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NEGATIVE MAGNETIC VISCOSITY IN TWO DIMENSIONS

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The occurrence of «negative viscosities» is studied within the framework of two-dimensional magnetohydrodynamics (MHD). We use assumptions which are typical when studying the effects of smaller-scale fields on larger-scale ones, namely, the small-scale MHD fields are assumed to be sufficiently weak, jointly stationary, homogeneous and maintained by external sources. The criteria of large-scale field generation due to negative viscosities are derived for various special forms of isotropic small-scale fields as well as anisotropic ones; the latter can be regarded as MHD stochastic analogs of the known Kolmogorov flow.

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1. INTRODUCTION

Problems of pattern formation and self-organization have been extensively studied in various hydrodynamic models of fluids and plasmas for at least three decades. Effects called «negative viscosity» belong to this wide class of the phenomena. This term was introduced by geophysicists and specialists in hydrodynamics in the 1950s, when analyzing large-scale geophysical processes [1, 2]. However, it is still not well known, and even seems paradoxical. Therefore, let us discuss its origin in more detail.

We first consider the classical theory of gases. In this framework, starting with the Boltzmann equation, one can derive hydrodynamic equations. The equation for momentum transport has the form

$$\frac{\partial v_i}{\partial t} + (\mathbf{v}\nabla)v_i = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} - \frac{1}{\rho}\frac{\partial \pi_{ij}}{\partial x_j}. \tag{1.1}$$

Here ρ, $\rho\mathbf{v}$ are the mass and momentum densities, respectively, $x_i = x, y, z$; p is pressure (here it is the ideal gas pressure), and π_{ij} is the viscosity tensor which describes irreversible «viscous» momentum transport. An explicit form of this tensor is obtained by invoking the smallness of the Knudsen number $Kn = l_{mfp}/L$, where l_{mfp} is the mean free path and L is an external scale (scale of inhomogeneity). In the first order of Kn, that is, in the 13-moment approximation of the Grad method [3], one has

$$\pi_{ij} = -\eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial v_k}{\partial x_k}\right), \tag{1.2}$$

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where η is the dynamical viscosity, which is of the order of $\rho v_{th} l_{mfp}$, v_{th} is the thermal velocity, and η is always positive; this is in agreement with the intuitively obvious fact that irreversible momentum transport proceeds from regions of higher velocity to those of lower velocity.

We now proceed with the thermodynamics of irreversible processes. In this framework the equation for momentum transport retains the form (1.1) (where the pressure p is determined by the motion of particles and their interaction). The form of the viscous tensor, however, cannot be determined from the kinetic equation, but instead has to be established from general principles [4]. Specifically, (i) viscous momentum transfer appears when different parts of the fluid move with different velocities, so π_{ij} must depend on the space derivatives of the velocity $\mathbf{v}$; (ii) it is assumed that these derivatives are not too large; thus, viscous momentum transfer depends on the first derivatives only; and (iii) π_{ij} must tend to zero when the fluid rotates uniformly as a whole at some angular frequency Ω. Since linear combinations of the type

$$\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}$$

tend to zero when $\mathbf{v}$ is equal to $[\Omega \times x]$, only those linear combinations are contained in π_{ij}. Finally, below we assume an incompressible fluid, $\rho = \text{const}$, and thus

$$\frac{\partial v_k}{\partial x_k} = 0.$$

Therefore, the viscous tensor retains the form (1.2) (without the last term in brackets), where η, of course, does not obey the simple gas law, but is instead a function of pressure and temperature. Assuming that the change in viscosity along the fluid is negligible, one can replace the term $-(1/\rho)\partial\pi_{ij}/\partial x_j$ in Eq. (1.1) with $\nu\Delta v_i$, where $\nu = \eta/\rho$ is the kinematic viscosity and Δ is the Laplacian. The positivity of η stems from the second law of thermodynamics for irreversible processes: the entropy S of a closed (isolated) system cannot decrease, $dS \geq 0$, whereas entropy production in a local equilibrium approximation has the same sign as η; see the detailed derivation in Ref. [3].

We now turn to turbulent processes that occur in open hydrodynamic systems. It is assumed for the turbulence of a fluid that Eqs. (1.1) and (1.2) are also valid for the stochastic velocity and pressure fields. Thus, we can define mean and fluctuating quantities

$$\mathbf{v} = \overline{\mathbf{v}} + \mathbf{v}^T, \quad p = \overline{p} + p^T,$$

where the bar denotes statistical averaging, and a superscript T denotes fluctuating (i. e., turbulent) components. In this paper we also use angle brackets $\langle\ldots\rangle$ to indicate statistical averaging in correlation functions. For the mean fields one has the equation

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x_j}\overline{v}_j\right)\overline{v}_i = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \nu\Delta\overline{v}_i - \frac{1}{\rho}\frac{\partial}{\partial x_j}\pi_{ij}^{turb}, \tag{1.3}$$

where

$$\pi_{ij}^{turb} = \rho\langle v_i^T v_j^T\rangle$$

are the Reynolds stresses.

Based on the analogy with the kinetic theory of gases, Boussinesq proposed to approximate the Reynolds stresses as

$$\langle v_i^T v_j^T\rangle = -\nu_{turb}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \tag{1.4}$$

where ν_{turb} is the turbulent kinematic viscosity. In contrast to the kinematic viscosity ν, which describes the physical properties of the gas (fluid), the turbulent kinematic viscosity describes the statistical properties of turbulent flows. An important fact is that ν_{turb} need not be positive, because there is no thermodynamic basis for its positivity. Indeed, we can speculate that when an external force (source) is added to the right hand side of Eq. (1.1), such that momentum (energy) is pumped into the small-scale component of the turbulence from this source (not from the mean flow), the momentum (energy) of fluctuations will be transferred to the mean flow. Assuming that ν_{turb} does not depend on x, Eq. (1.3) can be rewritten as

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x_j}\overline{v}_j\right)\overline{v}_i=-\frac{1}{\rho}\frac{\partial\overline{p}}{\partial x_i}+(\nu+\nu_{turb})\Delta\overline{v}_i, \tag{1.5}$$

where the sum $\nu+\nu_{turb}$ can be negative, and thus large-scale instability occurs, accompanied by an increase in the mean flow.

Approximation (1.3) assumes the locality of the turbulent mechanism of momentum transport. It implies a small ratio of the characteristic scale of turbulent fluctuations (vortices) to the scale of the mean flow (by analogy with the Knudsen number). Experimental data [5] suggest that in many cases the scale of turbulent fluctuations is of the order of that of the mean flow,, and the gradient approximation (1.4) becomes inadequate. In such cases, the contragradient transport is also observed frequently [6], but this case is much more involved from a theoretical standpoint. Thus, in this paper we restrict ourselves to the case in which the scales of fluctuation are much smaller than those of the mean quantities. We also note that in Ref. [7] the peculiarities of numerical simulation of nonlocal momentum and thermal turbulent transport are discussed in detail.

In Ref. [2] a qualitative analysis (with an accent on empirical data analysis) is carried out on a set of geophysical and astrophysical processes and laboratory experiments, in which negative viscosity effects have already been detected, or, can at least be suspected:

1) differential rotation of the Earth's atmosphere;
2) differential rotation of the Sun photosphere;
3) differential rotation of disks of spiral galaxies;
4) flows in some laboratory experiments;
5) Gulf Stream near American coast.

One can conclude that negative viscosity phenomena are ubiquitous; they occur in systems possessing various physical and geometrical properties, and on a very different range of scales, ranging from laboratory flows a few centimeters in diameter to galaxies that are kiloparsecs in diameter. It is worthwhile to note that a similar problem also occurs in a tokamak plasma, where the peaked profiles in H-mode regime can be attributed to negative turbulent dissipative coefficients and contragradient transport.

A review of the examples above suggests a necessary condition for maintenance of a (quasi)stationary regime with negative viscosity effects prevailing: turbulence must not be «passive» [2] in the sense that it is not fed by the kinetic energy of the mean flow, but instead has another source of energy, e.g., a thermal source, as occurs in Earth's atmosphere. Therefore, in papers dealing with a quantitative description of negative viscosity, the problem is formulated as follows. Suppose that we have a source giving rise to deterministic flows or stationary turbulence in a hydrodynamic medium. In the former case, the explicit form of the deterministic flows (or, at least, their symmetry properties) is assumed to be known, whereas in the latter case their statistical properties are known. The characteristic scale(s) of the given motion is assumed to be much smaller than the outer scale of the system of interest. The question is: can these small-scale motions act as a negative turbulent viscosity upon large-scale ones? From a theoretical standpoint, this implies the appearance of a negative dissipative factor in the equation for the mean flow, and its growth is understood as a manifestation of long-wavelength instability in a system of small-scale flows or vortices, the energy of small-scale motion being constant (it is mathematically convenient to treat the small-scale motion as being generated by an external source). This formulation is an example of inverse cascade problems in hydrodynamic systems. A similar (but not identical) example is related to the description of anomalous flows of the turbulent kinetic energy through the spectrum toward small wave numbers via local interaction between turbulent modes, and to the formation of stationary turbulent spectra [8, 9].

A number of analytic studies of the effects (characterized by the effective viscosity) of smaller-scale motion on larger-scale motion have begun since the late 1950s, when A. Kolmogorov proposed to study stability of a plane periodic flow sustained by a one-dimensional space-periodic external source in a viscous incompressible fluid [10]. This problem was first considered in Ref. [11], where the criterion of large-scale instability of one-dimensional space-periodic flow was found. Many subsequent papers are devoted to various aspects of the theory of Kolmogorov flow and its «relatives» in fluids; see, e.g., Refs. [12–15] and references therein. Being regarded as an «elementary object» of realistic turbulence with many degrees of freedom, Kolmogorov flow appears to be very useful in systematic studies of the peculiarities of the transition to turbulence, of the inverse cascade process in two-dimensional ($2D$) turbulent flow, and in coherent structure generation. Experimentally, a $2D$ flow subject to periodic forcing was studied in a thin layer of an electrolyte [16]. Two-dimensional hydrodynamic flows also attract considerable attention because the direct numerical solution of $2D$ fluid equations is a simpler problem than the solution of $3D$ problems [17].

For plasmas an analogous problem was considered in Ref. [18] where the stability of a gradient-drift wave was studied and coherent nonlinear structures formed as the result of instability were found.

Kolmogorov flow instability can be regarded as a simple manifestation of the negative viscosity effect when the small-scale basic flow is one-dimensional and space-periodic. Other basic forms of small-scale motion in fluids (isotropic time-independent [14] and δ-correlated in time [19]) have also been considered. A possible occurrence of the negative viscosity effect was studied in Ref. [20] for coherent wave motion, as well as for the small-scale Rossby turbulence and gradient drift-wave turbulence. We also mention the emergence of negative viscosity in a ferrofluid in an alternating magnetic field [21, 22].

Electrically conducting fluids exhibit a wide variety of turbulent phenomena. Here the concept of negative viscosities (both kinematic and magnetic) can also be useful for understanding the peculiarities of self-organizing processes. In Refs. [23, 24] it was pointed out that in a low-β plasma such as in a tokamak, small-scale magnetic turbulence acts as a negative effective magnetic viscosity on large-scale magnetic field perturbations. This leads to amplification of the large-scale field, and is a very likely mechanism in explosive magnetic phenomena, such as disruptions in tokamaks and solar flares. The reduced MHD equations [25] were taken as a starting point in these papers. It was found that the turbulent magnetic viscosity becomes negative if the magnetic energy of small-scale turbulence exceeds the kinetic energy, whereas the turbulent kinematic viscosity is positive. This problem was reconsidered in Ref. [19]. It was found that in $2D$ MHD, the conclusions are the same as in Refs. [23, 24], but for reduced MHD the results are inconsistent with those obtained in these articles. Furthermore, the role of cross-correlations and anisotropy of fluctuations remains unclear.

A more recent review [26], as well as numerical simulations [27, 28] (performed for freely decaying, not forced, turbulence), do not shed light on the problem of interest. Thus, it seems reasonable to study negative viscosity effects in MHD in more detail. In this paper we consider the problem in the context of $2D$ MHD, and based on the formulation of negative viscosities presented above.

This paper is organized as follows. In Sec. 2 we obtain general equations governing the evolution of large-scale MHD fields, and demonstrate explicitly how the negative magnetic viscosity term can appear in the equation for the mean magnetic potential. In Secs. 3–5, we study the influence (characterized by the turbulent viscosities) of various small-scale fields on large-scale fields. Specifically, in Sec. 3, small-scale turbulence is generated by a stationary white noise source; the transition to the results of Ref. [19] is demonstrated. In Sec. 4, more general forms of isotropic small-scale turbulence with correlation times spanning a broad range are used, and the transition to the results of Refs. [14, 24] is shown. Finally, in Sec. 5 we consider stochastic analogs of Kolmogorov flow for magnetohydrodynamics. In Secs. 3–5 the criteria for large-scale field growth are also derived. The results are summarized in Sec. 6. A detailed derivation of the equations governing the evolution of large-scale MHD fields is presented in the Appendix.

2. EQUATIONS FOR LARGE-SCALE FIELD EVOLUTION AND THE ORIGIN OF THE NEGATIVE MAGNETIC VISCOSITY TERM

We study a $2D$ incompressible conducting fluid with velocity field $\mathbf{v}(\mathbf{x},t) = [\mathbf{e}_z \nabla\psi]_z$ and magnetic field $\mathbf{B}(\mathbf{x},t) = [\mathbf{e}_z \nabla a]_z$, both in the xy plane, $\mathbf{x} = (x, y)$; ψ is the stream function, a is the magnetic potential, $\nabla \equiv \mathbf{e}_x \partial/\partial x + \mathbf{e}_y \partial/\partial y$, $[\ldots]_z$ implies z-projection of the vector product. The $2D$ MHD equations can be written as [26]

$$\begin{aligned} &\frac{\partial}{\partial t} W + \mathbf{v}\nabla W = \mathbf{B}\nabla j + \nu\Delta W, \\ &\frac{\partial}{\partial t} a + \mathbf{v}\nabla a = \mathbf{B}\nabla j + \eta\Delta a. \end{aligned} \tag{2.1}$$

Here $W = \nabla^2\psi$ is the vorticity, $j = \nabla^2 a$ is the current, $\nabla^2 \equiv \Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, and ν, η are the kinematic and the magnetic viscosities, respectively. In (2.1) the density of the fluid is assumed to be unity, and the magnetic field has the dimension of velocity. Furthermore, it is normally assumed that the kinematic and magnetic viscosities are of the same order. For our purposes it is convenient to rewrite (2.1) as

$$\begin{aligned} &\frac{\partial\Delta\psi}{\partial t} + [\nabla\psi \times \nabla\Delta\psi]_z = [\nabla a \times \nabla\Delta a]_z + \nu\Delta^2\psi, \\ &\frac{\partial a}{\partial t} + [\nabla\psi \times \nabla a]_z = \eta\Delta a. \end{aligned} \tag{2.2}$$

Now we divide ψ and a into mean and fluctuating (turbulent) components:

$$\psi = \overline{\psi} + \psi^T,$$

$$a = \overline{a} + a^T,$$

where the bar denotes statistical averaging, and T signifies «turbulent». Below we also use angle brackets $\langle\ldots\rangle$ for the statistical averaging.

From Eqs. (2.2) we obtain

$$\left(\frac{\partial}{\partial t}-\nu\Delta\right)\Delta\overline{\psi}= \\ =-\left\langle\left[\nabla\psi^T\times\nabla\Delta\psi^T\right]_z\right\rangle+\left\langle\left[\nabla a^T\times\nabla\Delta a^T\right]_z\right\rangle-\left[\nabla\overline{\psi}\times\nabla\Delta\overline{\psi}\right]_z+\left[\nabla\overline{a}\times\nabla\Delta\overline{a}\right]_z, \quad (2.3)$$

$$\left(\frac{\partial}{\partial t}-\eta\Delta\right)\overline{a}=-\left\langle\left[\nabla\psi^T\times\nabla a^T\right]_z\right\rangle-\left[\nabla\overline{\psi}\times\nabla\overline{a}\right]_z, \quad (2.4)$$

whereas for the fluctuating components we obtain, by subtracting Eqs. (2.3) and (2.4) from (2.2),

$$\left(\frac{\partial}{\partial t}-\nu\Delta\right)\Delta\psi^T+\left[\nabla\psi^T\times\nabla\Delta\overline{\psi}\right]_z+\left[\nabla\overline{\psi}\times\nabla\Delta\psi^T\right]_z+\left(\left[\nabla\psi^T\times\nabla\Delta\psi^T\right]_z- \\ -\left\langle\left[\nabla\psi^T\times\nabla\Delta\psi^T\right]_z\right\rangle\right)=\left[\nabla a^T\times\nabla\Delta\overline{a}\right]_z+\left[\nabla\overline{a}\times\nabla\Delta a^T\right]_z+\left(\left[\nabla a^T\times\nabla\Delta a^T\right]_z- \\ -\left\langle\left[\nabla a^T\times\nabla\Delta a^T\right]_z\right\rangle\right), \quad (2.5)$$

$$\left(\frac{\partial}{\partial t}-\eta\Delta\right)a^T+\left[\nabla\psi^T\times\nabla\overline{a}\right]_z+\left[\nabla\overline{\psi}\times\nabla a^T\right]_z+\left(\left[\nabla\psi^T\times\nabla a^T\right]_z- \\ -\left\langle\left[\nabla\psi^T\times\nabla a^T\right]_z\right\rangle\right)=0. \quad (2.6)$$

To derive a closed set of equations for $\overline{\psi}$, $\overline{a}$ it is necessary to express the quantities

$$Q_1=-\left\langle\left[\nabla\psi^T\times\nabla\Delta\psi^T\right]_z\right\rangle, \quad (2.7)$$

$$Q_2=\left\langle\left[\nabla a^T\times\nabla\Delta a^T\right]_z\right\rangle, \quad (2.8)$$

$$Q_3=-\left\langle\left[\nabla\psi^T\times\nabla a^T\right]_z\right\rangle \quad (2.9)$$

in terms of $\overline{\psi}$, $\overline{a}$ (see Eqs. (2.3) and (2.4)). Since we are interested in calculating turbulent viscosity (but not in the problems related to the evaluation of the turbulent spectrum), we use an approach developed in Ref. [20] to study negative viscosity in Rossby and drift-wave turbulence, and resembling an approach previously used in the dynamo problem [29, 30]. It also resembles the «quasilinear approximation» frequently employed to calculate the turbulent transport coefficients in magnetized inhomogeneous plasmas [31]. In so doing, we assume (in accordance with the discussion in Sec. 1) that the mean quantities vary on spatial and temporal scales that are larger than the characteristic scales of the fluctuating fields; that the statistical properties of the small-scale fields are known; and that quadratic terms in ψ^T and a^T can be neglected in Eqs. (2.5) and (2.6). Thus, instead of Eqs. (2.5) and (2.6) we obtain

$$\left(\frac{\partial}{\partial t}-\nu\Delta\right)\Delta\psi^T+\left[\nabla\psi^T\times\nabla\Delta\overline{\psi}\right]_z+\left[\nabla\overline{\psi}\times\nabla\Delta\psi^T\right]_z= \\ =\left[\nabla a^T\times\nabla\Delta\overline{a}\right]_z+\left[\nabla\overline{a}\times\nabla\Delta a^T\right]_z+F_\psi, \quad (2.10)$$

$$\left(\frac{\partial}{\partial t}-\eta\Delta\right)a^T+\left[\nabla\psi^T\times\nabla\overline{a}\right]_z+\left[\nabla\overline{\psi}\times\nabla a^T\right]_z=F_a, \quad (2.11)$$

where the stochastic sources $F_\psi(\mathbf{x},t)$ and $F_a(\mathbf{x},t)$ which maintain the stationary level of MHD fluctuations, are formally added to the right-hand side of the equations. We then solve Eqs. (2.10) and (2.11), insert the solutions into Eqs. (2.7)–(2.9) and then into Eqs. (2.3) and (2.4), and obtain the evolution equations for the mean fields $\overline{\psi}$, $\overline{a}$. Because we invoke a two-scale approximation for our problem, we introduce the «slow» variable $\mathbf{X}$ and the «fast» variable $\mathbf{x}$. The mean quantities depend on the slow variable only, whereas the fluctuating components depend on both fast and slow variables. We also have

$$\left|\frac{\partial}{\partial \mathbf{X}}\right| \approx |\mathbf{K}| \ll \left|\frac{\partial}{\partial \mathbf{x}}\right| \approx |\mathbf{k}|,$$

where $\mathbf{K}$ and $\mathbf{k}$ are large-scale and small-scale wave vectors, respectively.

We seek solutions of Eqs. (2.10) and (2.11) in powers of K, that is,

$$\psi^T = \psi^{(0)}(\mathbf{x},t) + \psi^{(1)}(\mathbf{x},\mathbf{X},t) + \ldots,$$

$$a^T = a^{(0)}(\mathbf{x},t) + a^{(1)}(\mathbf{x},\mathbf{X},t) + \ldots,$$

where the terms $\psi^{(0)}$, $a^{(0)}$ are sustained by external sources, whereas $\psi^{(1)}$, $\psi^{(2)}$ etc. appear because of the interaction between small- and large-scale components. Assuming that the small-scale fluctuations are jointly stationary and homogeneous (which, in turn, implies that both fields are stationary and homogeneous), we introduce their correlation and cross-correlation functions $C_{\psi\psi}$, C_{aa}, $C_{\psi a}$ as well as their space-time spectral functions $\hat{C}_{\psi\psi}$, $\hat{C}_{aa}$, $\hat{C}_{\psi a}$ e.g., as

$$\left\langle \psi^{(0)}(\mathbf{x},t)a^{(0)}(\mathbf{x}',t')\right\rangle = C_{\psi a}(\mathbf{x}-\mathbf{x}',t-t') = \int \frac{d\mathbf{k}\,d\omega}{(2\pi)^3}\,\hat{C}_{\psi a}(\mathbf{k},\omega)\exp\left[-i\omega(t-t') + i\mathbf{k}(\mathbf{x}-\mathbf{x}')\right].$$

The properties of the spectral functions of zeroth-order quantities are listed in Appendix, along with a detailed derivation of $Q_{1,2,3}$. Here we present the final result. For the evolution of the mean quantities $\overline{\psi}$, $\overline{a}$ instead of Eqs. (2.3) and (2.4), we have

$$\left(\frac{\partial}{\partial t} - \nu\Delta_s\right)\Delta_s\overline{\psi} = \varepsilon_{kl}\varepsilon_{mn}\left\{\delta^{(1)}_{lnp}\frac{\partial^3\overline{\psi}}{\partial X_k\partial X_m\partial X_p} + \delta^{(2)}_{lnp}\frac{\partial^3\overline{a}}{\partial X_k\partial X_m\partial X_p} + \right.$$
$$+\nu^{(1)}_{ln}\frac{\partial^2\Delta_s\overline{\psi}}{\partial X_k\partial X_m} + \nu^{(2)}_{ln}\frac{\partial^2\Delta_s\overline{a}}{\partial X_k\partial X_m} + \nu^{(1)}_{lnpr}\frac{\partial^4\overline{\psi}}{\partial X_k\partial X_m\partial X_p\partial X_r} +$$
$$\left. + \nu^{(2)}_{lnpr}\frac{\partial^4\overline{a}}{\partial X_k\partial X_m\partial X_p\partial X_r}\right\}, \tag{2.12}$$

$$\left(\frac{\partial}{\partial t} - \eta\Delta_s\right)\overline{a} = \varepsilon_{kl}\varepsilon_{mn}\left\{\eta^{(1)}_{ln}\frac{\partial^2\overline{a}}{\partial X_k\partial X_m} + \eta^{(2)}_{ln}\frac{\partial^2\overline{\psi}}{\partial X_k\partial X_m}\right\}, \tag{2.13}$$

where $\Delta_s \equiv \partial^2/\partial X^2 + \partial^2/\partial Y^2$, ε_{mn} is the unit antisymmetric tensor of the second rank, and

$$\delta^{(1)}_{lnp} = \int \frac{d\mathbf{k}d\omega}{(2\pi)^3} 2ik_lk_nk_p\left\{\frac{\hat{C}_{aa}}{-i\omega + \eta k^2} - \frac{\hat{C}_{\psi\psi}}{-i\omega + \nu k^2}\right\},$$

$$\delta^{(2)}_{lnp} = -\int \frac{d\mathbf{k}d\omega}{(2\pi)^3} 2ik_l k_n k_p \left\{ \frac{\hat{C}_{\psi a}}{-i\omega+\eta k^2} - \frac{\hat{C}^*_{\psi a}}{-i\omega+\nu k^2} \right\},$$

$$\nu^{(1)}_{ln} = \int \frac{d\mathbf{k}d\omega}{(2\pi)^3} k_l k_n \left\{ \frac{\hat{C}_{aa}}{-i\omega+\eta k^2} - \frac{\hat{C}_{\psi\psi}}{-i\omega+\nu k^2} \right\},$$

$$\nu^{(2)}_{ln} = -\int \frac{d\mathbf{k}d\omega}{(2\pi)^3} k_l k_n \left\{ \frac{\hat{C}_{\psi a}}{-i\omega+\eta k^2} - \frac{\hat{C}^*_{\psi a}}{-i\omega+\nu k^2} \right\},$$

$$\nu^{(1)}_{lnpr} = \int \frac{d\mathbf{k}d\omega}{(2\pi)^3} \frac{4k_l k_n k_p k_r}{k^2} \left\{ \frac{\hat{C}_{\psi\psi}}{-i\omega+\nu k^2} + \frac{\nu k^2}{(-i\omega+\nu k^2)^2} \hat{C}_{\psi\psi} - \frac{\eta k^2}{(-i\omega+\eta k^2)^2} \hat{C}_{aa} \right\},$$

$$\nu^{(2)}_{lnpr} = \int \frac{d\mathbf{k}d\omega}{(2\pi)^3} \frac{4k_l k_n k_p k_r}{k^2} \left\{ \frac{\eta k^2}{(-i\omega+\eta k^2)^2} \hat{C}_{\psi a} - \frac{\hat{C}^*_{\psi a}}{-i\omega+\nu k^2} - \frac{\nu k^2}{(-i\omega+\nu k^2)^2} \hat{C}^*_{\psi a} \right\},$$

$$\eta^{(1)}_{ln} = \int \frac{d\mathbf{k}d\omega}{(2\pi)^3} k_l k_n \left\{ \frac{\hat{C}_{\psi\psi}}{-i\omega+\eta k^2} - \frac{\hat{C}_{aa}}{-i\omega+\nu k^2} \right\},$$

$$\eta^{(2)}_{ln} = \int \frac{d\mathbf{k}d\omega}{(2\pi)^3} k_l k_n \left\{ \frac{\hat{C}_{\psi a}}{-i\omega+\nu k^2} - \frac{\hat{C}^*_{\psi a}}{-i\omega+\eta k^2} \right\}.$$

The reality of all coefficients in Eqs. (2.12) and (2.13) is easily demonstrated using the properties (A.9) of the space-time spectral functions.

The resulting equations enable us to study the influence (characterized by the turbulent viscosities) of various forms of small-scale fields on large-scale fields. However, in order to make the results more transparent, in this Section we also demonstrate explicitly how negative magnetic viscosity originates in Eqs. (2.4) and (2.10). For this purpose it is convenient to assume that the field a^T is given. Then the smallness of large-scale gradients enables us to derive from Eq. (2.10) the following relation between the small-scale Fourier-components $\hat{\psi}^T(\mathbf{k},\omega)$, $\hat{a}^T(\mathbf{k},\omega)$:

$$\hat{\psi}^T(\mathbf{k},\omega) = \varepsilon_{mn} \frac{ik_n}{-i\omega+\nu k^2} \frac{\partial \bar{a}}{\partial X_m} \hat{a}^T + (\text{other terms}). \tag{2.14}$$

Here we explicitly write the term that derives from $[\nabla\bar{a} \times \nabla\Delta a^T]_z$ in Eq. (2.10). Further, we note that the terms entering into $Q_3 = -\langle [\nabla\psi^T \times \nabla a^T]_z \rangle$ include

$$-\left\langle \left[\frac{\partial \psi^T}{\partial \mathbf{X}} \times \frac{\partial a^T}{\partial \mathbf{x}} \right] \right\rangle,$$

which, after inserting $\hat{\psi}^T$ from Eq. (2.14), gives rise to the term

$$-\varepsilon_{kl}\varepsilon_{mn} \frac{\partial^2 \bar{a}}{\partial X_k \partial X_m} \int \frac{d\mathbf{k}\,d\omega}{(2\pi)^3} \frac{k_l k_n}{-i\omega+\nu k^2} \left\langle \left(\hat{a}^T\right)^2 \right\rangle_{\mathbf{k},\omega} \tag{2.15}$$

on the right-hand side of Eq. (2.4); here $\left\langle \left(\hat{a}^T\right)^2 \right\rangle_{\mathbf{k},\omega}$ is the spectrum of the small-scale magnetic potential. Assuming isotropy and integrating over azimuthal angle in $\mathbf{k}$-space, one can easily show that the term (2.15) can be rewritten as

$$-\nu^T \Delta_s \bar{a}, \tag{2.16}$$

where

$$\nu^T = \int \frac{kdk\,d\omega}{8\pi^2}\frac{\nu k^4}{\omega^2+\nu^2k^4}\left\langle \left(\hat{a}^T\right)^2\right\rangle_{k,\omega}. \tag{2.17}$$

Equations (2.16) and (2.17) reveal a term of negative viscosity type in the equation for the mean component of the magnetic potential.

3. SMALL-SCALE ISOTROPIC TURBULENCE GENERATED BY STATIONARY WHITE NOISE SOURCE

It is mathematically convenient to treat small-scale turbulence as being generated by a stationary white noise source, to keep the statistical properties as simple as possible. This kind of the source, possessing zero cross-correlations, was considered in Ref. [19], so we are able not only to compare the results but also to clarify the role of cross-correlation terms. We define the properties of the sources F_ψ, F_a in Eqs. (2.10) and (2.11) as follows:

$$\langle F_\psi\rangle = \langle F_a\rangle = 0,$$

$$\begin{aligned}\langle F_\psi(\mathbf{x},t)F_\psi(\mathbf{x}',t')\rangle &= \Psi(\mathbf{x}-\mathbf{x}')\delta(t-t'),\\ \langle F_a(\mathbf{x},t)F_a(\mathbf{x}',t')\rangle &= A(\mathbf{x}-\mathbf{x}')\delta(t-t'),\\ \langle F_\psi(\mathbf{x},t)F_a(\mathbf{x}',t')\rangle &= H(\mathbf{x}-\mathbf{x}')\delta(t-t').\end{aligned} \tag{3.1}$$

Since fluctuations $\psi^{(0)}$, $a^{(0)}$ are related to the sources F_ψ, F_a by Eqs. (A.4), we find for the space-time spectral functions of the small-scale turbulence

$$\begin{aligned}\hat{C}_{\psi\psi}(\mathbf{k},\omega) &= \frac{\hat{\Psi}(\mathbf{k})}{k^4(\omega^2+\nu^2k^4)},\\ \hat{C}_{aa}(\mathbf{k},\omega) &= \frac{\hat{A}(\mathbf{k})}{\omega^2+\eta^2k^4},\\ \hat{C}_{\psi a}(\mathbf{k},\omega) &= -\frac{\hat{H}(\mathbf{k})}{k^2(i\omega+\nu k^2)(i\omega+\eta k^2)},\end{aligned} \tag{3.2}$$

where $\hat{\Psi}$, $\hat{A}$, $\hat{H}$ are the spatial Fourier transforms of Ψ, A, H, respectively, e.g.,

$$\hat{\Psi}(\mathbf{k}) = \int d\boldsymbol{\kappa}\,\Psi(\boldsymbol{\kappa})\exp(-i\mathbf{k}\boldsymbol{\kappa}).$$

It is also useful to express space-time spectral functions (3.2) in terms of spatial functions,

$$\hat{C}_{\psi a}(\mathbf{k}) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\hat{C}_{\psi a}(\mathbf{k},\omega).$$

Integrating over ω in Eqs. (3.2), we express the sources in terms of spatial spectral functions and obtain

$$\begin{aligned}
\hat{C}_{\psi\psi}(\mathbf{k},\omega) &= \frac{2\nu k^2}{\omega^2+\nu^2k^4}\hat{C}_{\psi\psi}(\mathbf{k}),\\
\hat{C}_{aa}(\mathbf{k},\omega) &= \frac{2\eta k^2}{\omega^2+\eta^2k^4}\hat{C}_{aa}(\mathbf{k}),\\
\hat{C}_{\psi a}(\mathbf{k},\omega) &= \frac{(\nu+\eta)k^2}{(-i\omega+\nu k^2)(i\omega+\eta k^2)}\hat{C}_{\psi a}(\mathbf{k}).
\end{aligned}\tag{3.3}$$

We then insert Eqs. (3.3) into the coefficients of Eqs. (2.12) and (2.13) and integrate over ω:

$$\begin{aligned}
\delta^{(1)}_{lnp} &= 0,\\
\delta^{(2)}_{lnp} &= -\int\frac{d\mathbf{k}}{4\pi^2}i\frac{k_lk_nk_p}{k^2}\left(\frac{\hat{C}_{\psi a}(\mathbf{k})}{\eta}-\frac{\hat{C}^*_{\psi a}(\mathbf{k})}{\nu}\right),\\
\nu^{(1)}_{ln} &= \int\frac{d\mathbf{k}}{4\pi^2}i\frac{k_lk_n}{2k^2}\left(\frac{\hat{C}_{aa}(\mathbf{k})}{\eta}-\frac{\hat{C}_{\psi\psi}(\mathbf{k})}{\nu}\right),\\
\nu^{(1)}_{lnpr} &= \int\frac{d\mathbf{k}}{4\pi^2}i\frac{k_lk_nk_pk_r}{k^4}\left(\frac{3\hat{C}_{\psi\psi}(\mathbf{k})}{\nu}-\frac{\hat{C}_{aa}(\mathbf{k})}{\eta}\right),\\
\nu^{(2)}_{ln} &= -\int\frac{d\mathbf{k}}{4\pi^2}\frac{k_lk_n}{2k^2}\left(\frac{\hat{C}_{\psi a}(\mathbf{k})}{\eta}-\frac{\hat{C}^*_{\psi a}(\mathbf{k})}{\nu}\right),\\
\nu^{(1)}_{lnpr} &= \int\frac{d\mathbf{k}}{4\pi^2}i\frac{k_lk_nk_pk_r}{k^4}\left(\frac{\hat{C}_{\psi a}(\mathbf{k})}{\eta}-\frac{3\hat{C}^*_{\psi a}(\mathbf{k})}{\nu}\right),\\
\eta^{(1)}_{ln} &= \int\frac{d\mathbf{k}}{4\pi^2}i\frac{k_lk_n}{k^2(\nu+\eta)}\left(\hat{C}_{\psi\psi}(\mathbf{k})-\hat{C}_{aa}(\mathbf{k})\right),\\
\eta^{(2)}_{ln} &= 0.
\end{aligned}\tag{3.4}$$

We note that if we set $\hat{C}_{\psi a}=0$, then Eqs. (2.12) and (2.13) together with Eqs. (3.4) appear to be in complete agreement with Eqs. (2.15) and (3.12)–(3.15) of Ref. [19] after some easy transforms. In this Section we assume isotropy of the small-scale spectra; the spatial spectral functions in Eqs. (3.4) therefore depend on $k\equiv|\mathbf{k}|$. Integrating over the azimuthal angle φ in $\mathbf{k}$-space in Eqs. (3.4) using the subsidiary integrals

$$\int_0^{2\pi}d\varphi\,k_mk_n=\pi k^2\delta_{mn},$$

$$\int_0^{2\pi}d\varphi\,k_kk_lk_mk_n=\frac{\pi k^4}{4}\left(\delta_{kl}\delta_{mn}+\delta_{kn}\delta_{lm}+\delta_{km}\delta_{ln}\right),$$

Eqs. (2.11) and (2.12) take the form

$$\left(\frac{\partial}{\partial t} - \nu\Delta_s\right)\Delta_s\overline{\psi} = \nu_{turb}\Delta_s^2\overline{\psi} + \delta\nu_{turb}\Delta_s^2\overline{a},$$
$$\left(\frac{\partial}{\partial t} - \eta\Delta_s\right)\overline{a} = \eta_{turb}\Delta_s\overline{a}, \tag{3.5}$$

where

$$\nu_{turb} = \frac{1}{8}\left(\frac{\left\langle\left(\psi^{(0)}\right)^2\right\rangle}{\nu} + \frac{\left\langle\left(a^{(0)}\right)^2\right\rangle}{\eta}\right),$$
$$\delta\nu_{turb} = -\left(\frac{1}{\nu} + \frac{1}{\eta}\right)\frac{\left\langle\left(\psi^{(0)}a^{(0)}\right)\right\rangle}{8}, \tag{3.6}$$
$$\eta_{turb} = \frac{1}{2(\nu+\eta)}\left(\left\langle\left(\psi^{(0)}\right)^2\right\rangle - \left\langle\left(a^{(0)}\right)^2\right\rangle\right),$$

$\left\langle\left(\psi^{(0)}\right)^2\right\rangle$, $\left\langle\left(a^{(0)}\right)^2\right\rangle$ are the stream-function and magnetic-potential variances of the small-scale fields, respectively:

$$\left\langle\left(\psi^{(0)}\right)^2\right\rangle = \int \frac{d\mathbf{k}}{4\pi^2}\,\hat{C}_{\psi\psi}(k),$$

and

$$\left\langle\left(\psi^{(0)}a^{(0)}\right)\right\rangle = \int \frac{d\mathbf{k}}{4\pi^2}\,\hat{C}_{\psi a}(k).$$

It follows from Eqs. (3.5) and (3.6) that for the case of small-scale turbulence generated by a stationary white noise source, the turbulent viscosity is always positive, the turbulent magnetic viscosity is negative if

$$\left\langle\left(a^{(0)}\right)^2\right\rangle > \left\langle\left(\psi^{(0)}\right)^2\right\rangle, \tag{3.7}$$

and if cross-correlations vanish, the large-scale magnetic field grows if $|\eta_{turb}|$ is large enough that $(\eta + \eta_{turb})$ becomes negative, while the large-scale velocity field does not grow. The existence of nonzero cross-correlations leads to amplification of both large-scale fields when $(\eta+\eta_{turb})$ becomes negative; the latter conclusion is unaffected by the sign of cross-correlations $\left\langle\left(\psi^{(0)}a^{(0)}\right)\right\rangle$.

4. MORE GENERAL FORMS OF ISOTROPIC SMALL-SCALE TURBULENCE

To provide a more general treatment of the isotropic case with zero cross-correlations, we define space-time spectral functions as follows:

$$\begin{aligned} \hat{C}_{\psi\psi}(k,\omega) &= \frac{2\gamma_{1k}}{\omega^2+\gamma_{1k}^2}\,\hat{C}_{\psi\psi}(k),\\ \hat{C}_{aa}(k,\omega) &= \frac{2\gamma_{2k}}{\omega^2+\gamma_{2k}^2}\,\hat{C}_{aa}(k),\\ \hat{C}_{\psi a}(k,\omega) &= 0. \end{aligned} \tag{4.1}$$

The ω-dependent part of the spectrum is taken in the Lorentzian form here. This form is frequently used in the literature; however, we choose it for convenience only. It can be easily verified that the results are changed only by a factor of order unity if one chooses other shapes, for example, the Gaussian shape or the rectangular one. Inserting Eqs. (4.1) into Eqs. (2.12) and (2.13) and integrating over ω and φ in the coefficients, we arrive at Eqs. (3.5), where

$$\begin{aligned} \nu_{turb} &= \int \frac{dk\,k^3}{4\pi}\left\{\frac{\nu k^2}{(\gamma_{1k}+\nu k^2)^2}\,\hat{C}_{\psi\psi}(k)+\frac{\gamma_{2k}}{(\gamma_{2k}+\eta k^2)^2}\,\hat{C}_{aa}(k)\right\},\\ \eta_{turb} &= \int \frac{dk\,k^3}{4\pi}\left\{\frac{\hat{C}_{\psi\psi}(k)}{\gamma_{1k}+\eta k^2}-\frac{\hat{C}_{aa}(k)}{\gamma_{2k}+\nu k^2}\right\}, \end{aligned} \tag{4.2}$$

whereas $\delta\nu_{turb}=0$. For $\gamma_{1k}=\nu k^2$, $\gamma_{2k}=\eta k^2$ we naturally obtain ν_{turb}, η_{turb} of Sec. 3; see Eqs. (3.6). Here we consider two special cases.

1. Long correlation times of the small-scale fluctuations, $\gamma_{1k},\gamma_{2k}\ll\nu k^2,\eta k^2$. In this limit

$$\begin{aligned} \nu_{turb} &= \frac{\left\langle\left(\psi^{(0)}\right)^2\right\rangle}{2\nu},\\ \eta_{turb} &= \frac{\left\langle\left(\psi^{(0)}\right)^2\right\rangle}{2\eta}-\frac{\left\langle\left(a^{(0)}\right)^2\right\rangle}{2\nu}, \end{aligned} \tag{4.3}$$

and ν_{turb} coincides with that obtained in Ref. [14] for time-independent random basic flow (which, in fact, corresponds to $\gamma_{1k}\to 0$) in the ordinary fluid, $B=0$. In the case considered, the turbulent magnetic viscosity is negative if

$$\left\langle\left(a^{(0)}\right)^2\right\rangle>\frac{\nu}{\eta}\left\langle\left(\psi^{(0)}\right)^2\right\rangle, \tag{4.4}$$

whereas ν_{turb} is always positive.

2. Short correlation times, γ_{1k}, $\gamma_{2k}\gg\nu k^2,\eta k^2$.

Assuming that

$$\gamma_{1k}\approx\gamma_{2k}\approx 1/\tau_c,$$

where τ_c is the correlation time, which is independent of k, we obtain from Eqs. (4.2)

$$\nu_{turb} = \frac{\tau_c}{2}\left\langle \left(B^{(0)}\right)^2\right\rangle,$$
$$\eta_{turb} = \frac{\tau_c}{2}\left(\left\langle \left(\mathbf{v}^{(0)}\right)^2\right\rangle - \left\langle \left(\mathbf{B}^{(0)}\right)^2\right\rangle\right), \tag{4.5}$$

where $\left\langle \left(\mathbf{v}^{(0)}\right)^2\right\rangle$ and $\left\langle \left(\mathbf{B}^{(0)}\right)^2\right\rangle$ are the velocity and magnetic-field variances, respectively. In this limit the turbulent viscosities are the same as those obtained in Ref. [24]. From the second equation, it follows that the turbulent magnetic viscosity is negative if

$$\left\langle \left(\mathbf{B}^{(0)}\right)^2\right\rangle > \left\langle \left(\mathbf{v}^{(0)}\right)^2\right\rangle, \tag{4.6}$$

whereas ν_{turb} is again always positive.

Now we consider the role of cross-correlations. Whereas the general form (4.1) of the space-time spectral functions $\hat{C}_{\psi\psi}$, $\hat{C}_{aa}$ is natural and widely used, it is not so easy, in the author's opinion, to choose an analogous general form of the cross-correlation spectrum. We therefore restrict attention to time-independent isotropic fluctuations (special case 1; see above) of $\psi^{(0)}$, $a^{(0)}$:

$$\hat{C}_{\psi\psi}(k,\omega) = 2\pi\delta(\omega)\hat{C}_{\psi\psi}(k),$$
$$\hat{C}_{aa}(k,\omega) = 2\pi\delta(\omega)\hat{C}_{aa}(k), \tag{4.7}$$
$$\hat{C}_{\psi a}(k,\omega) = 2\pi\delta(\omega)\hat{C}_{\psi a}(k).$$

Inserting Eqs. (4.7) into Eqs. (2.12) and (2.13), we have

$$\left(\frac{\partial}{\partial t} - \nu\Delta_s\right)\Delta_s\overline{\psi} = \nu_{turb}\Delta_s^2\overline{\psi} + \delta\nu_{turb}\Delta_s^2\overline{a},$$
$$\left(\frac{\partial}{\partial t} - \eta\Delta_s\right)\overline{a} = \eta_{turb}\Delta_s\overline{a} + \delta\eta_{turb}\Delta_s\overline{\psi}, \tag{4.8}$$

where ν_{turb} and η_{turb} are given by Eqs. (4.3), whereas

$$\delta\nu_{turb} = -\frac{\left\langle\left(\psi^{(0)}a^{(0)}\right)\right\rangle}{2\nu},$$
$$\delta\eta_{turb} = \frac{\left\langle\left(\psi^{(0)}a^{(0)}\right)\right\rangle}{2}\left(\frac{1}{\nu} - \frac{1}{\eta}\right). \tag{4.9}$$

Since we want to illustrate the role of cross-correlations, we consider the simplest case $\nu = \eta$. This is the usual assumption in numerical simulations; see, e.g., Ref. [28]. Equations (4.8) then take the form of Eqs. (3.5). In this case, both large-scale fields grow if $\eta + \eta_{turb} < 0$, regardless of the sign of the cross-correlations $\left\langle\left(\psi^{(0)}a^{(0)}\right)\right\rangle$, just as in the case of δ-correlated sources; see Sec. 3.

5. STOCHASTIC ANALOGS OF KOLMOGOROV FLOW FOR MAGNETOHYDRODYNAMICS

In this Section we consider specific examples of anisotropic time-independent fields, which can be regarded as magnetohydrodynamic stochastic analogs of Kolmogorov flow. In particular, we choose $\psi^{(0)}$ and $a^{(0)}$ to be

$$\begin{aligned} \psi^{(0)}(\mathbf{x}, t) &= A_1 \cos(\mathbf{k}_0 \mathbf{x} + \alpha), \\ a^{(0)}(\mathbf{x}, t) &= A_2 \cos(\mathbf{k}_0 \mathbf{x} + \varphi + \alpha), \end{aligned} \tag{5.1}$$

where A_1 and A_2 are the constant amplitudes of the zeroth-order stream function and magnetic potential, respectively; α is a random phase, uniformly distributed in $[0; 2\pi]$; and φ is a constant phase. The spectral functions are

$$\begin{aligned} &\hat{C}_{\psi\psi}(\mathbf{k}, \omega) = 2\pi^3 A_1^2 \delta(\omega) \left[\delta(\mathbf{k} + \mathbf{k}_0) + \delta(\mathbf{k} - \mathbf{k}_0)\right], \\ &\hat{C}_{aa}(\mathbf{k}, \omega) = 2\pi^3 A_2^2 \delta(\omega) \left[\delta(\mathbf{k} + \mathbf{k}_0) + \delta(\mathbf{k} - \mathbf{k}_0)\right], \\ &\hat{C}_{\psi a}(\mathbf{k}, \omega) = 2\pi^3 A_1 A_2 \delta(\omega) \left\{\cos\varphi \left[\delta(\mathbf{k}+\mathbf{k}_0)+\delta(\mathbf{k}-\mathbf{k}_0)\right] + i \sin\varphi \left[\delta(\mathbf{k}+\mathbf{k}_0)-\delta(\mathbf{k}-\mathbf{k}_0)\right]\right\}. \end{aligned} \tag{5.2}$$

It follows from Eqs. (5.1) and (5.2) that the zeroth-order fields so chosen are jointly homogeneous; that one is able to consider various forms of cross-correlations by varying φ; and that if there are different random phases (say, α and β) in Eqs. (5.1), then there are no cross-correlations. Thus, Eqs. (5.1) enable one to study a set of interesting consequences. Here we consider only the simple case,

$$\varphi = 0, \quad \mathbf{k}_0 = k_0 \mathbf{e}_x. \tag{5.3}$$

Inserting Eqs. (5.2) and (5.3) into Eqs. (2.12) and (2.13), we obtain

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \nu\Delta_s\right) \Delta_s \overline{\psi} = &- \left(\frac{A_1^2}{2\nu} - \frac{A_2^2}{2\eta}\right) \frac{\partial^2}{\partial Y^2} \Delta_s \overline{\psi} + \left(\frac{4A_1^2}{\nu} - \frac{2A_2^2}{\eta}\right) \frac{\partial^4 \overline{\psi}}{\partial X^2 \partial Y^2} - \\ &- A_1 A_2 \left(\frac{4}{\nu} - \frac{2}{\eta}\right) \frac{\partial^4 \overline{a}}{\partial X^2 \partial Y^2} + \frac{A_1 A_2}{2} \left(\frac{1}{\nu} - \frac{1}{\eta}\right) \frac{\partial^2}{\partial Y^2} \Delta_s \overline{a}, \\ \left(\frac{\partial}{\partial t} - \eta\Delta_s\right) \overline{a} = &\frac{A_1 A_2}{2} \left(\frac{1}{\nu} - \frac{1}{\eta}\right) \frac{\partial^2 \overline{\psi}}{\partial Y^2} + \left(\frac{A_1^2}{2\eta} - \frac{A_2^2}{2\nu}\right) \frac{\partial^2 \overline{a}}{\partial Y^2}. \end{aligned} \tag{5.4}$$

Examples resulting from subsequent simplifications are as follows.

1. $A_2 = 0$. There is no small-scale magnetic field. Equations (5.4) reduce to

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \nu\Delta_s\right) \Delta_s \overline{\psi} &= -\frac{A_1^2}{2\nu} \frac{\partial^2}{\partial Y^2} \Delta_s \overline{\psi} + \frac{4A_1^2}{\nu} \frac{\partial^4 \overline{\psi}}{\partial X^2 \partial Y^2}, \\ \left(\frac{\partial}{\partial t} - \eta\Delta_s\right) \overline{a} &= \frac{A_1^2}{2\eta} \frac{\partial^2 \overline{a}}{\partial Y^2}. \end{aligned} \tag{5.5}$$

It follows from Eqs. (5.5) that the «turbulent» magnetic viscosity is always positive, and thus no large-scale magnetic field is generated. The growth rate for large-scale perturbations of the stream function is

$$\gamma = -\nu K^2 + \frac{A_1^2}{2\nu} K_y^2 - \frac{4A_1^2}{\nu} \frac{K_x^2 K_y^2}{K^2}, \tag{5.6}$$

where $\mathbf{K}$ is the wave vector of large-scale perturbations, $K^2 = K_x^2 + K_y^2$. The growth rate (5.6) naturally coincides with that of Kolmogorov flow in an ordinary fluid; the latter has been calculated via the multi-scales expansion technique of Ref. [14]. The most «dangerous» (that is, those that appear first when the amplitude of the small-scale field increases) are the large-scale perturbations with a wave vector, perpendicular to that of the small-scale field, i.e., $K_x = 0$. Such perturbations grow if

$$A_1 > A_{1min} = \sqrt{2}\nu. \tag{5.7}$$

The criterion (5.7) is derived for an ordinary fluid in Ref. [11] for a regular small-scale velocity field.

2. $A_1 = 0$. There is no small-scale velocity field. Equations (5.4) take the form

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \nu\Delta_s\right)\Delta_s\overline{\psi} &= -\frac{2A_2^2}{\eta}\frac{\partial^4\overline{\psi}}{\partial X^2\partial Y^2} + \frac{A_2^2}{2\eta}\frac{\partial^2}{\partial Y^2}\Delta_s\overline{\psi}, \\ \left(\frac{\partial}{\partial t} - \eta\Delta_s\right)\overline{a} &= -\frac{A_2^2}{2\nu}\frac{\partial^2\overline{a}}{\partial Y^2}. \end{aligned} \tag{5.8}$$

The second equation signals the onset of negative magnetic viscosity. The most dangerous are large-scale magnetic field perturbations with a wave vector perpendicular to that of the small-scale field, $K_x = 0$. The instability criterion is then

$$A_2 > A_{2min} = \sqrt{2\nu\eta}. \tag{5.9}$$

A large-scale velocity field with only a K_y-component does not grow. However, it can be shown that $\overline{\psi}$ grows if its wave vector makes an acute angle with the x axis. This special case is studied in detail in Ref. [32] using multiple-scale methods.

3. $A_1 = A_2 = A$. We note that equal amplitudes (or energies) of the magnetic and velocity fields are frequently chosen at $t = 0$ in numerical simulations of freely decaying magnetohydrodynamic turbulence; see Refs. [27, 28]. Taking $\overline{\psi}$, $\overline{a}$ in the form

$$\begin{aligned} \overline{\psi} &= R\exp\left(\gamma t + iK_xX + iK_yY\right), \\ \overline{a} &= P\exp\left(\gamma t + iK_xX + iK_yY\right), \end{aligned} \tag{5.10}$$

and inserting Eqs. (5.10) into Eqs. (5.4), we obtain a linear system with unknown P and R. The equation for the growth rate then follows by equating the determinant to zero:

$$a\gamma^2 + b\gamma + c = 0, \tag{5.11}$$

where

$$a = K^2,$$

$$b = (\nu+\eta)K^4 - 2\Lambda K^2K_y^2 + \Lambda_1K_x^2K_y^2,$$

$$c = \nu\eta K^6 - \Lambda(\nu+\eta)K^4K_y^2 + \Lambda_1\eta K^2K_x^2K_y^2,$$

$$\Lambda = \frac{A^2}{2}\left(\frac{1}{\nu} - \frac{1}{\eta}\right), \quad \Lambda_1 = 2A^2\left(\frac{2}{\nu} - \frac{1}{\eta}\right).$$

Introducing polar coordinates

$$K_x = K\cos\theta, \quad K_y = K\sin\theta$$

we obtain an equation for the neutral curve $\gamma = 0$ in terms of A, θ:

$$\frac{A^2}{2\nu^2}\left[1 - (\mathrm{Pr}_m)^2 - 4(2 - \mathrm{Pr}_m)\cos^2\theta\right]\sin^2\theta = 1, \tag{5.12}$$

where $\mathrm{Pr}_m = \nu/\eta$ is the magnetic Prandtl number. Instability is possible for

$$\mathrm{Pr}_m < 1$$

and

$$|\cos\theta| < \frac{1}{2}\left[\frac{1 - (\mathrm{Pr}_m)^2}{2 - \mathrm{Pr}_m}\right]^{1/2}.$$

The most dangerous perturbations are those with $K_x = 0$ (as can be seen by comparing $A^2(\theta)$ with $A^2(\pi/2)$ estimated from Eq. (5.12)). In this case, the instability criterion is

$$A > A_{min} = \frac{\sqrt{2}\,\nu}{\sqrt{1 - (\mathrm{Pr}_m)^2}}, \tag{5.13}$$

whereas the growth rate takes the form

$$\gamma = \left\{\Lambda - \frac{\nu + \eta}{2} + \sqrt{\Lambda^2 + \frac{(\nu - \eta)^2}{4}}\right\}K^2. \tag{5.14}$$

Because both large-scale fields increase, this case is of interest for subsequent nonlinear analysis and numerical simulation, which will be the subject of future research.

6. RESULTS

In this paper, in the framework of $2D$ magnetohydrodynamics, we have studied the possible occurrence of large-scale mean velocity and magnetic fields generated by small-scale random fields. The latter are assumed jointly stationary, homogeneous, and maintained by an external source.

The random fields lead to negative dissipative factors in the equations for the mean fields, which is why the term «negative viscosity» is used. Viscous damping of large-scale fields is thus replaced by growth, which is limited due to nonlinear effects in the amplitudes of the large-scale fields. This picture, being simplified, though, is fruitful for studying the effects of smaller-scale fields on large-scale ones.

Our results are as follows.

1. Using a two-scale expansion, we obtain equations, that describe the evolution of the mean stream function and the mean magnetic potential in the presence of small-scale MHD fluctuations. These expressions enable us to study the evolution of large-scale MHD perturbations with the assumption that the statistical properties of the small-scale fields are known.

2. With our approach we easily demonstrate how a negative magnetic viscosity term can appear in the equation for the mean magnetic potential.

3. The general expressions also enable us to recover previous results on the eddy (turbulent) viscosity of the ordinary fluid and on turbulent viscosities in the presence of small-scale MHD fluctuations.

4. For isotropic small-scale fluctuations, we estimate turbulent viscosities and find criteria for the onset of negative magnetic viscosity, as well as for the growth of large-scale MHD fields, in three cases:

(i) fluctuations are generated by white noise source;

(ii) fluctuations possess long correlation times;

(iii) fluctuations possess short correlation times (in comparison with characteristic dissipation times associated with molecular kinematic and magnetic viscosities).

In particular, it is shown when the cross-correlations among small-scale fields vanish, the turbulent viscosity is always positive, whereas the turbulent magnetic viscosity can be negative, thus giving rise to the growth of large-scale magnetic perturbations. When cross-correlations are nonvanishing, both large-scale fields can be amplified.

5. We also consider how large-scale fields are influenced by anisotropic small-scale random fields, which can be regarded as stochastic analogs of Kolmogorov flow. We find that

(i) if there is only a small-scale velocity field, the growth rate of the large-scale velocity field corresponds to that of a Kolmogorov flow, whereas no magnetic field is generated;

(ii) if there is a small-scale magnetic field only, then the large-scale fields increases fastest for perturbations transverse to the small-scale anisotropic ones;

(iii) finally, if the random anisotropic fields are of equal amplitude, then both large-scale fields grow; again, the growth rate is greatest for large-scale perturbations transverse to the small-scale ones.

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APPENDIX

Derivation of Q_1, Q_2, Q_3

In this Appendix, the terms Q_1, Q_2, Q_3 (see Eqs. (2.7)–(2.9)) are expressed in terms of the mean components $\overline{\psi}$, $\overline{a}$ and space-time spectral functions of the fluctuating components ψ^T, a^T obtained from Eqs. (2.10) and (2.11). Taking the approach outlined in Sec. 2, we introduce slow and fast spatial variables $\mathbf{X}$ and $\mathbf{x}$. The spatial operators are then written in the form

$$\begin{gathered}\nabla \to \frac{\partial}{\partial \mathbf{x}} + \frac{\partial}{\partial \mathbf{X}}, \\ \Delta \to \Delta + 2\frac{\partial^2}{\partial x_p \partial X_p} + \Delta_s, \\ \Delta^2 \to \Delta^2 + 2\Delta\Delta_s + 4\frac{\partial^2}{\partial x_p \partial X_p}\Delta + 4\frac{\partial^2}{\partial x_p \partial X_p}\Delta_s + 4\frac{\partial^4}{\partial x_p \partial X_p \partial x_r \partial X_r}.\end{gathered} \tag{A.1}$$

We start by deriving Q_3 since it has a simpler form than Q_1 and Q_2. According to Eqs. (2.9) and (A.1), this term can be written

$$Q_3 = -\left\langle \left[\nabla\psi^T \times \nabla a^T\right]_z \right\rangle = Q_3^{(00)} + Q_3^{(01)} + Q_3^{(10)} + Q_3^{(11)} + Q_3^{(02)} + Q_3^{(20)} + O(K^3, K^4 \ldots), \quad (A.2)$$

where

$$Q_3^{(00)} = -\varepsilon_{mn} \left\langle \frac{\partial\psi^{(0)}}{\partial x_m} \frac{\partial a^{(0)}}{\partial x_n} \right\rangle = 0$$

due to homogeneity of the turbulence,

$$Q_3^{(01)} = -\varepsilon_{mn} \left\langle \frac{\partial\psi^{(0)}}{\partial x_m} \left(\frac{\partial}{\partial x_n} + \frac{\partial}{\partial X_n} \right) a^{(1)} \right\rangle,$$

$$Q_3^{(10)} = -\varepsilon_{mn} \left\langle \left(\frac{\partial}{\partial x_m} + \frac{\partial}{\partial X_m} \right) \psi^{(1)} \frac{\partial a^{(0)}}{\partial x_n} \right\rangle, \quad (A.3)$$

and the remaining terms in Eq. (A.2) have a similar structure, which is now obvious. We retain only those terms in Eq. (A.2), that are of order K, K^2. As will be seen below, it is just these terms that give rise to negative magnetic viscosity.

To calculate the terms in Eq. (A.2), it is necessary to derive expressions for $\psi^{(i)}$ and $a^{(i)}$, $i = 0, 1, 2$. In the zeroth approximation, Eqs. (2.10) and (2.11) yield

$$\left(\frac{\partial}{\partial t} - \nu\Delta \right) \Delta\psi^{(0)} = F_\psi(\mathbf{x}, t),$$

$$\left(\frac{\partial}{\partial t} - \eta\Delta \right) a^{(0)} = F_a(\mathbf{x}, t). \quad (A.4)$$

Before solving the equations in various orders of approximation, we introduce correlation functions and Fourier spectra of the zero-order fields. Since we assume joint homogeneity and stationarity of the small-scale fields, we have

$$C_{\psi\psi}(\mathbf{x} - \mathbf{x}', t - t') \equiv \left\langle \psi^{(0)}(\mathbf{x}, t) \psi^{(0)}(\mathbf{x}', t') \right\rangle,$$

$$C_{aa}(\mathbf{x} - \mathbf{x}', t - t') \equiv \left\langle a^{(0)}(\mathbf{x}, t) a^{(0)}(\mathbf{x}', t') \right\rangle, \quad (A.5)$$

$$C_{\psi a}(\mathbf{x} - \mathbf{x}', t - t') \equiv \left\langle \psi^{(0)}(\mathbf{x}, t) a^{(0)}(\mathbf{x}', t') \right\rangle.$$

Using the Fourier transform over the fast variables $\mathbf{x}$, t,

$$\psi^{(0)}(\mathbf{x}, t) = \int \frac{d\mathbf{k}\, d\omega}{(2\pi)^3} \hat{\psi}^{(0)}(\mathbf{k}, \omega) \exp(-i\omega t + i\mathbf{k}\mathbf{x}), \quad (A.6)$$

the corresponding space-time spectral functions are defined as

$$C_{\psi a}(\boldsymbol{\kappa}, \tau) = \int \frac{d\mathbf{k}\, d\omega}{(2\pi)^3} \hat{C}^{(0)}_{\psi a}(\mathbf{k}, \omega) \exp(-i\omega\tau + i\mathbf{k}\boldsymbol{\kappa}). \quad (A.7)$$

Since the small-scale fluctuations are stationary and homogeneous,

$$\left\langle \hat{\psi}^{(0)}(\mathbf{k}, \omega) \hat{a}^{(0)}(\mathbf{k}', \omega') \right\rangle = (2\pi)^3 \delta(\omega + \omega') \delta(\mathbf{k} + \mathbf{k}') \hat{C}_{\psi a}(\mathbf{k}, \omega). \quad (A.8)$$

The properties of the spectral functions are

$$\begin{aligned} &\text{(i)}\quad \hat{C}^{*}_{\psi\psi}(\mathbf{k},\omega)=\hat{C}_{\psi\psi}(\mathbf{k},\omega)=\hat{C}_{\psi\psi}(-\mathbf{k},-\omega),\\ &\text{(ii)}\quad \hat{C}^{*}_{\psi a}(\mathbf{k},\omega)=\hat{C}_{\psi a}(-\mathbf{k},-\omega)=\hat{C}_{a\psi}(\mathbf{k},\omega), \end{aligned} \tag{A.9}$$

where the asterisk denotes the complex conjugate. The properties (ii) stem from the reality of $C_{\psi a}(\boldsymbol{\kappa},\tau)$ and the condition

$$C_{\psi a}(\boldsymbol{\kappa},\tau)=C_{a\psi}(-\boldsymbol{\kappa},-\tau).$$

It is also useful to introduce the spectra of the fields $\mathbf{v}^{(0)}(x,t)$ and $\mathbf{B}^{(0)}(x,t)$ for the Fourier components

$$\begin{aligned} \hat{v}_i(\mathbf{k},\omega)&=-i\varepsilon_{ij}k_j\hat{\psi}(\mathbf{k},\omega),\\ \hat{B}_i(\mathbf{k},\omega)&=-i\varepsilon_{ij}k_j\hat{a}(\mathbf{k},\omega), \end{aligned} \tag{A.10}$$

where ε_{ij} is the unit antisymmetric tensor of the second rank. Since

$$\left\langle \hat{v}_i^{(0)}(\mathbf{k},\omega)\hat{v}_j^{(0)}(\mathbf{k}',\omega')\right\rangle=(2\pi)^3\delta(\omega+\omega')\delta(\mathbf{k}+\mathbf{k}')\langle v_i^{(0)}v_j^{(0)}\rangle_{\mathbf{k},\omega}, \tag{A.11}$$

where $\langle v_i^{(0)}v_j^{(0)}\rangle_{\mathbf{k},\omega}$ is the space-time spectral tensor of the zero-order velocity field, we obtain with the help of Eqs. (A.10) and (A.11)

$$\begin{aligned} \left\langle\left(\mathbf{v}^{(0)}\right)^2\right\rangle_{\mathbf{k},\omega}&=k^2\hat{C}_{\psi\psi}(\mathbf{k},\omega),\\ \left\langle\left(\mathbf{B}^{(0)}\right)^2\right\rangle_{\mathbf{k},\omega}&=k^2\hat{C}_{aa}(\mathbf{k},\omega),\\ \left\langle\left(\mathbf{v}^{(0)}\mathbf{B}^{(0)}\right)\right\rangle_{\mathbf{k},\omega}&=k^2\hat{C}_{\psi a}(\mathbf{k},\omega). \end{aligned} \tag{A.12}$$

Now we return to Eqs. (2.10) and (2.11). In the first approximation we have

$$\begin{aligned} &\left(\frac{\partial}{\partial t}-\nu\Delta\right)\Delta\psi^{(1)}+\varepsilon_{mn}\frac{\partial\overline{\psi}}{\partial X_m}\frac{\partial}{\partial x_n}\Delta\psi^{(0)}=\varepsilon_{mn}\frac{\partial\overline{a}}{\partial X_m}\frac{\partial}{\partial x_n}\Delta a^{(0)},\\ &\left(\frac{\partial}{\partial t}-\eta\Delta\right)a^{(1)}+\varepsilon_{mn}\frac{\partial\psi}{\partial x_m}\frac{\partial\overline{a}}{\partial X_n}+\varepsilon_{mn}\frac{\partial\overline{\psi}}{\partial X_m}\frac{\partial a^{(0)}}{\partial x_n}=0. \end{aligned} \tag{A.13}$$

Taking the Fourier transform, we obtain

$$\begin{aligned} \hat{\psi}^{(1)}(\mathbf{k},\omega)&=\varepsilon_{mn}\frac{ik_n}{-i\omega+\nu k^2}\left(\frac{\partial\overline{a}}{\partial X_m}\hat{a}^{(0)}(\mathbf{k},\omega)-\frac{\partial\overline{\psi}}{\partial X_m}\hat{\psi}^{(0)}(\mathbf{k},\omega)\right),\\ \hat{a}^{(1)}(\mathbf{k},\omega)&=\varepsilon_{mn}\frac{ik_n}{-i\omega+\eta k^2}\left(\frac{\partial\overline{a}}{\partial X_m}\hat{\psi}^{(0)}(\mathbf{k},\omega)-\frac{\partial\overline{\psi}}{\partial X_m}\hat{a}^{(0)}(\mathbf{k},\omega)\right). \end{aligned} \tag{A.14}$$

In the second approximation we have from Eqs. (2.10) and (2.11)

$$\left(\frac{\partial}{\partial t} - \nu\Delta\right)\Delta\psi^{(2)} + 2\frac{\partial^3\psi^{(1)}}{\partial t\,\partial x_p\partial X_p} - 4\nu\frac{\partial^2}{\partial x_p\partial X_p}\Delta\psi^{(1)} + \varepsilon_{mn}\frac{\partial\overline{\psi}}{\partial X_m}\frac{\partial}{\partial x_n}\Delta\psi^{(1)} =$$
$$= \varepsilon_{mn}\frac{\partial\overline{a}}{\partial X_m}\frac{\partial}{\partial x_n}\Delta a^{(1)},$$
$$\left(\frac{\partial}{\partial t} - \eta\Delta\right)a^{(2)} - 2\eta\frac{\partial^2 a^{(1)}}{\partial x_p\partial X_p} + \varepsilon_{mn}\frac{\partial\psi^{(1)}}{\partial x_m}\frac{\partial\overline{a}}{\partial X_n} + \varepsilon_{mn}\frac{\partial\overline{\psi}}{\partial X_m}\frac{\partial a^{(1)}}{\partial x_n} = 0. \qquad \text{(A.15)}$$

Taking the Fourier transform, we obtain

$$\hat{\psi}^{(2)}(\mathbf{k},\omega) = 2\frac{\omega k_p}{k^2}\frac{1}{-i\omega+\nu k^2}\frac{\partial\hat{\psi}^{(1)}}{\partial X_p} + \frac{4i\nu k_p}{-i\omega+\nu k^2}\frac{\partial\hat{\psi}^{(1)}}{\partial X_p} -$$
$$- \varepsilon_{mn}\frac{ik_n}{-i\omega+\nu k^2}\left(\frac{\partial\overline{\psi}}{\partial X_m}\hat{\psi}^{(1)} - \frac{\partial\overline{a}}{\partial X_m}\hat{a}^{(1)}\right),$$
$$\hat{a}^{(2)}(\mathbf{k},\omega) = \frac{2i\eta k_p}{-i\omega+\eta k^2}\frac{\partial\hat{a}^{(1)}}{\partial X_p} + \varepsilon_{mn}\frac{ik_m}{-i\omega+\eta k^2}\left(\frac{\partial\overline{\psi}}{\partial X_n}\hat{a}^{(1)} - \frac{\partial\overline{a}}{\partial X_n}\hat{\psi}^{(1)}\right). \qquad \text{(A.16)}$$

We do not calculate terms of third and fourth order, since they do not contribute to $Q_{1,2,3}$, as will be seen below. Furthermore, since we are interested in negative viscosity effects (linear in the mean quantities $\overline{\psi}$, $\overline{a}$), we neglect terms nonlinear in $\overline{\psi}$, $\overline{a}$ in $\hat{\psi}^{(2)}$, $\hat{a}^{(2)}$, namely, the last terms on the right-hand side of Eqs. (A.16). These can be taken into account in the same manner as the linear terms, and this was done for the more straightforward case in Ref. [20]. However, in this paper we do not consider nonlinear effects in the mean quantities.

Now we are ready to calculate all terms in Eq. (A.2). Note that the term $Q_3^{(11)}$ need not be taken into account because it is nonlinear in $\overline{\psi}$ and $\overline{a}$. Then, it can easily be seen by explicitly writing the terms $Q_3^{(20)}$ and $Q_3^{(02)}$ that they yield zero to the second order inclusive. Therefore, only the sum $Q_3^{(01)} + Q_3^{(10)}$ has to be evaluated; see Eqs. (A.3). Using Eqs. (A.14) and the properties (A.5)–(A.9) of the zero-order spectral functions, we obtain to order K^2:

$$Q_3^{(01)} = -\varepsilon_{kl}\varepsilon_{mn}\int\frac{d\mathbf{k}\,d\omega}{(2\pi)^3}\frac{k_l k_m}{-i\omega+\eta k^2}\left\{\hat{C}_{\psi\psi}\frac{\partial^2\overline{a}}{\partial X_k\partial X_n} - \hat{C}_{a\psi}\frac{\partial^2\overline{\psi}}{\partial X_k\partial X_n}\right\},$$
$$Q_3^{(10)} = -\varepsilon_{kl}\varepsilon_{mn}\int\frac{d\mathbf{k}\,d\omega}{(2\pi)^3}\frac{k_l k_n}{-i\omega+\nu k^2}\left\{\hat{C}_{aa}\frac{\partial^2\overline{a}}{\partial X_k\partial X_m} - \hat{C}_{\psi a}\frac{\partial^2\overline{\psi}}{\partial X_k\partial X_m}\right\},$$

and then obtain Eq.(2.13) with the right hand side being the sum $Q_3^{(01)} + Q_3^{(10)}$.

Now we calculate

$$Q_1 = -\left\langle\left[\nabla\psi^T\times\nabla\Delta\psi^T\right]_z\right\rangle = Q_1^{(00)} + Q_1^{(01)} + Q_1^{(10)} + Q_1^{(02)} + Q_1^{(20)} + Q_1^{(03)} + Q_1^{(30)} +$$
$$+ Q_1^{(04)} + Q_1^{(40)} + Q_1^{(11)} + Q_1^{(12)} + Q_1^{(21)} + Q_1^{(13)} + Q_1^{(31)} + O(K^5, K^6, \ldots), \qquad \text{(A.17)}$$

where

$$Q_1^{(00)} = -\varepsilon_{mn}\left\langle\frac{\partial\psi^{(0)}}{\partial x_m}\frac{\partial}{\partial x_n}\Delta\psi^{(0)}\right\rangle = 0$$

due to homogeneity of the turbulence, and

$$Q_1^{(01)} = -\varepsilon_{mn}\left\langle \frac{\partial\psi^{(0)}}{\partial x_m}\left(\frac{\partial}{\partial x_n}+\frac{\partial}{\partial X_m}\right)\left(\Delta+2\frac{\partial^2}{\partial x_p\partial X_p}+\Delta_s\right)\psi^{(1)}\right\rangle,$$
$$Q_1^{(10)} = -\varepsilon_{mn}\left\langle\left(\frac{\partial}{\partial x_n}+\frac{\partial}{\partial X_m}\right)\psi^{(1)}\frac{\partial}{\partial x_n}\Delta\psi^{(0)}\right\rangle. \tag{A.18}$$

The remaining terms in Eq. (A.17) have similar structure. We retain only terms of order $K^1,\ldots,K^4$. We naturally expect that among these are terms that give rise to the effective viscosity term in the equation for the mean flow.

As in case of Q_3, we omit all nonlinear terms in $\overline{\psi}$, $\overline{a}$, which can be taken into account on the same basis as the linear terms. Then, starting with expressions similar to those in Eq. (A.18), it can be easily verified that $Q_1^{(03)}+Q_1^{(30)}=0$ to order K^4; the same conclusion holds $Q_1^{(04)}+Q_1^{(40)}$. Thus, we only need to calculate

$$Q_1^{(01)}+Q_1^{(10)}+Q_1^{(02)}+Q_1^{(20)}.$$

It is convenient to divide this sum into two terms, $Q_1^{(01)}+Q_1^{(10)}$ and $Q_1^{(02)}+Q_1^{(20)}$, and evaluate them separately. Using Eqs. (A.14) and the properties (A.5)–(A.9), we have for the first sum

$$Q_1^{(01)}+Q_1^{(10)} = \varepsilon_{kl}\varepsilon_{mn}\left\{\delta^{(1)}_{1lnp}\frac{\partial^3\overline{\psi}}{\partial X_k\partial X_m\partial X_p}+\delta^{(2)}_{1lnp}\frac{\partial^3\overline{a}}{\partial X_k\partial X_m\partial X_p}+\right.$$
$$\left.+\nu^{(1)}_{1ln}\frac{\partial^2}{\partial X_k\partial X_m}\Delta_s\overline{\psi}+\nu^{(2)}_{1ln}\frac{\partial^2}{\partial X_k\partial X_m}\Delta_s\overline{a}\right\}, \tag{A.19}$$

where

$$\delta^{(1)}_{1lnp} = -\int\frac{d\mathbf{k}\,d\omega}{(2\pi)^3}\frac{2ik_lk_nk_p}{-i\omega+\nu k^2}\hat{C}_{\psi\psi},$$
$$\delta^{(2)}_{1lnp} = \int\frac{d\mathbf{k}\,d\omega}{(2\pi)^3}\frac{2ik_lk_nk_p}{-i\omega+\nu k^2}\hat{C}_{a\psi},$$
$$\nu^{(1)}_{1ln} = -\int\frac{d\mathbf{k}\,d\omega}{(2\pi)^3}\frac{k_lk_n}{-i\omega+\nu k^2}\hat{C}_{\psi\psi},$$
$$\nu^{(2)}_{1ln} = \int\frac{d\mathbf{k}\,d\omega}{(2\pi)^3}\frac{k_lk_n}{-i\omega+\nu k^2}\hat{C}_{a\psi}.$$

Using Eqs. (A.16) and the properties of Eqs. (A.5)–(A.9), we also obtain

$$Q_1^{(02)}+Q_1^{(20)} = \varepsilon_{kl}\varepsilon_{mn}\left\{\nu^{(1)}_{1lnpr}\frac{\partial^4\overline{\psi}}{\partial X_k\partial X_m\partial X_p\partial X_r}+\nu^{(2)}_{1lnpr}\frac{\partial^4\overline{a}}{\partial X_k\partial X_m\partial X_p\partial X_r}\right\}, \tag{A.20}$$

where

$$\nu^{(1)}_{1lnpr} = \int\frac{d\mathbf{k}\,d\omega}{(2\pi)^3}\frac{4k_lk_nk_pk_r}{k^2}\left\{\frac{1}{-i\omega+\nu k^2}+\frac{\nu k^2}{(-i\omega+\nu k^2)^2}\right\}\hat{C}_{\psi\psi},$$
$$\nu^{(2)}_{1lnpr} = -\int\frac{d\mathbf{k}\,d\omega}{(2\pi)^3}\frac{4k_lk_nk_pk_r}{k^2}\left\{\frac{1}{-i\omega+\nu k^2}+\frac{\nu k^2}{(-i\omega+\nu k^2)^2}\right\}\hat{C}_{a\psi}.$$

Q_2 is evaluated in exactly the same way as Q_1. As a result, terms with $\delta^{(1)}_{2lnp}$, $\delta^{(2)}_{2lnp}$, $\nu^{(1)}_{2ln}$, $\nu^{(2)}_{2ln}$ appear in the sum $Q_2^{(01)}+Q_2^{(10)}$, which differ from their counterparts in Eq. (A.19) by the interchange of ν and η, $\hat{C}_{\psi\psi}$ and $-\hat{C}_{aa}$, and $\hat{C}_{a\psi}$ and $-\hat{C}_{\psi a}$. Then, for the sum $Q_2^{(02)}+Q_2^{(20)}$ we obtain

$$Q_2^{(02)}+Q_2^{(20)} = \varepsilon_{kl}\varepsilon_{mn}\left\{\nu^{(1)}_{2lnpr}\frac{\partial^4\overline{\psi}}{\partial X_k\partial X_m\partial X_p\partial X_r} + \nu^{(2)}_{2lnpr}\frac{\partial^4\overline{a}}{\partial X_k\partial X_m\partial X_p\partial X_r}\right\}, \quad \text{(A.21)}$$

where

$$\nu^{(1)}_{2lnpr} = -\int\frac{d\mathbf{k}\,d\omega}{(2\pi)^3}\frac{4k_lk_nk_pk_r}{k^2}\frac{\eta k^2}{(-i\omega+\eta k^2)^2}\hat{C}_{aa},$$

$$\nu^{(2)}_{2lnpr} = \int\frac{d\mathbf{k}\,d\omega}{(2\pi)^3}\frac{4k_lk_nk_pk_r}{k^2}\frac{\eta k^2}{(-i\omega+\eta k^2)^2}\hat{C}_{\psi a}.$$

Finally, summing the terms $Q_1^{(01)}+Q_1^{(10)}$, $Q_2^{(01)}+Q_2^{(10)}$, $Q_1^{(02)}+Q_1^{(20)}$, and $Q_2^{(02)}+Q_2^{(20)}$, and introducing $\delta^{(1)}_{lnp}=\delta^{(1)}_{1lnp}+\delta^{(1)}_{2lnp}$, $\delta^{(2)}_{lnp}=\delta^{(2)}_{1lnp}+\delta^{(2)}_{2lnp}$, $\nu^{(1)}_{ln}=\nu^{(1)}_{1ln}+\nu^{(1)}_{2ln}$, etc., we obtain the final expression for the right-hand side of Eq. (2.12).

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