{Pe} = L/r_{dif}$ is large enough; we will assume this condition. Since all scales from the convective interval are assumed to be smaller than L_v, we will discuss advection by a large-scale turbulent flow. The problem is of physical interest for dimensionalities $d = 2, 3$, but formally it can be treated for an arbitrary dimensionality d of space. Below we will treat d as a parameter. In particular, all expressions will be true for a space of high dimensionality d.

Description of the small-scale statistics of a passive scalar advected by a large-scale solenoidal velocity field is a special problem in turbulence theory. This problem was treated consistently from the very beginning and some rigorous results have been obtained, which is quite unusual for a turbulence problem. Batchelor (see Ref. [1]) examined the case of an external velocity field being so slow that it does not change during the time of the spectral transfer of the scalar from the external scale to the diffusion scale. Then Kraichnan (see Ref. [2]) obtained

*E-mail: lebede@cpd.landau.ac.ru; lwlebede@wicc.weizmann.ac.il

plenty of results in the opposite limit of a velocity field delta-correlated in time. The pair correlation function of the passive scalar $\langle\theta(\mathbf{r})\theta(0)\rangle$ was found to be proportional to the logarithm $\ln(L/r)$, and the pair correlation function of the passive scalar difference $\langle[\theta(\mathbf{r})-\theta(0)]^2\rangle$ was found to be proportional to $\ln(r/r_{dif})$ in both cases. The assertions are really correct for any temporal statistics of the velocity field (see Refs. [3, 4]). Thus we are dealing with the logarithmic case which is substantially simpler than cases with power-like correlation functions usually encountered in turbulence problems (see Refs. [5–7]).

Now about high-order correlation functions of the passive scalar. As long as all distances between the points are much less than L, the $2n$-point correlation functions of θ are given by their reducible parts (that is, are expressed via products of the pair correlation function) up to $n \sim \ln(L/r)$, where r is either the smallest distance between the points or r_{dif}, depending on which is larger (see Ref. [4]). The reason for such Wick decoupling is simply the fact that reducible parts contain more logarithmic factors (which are considered as the large ones) than non-reducible parts do. Consistent calculations of the fourth-order correlation function of the passive scalar at $d = 2$ (see Ref. [8]) confirm the assertion. Therefore, e.g., the single-point PDF of θ has a Gaussian core (that describes the first moments with $n < \ln \mathrm{Pe}$) and a non-Gaussian tail (that describes moments with $n > \ln \mathrm{Pe}$). The tail appears to be exponential (see Refs. [3, 4]). The same is true of the passive scalar difference $\Delta\theta = \theta(\mathbf{r}) - \theta(0)$, where instead of $\ln \mathrm{Pe}$ we should take $\ln(r/r_{dif})$. The tails do not depend on $\ln \mathrm{Pe}$ or on $\ln(r/r_{dif})$, and contain only coefficients that depend on the statistics of the advecting velocity.

Correlation functions of the passive scalar can be written as averages of integrals of the pumping along Lagrangian trajectories (see, e.g., Ref. [9]). For example, the pair correlation function $\langle\theta(\mathbf{r})\theta(0)\rangle$ is proportional to the average time needed for two points moving along Lagrangian trajectories to run from the distance r to the distance L. Generally, correlation functions of a passive scalar are determined by spectral transfer via evolution of Lagrangian separations up to the scale L. For the large-scale velocity field, the Lagrangian dynamics is determined by the stretching matrix $\sigma_{\alpha\beta} = \nabla_\beta v_\alpha$ and, consequently, the statistics of the matrix determines correlation functions of the passive scalar. For example, the coefficient of the logarithm in the pair correlation function of the passive scalar is $P_2/\bar\lambda$ (see Refs. [1–4]) where P_2 is the pumping rate of θ^2 and $\bar\lambda$ is Lyapunov exponent that is the average of the largest eigenvalue of the matrix $\hat\sigma$. The coefficients in the exponential tails are more sensitive to the statistics of $\hat\sigma$; specifically, they depend on the dimensionless parameter $\bar\lambda\tau$ (see Ref. [4]) where τ is the correlation time of $\hat\sigma$. The motion of the fluid particles in the random velocity field resembles in some respects random walks, but one should remember that correlation lengths of both the advecting velocity and of the pumping are much larger than scales from the convective interval we are interested in. Thus the situation is opposite to one usually encountered in solid state physics, where, e.g., random potential is short-range correlated in space.

Since $\ln(L/r)$ is really not very large, it is of interest to find all PDF's for the single-point statistics of θ and for the passive scalar difference $\Delta\theta$. It is possible to do this for the Kraichnan short-correlated case $\bar\lambda\tau \ll 1$ when the statistics of $\hat\sigma$ can be regarded to be Gaussian. An attempt to do this was made in Refs. [10, 11] in terms of the statistics of the main eigenvalue of the matrix $\hat\sigma$. Unfortunately, the scheme works only for the dimensionality $d = 2$ where the matrix $\hat\sigma$ has a single eigenvalue. This was noted in Ref. [12] where also the correct coefficient in the exponential tails for an arbitrary dimensionality of space d was found. Here, we develop a scheme enabling one to obtain all PDF's for arbitrary d. The scheme is also interesting from a methodological point of view. For example, its modification enables one to calculate the statistics of local dissipation (see Ref. [13]).

The paper is organized as follows. In Sec. 1 we find a path integral representation for the simultaneous statistics of the passive scalar. In Sec. 2 we analyze the generating functional for correlation functions of the passive scalar in the convective interval of scales. Using different approaches we obtain the functional and establish the applicability conditions of our consideration. In Sec. 3 we find explicit expressions for the single-point PDF and for the PDF of the passive scalar difference. In the Conclusion we briefly discuss the results obtained.

1. GENERAL RELATIONS

The dynamics of the passive scalar θ advected by the velocity field $\mathbf{v}$ is described by Eq.

$$\partial_t\theta + \mathbf{v}\nabla\theta - \kappa\nabla^2\theta = \phi\,. \tag{1.1}$$

Here, the term with the velocity $\mathbf{v}$ describes the advection of the passive scalar, the next term is diffusive (κ is the diffusion coefficient), and ϕ describes a pumping source of the passive scalar. Both $\mathbf{v}(t,\mathbf{r})$ and $\phi(t,\mathbf{r})$ are assumed to be random functions of t and $\mathbf{r}$. We regard the statistics of the velocity and source to be independent. Therefore, all correlation functions of θ are to be treated as averages over both statistics.

A. Simultaneous Statistics

The source ϕ is believed to possess Gaussian statistics and to be δ-correlated in time. The statistics is entirely characterized by the pair correlation function

$$\langle\phi(t_1,\mathbf{r}_1)\phi(t_2,\mathbf{r}_2)\rangle = \delta(t_1-t_2)\chi\left(|\mathbf{r}_1-\mathbf{r}_2|\right)\,, \tag{1.2}$$

where we assume that the pumping is isotropic. The function $\chi(r)$ is assumed to have a characteristic scale L, which is the pumping length. We will be interested in the statistics of the passive scalar on scales much smaller than L.

Simultaneous correlation functions of the passive scalar θ can be represented as coefficients in the expansion over y of the generating functional

$$\mathcal{Z}(y) = \left\langle \exp\left\{iy\int d\mathbf{r}\,\beta(\mathbf{r})\theta(0,\mathbf{r})\right\}\right\rangle\,, \tag{1.3}$$

where β is a function of the coordinates and angular brackets denote averaging over both the statistics of the pumping ϕ and the statistics of the velocity $\mathbf{v}$. The generating functional $\mathcal{Z}(y)$ contains complete information about the simultaneous statistics of the passive scalar ϑ. Specifically, knowing $\mathcal{Z}(y)$ one can reconstruct the simultaneous PDF of the passive scalar; the problem is discussed in Sec. 3.

If characteristic scales of β in (1.7) are much larger than the diffusion scale r_{dif}, then it is possible to neglect diffusion when treating the generating functional (1.3). Then the left-hand side of Eq. (1.1) describes simple advection, and it is reasonable to consider a solution of Eq. in terms of Lagrangian trajectories $\varrho(t)$ introduced by Eq.

$$\partial_t\varrho = \mathbf{v}(t,\varrho)\,. \tag{1.4}$$

We label the trajectories with $\mathbf{r}$, which are the positions of the Lagrange particles at $t=0$: $\varrho(0,\mathbf{r})=\mathbf{r}$. Next, introducing $\tilde\theta(t,\mathbf{r})=\theta(t,\varrho)$, we rewrite Eq. (1.1) as $\partial_t\tilde\theta=\phi$, which leads to

$$\theta(0,\mathbf{r}) = \int_{-\infty}^{0} dt\, \phi(t,\boldsymbol{\varrho}) . \tag{1.5}$$

Here we have taken into account that at $t = 0$ the functions θ and $\tilde{\theta}$ coincide. Starting with (1.5) and exploiting Gaussian pumping statistics, we can average the generating functional (1.3) explicitly over the statistics. The result is

$$\mathcal{Z}(y) = \left\langle \exp\left[-\frac{y^2}{2}\int_{-\infty}^{0} dt\, U\right]\right\rangle , \tag{1.6}$$

$$U = \int d\mathbf{r}_1 d\mathbf{r}_2\, \beta(\mathbf{r}_1)\beta(\mathbf{r}_2)\chi(|\boldsymbol{\varrho}_1 - \boldsymbol{\varrho}_2|) , \tag{1.7}$$

where angular brackets mean averaging over the statistics of the velocity field only.

Being interested in the single-point statistics of θ we should take $\beta(\mathbf{r}) = \delta(\mathbf{r})$. But this is impossible since we have neglected diffusion. We take $\beta(\mathbf{r}) = \delta_\Lambda(\mathbf{r})$ instead, where the function $\delta_\Lambda(\mathbf{r})$ tends to zero at $\Lambda r > 1$ fast enough, and is normalized by the condition

$$\int d\mathbf{r}\, \delta_\Lambda(\mathbf{r}) = 1 .$$

Then the generating functional (1.6) will describe the statistics of an object

$$\theta_\Lambda = \int d\mathbf{r}\, \delta_\Lambda(\mathbf{r})\theta(\mathbf{r}) , \tag{1.8}$$

smeared over a spot of size Λ^{-1}. If $r_{dif}\Lambda \ll 1$, then the statistics of the object is not sensitive to diffusivity. On the other hand, if $\Lambda L \gg 1$, then knowing the correlation functions of θ_Λ, we can reconstruct single-point statistics due to the logarithmic character of the correlation functions. To obtain single-point correlation functions one should substitute simply $\Lambda \to r_{dif}^{-1}$ into the correlation functions of θ_Λ. The above inequalities $\Lambda r_{dif} \ll 1$ and $\Lambda L \gg 1$ are compatible because of $\mathrm{Pe} \gg 1$. If we are interested in the statistics of the passive scalar differences in points with a separation $\mathbf{r}_0$ (where $r_0 \gg r_{dif}$) then instead of $\delta_\Lambda(\mathbf{r})$ we should take

$$\beta(\mathbf{r}) = \delta_\Lambda(\mathbf{r} - \mathbf{r}_0/2) - \delta_\Lambda(\mathbf{r} + \mathbf{r}_0/2) . \tag{1.9}$$

Then the generating functional (1.6) will describe the statistics of an object

$$\Delta\theta_\Lambda = \theta_\Lambda(\mathbf{r}_0/2) - \theta_\Lambda(-\mathbf{r}_0/2) . \tag{1.10}$$

Again, correlation functions of the passive scalar differences can be found from correlation functions of $\Delta\theta_\Lambda$ after the substitution $\Lambda \to r_{dif}^{-1}$.

B. Path Integral

Below, we treat advection of the passive scalar by a large-scale velocity field, that is, we assume that the velocity correlation length L_v is larger than the scales from the convective interval. Then for the scales one can expand the difference

$$v_\alpha(\mathbf{r}_1) - v_\alpha(\mathbf{r}_2) = \sigma_{\alpha\beta}(t)(r_{1\beta} - r_{2\beta}) , \quad \sigma_{\alpha\beta} = \nabla_\beta v_\alpha . \tag{1.11}$$

Here $\sigma_{\alpha\beta}(t)$ can be treated as an **r**-independent matrix field. Then Eq. (1.4) leads to

$$\partial_t(\varrho_{1,\alpha} - \varrho_{2,\alpha}) = \sigma_{\alpha\beta}(t)(\varrho_{1,\beta} - \varrho_{2,\beta}). \tag{1.12}$$

A formal solution of Eq. (1.12) is

$$\varrho_{1,\alpha} - \varrho_{2,\alpha} = W_{\alpha\beta}(r_{1,\beta} - r_{2,\beta}),$$
$$\partial_t \hat{W} = \hat{\sigma}\hat{W}, \quad \hat{W} = \mathcal{T}\exp\left(-\int_t^0 dt\,\hat{\sigma}\right), \tag{1.13}$$

where $\mathcal{T}$ denotes antichronological ordering. Note that $\det\hat{W} = 1$; this property is a consequence of $\mathrm{Tr}\,\hat{\sigma} = 0$ and the initial condition $\hat{W} = 1$ at $t = 0$. The Lagrangian difference in (1.7) is now rewritten as

$$|\boldsymbol{\varrho}_1 - \boldsymbol{\varrho}_2| = \sqrt{(r_{1\alpha} - r_{2\alpha})B_{\alpha\beta}(r_{1\beta} - r_{2\beta})}, \quad \hat{B} = \hat{W}^T\hat{W}, \tag{1.14}$$

where the subscript T denotes a matrix transpose. Note that $\det\hat{B} = 1$ since $\det\hat{W} = 1$.

The generating functional $\mathcal{Z}(y)$ (1.6) can be explicitly calculated in the Kraichnan case (see Ref. [2]) when the statistics of the velocity is δ-correlated in time. Then the velocity statistics is Gaussian and is entirely determined by the pair correlation function, which in the convective interval is written as

$$\langle v_\alpha(t_1, \mathbf{r}_1)v_\beta(t_2, \mathbf{r}_2)\rangle = \delta(t_1 - t_2)\left[\mathcal{V}_0\delta_{\alpha\beta} - \mathcal{K}_{\alpha\beta}(\mathbf{r}_1 - \mathbf{r}_2)\right], \tag{1.15}$$

$$\mathcal{K}_{\alpha\beta}(\mathbf{r}) = D(r^2\delta_{\alpha\beta} - r_\alpha r_\beta) + \frac{(d-1)D}{2}\delta_{\alpha\beta}r^2. \tag{1.16}$$

Here $\mathcal{V}_0$ is a huge **r**-independent constant and D is a parameter characterizing the amplitude of the strain fluctuations. The structure of Expr. (1.16) is determined by the assumed isotropy and spacial homogeneity, and by the incompressibility condition $\nabla\mathbf{v} = 0$. Then the statistics of $\hat{\sigma}$ is Gaussian and is determined by the pair correlation function, which can be found from Eqs. (1.15), (1.16):

$$\langle\sigma_{\alpha\beta}(t_1)\sigma_{\mu\nu}(t_2)\rangle = D\left[(d+1)\delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu} - \delta_{\alpha\beta}\delta_{\mu\nu}\right]\delta(t_1 - t_2). \tag{1.17}$$

Note that the correlation function (1.17) is **r**-independent, as it should be. We see from (1.17) that the parameter D characterizes the amplitude of $\hat{\sigma}$ fluctuations.

Averaging over the statistics of $\hat{\sigma}$ can be replaced by a path integral over unimodular matrices $\hat{W}(t)$ with a weight $\exp(i\mathcal{I})$. The effective action $\mathcal{I} = \int dt\,\mathcal{L}_0$ is determined by (1.17):

$$i\mathcal{L}_0 = -\frac{1}{2d(d+2)D}\left[(d+1)\mathrm{Tr}(\hat{\sigma}^T\hat{\sigma}) + \mathrm{Tr}\hat{\sigma}^2\right]. \tag{1.18}$$

Then the generating functional (1.7) can be rewritten as the following functional integral over unimodular matrices

$$\mathcal{Z}(y) = \int \mathcal{D}\hat{W}\,\exp\left[\int_{-\infty}^0 dt\,\left(i\mathcal{L}_0 - \frac{y^2}{2}U\right)\right], \tag{1.19}$$

$$U = \int d\mathbf{r}_1 d\mathbf{r}_2\,\beta(\mathbf{r}_1)\beta(\mathbf{r}_2)\chi\left[\sqrt{(r_{1\alpha} - r_{2\alpha})B_{\alpha\beta}(r_{1\beta} - r_{2\beta})}\right]. \tag{1.20}$$

Here, we should substitute $\hat{\sigma} = \partial_t \hat{W}(\hat{W})^{-1}$ and recall the boundary condition $\hat{W} = 1$ at $t = 0$.

Some words about the «potential» U (1.7) figuring in (1.20). The characteristic value of $\mathbf{r}_1 - \mathbf{r}_2$ in the integral (1.7) is of order Λ^{-1} for $\beta(\mathbf{r}) = \delta_\Lambda(\mathbf{r})$. Since we assume $\Lambda L \gg 1$, then for single-point statistics $U \approx P_2$, where $P_2 = \chi(0)$, if B is not very large. In particular, it is correct at moderate times $|t|$, since $\hat{B} = \hat{1}$ at $t = 0$. With increasing $|t|$ the argument of χ in (1.20) grows and U tends to zero when the argument of χ becomes greater than L. For the passive scalar difference when β is determined by (1.9) the situation is a bit more complicated. Then U is a difference of two contributions. The first contribution behaves as for single-point statistics. The second contribution contains χ with the argument determined by $\mathbf{r}_1 - \mathbf{r}_2 \approx \pm\mathbf{r}_0$. Then at $t = 0$ the meaning of the second contribution is determined again by P_2, but it vanishes with increasing $|t|$ earlier than the first contribution.

The path integral representation (1.19) indicates that we reduced our problem to the quantum mechanics with $d^2 - 1$ degrees of freedom. Nevertheless to solve the problem we should perform an additional reduction of the degrees of freedom. The conventional way to do this is passing to eigenvalues, say, of the matrix $\hat{B}$ figuring in (1.20) (see, e.g., Ref. [14]) and excluding angular degrees of freedom. Just this way was used by Bernard, Gawedzki and Kupiainen (see Ref. [12]). Then the authors using known facts about the quantum mechanics associated with the eigenvalues (see, e.g., Ref. [15]) have found the coefficient in the exponential tail of the single-point PDF of θ. Unfortunately this way is not very convenient to find the whole PDF. To do this we will use a special representation of the matrix $\hat{W}$ in the spirit of the nonlinear substitution introduced by Kolokolov (see Ref. [16]). That is the subject of the next subsection.

C. Choice of Parametrization

To examine the generating functional $\mathcal{Z}(y)$ we use a mixed rotational-triangle parametrization

$$\hat{W} = \hat{R}\hat{T}, \quad \hat{B} = \hat{T}^T\hat{T}, \tag{1.21}$$

where $\hat{R}$ is an orthogonal matrix and $\hat{T}$ is a triangular matrix; $T_{ij} = 0$ for $i > j$. The parametrization (1.21) is the direct generalization of the $2d$ substitution suggested in Ref. [17]. Note that $\det\hat{T} = 1$ since $\det\hat{W} = 1$. Note also that the matrix $\hat{B}$ introduced by (1.14) does not depend on $\hat{R}$, as is seen from (1.21). That is a motivation to exclude the matrix $\hat{R}$ from consideration, integrating over the corresponding degrees of freedom in the path integral (1.19). A Jacobian appears in the integration. To avoid an explicit calculation of the Jacobian, which needs a discretization over time and then an analysis of an infinite matrix (see Ref. [10]), we use an alternative procedure described below.

Let us examine the dynamics of the matrix $\hat{T}$. It is determined by the equation

$$\partial_t T_{ij} = \Sigma_{ii} T_{ij} + \sum_{i<k\le j} (\Sigma_{ik} + \Sigma_{ki}) T_{kj}, \tag{1.22}$$

following from Eqs. (1.13) and (1.21). Here we used the notations

$$\hat{\Sigma} = \hat{R}^T \hat{\sigma} \hat{R}. \tag{1.23}$$

Next introducing the quantities

$$T_{ii} = \exp(\rho_i), \quad T_{ij} = \exp(\rho_i)\eta_{ij}, \quad \text{if } i < j, \tag{1.24}$$

we rewrite Eq. (1.22) as

$$\partial_t \rho_i = \Sigma_{ii}\,, \tag{1.25}$$

$$\partial_t \eta_{ij} = (\Sigma_{ij} + \Sigma_{ji}) \exp(\rho_j - \rho_i) + \sum_{i<k<j} (\Sigma_{ik} + \Sigma_{ki}) \exp(\rho_k - \rho_i) \eta_{kj}\,. \tag{1.26}$$

Comparing (1.13) with (1.21), one can find the following expression for $\hat{A} = \hat{R}^T \partial_t \hat{R}$:

$$A_{ij} = \Sigma_{ij} \quad \text{if } i > j, \qquad A_{ij} = -\Sigma_{ji} \quad \text{if } i < j\,. \tag{1.27}$$

One can easily check that the irreducible pair correlation function of Σ_{ij} has the same form as for σ_{ij} [see Eq. (1.17)]:

$$\langle \Sigma_{ij}(t_1) \Sigma_{mn}(t_2) \rangle = D[(d+1)\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm} - \delta_{ij}\delta_{mn}]\delta(t_1 - t_2)\,. \tag{1.28}$$

Furthermore, the average value of Σ_{ij} is nonzero (see Ref. [10]):

$$\langle \Sigma_{ij} \rangle = -D \frac{d(d-2i+1)}{2} \delta_{ij}\,. \tag{1.29}$$

Nonzero averages of Σ_{ij} are related to Lyapunov exponents (not only the first one), see Ref. [18] (for our model see also Ref. [19]). To obtain (1.29) one should take into account that the matrix $\hat{R}$ propogates backward in time since $\hat{R} = 1$ is fixed at $t = 0$ and we treat negative t. Solving Eq. $\hat{A} = \hat{R}^T \partial_t \hat{R}$ for $\hat{R}$ on a small interval τ we get

$$\hat{R}(t - \tau) \approx \hat{R}(t) \left[1 - \int\limits_{t-\tau}^{t} dt'\, \hat{A}(t') \right]\,.$$

Then with the same accuracy we get from Eq. (1.23)

$$\hat{\Sigma}(t - \tau) \approx \hat{R}^T(t) \hat{\sigma}(t - \tau) \hat{R}(t) - \left[\hat{\Sigma}(t - \tau), \int\limits_{t-\tau}^{t} dt'\, \hat{A}(t') \right]\,. \tag{1.30}$$

The average value of $\hat{\Sigma}$ arises from the second term on the right-hand side of (1.30). The explicit form of the average can be found using

$$\left\langle \Sigma_{ij}(t - \tau) \int\limits_{t-\tau}^{t} dt'\, \Sigma_{mn}(t') \right\rangle = \frac{D}{2} [(d+1)\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm} - \delta_{ij}\delta_{mn}]\,. \tag{1.31}$$

Here we utilized Eq. (1.28) and replaced the integral

$$\int\limits_{t-\tau}^{t} dt'\, \delta(t - \tau - t')$$

by $1/2$. The reason is that the correlation function of $\hat{\sigma}$ actually has a finite correlation time, and therefore $\delta(t)$ (representing this correlation function) should be replaced by a narrow function symmetric under $t \to -t$. Then we will get $1/2$. Expressing $\hat{A}$ via $\hat{\Sigma}$ from (1.27) in (1.30) and calculating its average using (1.31) we get the answer (1.29).

The expressions (1.25), (1.26), (1.28), and (1.29) entirely determine the stochastic dynamics of ρ_i and η_{ij}. Using the conventional approach (see Refs. [20–24]) correlation functions of these degrees of freedom can be described in terms of a path integral over ρ_i, η_{ij} and over auxiliary fields which we denote by m_i and μ_{in} $(i < n)$. This integral should be taken with the weight $\exp(i \int dt\, \mathscr{L})$, where the Lagrangian is

$$\mathscr{L} = \sum_{a=1}^{d} m_a \left[\partial_t \rho_a + D\frac{d(d-2a+1)}{2}\right] + \frac{iD}{2}\left[d\sum_a m_a^2 - \left(\sum_a m_a\right)^2\right] +$$
$$+ iDd\sum_{i<j}\exp(2\rho_j - 2\rho_i)\mu_{ij}^2 + 2iDd\sum_{i<k<j}\mu_{ij}\mu_{ik}\exp(2\rho_k - 2\rho_i)\eta_{kj} +$$
$$+ \sum_{i<j}\mu_{ij}\partial_t\eta_{ij} + iDd\sum_{i<k<m,n}\mu_{im}\mu_{in}\eta_{km}\eta_{kn}\exp(2\rho_k - 2\rho_i). \qquad (1.32)$$

Since the matrix $\hat{B}$ in accordance with (1.21) does not depend on $\hat{R}$ it is enough to know the statistics of ρ_a and η_{ij} to determine the average (1.6). Therefore, instead of (1.19) we get

$$\mathscr{K}(y) = \int \mathscr{D}\rho\mathscr{D}\eta\mathscr{D}m\mathscr{D}\mu \exp\left[\int_{-\infty}^{0} dt \left(i\mathscr{L} - \frac{y^2}{2}U\right)\right]. \qquad (1.33)$$

Here U is determined by (1.20), where the matrix $\hat{B}$ is determined by Eqs. (1.21), (1.24).

Thus we obtained the expression for the generating functional (1.3) in terms of the functional (path) integral which is convenient for the analysis presented in the subsequent section.

2. GENERATING FUNCTIONAL

Here we calculate the generating functional (1.3) for a single-point statistics of θ that is of the object (1.8) corresponding to $\beta(\mathbf{r}) = \delta_\Lambda(\mathbf{r})$, and also the statistics of the difference that is of the object (1.10) corresponding to (1.9). The starting point for the subsequent consideration is the expression (1.33). There are different ways to examine $\mathscr{K}(y)$. We will describe two schemes leading to the same answer but carrying in some sense complementary information. We also believe that consideration of the different schemes is useful from a methodological point of view. A modification of the second scheme is presented in the Appendix.

A. Saddle-Point Approach

The first way to obtain the answer for the generating functional (1.3) is by using the saddle-point approximation for the path integral (1.33). The inequalities justifying the approximation are $\Lambda L \gg 1$ for the object (1.8) and $\Lambda r \gg 1$ for the object (1.10).

As we will see, large values of the differences $\rho_i - \rho_k$ $(i < k)$ will be relevant for us. Then fluctuations of η and μ are suppressed and it is possible to neglect the fluctuations. Therefore we can omit the integration over η and μ in (1.33), substituting $\eta = \mu = 0$ into (1.32). After that we obtain a reduced Lagrangian:

$$\mathcal{L}_r = \sum_{a=1}^{d} m_a \left[\partial_t \rho_a + D\frac{d(d+1-2a)}{2}\right] + \frac{iD}{2}\left[d\sum_a m_a^2 - \left(\sum_a m_a\right)^2\right]. \tag{2.1}$$

Now, to obtain $\mathcal{K}(y)$ one should integrate the exponent in (1.33) (with $\mathcal{L}_r$) over ρ_a and m_a. To examine (2.1) it is convenient to pass to new variables $\phi_a = O_{ab}\rho_b$ and $\tilde{m}_a = O_{ab}m_b$, where $\hat{O}$ is an orthogonal matrix. We make the following transformation:

$$\begin{gathered} \phi_1 = \sqrt{\frac{3}{d(d^2-1)}}\,[(d-1)\rho_1 + (d-3)\rho_2 + \ldots + (1-d)\rho_d]\,, \\ \phi_2 = \ldots, \quad \ldots, \quad \phi_d = \frac{1}{\sqrt{d}}\,[\rho_1 + \rho_2 + \ldots + \rho_d]\,. \end{gathered} \tag{2.2}$$

Then the expression (2.1) will be rewritten as

$$i\mathcal{L}_r = i\sum_{a=1}^{d} \tilde{m}_a \partial_t \phi_a - \frac{Dd}{2}\sum_{a=1}^{d-1}\tilde{m}_a^2 + i\frac{Dd}{2}\sqrt{\frac{d(d^2-1)}{3}}\tilde{m}_1\,. \tag{2.3}$$

The Lagrangian (2.3) is a sum over different degrees of freedom. The dynamics of ϕ_1 is ballistic, whereas the dynamics of ϕ_a for $d > a > 1$ is purely diffusive. The condition $\det\hat{T} = 1$ means $\phi_d = 0$, correspondinly the dynamics of ϕ_d determined by the Lagrangian (2.3) is trivial: $\partial_t \phi_t = 0$. We will see that times determining the main contribution to the generating functional are large enough that $\phi_1 \gg \phi_a$ for the relevant region. Therefore, the potential U (1.20) depends essentially only on ϕ_1, and it is possible to integrate explicitly over ϕ_a and $\tilde{m}_a$ for $a > 1$. After that we are left with only one degree of freedom, which is described by the Lagrangian

$$i\mathcal{L}_1 = i\tilde{m}_1\left(\partial_t\phi_1 + \frac{Dd}{2}\sqrt{\frac{d(d^2-1)}{3}}\right) - \frac{Dd}{2}\tilde{m}_1^2\,. \tag{2.4}$$

Neglecting all ϕ_a for $a > 1$ and inverting transformation (2.2) we obtain

$$\rho_1 \approx \sqrt{\frac{3(d-1)}{d(d+1)}}\,\phi_1\,, \qquad \rho_a \approx \frac{d-2a+1}{d-1}\rho_1\,. \tag{2.5}$$

We will see below that the characteristic value $\phi_1 \gg 1$. Therefore the characteristic value of e^{ρ_1} is much larger than other e^{ρ_a}, and we conclude that the potential U depends really only on ρ_1. For the case of the single-point statistics, the characteristic value of the difference $\mathbf{r}_1 - \mathbf{r}_2$ in (1.20) is Λ^{-1}. Then it follows from (1.21) and (1.24) that the potential U decreases from P_2 to zero near the point $\rho_1 = \ln(L\Lambda)$, which is near the point $\phi_1 = \phi_\Lambda$, where

$$\phi_\Lambda = \sqrt{\frac{d(d+1)}{3(d-1)}}\ln(L\Lambda)\,. \tag{2.6}$$

For the difference the potential increases from zero to $2P_2$ at $\phi_1 = \phi_R$, where

$$\phi_R = \sqrt{\frac{d(d+1)}{3(d-1)}}\ln\frac{L}{r_0}\,, \tag{2.7}$$

and then decreases from $2P_2$ to zero near $\phi_1 = \phi_\Lambda$. The expressions (2.6) and (2.7) determine the characteristic values of ϕ_1, which are actually large, since $L\Lambda \gg 1$ or $L/r_0 \gg 1$; this justifies our conclusions.

Now we can employ the saddle-point approximation:

$$\ln \mathcal{X}(y) \approx \int_{-\infty}^{0} dt \left(i\mathcal{L}_1 - \frac{y^2}{2}U \right) \Bigg|_{inst}, \tag{2.8}$$

where we should substitute solutions of the extremal conditions, which we will call instantonic equations. The instantonic equations, which can be found from extremal conditions for $i\mathcal{L}_1 - y^2U/2$, are

$$\partial_t \phi_1 + \frac{Dd}{2}\sqrt{\frac{d(d^2-1)}{3}} = -iDd\tilde{m}_1, \tag{2.9}$$

$$\partial_t \tilde{m}_1 = i\frac{y^2}{2}\frac{\partial U}{\partial \phi_1}. \tag{2.10}$$

Eqs. conserve the «energy»

$$-i\frac{Dd}{2}\tilde{m}_1\sqrt{\frac{d(d^2-1)}{3}} + \frac{Dd}{2}\tilde{m}_1^2 + \frac{y^2}{2}U. \tag{2.11}$$

The conservation law is satisfied since $i\mathcal{L}_1 - y^2U/2$ does not explicitly depend on t. The «energy» (2.11) is equal to zero, since as $t \to -\infty$ the value of $\tilde{m}_1$ should tend to zero. This property can be treated as the extremal condition when $i\mathcal{L}_r - y^2U/2$ is varied over the initial value of ϕ_1. Equating the «energy» (2.11) to zero, we can express $\tilde{m}_1$ via ϕ_1. Next, since (2.11) is zero, the saddle-point value of $\mathcal{X}(y)$ (2.8) can be written as $i\int d\phi_1\, \tilde{m}_1$, where the integral over ϕ_1 goes from zero to infinity.

Substituting the expression for $\tilde{m}_1$ in terms of ϕ_1 into $i\int d\phi_1\, \tilde{m}_1$, we get for the single-point statistics

$$\ln \mathcal{X}(y) \simeq \frac{d(d+1)}{6}\left[1 - \sqrt{1 + \frac{12y^2P_2}{Dd^2(d^2-1)}}\right]\ln(L\Lambda). \tag{2.12}$$

Note that the expression (2.12) has (as a function of y) two branch points $y = \pm i y_{sing}$, where

$$y_{sing}^2 = \frac{Dd^2(d^2-1)}{12P_2}. \tag{2.13}$$

The same procedure can be done for the passive scalar difference, or, more precisely, for the object (1.10). Taking into account the presence of the jumps (2.6) and (2.7) in the potential U, we get an answer slightly different from (2.12):

$$\ln \mathcal{X}(y) \simeq \frac{d(d+1)}{6}\left[1 - \sqrt{1 + \frac{24y^2P_2}{Dd^2(d^2-1)}}\right]\ln(r_0\Lambda), \tag{2.14}$$

$$y_{sing}^2 = \frac{Dd^2(d^2-1)}{24P_2}. \tag{2.15}$$

Note that (2.14) does not depend on the pumping scale L, but still depends on the cutoff Λ.

The characteristic value of ϕ_1 is determined by the quantity (2.6) which is much larger than unity. Then it follows from (2.5) that $\exp(2\rho_j - 2\rho_i) \ll 1$, $i > j$, (excluding a short initial stage of evolution) and we see from (1.32) that fluctuations of the fields η are suppressed in comparison, say, with ρ_a. This justifies neglecting the fields η and μ leading to the reduced Lagrangian (2.1). Next, the dynamics of ϕ_a for $a > 1$ is diffusive, and it follows from (2.3) that the characteristic value of ϕ_a can be estimated to be $\sqrt{Dd|t|}$. As follows from (2.3), $\partial_t \phi_1 \sim Dd^{5/2}$, and we find from (2.6) the instantonic lifetime

$$t_{lt} = D^{-1} d^{-2} \ln(L\Lambda), \tag{2.16}$$

which determines times producing nonzero contributions to the effective action. At $|t| \sim t_{lt}$, the characteristic values of ϕ_a for $a > 1$ are of order $\sqrt{\ln(L\Lambda)/d}$, and we conclude that

$$\frac{\phi_a}{\phi_1} \sim \frac{1}{d\sqrt{\ln(L\Lambda)}} \ll 1 \tag{2.17}$$

at times $|t| \sim t_{lt}$. The inequality (2.17) justifies passing to the Lagrangian (2.4). The same arguments can be applied to the generating functional for the passive scalar difference; the only modification is in the substitution $\ln(L\Lambda) \to \ln(r_0\Lambda)$.

There are also additional applicability conditions for the results (2.12) and (2.14). To establish the conditions, one should go beyond the main order of the saddle-point approximation. It will be more convenient for us to develop an alternative scheme, which enables one to find the conditions more simply. That is the subject of the next subsection.

B. Schrödinger equation

Here we present another way to get the answers (2.12) and (2.14). As before, we start with the path integral representation (1.33) for the generation functional $\mathcal{Z}(y)$.

Unfortunately it is impossible to get a closed equation for $\mathcal{Z}(y)$. To avoid the difficulty we introduce an auxiliary quantity

$$\Psi(t, y, \rho_0, \eta_0) = \int \mathcal{D}\rho\mathcal{D}\eta\mathcal{D}m\mathcal{D}\mu \, \exp\left[\int_{-t}^{0} dt' \left(i\mathcal{L} - \frac{y^2}{2}U\right)\right]\Bigg|_{\rho(-t)=\rho_0, \eta(-t)=\eta_0} . \tag{2.18}$$

It follows from the definition (2.18) that

$$\mathcal{Z}(y) = \lim_{t\to\infty} \int \prod d\rho_a \, d\eta_{ij} \, \Psi(t, y, \rho, \eta). \tag{2.19}$$

Eq. for the function Ψ can be obtained from the expression (1.32) and the definition (2.18):

$$\begin{aligned}\partial_t \Psi = \frac{Dd}{2}\Bigg[&\sum_{i=1}^{d} \frac{\partial^2}{\partial\rho_i^2} - \frac{1}{d}\left(\sum_{i=1}^{d}\frac{\partial}{\partial\rho_i}\right)^2 - \sum_{i=1}^{d}(d-2i+1)\frac{\partial}{\partial\rho_i} + \\ &+ 2\sum_{i<j}\exp(2\rho_j - 2\rho_i)\frac{\partial^2}{\partial\eta_{ij}^2} + 4\sum_{i<k<j}\exp(2\rho_k - 2\rho_i)\frac{\partial}{\partial\eta_{ij}}\frac{\partial}{\partial\eta_{ik}}\eta_{kj} + \\ &+ 2\sum_{i<k<m,n}\exp(2\rho_k - 2\rho_i)\frac{\partial}{\partial\eta_{im}}\frac{\partial}{\partial\eta_{in}}\eta_{km}\eta_{kn}\Bigg]\Psi - \frac{y^2 U}{2}\Psi.\end{aligned} \tag{2.20}$$

We see that Eq. (2.20) for Ψ resembles the Schrödinger equation. The initial condition for the equation can be found directly from the definition (2.18):

$$\Psi(t=0, y, \rho, \eta) = \prod \delta(\rho_a)\delta(\eta_{ij}) . \qquad (2.21)$$

The value of $\mathcal{X}$, in accordance with (2.19), is determined by the integral of Ψ over η and ρ. This integral is equal to unity at $t = 0$, and then varies with increasing time t due to $U \neq 0$, since only the term with U in (2.21) breaks the conservation of the integral. Thus, to find $\mathcal{X}$ we must establish the evolution of the function Ψ from $t = 0$ to large t.

Below we concentrate on the single-point statistics. The scheme can obviously be generalized for the passive scalar difference.

Let us first describe the evolution qualitatively. The initial condition (2.21) shows that at $t = 0$ the function Ψ is concentrated at the origin. Then it undergoes spreading in all directions, except for $\rho_1 + \ldots + \rho_d$, since the operator on the right-hand side of (2.20) commutes with $\rho_1+\ldots+\rho_d$. This is a consequence of the condition $\det \hat{T} = 1$ (to be satisfied), which implies that during evolution $\rho_1 + \ldots + \rho_d = 0$. This means that a solution of (2.20) is $\Psi \propto \delta(\rho_1 + \ldots + \rho_d)$. The function Ψ is smeared diffusively with time, and also moves as a whole in some direction, which is determined by the term with the first derivative in (2.20). The rate of ballistic motion is

$$\langle \partial_t \rho_i \rangle = D\,\frac{d(d-2i+1)}{2} . \qquad (2.22)$$

Therefore Ψ describes a cloud, the center of which moves according to the law

$$\rho_i = D\,\frac{d(d-2i+1)}{2}t . \qquad (2.23)$$

Effective diffusion coefficients for the η's decrease with increasing t, since in accordance with (2.23) the differences $\rho_k - \rho_i$, figuring in (2.20), are negative and grow in absolute value. Therefore diffusion over η stops when the characteristic values of $\rho_i - \rho_k$ becomes greater than unity. Note that the «frozen» values of η do not depend on y, since U can be considered uniform during the initial stage of evolution. After that the η's are frozen, diffusion continues only over the ρ's. If the cloud is inside the region where $U \simeq P_2$, then evolution of the cloud is not influenced by U. After a period of time t_{lt} (2.16), the cloud reaches a barrier, where the potential U decreases from P_2 to 0. The subsequent history depends on the value of y. For moderate y the cloud passes this barrier and continues to move at the same rate. After this, the integral of Ψ will not change in time, and its value will determine the generating functional $\mathcal{X}(y)$. Naive estimates yield $\ln \mathcal{X}(y) = -y^2 t_{lt}/2$, which reproduces the pair correlation function of θ.

Special consideration is needed if $|y| \gg y_{sing}$, or if y is close to $\pm i y_{sing}$, where y_{sing} is defined by (2.13). Just this region determines the PDF's, and is consequently of special interest. Note that $y = \pm i y_{sing}$ corresponds to the appearance of a bound state near the pumping boundary (where U decreases from P_2 to zero). If $y \gg y_{sing}$, then the front of the cloud reaches the jump of the potential much earlier than t_{lt}. The remainder of the cloud (inside the potential well) is damped due to the term with y, and does not contribute to $\mathcal{X}(y)$. If $|y| \gg y_{sing}$ then $\mathcal{X}(y) \gg \exp(-y^2 t_{lt}/2)$; the asymptotics of $\mathcal{X}(y)$ is actually exponential in the case. If $|y \pm i y_{sing}| \ll y_{sing}$ then the cloud stays near the pumping boundary for a long time, that is the shape of Ψ inside the region $U \simeq P_2$ varies in time comparatively slowly. Furthermore, a part of Ψ percolates out to the region where $U \simeq 0$, and the integral of Ψ grows with increasing $|t|$. As y approaches $i y_{sing}$, this stage lasts longer. One can say that the back of the cloud Ψ gives the right answer for $\mathcal{K}(y)$. The important point is that if y is not very close to iy_{sing}, then during the time Ψ leaves the potential, the width of Ψ in terms of diffusive degrees of freedom is much less than $\ln L\Lambda$. This means that the function Ψ is really narrow, which justifies our consideration.

For a quantitative analysis it is convenient to pass to the variables ϕ_i (2.2). Since the η-dependence of Ψ is frozen after the initial evolution, it is possible to obtain an equation for the integral of Ψ over η:

$$\tilde{\Psi}(\phi_1, \ldots, \phi_{d-1}) = \int d\phi_d \prod d\eta_{ij}\, \Psi, \tag{2.24}$$

where we also included an integration over ϕ_d to remove the factor $\delta(\rho_1 + \ldots + \rho_d)$. Eq. for the function (2.24) is

$$\partial_t \tilde{\Psi} = \frac{Dd}{2}\left[\sum_{i=1}^{d-1} \frac{\partial^2}{\partial \phi_i^2} - \sqrt{\frac{d(d^2-1)}{3}}\frac{\partial}{\partial \phi_1}\right]\tilde{\Psi} - \frac{y^2 \tilde{U}}{2}\tilde{\Psi}, \tag{2.25}$$

where $\tilde{U}$ is function of ϕ_a only which can be found by substituting into U the «frozen» values of η's. Qualitatively $\tilde{U}$ has the same structure as U itself. One can conclude from (2.25) that the cloud described by $\tilde{\Psi}$ moves ballistically in the ϕ_1 direction and spreads along other directions. We are going to treat the situation when the cloud remains narrow during the relevant part of the evolution. Then one can integrate $\tilde{\Psi}$ over all ϕ_i, $i > 1$ in a similar way as in the case with η's, and get a $1d$ equation for

$$\bar{\Psi}(\phi_1) = \int \prod_2^{d-1} d\phi_i\, \tilde{\Psi}.$$

The function $\bar{\Psi}$ satisfies Eq.

$$\partial_t \bar{\Psi} = \frac{Dd}{2}\left[\frac{\partial}{\partial \phi_1} - \sqrt{\frac{d(d^2-1)}{3}}\right]\frac{\partial}{\partial \phi_1}\bar{\Psi} - \frac{y^2 \bar{U}}{2}\bar{\Psi}. \tag{2.26}$$

The initial condition for Eq. (2.26) is $\bar{\Psi}(t=0) = \delta(\phi_1)$. The potential $\bar{U}$ is obtained from $\tilde{U}$ by the substitution $\phi_a \to 0$ for $a > 0$. In fact, for the direction (2.23) the potential $\bar{U}$ depends only on ρ_1. The barrier is reached when $\rho_1 \simeq \ln L\Lambda$. Passing to the variables ϕ_i, we conclude that the potential $\bar{U}$ diminishes from P_2 at $\phi_1 < \phi_\Lambda$ to zero at $\phi_1 > \phi_\Lambda$, where ϕ_Λ is defined by (2.6).

The character of the solution of Eq. (2.26) can be analyzed semiqualitatively in terms of the width l of $\bar{\Psi}$ over ϕ_1 and its amplitude h. When $\bar{\Psi}$ reaches the pumping boundary, it stops there for a period of time. Then the width l and the amplitude h are governed by the equations

$$\frac{dl}{dt} = -Dd\lambda + \frac{Dd}{l}, \quad \frac{dh}{dt} = -\frac{Ddh}{l^2} - \frac{y^2 P_2 h}{2}, \tag{2.27}$$

where $\lambda = \sqrt{d(d^2-1)/12}$, $Dd\lambda$ is the rate of cloud motion along the ϕ_1 direction (when $U = \text{const}$), and Dd is the diffusion coefficient for the ϕ_1 direction. One can estimate from the first equation the width $l \sim 1/\lambda$. Then from the second equation the height h decreases or grows in time depending on y. The characteristic y where the regime changes is of the order $|y_{sing}|^2 \sim Dd\lambda^2/P_2$. We show this by consistent calculations.

Eqution (2.26) can be solved analytically, e.g., by the Laplace transform over time t. Taking the Laplace transform, one gets

$$p\bar{\Psi}(p) - \delta(\phi_1) = \frac{Dd}{2}\left[\frac{\partial}{\partial\phi_1} - \sqrt{\frac{d(d^2-1)}{3}}\right]\frac{\partial}{\partial\phi_1}\bar{\Psi}(p) - \frac{y^2}{2}\bar{U}(\phi_1)\bar{\Psi}(p). \tag{2.28}$$

We are interesting in the bound state described by this equation. Solutions for $\bar{\Psi}(p)$ in the intervals $(-\infty, 0)$, $(0, \phi_\Lambda)$, (ϕ_Λ, ∞) are exponential, and must be matched. The function $\bar{\Psi}(p)$ as a function of p has two branch points at

$$p_1 = -\frac{Dd^2(d^2-1)}{24} - \frac{y^2P_2}{2}, \qquad p_2 = -\frac{Dd^2(d^2-1)}{24}, \tag{2.29}$$

coming from the regions $\phi_1 < \phi_\Lambda$ and $\phi_1 > \phi_\Lambda$, respectively. When one of these branch points passes $p = 0$, $\bar{\Psi}$ starts to grow exponentially in time. This happens when y passes $\pm iy_{sing}$, moving along the imaginary axis.

The value of the generating functional is determined in accordance with (2.19) by the large-time behavior of $\Psi(t)$. This means that we should be interested in the behavior of $\Psi(p)$ at small p. The function $\int d\phi_1\,\hat{\Psi}(p)$ in (2.19) has a pole at $p = 0$ related to the asymptotic behavior

$$\bar{\Psi}(p) \propto \exp\left(-\frac{2p}{Dd}\sqrt{\frac{3}{d(d^2-1)}}\phi_1\right),$$

at $\phi_1 > \phi_\Lambda$ and small p; the behavior can be found from (2.28). The residue of $\int d\phi_1\,\bar{\Psi}(p)$ at the pole determines $\mathcal{Z}(y)$. To find the residue we must analyze the behavior of $\bar{\Psi}(p)$ at $0 < \phi_1 < \phi_\Lambda$. At small p there are two contributions to $\bar{\Psi}$, proportional to

$$\exp\left\{\left(\sqrt{\frac{d(d^2-1)}{12}} \pm \sqrt{\frac{d(d^2-1)}{12} + \frac{y^2P_2}{Dd}}\right)\phi_1\right\}, \tag{2.30}$$

as follows from (2.28) at $p = 0$. Therefore the residue, which is determined by the integral $\int d\phi_1\,\bar{\Psi}(p)$ over the region $\phi_1 > \phi_\Lambda$, is proportional to

$$\exp\left\{\left(\sqrt{\frac{d(d^2-1)}{12}} + \sqrt{\frac{d(d^2-1)}{12} + \frac{y^2P_2}{Dd}}\right)\phi_\Lambda\right\}. \tag{2.31}$$

Substituting (2.6) here, we reproduce (2.12).

Let us now establish the applicability condition for the above procedure. The expression (2.31) implies that the exponent with the minus sign in (2.30) makes a negligible contribution to $\Psi(p)$ at $\phi_1 = \phi_\Lambda$. The condition is satisfied if

$$|y^2 + y_{sing}^2|\phi_\Lambda^2 \gg \frac{Dd}{P_2}.$$

Substituting (2.6) and (2.13) here, we obtain

$$\left|\frac{y \pm iy_{sing}}{y_{sing}}\right| \gg \left(d^4 \ln^2 L\Lambda\right)^{-1} . \tag{2.32}$$

For y close to $\pm iy_{sing}$, one must be careful, since then the subtle analytic structure of $\mathcal{K}(y)$ will be relevant. As an analysis for $d = 2$ shows $\mathcal{K}(y)$ has a system of poles along the imaginary semiaxis starting from $\pm iy_{sing}$, and the parameter $\left(d^4 \ln^2 L\Lambda\right)^{-1}$ determines the separation between the poles. The poles correspond to bound states. The assertion about the cut made in the previous subsection is related to the restrictions of the saddle-point approximation which cannot feel this fine pole structure; it yields the cut, which is a picture averaged over the interpole distances. This averaged picture is acceptable at the condition (2.32).

Note that the same criterion (2.32) justifies our assumption that the cloud described by Ψ is narrow during the relevant part of the evolution. Namely, the duration of the part is determined by the time $t_{exit} = p_1^{-1}$ (see (2.29)). This is the time that the cloud stays near the barrier. For y close to $\pm iy_{sing}$, the time can be estimated to be $t_{exit}^{-1} \sim P_2|y_{sing}||y \mp iy_{sing}|$. Then the diffusive width $\sqrt{Ddt_{exit}}$ of Ψ in the directions ϕ_a for $a > 1$ is much less than ϕ_Λ precisely if (2.32) is satisfied. In principle the diffusive dynamics at $d > 2$ could modify the noted fine pole structure of $\mathcal{K}$; this problem requires additional investigation.

The same procedure can be done for the passive scalar differences. The cloud Ψ should pass the region $\rho_1 < \ln(L/r_0)$ before it reaches the potential. Then it enters the region $\bar{U} = 2P_2$ with some finite diffusive width. One can note, however, that this is irrelevant. The only characteristics of the potential that are needed are its value (here $2P_2$ instead of P_2) and the length of the path inside it (which is $\Delta\rho_1 = \ln(r_0\Lambda)$ instead of $\ln(L\Lambda)$). The evolution of $\bar{\Psi}$ goes in the same way as in the case of single-point statistics. Again, we get (2.14) and the criterion analogous to (2.32).

In this subsection we presented an analysis based on the dynamical equation (2.20) for the auxiliary object Ψ. The results obtained can be reproduced also in alternative language: for this we must introduce another auxiliary object, the equation for which is stationary. The corresponding scheme, which might be interesting from a methodological point of view, is sketched in the Appendix.

3. CALCULATION OF PDF

In this section we calculate the PDF's $\mathcal{P}$ for the objects (1.8) and (1.10). The most convenient way to do so is by using the relation

$$\mathcal{P}(\vartheta) = \int \frac{dy}{2\pi} \exp(-iy\vartheta)\mathcal{K}(y) , \tag{3.1}$$

where ϑ is

$$\vartheta = \int d\mathbf{r}\, \beta(\mathbf{r})\theta(0, \mathbf{r}) . \tag{3.2}$$

Let us recall that knowing $\mathcal{P}(\vartheta)$, one can also restore the moments of ϑ:

$$\langle|\vartheta|^n\rangle = \int d\vartheta\, |\vartheta|^n \mathcal{P}(\vartheta) . \tag{3.3}$$

The generating functional in (3.1) is determined by (2.12) or (2.14). Being interested in the main exponential dependence of the PDF's for the objects (1.8) and (1.10), we can forget about preexponents. Then

$$\mathcal{P}(\vartheta) = \int \frac{dy}{2\pi} \exp\left(-iy\vartheta + q\left[1 - \sqrt{1 + y^2/y_{sing}^2}\right]\right), \tag{3.4}$$

where for the single-point statistics and for the statistics of the passive scalar difference respectively

$$y_{sing}^2 = \frac{Dd^2(d^2-1)}{12P_2}, \quad y_{sing}^2 = \frac{Dd^2(d^2-1)}{24P_2}, \tag{3.5}$$

$$q = \frac{d(d+1)}{6}\ln(L\Lambda), \quad q = \frac{d(d+1)}{6}\ln(r_0\Lambda). \tag{3.6}$$

Since both q defined by (3.6) are regarded to be much larger than unity, the integral (3.4) can be calculated in the saddle-point approximation. The saddle-point value is

$$y_{sp} = i\frac{y_{sing}}{1 + q^2/y_{sing}^2\vartheta^2}. \tag{3.7}$$

Then

$$\ln\mathcal{P}(\vartheta) \simeq q\left(1 - \sqrt{1 + \frac{y_{sing}^2\vartheta^2}{q^2}}\right). \tag{3.8}$$

This expression leads to the exponential tail

$$\ln\mathcal{P}(\vartheta) \simeq -y_{sing}|\vartheta|, \tag{3.9}$$

realized at $|\vartheta| \gg q/y_{sing}$. The coefficient y_{sing} in (3.9) determined by (2.13) is in agreement with the result obtained in Ref. [12].

The expression (3.8) enables one to find the following averages in accordance with (3.3):

$$\langle\theta_\Lambda^2\rangle = \frac{2P_2}{d(d-1)D}\ln(L\Lambda), \quad \langle(\Delta\theta_\Lambda)^2\rangle = \frac{4P_2}{d(d-1)D}\ln(r_0\Lambda). \tag{3.10}$$

The expressions (3.10) can also be obtained by direct expansion of $\mathcal{K}(y)$ from (2.12) or (2.14). The universal tail (3.9) is realized if

$$\theta_\Lambda \gg \sqrt{\langle\theta_\Lambda^2\rangle}\, d\ln(L\Lambda), \quad \Delta\theta_\Lambda \gg \sqrt{\langle(\Delta\theta_\Lambda)^2\rangle}\, d\ln(r_0\Lambda). \tag{3.11}$$

Since both logarithms are assumed to be large, we conclude that there exists a relatively wide region where the statistics of ϑ is approximately Gaussian; the region is determined by the inequalities inverse to (3.11).

Let us discuss the applicability conditions of the expression (3.8). First, if one calculates the passive scalar PDF by the saddle point method, then the position of the saddle point is determined by (2.32) if

$$\vartheta \ll d^2\sqrt{\frac{P_2}{D}}\ln^2(L\Lambda). \tag{3.12}$$

The applicability domain of the saddle-point method overlaps the region of validity of (2.12) for the generation function $\mathcal{Z}(y)$. The above inequalities are correct for θ_Λ; for $\Delta\theta_\Lambda$ one must replace $\ln(L\Lambda)$ with $\ln(r_0\Lambda)$. Second, fluctuations of y have to be small compared to the distance between y_{sp} and y_{sing}. This gives the same criterion (3.12).

Let us stress that though formally our procedure is incorrect at $\vartheta \gtrsim d^2\sqrt{P_2/D}\ln^2(L\Lambda)$ the answer will be the same: the PDF will be determined by the exponential tail (3.9). The point is that the character of the integral (3.1) at such extremely large ϑ will be determined by the position of the singular point of $\mathcal{Z}(y)$ nearest to the real axis. This is just iy_{sing}, leading to (3.9). To conclude, only the character of the preexponent in $\mathcal{P}(\vartheta)$ is changed at $\vartheta \sim \sim d^2\sqrt{P_2/D}\ln^2(L\Lambda)$, whereas the principal exponential behavior of $\mathcal{P}(\vartheta)$ remains unchanged there.

4. CONCLUSION

The single-point statistics of the passive scalar θ and the statistics of its difference $\Delta\theta$ are traditional objects which carry essential information about correlation functions of the passive scalar in the convective interval. We examined the passive scalar in the large-scale turbulent flow, where the correlation functions logarithmically depend on scale. Since the logarithms are actually not very large, it is useful to have all the PDF's of θ and $\Delta\theta$. That was the main purpose of our investigation, which was performed in the context of the Kraichnan model. The single-point PDF for the passive scalar and the PDF for the passive scalar differences can be obtained from (3.8) if we substitute $\Lambda \to r_{dif}^{-1}$ where r_{dif} is the diffusive length. Though both the advecting velocity and the pumping force in the Kraichnan model are considered δ-correlated in time, we hope that our results are universal, that is, are true in the limit when the size of the convective interval tends to infinity for arbitrary temporal behavior of the velocity and pumping. The reason is that the spectral transfer time grows with increasing convective interval, and in the limit is much larger than the correlation times of the velocity and pumping.

We believe also that the analytic scheme proposed in our work could be extended for other problems related to the passive scalar statistics. Note as an example Ref. [13] where a modification of the scheme enabled one to find the statistics of the passive scalar dissipation. It is also useful for investigating the large-scale statistics (on scales larger that the pumping length) of the passive scalar see Ref. [25]. We also hope that it is possible to go beyond the case of the large-scale velocity field using a perturbation technique of the type proposed in Refs. [26–28].

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APPENDIX

Here we present an alternative way to obtain the results (2.12) and (2.14). We use an auxiliary quantity

$$\Xi(y, \rho_0, \eta_0) = \int \mathcal{D}\rho\mathcal{D}\eta\mathcal{D}m\mathcal{D}\mu \, \exp\left[\int_{-\infty}^{0} dt \left(i\mathcal{L} - \frac{y^2}{2}U\right)\right]\Bigg|_{\rho(0)=\rho_0, \eta(0)=\eta_0}, \tag{A.1}$$

so

$$\mathcal{K}(y) = \Xi(y, 0, 0). \tag{A.2}$$

The function Ξ can be also defined as

$$\Xi(y, \rho_0, \eta_0) = \lim_{t\to\infty} \int \prod d\rho_a \, d\eta_{ij} \, \Psi(t, y, \rho, \eta), \tag{A.3}$$

where Ψ is governed by Eq. (2.20) with initial condition $\Psi(t=0, y, \rho, \eta) = \delta(\rho - \rho_0)\delta(\eta - \eta_0)$. The equation for Ξ can be found from Eqs. (1.32) and (A.1):

$$\left[\sum_{i=1}^{d} \frac{\partial^2}{\partial\rho_i^2} - \frac{1}{d}\left(\sum_{i=1}^{d}\frac{\partial}{\partial\rho_i}\right)^2 + \sum_{i=1}^{d}(d - 2i + 1)\frac{\partial}{\partial\rho_i} + \right.$$

$$+2\sum_{i<j} \exp(2\rho_j - 2\rho_i)\frac{\partial^2}{\partial\eta_{ij}^2} + 4\sum_{i<k<j} \exp(2\rho_k - 2\rho_i)\eta_{kj}\frac{\partial}{\partial\eta_{ij}}\frac{\partial}{\partial\eta_{ik}} +$$

$$\left. +2\sum_{i<k<m,n} \exp(2\rho_k - 2\rho_i)\eta_{km}\eta_{kn}\frac{\partial}{\partial\eta_{im}}\frac{\partial}{\partial\eta_{in}}\right]\Xi - \frac{y^2U}{Dd}\Xi = 0. \tag{A.4}$$

The boundary condition for Eq. (A.4) follows from the definition (A.1): for large enough ρ_i, η_i the potential $U = 0$ at $t = 0$ and also remains zero at finite times t. Therefore the integral (A.1) must be equal to unity in the case. Thus $\Xi(y, \rho, \eta)$ must tend to unity where ρ, $\eta \to \infty$.

Let us rewrite Eq. (A.4) in terms of the variables (2.2):

$$(\hat{\Gamma}_1 + \hat{\gamma})(\Xi_1 + \xi) = 0, \quad \Xi = \Xi_1 + \xi, \tag{A.5}$$

$$\hat{\Gamma}_1 = \frac{\partial^2}{\partial\phi_1^2} + \sqrt{\frac{d(d^2-1)}{3}}\frac{\partial}{\partial\phi_1} - \frac{y^2U}{Dd}, \tag{A.6}$$

$$\hat{\gamma} = \sum_{i=2}^{d-1}\frac{\partial^2}{\partial\phi_i^2} + 2\sum_{i<k}\exp(2\rho_k - 2\rho_i)\frac{\partial^2}{\partial\eta_{ik}^2} + 4\sum_{i<k<n}\exp(2\rho_k - 2\rho_i)\eta_{kn}\frac{\partial}{\partial\eta_{in}}\frac{\partial}{\partial\eta_{ik}} +$$

$$+2\sum_{i<k<m,n}\exp(2\rho_k - 2\rho_i)\eta_{km}\eta_{kn}\frac{\partial}{\partial\eta_{im}}\frac{\partial}{\partial\eta_{in}}. \tag{A.7}$$

Here U as a function of ϕ_1 is equal to P_2 inside a region restricted by ϕ_Λ^- and ϕ_Λ^+ (where $\phi_\Lambda^\pm$ are functions of variables $\phi_2, \ldots, \phi_d, \eta$) and tends to zero outside the region. We solve Eq. (A.5) using perturbation theory over $\hat{\gamma}$, ξ. Then the zero-order equation is

$$\hat{\Gamma}_1\Xi_1 = 0. \tag{A.8}$$

Equation (A.8) can easily be solved at $\phi_\Lambda^- < \phi_1 < \phi_\Lambda^+$; the answer is

$$\Xi_1 \simeq \frac{2\lambda}{\sqrt{\lambda^2 + \frac{y^2P_2}{Dd}} + \lambda}\exp\left\{-\left(\sqrt{\lambda^2 + y^2P_2/Dd} - \lambda\right)(\phi_\Lambda^+ - \phi_1)\right\}, \tag{A.9}$$

where $\lambda = \sqrt{d(d^2-1)/12}$, $Dd\lambda$ is the rate of the cloud motion along the ϕ_1 direction. The result (A.9) can be obtained using the inequality $\sqrt{\lambda^2 + y^2 P_2/Dd}\ln L\Lambda \gg 1$. The derivative $\partial\Xi_1/\partial\phi_1 = 0$ at $\phi_1 < \phi_\Lambda^-$. However, $\Xi_1 \neq 1$ in this region. This is due to the following fact: this region corresponds to the evolution of Ψ when its initial position is to the left of potential U (see (A.3)). During evolution, cloud Ψ passes the region of U and its integral over ρ, η changes. Then Ξ is not equal to 1. Only when the distance between the initial position and potential is of order $\ln^2 L\Lambda$ will the diffusion of the cloud lead to smallness of the part of Ψ that passes the potential U, and Ξ becomes closer to unity. Thus, function Ξ has a long tail from the potential pointing toward negative ϕ_1, where it is not equal to 1. The procedure of finding Ξ from Eq. (A.8) corresponds to the geometrical optics approximation (taking into account only derivatives in propagation direction; this allows one to get the fact of propagation). This tail of Ξ in this approximation is none other than the shadow of potential U. Higher orders of perturbation theory over the transverse derivatives correspond to diffraction corrections.

Now let us consider the correction ξ. Eq. for it looks like $(\hat\Gamma_1 + \hat\gamma)\xi = -\hat\gamma\Xi_1$. Again let us neglect $\hat\gamma$ on the left-hand side and solve the equation. Ξ_1 is some exponential function with scale of the order 1. Then $\hat\gamma\Xi_1 \sim \Xi_1$. Note that $\hat\gamma\Xi_1$ is almost equal to zero at $\phi_1 > \phi_\Lambda^+$. To estimate ξ one must construct the Green function $G(\phi_1|\phi_0)$ for operator $\hat\Gamma_1$:

$$G(0|\phi_0) \simeq \frac{1}{2\lambda}\exp\left(-\left(\sqrt{\lambda^2+\frac{y^2P_2}{Dd}}-\lambda\right)\phi_0\right)\left(1-C\exp\left(-2\sqrt{\lambda^2+\frac{y^2P_2}{Dd}}(\phi_\Lambda^+-\phi_0)\right)\right), \quad \text{(A.10)}$$

where

$$C = \left(\sqrt{\lambda^2 + y^2P_2/Dd} - \lambda\right) / \left(\sqrt{\lambda^2 + y^2P_2/Dd} + \lambda\right).$$

The unity in the parentheses in (A.10) gives the correction for Ξ, which has the same exponential factor as Ξ_1. Thus ξ does not change the answer, to logarithmic accuracy. The second term in the parentheses gains while ϕ_0 is close to ϕ_Λ^+. This is due to the nonzero width of the cloud Ξ and to the dependence of t_{lt} on other variables. Again, it does not change the exponent.

To get $\mathcal{X}$ from Ξ we in accordance with (A.2) have to substitute zero values of ρ and η into Ξ. Then $\phi_1 = 0$ and $\phi_\Lambda^+ = \phi_\Lambda$ where ϕ_Λ is defined by (2.6). Substituting the values into (A.9) we reproduce (2.12). The case of the passive scalar differences can be considered in a similar way.

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