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MULTIPETAL VORTEX STRUCTURES IN TWO-DIMENSIONAL MODELS OF GEOPHYSICAL FLUID DYNAMICS AND PLASMA

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A new class of strongly nonlinear steadily rotating vortices is found. The Hamiltonian contour dynamics is proposed as a new approach for their study in some models of geophysical fluid dynamics and plasma. Using the Euler description as a starting point, we present a systematic procedure to reduce the two-dimensional dynamics of constant-vorticity and constant-density patches to the Hamiltonian dynamics of their contours for various parametrizations of the contour. The special Dirac procedure is used to eliminate the constraints arising in the Hamiltonian formulations with the Lagrangian parametrization of the contour. Numerical estimations illustrating the physical significance of the results and the range of model parameters where these results can be applicable are presented. Possible generalizations of the approach based on the application of the Hamiltonian contour dynamics to nonplanar and $3D$ flows are discussed.

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1. INTRODUCTION

The purpose of this paper is the analytical and numerical study of a new class of strongly nonlinear steadily rotating vortices that can exist in two-dimensional flows with the internal scale similar to the Rossby deformation radius in quasigeostrophic models of geophysical fluid dynamics [1]. We show that these vortices can have a nontrivial multipetal structure and must rotate with comparatively small velocities under the assumption that their characteristic scales are sufficiently large compared to the internal one.

We also present a new approach based on Hamiltonian versions of the contour dynamics. The fact that equations of contour dynamics are strongly nonlinear and genuinely nonlocal gave impetus to the progress and application mainly of numerical methods for their solution [2]. The analytical versions involving small parameters used for deriving and solving the approximate (local) equations of contour dynamics are only applicable in fluid models with an exterior characteristic scale (e.g., the depth of the unperturbed layer [3]) or with an internal one (e.g., the Rossby radius [4]). Because the solution of problems of this type essentially depends on choosing dynamic variables parametrizing the boundary, it is desirable to have a sufficiently flexible formulation of the equations of contour dynamics such that these equations could be easily reformulated from one phase space into another. In using approximate methods, it is important to keep in mind that all the information on the internal symmetry properties responsible for the dynamical individuality of the Hamiltonian system is contained in the Poisson brackets. Thus, in order to prevent the loss of internal symmetry properties of the system, we must use the approximations where one quantity—the Hamiltonian of system—is subjected to these approximations but the original Poisson brackets remain intact. The need to use asymptotic methods is the principal reason for refusing traditional formulations, which are not only incompatible with these requirements but also not infrequently lead to cumbersome and recurrent calculations.

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This paper is organized as follows. In Sec. 2, we construct local Poisson brackets for an incompressible nonuniform fluid. Relying heavily on this result as a fundamental principle, in Sec. 3 we derive a hierarchy of the reduced Poisson brackets specially adapted to the Hamiltonian description of models of the contour dynamics. The contour parametrization plays a decisive role. The occurrence of constraints is the indispensable feature of those Hamiltonian formulations that use the Lagrangian coordinates for this purpose. To eliminate the constraints, Dirac's procedure is used. In Sec. 4, we consider multipetal vortex structures in the Hasegawa–Mima model and the axial model of electronic fluid as examples of models admitting a direct application of the obtained results. We focus our attention on the study of steadily rotating multipetal vortex structures without contour self-intersections. Some numerical estimates and concluding remarks are presented in Sec. 5.

2. POISSON BRACKETS FOR AN INCOMPRESSIBLE NONUNIFORM EULERIAN FLUID

The equations of motion for a nonuniform incompressible fluid are formulated in terms of the Eulerian variables: the mass density ρ, the velocity $\mathbf{v}$, and the pressure p, as

$$\partial_t v_i + v_k \partial_k v_i = -\frac{1}{\rho}\partial_i p + \frac{1}{\rho} f_i, \tag{2.1}$$

$$\partial_t \rho + v_k \partial_k \rho = 0, \tag{2.2}$$

$$\partial_k v_k = 0, \tag{2.3}$$

where $\mathbf{f}$ is the resultant of exterior forces that do not violate conservativeness of the fluid. This means that equations of motion (2.1)–(2.3) preserve the total energy H given by the sum of the kinetic energy T and the potential energy U of the fluid,

$$H = T + U,$$
$$T = \int \frac{\mathbf{v}^2}{2\rho} d\mathbf{x}, \quad U = U[\rho], \tag{2.4}$$

where U is in general an arbitrary functional of the density ρ. For simplicity, we assume that the fluid is unbounded.

We now find the evolution equation for the momentum density $\boldsymbol{\pi} = \rho\mathbf{v}$. Equations (2.1) and (2.2) imply

$$\partial_t \pi_i + v_k(\partial_k \pi_i - \partial_i \pi_k) = \\ = -\partial_i\left(p + \rho\frac{\mathbf{v}^2}{2}\right) + \frac{\mathbf{v}^2}{2}\partial_i \rho + f_i. \tag{2.5}$$

Taking the curl of (2.5) and thereby eliminating the gradient term involving the pressure, we obtain the equation

$$\partial_t \gamma_i = e^{imn}\partial_m\left[e^{nkl} v_k \gamma_l - \frac{\mathbf{v}^2}{2}\partial_n \rho + f_n\right] \tag{2.6}$$

that describes the evolution law for the vorticity of the momentum density $\boldsymbol{\gamma} = \nabla \times \boldsymbol{\pi}$ under the action of exterior conservative forces.

We now show that the equations of motion for the incompressible inhomogeneous fluid reformulated in terms of the momentum density vorticity are Hamiltonian with the local Poisson brackets $\{\gamma_i, \gamma'_k\}$ and $\{\rho, \gamma'_k\}$. First, we compute the Poisson bracket $\{\rho, \gamma'_k\}$. Because the model is expected to be Hamiltonian, we have every reason to write

$$\partial_t \rho = \{\rho, H\} = \\ = \int \left[\{\rho, \gamma'_k\}\frac{\delta T}{\delta \gamma'_k} + \{\rho, \rho'\}\frac{\delta U}{\delta \rho'}\right] d\mathbf{x}'. \tag{2.7}$$

Comparing (2.7) with continuity condition (2.2) leads us to

$$\int \left[\{\rho, \gamma'_k\}\frac{\delta T}{\delta \gamma'_k} + \{\rho, \rho'\}\frac{\delta U}{\delta \rho'}\right] d\mathbf{x}' + v_k \partial_k \rho = 0. \tag{2.8}$$

We next introduce a local term in the integrand using the δ-function and express the velocity components v_l in terms of the functional derivatives $\delta T/\delta\gamma_k$ as

$$v_l = \frac{\delta T}{\delta \pi_l} = \int \frac{\delta T}{\delta \gamma'_k}\frac{\delta \gamma'_k}{\delta \pi_l} d\mathbf{x}' = e^{lki}\partial_k \frac{\delta T}{\delta \gamma_i}, \tag{2.9}$$

which can be directly obtained from (2.4). Upon integrating by parts and after some algebra in (2.8), we obtain

$$\int \frac{\delta T}{\delta \gamma'_k}\left[\{\rho, \gamma'_k\} - e^{kml}\partial_l \rho \partial_m \delta(\mathbf{x} - \mathbf{x}')\right] d\mathbf{x}' + \\ + \int \{\rho, \rho'\}\frac{\delta U}{\delta \rho'} d\mathbf{x}' = 0.$$

This implies that

$$\{\rho, \gamma'_k\} = e^{kml}\partial_l \rho \partial_m \delta(\mathbf{x} - \mathbf{x}'), \quad \{\rho, \rho'\} = 0. \tag{2.10}$$

It remains to compute the Poisson bracket $\{\gamma_i, \gamma'_k\}$. Using the same reasoning as for the density, we can write the equation of motion for the vorticity of the momentum density $\boldsymbol{\gamma}$ as

$$\partial_t \gamma_i = \{\gamma_i, H\} = \int \left[\{\gamma_i, \gamma'_k\}\frac{\delta T}{\delta \gamma'_k} + \\ + \{\gamma_i, \rho'\}\frac{\delta T}{\delta \rho'}\right] d\mathbf{x}' + \{\gamma_i, U\}. \tag{2.11}$$

With the bracket $\{\rho, \gamma'_k\}$ already computed and

$$\frac{\delta T}{\delta \rho} = \frac{1}{2} v_k^2,$$

equation (2.11) can be rewritten as

$$\partial_t \gamma_i = \int \{\gamma_i, \gamma'_k\} \frac{\delta T}{\delta \gamma'_k} d\mathbf{x}' - - e^{iml} \partial_m \left(\frac{1}{2} v_k^2 \partial_l \rho \right) + \{\gamma_i, U\}. \quad (2.12)$$

Comparing (2.12) and (2.6), we obtain

$$\int \{\gamma_i, \gamma'_k\} \frac{\delta T}{\delta \gamma'_k} d\mathbf{x}' - e^{imn} \partial_m \left(e^{nkl} v_k \gamma_l \right) + + \{\gamma_i, U\} - e^{imn} \partial_m f_n = 0.$$

If we introduce the local term $e^{imn}\partial_m \left(e^{nkl} v_k \gamma_l \right)$ into the integral using the δ-function and replace the velocity components v_l in accordance with (2.9), after the integration by parts we obtain

$$\int \frac{\delta T}{\delta \gamma'_k} \left[\{\gamma_i, \gamma'_k\} - e^{ipj} e^{jln} e^{knm} \partial_p \gamma_l \partial_m \delta \left(\mathbf{x} - \mathbf{x}' \right) \right] d\mathbf{x}' + + \{\gamma_i, U\} - e^{imn} \partial_m f_n = 0.$$

This immediately implies that the Poisson bracket for the vector field $\boldsymbol{\gamma}$ and the relation between the exterior force and the potential energy are given by

$$\{\gamma_i, \gamma'_k\} = e^{ipj} e^{jln} e^{knm} \partial_p \gamma_l \partial_m \delta, \quad (2.13)$$

$$\{\gamma_i, U\} = e^{imn} \partial_m f_n. \quad (2.14)$$

We note that the resulting force $\mathbf{f}$ can be found from (2.14) up to a gradient term. This fact is a consequence of the invariance of the equations of motion (2.1)–(2.3) under the transformation

$$p \to p + \phi, \quad f_i \to f_i - \partial_i \phi,$$

where ϕ is an arbitrary function whose choice has no influence on physical implications of the theory. Thus, it follows from (2.14) that no structure other than

$$f_i = \frac{\partial \rho}{\partial x_i} \frac{\delta U}{\delta \rho}$$

is admissible for the external forces in the case where $U = U[\rho]$.

Collecting Eqs. (2.10) and (2.13), we find the complete system of Poisson brackets in the phase space $(\boldsymbol{\gamma}, \rho)$,

$$\{\rho, \rho'\} = 0, \quad (2.15)$$

$$\{\rho, \gamma'_k\} = e^{kml} \partial_l \rho \partial_m \delta, \quad (2.16)$$

$$\{\gamma_i, \gamma'_k\} = e^{ipj} e^{jln} e^{knm} \partial_h \gamma_l \partial_m \delta. \quad (2.17)$$

Therefore, the equations of motion for the incompressible nonuniform fluid corresponding to these Poisson brackets take the form

$$\partial_t \boldsymbol{\gamma} = \{\boldsymbol{\gamma}, H\} = = \nabla \times \left(\left[\boldsymbol{\gamma}, \nabla \times \frac{\delta H}{\delta \boldsymbol{\gamma}} \right] + \frac{\delta H}{\delta \rho} \nabla \rho \right), \quad (2.18)$$

$$\partial_t \rho = \{\rho, H\} = - \left(\nabla \times \frac{\delta H}{\delta \boldsymbol{\gamma}} \right) \cdot \nabla \rho. \quad (2.19)$$

The results obtained in Eqs. (2.15)–(2.19) can be considered as a generalization of the well-known Hamiltonian description of the incompressible homogeneous fluid (see, for example, [5–10]) and are used in what follows as a fundamental principle in constructing a hierarchy of reduced Poisson brackets for various models of contour dynamics.

3. HAMILTONIAN VERSION OF THE CONTOUR DYNAMICS

We begin with a two-dimensional plane flow where the curl of the momentum is normal to the flow plane and hence has the only component

$$\boldsymbol{\gamma} = \{0, 0, \gamma\}, \quad \gamma = \varepsilon^{ik} \partial_i \pi_k, \quad (3.1)$$

where ε^{ik} is the unit antisymmetric tensor (with $\varepsilon^{12} = 1$). In this case, Poisson brackets (2.15)–(2.17) for the incompressible inhomogeneous fluid can be reformulated for the dynamical variables γ and ρ as

$$\{\rho, \rho'\} = 0, \quad (3.2)$$

$$\{\rho, \gamma'\} = \varepsilon^{ki} \partial_i \rho \partial_k \delta \left(\mathbf{x} - \mathbf{x}' \right), \quad (3.3)$$

$$\{\gamma, \gamma'\} = \varepsilon^{ki} \partial_i \gamma \partial_k \delta \left(\mathbf{x} - \mathbf{x}' \right). \quad (3.4)$$

It is well known that two-dimensional dynamics of patches of a constant vorticity and density can be reduced to dynamics of their contours, ignoring the description of the rest of the fluid. However, it is a nontrivial fact that the description of the contour evolution can take various forms depending on the variables used; this deserves attention from both practical and theoretical standpoints.

For simplicity, we consider a single domain G^+ bounded by a closed fluid contour that separates it from the rest of the fluid in an exterior region G^-. Denoting the vorticity and the density inside and outside accordingly as ω^+, ρ^+, and ω^-, ρ^-, we use the respective + and − superscripts for labeling variables in the internal domain G^+ and in the exterior region G^-. Using this notation, we can write the momentum and the mass density as

$$\boldsymbol{\pi} = \rho^+ \mathbf{v}^+ \theta^+ + \rho^- \mathbf{v}^- \theta^-, \quad \rho = \rho^+ \theta^+ + \rho^- \theta^-, \quad (3.5)$$

where θ^+ and θ^- are the mutually complementary substantive functions

$$\theta^+ = \begin{cases} 1 & \text{if } \mathbf{x} \in G^+, \\ 0 & \text{if } \mathbf{x} \in G^-, \end{cases} \qquad \theta^- = \begin{cases} 1 & \text{if } \mathbf{x} \in G^-, \\ 0 & \text{if } \mathbf{x} \in G^+, \end{cases}$$

such that

$$\theta^+ + \theta^- = 1, \quad \theta^+ \theta^- = 0. \quad (3.6)$$

We note that by definition, a substantive θ-function characterizing a fluid domain has the dynamical property

$$\partial_t \theta + v_k \partial_k \theta = 0$$

implying that the corresponding domain moves together with the fluid.

Inserting $\boldsymbol{\pi}$-representation (3.5) in (3.1) yields

$$\gamma = \rho^+ \omega^+ \theta^+ - \rho^- \omega^- \theta^- + \beta, \quad (3.7)$$

where the variable β can be expressed as

$$\beta = \left(\rho^+ v_k^+ - \rho^- v_k^-\right) \varepsilon^{ik} \partial_i \theta^+. \quad (3.8)$$

It is easily seen that β has a δ-functional character and thus describes a vortex sheet whose density is specified by the jump of the tangential momentum across the contour.

As the first step, we transform Poisson brackets (3.2)–(3.4) from the phase space (γ, ρ) into the space of dynamical variables (β, θ^+). In accordance with (3.5), (3.6), and (3.7), we have

$$\rho = \rho^- + \left(\rho^+ - \rho^-\right) \theta^+, \quad (3.9)$$

$$\gamma = \rho^- \omega^- + \left(\rho^+ \omega^+ - \rho^- \omega^-\right) \theta^+ + \beta. \quad (3.10)$$

Depending on the existence of a mass density jump across the contour, insertion of (3.9) and (3.10) into (3.2)–(3.4) leads to two types of Poisson brackets.

3.1. Piecewise-constant vortex models without mass density jumps

We first consider the degenerate case where the mass density jump is absent, and therefore $\rho^+ = \rho^- = \rho_0$. In this case, the vortex sheet density is a constant of motion and its presence modifies the Hamiltonian of the model but has no influence on the Poisson bracket $\{\theta^+, \theta^{+\prime}\}$ that completely determines the contour evolution. Taking this into account, we can set $\beta = 0$ for simplicity of computing. Inserting (3.10) in (3.4), we then obtain

$$\{\theta^+, \theta^{+\prime}\} = \nu^{-1} \varepsilon^{ik} \partial_k \theta^+ \partial_i \delta\left(\mathbf{x} - \mathbf{x}'\right), \quad (3.11)$$

where $\nu = \rho_0 \left(\omega^+ - \omega^-\right)$.

Which of the Hamiltonian versions of contour dynamics follows from (3.11) depends on how we parameterize the substantive θ^+-function. The simplest parameterization can be achieved with the Heaviside function

$$\theta^+ \left(\eta - x_2\right) = \begin{cases} 1 & \text{if } \eta \geq x_2, \\ 0 & \text{if } \eta < x_2, \end{cases}$$

where the variable $\eta = \eta(x_1, t)$ specifies the contour shape. The corresponding version of the Hamiltonian description defined by the Poisson bracket $\{\eta, \eta'\}$ can be derived directly from (3.11) if we use the trivial relation

$$\eta = \int x_2 \frac{d}{dx_2} \theta^+ \left(\eta - x_2\right) dx_2$$

that maps the dynamics in the phase space of $\boldsymbol{\gamma}$ into the phase space of η. After some algebra, we then find

$$\{\eta, \eta'\} = \int x_2 x_2' \frac{d^2}{dx_2 dx_2'} \{\theta^+, \theta^{+\prime}\} dx_2 dx_2' = = -\nu^{-1} \frac{\partial}{\partial x_1} \delta\left(x_1 - x_1'\right).$$

It is noteworthy that the same Poisson bracket characterizes the KdV-type equations. Hamiltonian formulations based on this version of Poisson brackets are preferable for the study of multilayer models [3].

A more general parameterization can be realized when the contour C bounding the domain G^+ is given in the parametric form

$$\mathbf{x} = \hat{\mathbf{x}}(s, t),$$

where s is the contour arc length. The vector $\mathbf{t} = \partial \hat{\mathbf{x}} / \partial s$ tangential to the contour satisfies the normalization condition

$$\mathbf{t}^2 = 1. \quad (3.12)$$

We note that the θ^+-functions admit an analytical representation through the contour integral,

$$\theta^+ = \frac{i}{2\pi} \int\limits_C \frac{\hat{z}_s ds}{z - \hat{z}}, \quad (3.13)$$

where $z = x_1 + ix_2$ and $\hat{z} = \hat{x}_1 + i\hat{x}_2$ are complex-valued notations for the vectors $\mathbf{x} = (x_1, x_2)$ and $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2)$, and i is the imaginary unit. Representation (3.13) can be obtained as a consequence of the Cauchy formula that is well known in the theory of functions of a complex variable. Using another formula [11]

$$\frac{\partial}{\partial \bar{z}} \frac{1}{z} = \pi \delta(\mathbf{x}),$$

the z-derivative of the θ^+-function can be easily calculated from (3.13) as

$$\frac{\partial \theta^+}{\partial \bar{z}} = \frac{i}{2} \int\limits_C \hat{z}_s \delta(\mathbf{x} - \hat{\mathbf{x}})\, ds.$$

With this result, we can find the usual and variational derivatives of the θ^+-function,

$$\partial_i \theta^+ = \int\limits_C n_i \delta(\mathbf{x} - \hat{\mathbf{x}})\, ds, \tag{3.14}$$

$$\frac{\delta \theta^+}{\delta \hat{x}_i} = -n_i \delta(\mathbf{x} - \hat{\mathbf{x}}), \tag{3.15}$$

where $\mathbf{n}$ is the unit normal vector related to the unit tangent vector $\mathbf{t}$ as $n_i = \varepsilon^{ki} t_k$.

We now find the expression for Poisson bracket (3.11) in the phase space of the dynamic variables $\hat{\mathbf{x}}(s,t)$. We first express the left-hand side of (3.11) in terms of the bracket $\{\hat{x}_i, \hat{x}'_k\}$,

$$\{\theta^+, \theta^{+\prime}\} = \iint\limits_C \frac{\delta \theta^+(\mathbf{x})}{\delta \hat{x}_i(s)} \frac{\delta \theta^+(\mathbf{x}')}{\delta \hat{x}_k(s')} \{\hat{x}_i, \hat{x}'_k\} ds\, ds'.$$

Using (3.15), we obtain

$$\{\theta^+, \theta^{+\prime}\} = \iint\limits_C \delta(\mathbf{x} - \hat{\mathbf{x}})\, \delta(\mathbf{x}' - \hat{\mathbf{x}}') \times \\ \times n_i n'_k \{\hat{x}_i, \hat{x}'_k\} ds\, ds'. \tag{3.16}$$

On the other hand, using (3.14), we can represent the right-hand side of (3.11) as

$$\nu^{-1} \varepsilon^{ik} \partial_k \theta^+ \partial_i \delta(\mathbf{x} - \mathbf{x}') = \\ = \nu^{-1} \iint\limits_C \delta(\mathbf{x} - \hat{\mathbf{x}})\, \delta(\mathbf{x}' - \hat{\mathbf{x}}') \frac{\partial \delta(s - s')}{\partial s} ds ds'. \tag{3.17}$$

Comparing (3.16) and (3.17) yields the integral equality

$$\iint\limits_C \delta(\mathbf{x} - \hat{\mathbf{x}})\, \delta(\mathbf{x}' - \hat{\mathbf{x}}') \times \\ \times [\nu n_i n'_k \{\hat{x}_i, \hat{x}'_k\} - \partial_s \delta(s - s')]\, ds\, ds' = 0,$$

whence it follows that

$$\nu n_i n'_k \{\hat{x}_i, \hat{x}'_k\} = \partial_s \delta(s - s'). \tag{3.18}$$

Because the bracket is skew-symmetric, the general solution of (3.18) for $\{\hat{x}_i, \hat{x}'_k\}$ can be written as

$$\nu \{\hat{x}_i, \hat{x}'_k\} = n_i n'_k \partial_s \delta(s - s') + t_i n'_k a(s, s') - \\ - t'_k n_i a(s', s) + t_i t'_k b(s, s'), \tag{3.19}$$

where $a(s', s)$ and $b(s, s')$ are some structure functions and in addition, $b(s, s')$ must be antisymmetric,

$$b(s, s') = -b(s', s).$$

The choice of the structure functions $a(s', s)$ and $b(s, s')$ cannot be arbitrary but must be matched with constraint (3.12) that means that $\mathbf{t}^2$ is the integral of motion for contour dynamics models with any Hamiltonian. Geometrically, Eq. (3.12) specifies a surface in the phase space $\hat{\mathbf{x}}(s,t)$ such that all the trajectories of real motions lie on this surface. Similar integrals of motion are known as Casimir invariants, or annihilators, of Poisson brackets, i.e., $\{\mathbf{t}^2, \hat{x}'_k\} = 0$. This immediately implies

$$t_i \partial_s \{\hat{x}_i, \hat{x}'_k\} = 0. \tag{3.20}$$

Inserting (3.19) into this condition, we obtain

$$\partial_s a(s, s') = -t_i \frac{\partial n_i}{\partial s} \partial_s \delta(s - s'), \tag{3.21}$$

$$\partial_s b(s, s') = t_i \frac{\partial n_i}{\partial s} a(s', s). \tag{3.22}$$

Solving (3.21) and (3.22) for the structure functions $a(s', s)$ and $b(s, s')$, we find

$$a(s, s') = \frac{\partial}{\partial s'} [\kappa' \sigma(s' - s)],$$

$$b(s, s') = \frac{1}{2} \left(\kappa'^2 + \kappa^2\right) \sigma(s' - s),$$

where

$$\kappa = n_i \partial_s t_i = -t_i \partial_s n_i$$

is the contour curvature and

$$\sigma(s - s') = \frac{1}{2} \operatorname{sign}(s - s').$$

Thus, the Poisson bracket in the phase space $\hat{\mathbf{x}}(s,t)$ is expressible as

$$\{\hat{x}_i, \hat{x}'_k\} = \nu^{-1} \Bigg[n_i n'_k \partial_s \delta(s - s') - \\ - t_i n'_k \frac{\partial}{\partial s'} [\kappa' \sigma(s - s')] + t'_k n_i \frac{\partial}{\partial s} [\kappa \sigma(s' - s)] + \\ + \frac{1}{2} t_i t'_k \left(\kappa'^2 + \kappa^2\right) \sigma(s' - s) \Bigg]. \tag{3.23}$$

Now, the equations of contour dynamics can be written in the Hamiltonian form as

$$\partial_t\hat{x}_i = \{\hat{x}_i, H\} = \nu^{-1}\left[n_i\frac{\partial}{\partial s}\left(n_k\frac{\delta H}{\delta\hat{x}_k}\right) + \right. + t_i\int_C \kappa'\sigma(s-s')\frac{\partial}{\partial s'}\left(n'_k\frac{\delta H}{\delta\hat{x}'_k}\right)ds' + + n_i\frac{\partial}{\partial s}\kappa\int_C\sigma(s'-s)\,t'_k\frac{\delta H}{\delta\hat{x}'_k}ds' + \left. + \frac{1}{2}t_i\int_C\left(\kappa'^2+\kappa^2\right)\sigma(s'-s)\,t'_k\frac{\delta H}{\delta\hat{x}'_k}ds'\right]. \quad (3.24)$$

We emphasize that constraint (3.12) must be used only after all the variational derivatives are taken in (3.24).

In most fluid dynamics models arising commonly in applications, the Hamiltonians are constructed such that

$$t_i\frac{\delta H}{\delta\hat{x}_k} = 0.$$

In this case, Eqs. (3.24) can be presented as

$$n_i\left(\partial_t\hat{x}_i - \nu^{-1}\frac{\partial}{\partial s}\frac{\delta H}{\delta\hat{x}_i}\right) = 0. \quad (3.25)$$

Recalling that in these models

$$\frac{\delta H}{\delta\hat{x}_k} = n_k\nu\hat{\psi},$$

where $\hat{\psi}$ is the streamfunction given on the contour, we obtain from (3.25) the equations of contour dynamics in the traditional form

$$n_i\partial_t\hat{x}_i = \frac{\partial\hat{\psi}}{\partial s}.$$

Equation of motion of this type was used in [4] to derive equation of contour dynamics in the weak-curvature approximation for the Hasegawa–Mima model of plasma.

To eliminate the constraint from the Hamiltonian formulation of the contour dynamics, we introduce two new variables φ and ρ as

$$t_1 = \rho\cos\varphi, \quad t_2 = \rho\sin\varphi, \quad (3.26)$$

where $\varphi(t,s)$ is the inclination angle of the unit tangent vector $\mathbf{t}$ to the axis x_1. In terms of the new variables, constraint (3.12) becomes

$$\rho = 1.$$

Following [12], we define the total Hamiltonian as the superposition

$$H_D = H + \lambda_i I_i$$

involving the original Hamiltonian H and a linear combination of the constraints

$$I_i = \int_C t_i ds = 0$$

with λ_i being some multipliers that must be determined. The constraints of this type are not a prerogative of closed contours for which the identities

$$\int_C t_i ds \equiv \int_C \frac{\partial\hat{x}_i}{\partial s}ds \equiv 0$$

are quite evident. The same constraints are also valid for open contours if we assume that the contours are closed at infinity. In what follows, for simplicity, we consider an open contour C running in the x_1-direction from $-\infty$ to $+\infty$. We note that in the weak-curvature approximation, the descriptions of models with closed and open contours are locally equivalent. In this situation, the results obtained for open contours can be extended to closed ones.

The multipliers λ_i can be determined from the requirement that the equation of motion for the variable φ on the surface of the constraint $\rho = 1$ must be defined by the Poisson bracket $\{\varphi, \varphi'\}$ as

$$\partial_t\varphi = \{\varphi, H_D\} = \int_{-\infty}^{\infty}\{\varphi,\varphi'\}\frac{\delta H_D}{\delta\varphi'}ds'. \quad (3.27)$$

Using the formulas for the variational derivatives

$$\frac{\delta\varphi}{\delta\hat{x}'_i} = \frac{n_i}{\rho^2}\partial_s\delta(s-s'), \qquad \frac{\delta\rho}{\delta\hat{x}'_i} = \frac{t_i}{\rho}\partial_s\delta(s-s'), \quad (3.28)$$

we find that

$$\partial_t\varphi = \{\varphi, H_D\} = \int_{-\infty}^{\infty}\frac{\delta\varphi}{\delta\hat{x}''_i}\{\hat{x}''_i,\hat{x}'_k\}\frac{\delta H_D}{\delta\hat{x}'_k}ds''ds' = = -\int_{-\infty}^{\infty}n_i\frac{\partial\{\hat{x}_i,\hat{x}'_k\}}{\partial s}\frac{\partial}{\partial s'}\left(n'_k\frac{\delta H_D}{\delta\varphi'} + t'_k\frac{\delta H_D}{\delta\rho'}\right)ds'. \quad (3.29)$$

Integration by parts brings Eq. (3.29) to the form

$$\partial_t\varphi = \int_{-\infty}^{\infty} n_i n'_k \frac{\partial^2\{\hat{x}_i, \hat{x}'_k\}}{\partial s \partial s'} \frac{\delta H_D}{\delta \varphi'} ds' - - n_i \frac{\partial\{\hat{x}_i, \hat{x}'_k\}}{\partial s} \left(n'_k \frac{\delta H_D}{\delta \varphi'} + t'_k \frac{\delta H_D}{\delta \rho'} \right) \Bigg|_{-\infty}^{+\infty}. \quad (3.30)$$

Under the assumption that the perturbation on the contour vanishes at infinity, and therefore, φ and its derivatives tend to zero as $s \to \pm\infty$, the last term in (3.30) can be written as

$$n_i \frac{\partial\{\hat{x}_i, \hat{x}'_k\}}{\partial s} \left(n'_k \frac{\delta H_D}{\delta \varphi'} + t'_k \frac{\delta H_D}{\delta \rho'} \right) \Bigg|_{\pm\infty} = = \nu^{-1} \left(\varphi_{sss} + \frac{1}{2}\varphi_s^3 \right) \left(\frac{\delta H}{\delta \rho} \Bigg|_{\pm\infty} + \lambda_1 \right).$$

In accordance with (3.28), we have

$$\{\varphi, \varphi'\} = \int_{-\infty}^{\infty} \frac{\delta \varphi}{\delta \hat{x}''_i} \frac{\delta \varphi'}{\delta \hat{x}'''_k} \{\hat{x}''_i, \hat{x}'''_k\} ds'' ds''' = = n_i n'_k \frac{\partial^2\{\hat{x}_i, \hat{x}'_k\}}{\partial s \partial s'}, \quad (3.31)$$

and it is therefore easy to conclude that Eq. (3.30) can be rewritten in form (3.27) only if the last term in (3.30) can be eliminated. There is no way of doing this except by setting

$$\lambda_1 = - \frac{\delta H}{\delta \rho} \Bigg|_{\pm\infty}.$$

Because the theory is independent of λ_2, this multiplier can be chosen arbitrarily without affecting the equation of motion. For simplicity, we put $\lambda_2 = 0$.

The explicit form of the Poisson bracket $\{\varphi, \varphi'\}$ can be found by inserting Poisson bracket (3.23) in (3.31) and by using the Frenet formulas

$$\partial_s t_i = \kappa n_i, \quad \partial_s n_i = -\kappa t_i, \quad \kappa = \varphi_s. \quad (3.32)$$

By a direct calculation, we obtain

$$\{\varphi, \varphi'\} = -\nu^{-1} \Bigg[\partial_s^3 \delta(s - s') + 2\varphi_s \partial_s (\varphi_s \delta(s - s')) + +\sigma(s - s') \left(\varphi'_s \left(\varphi_{sss} + \frac{1}{2}\varphi_s^3 \right) + \varphi_s \left(\varphi'_{sss} + \frac{1}{2}\varphi_s'^3 \right) \right) \Bigg].$$

Thus, we have obtained the Poisson bracket for one more Hamiltonian version of contour dynamics. The corresponding equation of motion (3.29) can now be written as

$$\partial_t\varphi = \{\varphi, H_D\} = -\nu^{-1} \Bigg[\partial_s^3 \frac{\delta H_D}{\delta \varphi} + 2\varphi_s \partial_s \varphi_s \frac{\delta H_D}{\delta \varphi} + + \left(\varphi_{sss} + \frac{1}{2}\varphi_s^3 \right) \int_{-\infty}^{\infty} \sigma(s - s') \varphi'_s \frac{\delta H_D}{\delta \varphi'} ds' + + \varphi_s \int_{-\infty}^{\infty} \sigma(s - s') \left(\varphi'_{sss} + \frac{1}{2}\varphi_s'^3 \right) \frac{\delta H_D}{\delta \varphi'} ds' \Bigg]. \quad (3.33)$$

Because the constraint $\rho = 1$ can now be imposed directly on the total Hamiltonian H_D before evaluating the Poisson bracket, Dirac's total Hamiltonian is given by

$$H_D = \left[H - \frac{\delta H}{\delta \rho} \Bigg|_{s=\infty} \int_{-\infty}^{\infty} \cos\varphi \, ds \right]_{\rho=1}. \quad (3.34)$$

3.2. Piecewise-uniform models with vorticity and density jumps

When a piecewise-uniform model admits density jumps, i.e., $\rho^+ \neq \rho^-$, the vortex sheet density

$$\mu(s,t) = \left(\rho^- \hat{v}_i^- - \rho^+ \hat{v}_i^+ \right) t_i, \quad \hat{v}_i^\pm = v_i^\pm \big|_{\mathbf{x}=\hat{\mathbf{x}}}$$

is no longer a constant of motion. In this case, the evolution of the contour is therefore defined in the phase space of two variables θ^+ and β, where in accordance with (3.8) and (3.14), β is related to μ as

$$\beta = \left(\rho^+ v_k^+ - \rho^- v_k^- \right) \varepsilon^{ik} \partial_i \theta^+ = = \int_C \mu(s,t) \, \delta(\mathbf{x} - \hat{\mathbf{x}}) \, ds.$$

Inserting (3.9)–(3.10) in (3.2)–(3.4) gives the Poisson brackets

$$\{\theta^+, \theta^{+\prime}\} = 0, \quad (3.35)$$

$$\{\theta^+, \beta'\} = \varepsilon^{ik} \partial_k \theta^+ \partial_i \delta(\mathbf{x} - \mathbf{x}'), \quad (3.36)$$

$$\{\beta, \beta'\} = \nu \varepsilon^{ik} \partial_k \theta^+ \partial_i \delta(\mathbf{x} - \mathbf{x}') + +\varepsilon^{ik} \partial_k \beta \partial_i \delta(\mathbf{x} - \mathbf{x}'), \quad (3.37)$$

where $\nu = \rho^+\omega^+ - \rho^-\omega^-$.

The reformulation of contour dynamics from the (θ^+, β) phase space into the ($\hat{\mathbf{x}}$, μ) phase space is carried out in much the same way as in the previous subsection. Following this procedure, we obtain from (3.35)–(3.37) that the Poisson brackets $\{\hat{x}_i, \hat{x}'_k\}$ and $\{\hat{x}_i, \mu'\}$ satisfy

$$n_i n'_k \{\hat{x}_i, \hat{x}'_k\} = 0, \quad (3.38)$$

$$n_i \partial_s \left[\mu' t'_k \{\hat{x}_i, \hat{x}'_k\}\right] - n_i \{\hat{x}_i, \mu'\} = \partial_s \delta(s - s'), \quad (3.39)$$

$$\partial_s \partial'_s \left[\mu' \mu t_i t'_k \{\hat{x}_i, \hat{x}'_k\}\right] - \partial_s \left[\mu t_i \{\hat{x}_i, \mu'\}\right] - -\partial'_s \left[\mu' t'_i \{\mu, \hat{x}'_i\}\right] + \{\mu, \mu'\} = \nu \partial_s \delta(s - s'). \quad (3.40)$$

Finding the Poisson brackets must be matched with constraint (3.12). As noted above, this constraint means that the quantity $\mathbf{t}^2$ is a Casimir invariant and hence commutes with all the variables making up a basis of the phase space. Therefore, condition (3.20) must be complemented by one more condition

$$t_i \partial_s \{\hat{x}_i, \mu'\} = 0. \quad (3.41)$$

Solving (3.38)–(3.40) with conditions (3.20) and (3.41), we obtain

$$\{\hat{x}_i, \hat{x}'_k\} = 0, \quad (3.42)$$

$$\{\hat{x}_i, \mu'\} = -n_i \partial_s \delta(s - s') + t_i \partial'_s \left[\kappa' \sigma(s - s')\right], \quad (3.43)$$

$$\{\mu, \mu'\} = \nu \partial_s \delta(s - s') + \partial_s \partial'_s \times \times \left[(\kappa' \mu + \kappa \mu') \sigma(s - s')\right]. \quad (3.44)$$

To eliminate the constraints, by analogy with the previous subsection, we introduce two new variables φ and ρ in accordance with (3.26) under the constraint $\rho = 1$. The Poisson brackets on the $(\hat{x}_i, \mu)$ phase space can be easily transformed into the (φ, μ) space. In fact, only the first two brackets (3.42) and (3.43), where the dynamical variables $\hat{x}_i$ appear, must be reformulated. The required formulas can be obtained using (3.28) and take the form

$$\{\varphi, \varphi'\} = n_i n'_k \frac{\partial}{\partial s} \frac{\partial}{\partial s'} \{\hat{x}_i, \hat{x}'_k\}, \quad (3.45)$$

$$\{\varphi, \mu'\} = n_i \frac{\partial}{\partial s} \{\hat{x}_i, \mu'\}. \quad (3.46)$$

Inserting the Poisson bracket in Eqs. (3.42) and (3.43) in (3.45) and (3.46) and using Frenet formulas (3.32), we obtain

$$\{\varphi, \varphi'\} = 0,$$

$$\{\varphi, \mu'\} = -\partial_s^2 \delta(s - s') + \varphi_s \partial'_s \left[\varphi'_s \sigma(s - s')\right],$$

$$\{\mu, \mu'\} = \nu \partial_s \delta(s - s') + +\partial_s \partial'_s \left[(\varphi'_s \mu + \varphi_s \mu') \sigma(s - s')\right].$$

If we restrict our consideration to open contours running from $-\infty$ to $+\infty$ in the x_1-direction, the corresponding Dirac's total Hamiltonian H_D can be determined in the same way as in the previous subsection, with the same result as in Eq. (3.34). Thus, contour dynamics corresponding to a given system of the Poisson brackets is described by the equations

$$\partial_t \varphi = \{\varphi, H_D\} = -\frac{\partial^2}{\partial s^2} \frac{\delta H_D}{\delta \mu} - \varphi_s \int_{-\infty}^{\infty} \varphi'_s \sigma(s - s') \frac{\partial}{\partial s'} \frac{\delta H_D}{\delta \mu'} ds',$$

$$\partial_t \mu = \{\mu, H_D\} = \frac{\partial^2}{\partial s^2} \frac{\delta H_D}{\delta \varphi} + \frac{\partial}{\partial s} \left[\varphi_s \int_{-\infty}^{\infty} \varphi'_s \sigma(s - s') \frac{\delta H_D}{\delta \mu'} ds'\right] + \nu \frac{\partial}{\partial s} \frac{\delta H_D}{\delta \mu} - \frac{\partial}{\partial s} \left[\int_{-\infty}^{\infty} (\varphi'_s \mu + \varphi_s \mu') \sigma(s - s') \frac{\partial}{\partial s'} \frac{\delta H_D}{\delta \mu'} ds'\right].$$

4. N-PETAL STRUCTURES IN TWO-DIMENSIONAL FLUID MODELS

4.1. Hamiltonian formulation of the problem

The simplest models that admit a direct application of the obtained results are a quasigeostrophic barotropic model, a model of plasma based on the Hasegawa–Mima equation, and an axial model of electronic vortices. These models are known [1, 13] to belong to vorticity-like systems governed by the equation

$$\partial_t \omega + (\partial_1 \psi) \partial_2 \omega - (\partial_2 \psi) \partial_1 \omega = 0, \quad (4.1)$$

where the potential vorticity ω and the streamfunction ψ are functions of the x_1 and x_2 coordinates in the horizontal plane and are related by

$$\omega = \left(\Delta - \frac{1}{r^2}\right) \psi,$$

where r is an internal scale treated as the Rossby deformation radius and $\Delta = \partial_1^2 + \partial_2^2$ is the two-dimensional Laplace operator. For the Hasegawa–Mima model, the parameter r is treated as the Larmor ion radius r_L given by

$$r_L = \left(\frac{m_i T_e c^2}{B_0^2 e^2}\right)^{1/2}, \quad (4.2)$$

where m_i is the ion mass, T_e is the electron temperature, e is the electron charge, c is the light velocity, and B_0 is the induction of an ambient uniform magnetic field. The electric potential Φ and the electron number density n_e can be expressed in terms of the streamfunction as

$$\Phi = \frac{B_0}{c}\psi, \quad n_e = n_0 \exp\left(\frac{B_0 e}{T_e c}\psi\right), \tag{4.3}$$

where n_0 is the unperturbed plasma density.

In the axial model of electronic fluid with constant density, the parameter r must be chosen as the skin layer width r_S given by

$$r_S = c\left(\frac{m_e}{4\pi n e^2}\right)^{1/2},$$

where m_e is the electron mass and n is the constant plasma density. In this model, the magnetic field B is related to the streamfunction ψ by

$$B = -\frac{4\pi n e}{c}\psi.$$

It is easy to verify that the vorticity-like models governed by equation of motion (4.1) are Hamiltonian, namely, are characterized by the Poisson bracket of the same type as (3.4),

$$\{\omega, \omega'\} = \varepsilon^{ki}\partial_i\omega\partial_k\delta(\mathbf{x}-\mathbf{x}'),$$

and have the Hamiltonian

$$H = -\frac{1}{2}\int \psi\omega d\mathbf{x}$$

that can be rewritten solely in terms of the potential vorticity as

$$H = -\frac{1}{2}\int \omega\omega' G(\mathbf{x},\mathbf{x}')\, d\mathbf{x}\, d\mathbf{x}'.$$

Green's function G is found as the solution of the problem

$$\left(\Delta - \frac{1}{r^2}\right) G = \delta(\mathbf{x}-\mathbf{x}')$$

and has the explicit form

$$G(\mathbf{x},\mathbf{x}') = -\frac{1}{2\pi}K_0\left(\frac{|\mathbf{x}-\mathbf{x}'|}{r}\right),$$

where K_0 denotes the modified zero-order Bessel function.

As already proved, the reduction of the description of vorticity-like systems in Eq. (4.1) to contour dynamics becomes possible if the entire fluid can be decomposed into domains each of which moves with the fluid and has a constant potential vorticity. For the unbounded fluid with a single vortex patch embedded in a background shear flow, the distribution of the potential vorticity ω can be presented as

$$\omega = \omega^+\theta^+ + \omega^-\theta^-, \quad \theta^+ + \theta^- = 1,$$

where ω^+, θ^+ and ω^-, θ^- have the same meaning as before. The corresponding Hamiltonian is then given by

$$H = -\frac{\nu^2}{2}\int \theta^+\theta^{+\prime} G(\mathbf{x},\mathbf{x}')\, d\mathbf{x}\, d\mathbf{x}', \quad \nu = \omega^+ - \omega^-.$$

After some manipulations, this can be expressed in terms of contour-dynamical variables as

$$\begin{aligned} H &= \frac{(r\nu)^2}{2}\int \theta^+\theta^{+\prime}\left[\delta(\mathbf{x}-\mathbf{x}') - \Delta G\right] d\mathbf{x}\, d\mathbf{x}' = \\ &= \frac{(r\nu)^2}{2}\left(\int \theta^+ d\mathbf{x} + \int G\frac{\partial\theta^+}{\partial x_i}\frac{\partial\theta^{+\prime}}{\partial x_i'} d\mathbf{x} d\mathbf{x}'\right) = \\ &= \frac{(r\nu)^2}{2}\left(\int \theta^+ d\mathbf{x} + \int_C G(\hat{\mathbf{x}},\hat{\mathbf{x}}')\, t_i t_i' ds\, ds'\right). \end{aligned} \tag{4.4}$$

We note that the first integral

$$I = \int \theta^+ d\mathbf{x} = -\frac{1}{2}\int_C \hat{x}_i n_i ds,$$

has a simple geometric meaning of the vortex patch area and is a Casimir invariant (belongs to the annihilator of Poisson bracket (3.23)). Therefore, it does not affect the equation of motion and can be omitted in defining the Hamiltonian. Thus, we obtain from (4.4)

$$H = -\frac{(r\nu)^2}{4\pi}\iint_C K_0\left(\frac{|\hat{\mathbf{x}}-\hat{\mathbf{x}}'|}{r}\right) t_i t_i' ds\, ds'. \tag{4.5}$$

The following analysis is carried out in the weak curvature approximation where the characteristic curvature radius R of the contour is much larger than the internal scale (deformation radius) r, which allows introducing a small parameter $\varepsilon = r/R$. In this case, it is possible to develop the local presentation for the Hamiltonian in Eq. (4.5),

$$H = \int_C h[s;\rho,\varphi]\, ds, \tag{4.6}$$

where the Hamiltonian density h is expressible as a power series in the small parameter ε,

$$h = \frac{(r\nu)^2}{4\pi}\times \left(-\pi r\rho + \frac{3}{8}\pi\frac{r^3}{\rho}\varphi_s^2 - \frac{8}{3}\frac{r^4}{\rho^3}\varphi_s(\rho_s\varphi_s - \rho\varphi_{ss}) + O(\varepsilon^4)\right).$$

Inserting (4.6) in (3.34) and neglecting the fourth-order terms in ε, we find Dirac's Hamiltonian for contour dynamics in vorticity-like systems under consideration,

$$H_D = \frac{r^3\nu^2}{4}\int\limits_C \left(\cos\varphi + \frac{3}{8}r^2\varphi_s^2\right) ds. \tag{4.7}$$

It is interesting to note that because $H \sim O(\varepsilon^2)$, the main contribution to Dirac's Hamiltonian is given solely by the constraint functional. In the leading-order approximation, therefore, Eq. (4.7) becomes

$$H_D = \frac{r^3\nu^2}{4}\int\limits_C \cos\varphi ds. \tag{4.8}$$

In accordance with (3.33), we now obtain the contour dynamics equation

$$\partial_t\varphi = \{\varphi, H_D\} = -\frac{r^3\nu}{4}\left(\varphi_{sss} + \frac{1}{2}\varphi_s^3\right). \tag{4.9}$$

4.2. Steadily rotating localized vortex structures

We consider solutions of Eq. (4.9) that manifest themselves as stationary vortex structures rotating with a constant angular velocity ω_0. These solutions have the form

$$\varphi(t,s) = \tilde\varphi(s - ct) - \omega_0 t, \tag{4.10}$$

where $\omega_0 > 0$ for the clockwise rotation and $\omega_0 < 0$ for the counterclockwise rotation. Inserting (4.10) in (4.9) and choosing the spatial scale R as

$$R = \frac{r}{2}\left(\frac{\nu}{\omega_0}\right)^{1/3}, \tag{4.11}$$

we introduce the dimensionless variables

$$\tilde s = \frac{s - ct}{R}, \quad \tilde\kappa = \frac{\partial\tilde\varphi}{\partial\tilde s}$$

and obtain the equation

$$\left(\frac{\partial\tilde\kappa}{\partial\tilde s}\right)^2 = -\frac{1}{4}\tilde\kappa^4 + c_1\tilde\kappa^2 + \tilde\kappa + c_2, \tag{4.12}$$

where c_2 is an integration constant and $c_1 = c(2\omega_0 R)^{-1}$.

According to the theory of elliptic functions [14], Eq. (4.12) has two sets of periodic solutions expressed in terms of elliptic functions,

$$\tilde\kappa = b + \frac{a - b}{1 - \alpha\,\mathrm{F}(\lambda\tilde{\mathrm{s}}|\mathrm{m})}, \tag{4.13}$$

where F is one of the Jacobi elliptic functions (either sn or dn) and m is the parameter of these functions, with the vertical line symbolizing the m-dependence. We note that depending on the type of the Jacobi elliptic functions, the independent basic parameters α and m parametrize all the others parameters a, b, λ, and consequently, c_1 and c_2.

To derive the equations describing the boundary shape of vortex structures rotating in the horizontal z-plane, we must integrate the equation

$$\frac{\partial\hat z}{\partial\tilde s} = \exp(i\tilde\varphi), \tag{4.14}$$

where $\hat z = (\hat x_1 + i\hat x_2)/R$ is the dimensionless complex coordinate of the contour. It can be directly verified that if $\tilde\kappa$ satisfies (4.12), the solution of (4.14) is given by

$$\hat z(\tilde s) = 2\left[\frac{\partial\tilde\kappa}{\partial\tilde s} + i\left(c_1 - \frac{\tilde\kappa^2}{2}\right)\right]\exp(i\tilde\varphi). \tag{4.15}$$

4.3. Classification of solutions

In this subsection, we focus our attention on the classification of those solutions of Eq. (4.12) that correspond to multipetal vortex structures without self-intersection of the contour. For this purpose, we perform both analytical and numerical investigation of the problem in Eqs. (4.13) and (4.15) restricting our study to the case where F = sn. As becomes apparent after a close examination, the solutions of the second type with F = dn do not contain vortices without contour self-intersections.

With F = sn, the periodic solution for the contour curvature (4.13) takes the form

$$\tilde\kappa = b + \frac{a - b}{1 - \alpha\,\mathrm{sn}(\lambda\tilde{\mathrm{s}}|\mathrm{m})}. \tag{4.16}$$

If the independent parameters α and m are considered as basic, all the other parameters a, b, and λ can be expressed as

$$a = -2^{-1/3}\frac{\alpha\left(1 + m - 2\alpha^2\right)}{\left[(1-m)^2\alpha(m - \alpha^4)\right]^{1/3}},$$

$$b = 2^{-1/3}\frac{\alpha^2 + m(\alpha^2 - 2)}{\alpha\left[(1-m)^2\alpha(m - \alpha^4)\right]^{1/3}},$$

$$\lambda = 2^{-1/3}\frac{\sqrt{(\alpha^2 - m)(1 - \alpha^2)}}{\left[(1-m)^2\alpha(m - \alpha^4)\right]^{1/3}}.$$

The parameters c_1 and c_2 are expressed in terms of a and b as

$$c_1 = \frac{ba}{2} - \frac{1}{a+b}, \quad c_2 = -\frac{1}{4}\left(b + a + b^2a^2\right).$$

We emphasize that the conditions of the contour continuity (smoothness) and reality of solutions to (4.12) impose the following restrictions on the parameters α and m:

$$0 \le \alpha \le 1, \quad m < \alpha^2. \tag{4.17}$$

It follows from (4.15) that in order to find the boundary shape we must know the slope angle $\tilde{\varphi}(\tilde{s})$ in addition to the variable $\tilde{\kappa}$. This can be computed by integrating (4.16) along the contour line,

$$\tilde{\varphi}(\tilde{s}) = \int_0^{\tilde{s}} \tilde{\kappa}(s)ds = b\tilde{s} + \frac{a-b}{\lambda}\Pi\left(\alpha^2; \mathrm{am}\left(\lambda\tilde{s}|m\right)|m\right) - \\ - 2\,\mathrm{Im}\left\{\ln\left[\mathrm{cn}\left(\lambda\tilde{s}|m\right)\sqrt{\alpha^2 - m} + \right.\right. \\ \left.\left. + \,i\mathrm{dn}\left(\lambda\tilde{s}|m\right)\sqrt{1-\alpha^2}\right]\right\}, \tag{4.18}$$

where $\Pi(u;\vartheta|m)$ is the incomplete elliptic integral of the third kind and the Jacobi amplitude $\mathrm{am}(u|m)$ is defined by

$$\mathrm{am}(u|m) = \arcsin\left(\mathrm{sn}(u|m)\right).$$

As mentioned above, our study is restricted to vortex structures with a finite area bounded by a closed contour without self-intersections. It is worth noting that the elimination of self-intersecting contours corresponding to rather exotic vortex formations from the consideration is motivated by the weak-curvature approximation used in deriving Eq. (4.9), but is not at all dictated by intrinsic reasons of fluid dynamics. In other words, the exact equations of motion for the two-dimensional ideal fluid admit the existence of solutions with such a contour topology.

Obviously, considering such contours requires a generalization of model assumptions in the initial statement of the problem. Because the vortex region becomes multiply connected when the contour admits self-intersections, the corresponding piecewise-constant vorticity distribution can be rather specific. If the topology of the contour self-intersection is known, the vorticity distribution can be easily reproduced because the vorticity jump must remain invariant when going around the contour in one of the directions (see Fig. 1). In essence, the question of whether to include solutions of this type into the framework of our scheme is the question of whether a global behavior of solutions is sensitive to a local violation of the weak-curvature approximation. The answer can be found by comparing numerical and analytical solutions. If these solutions are insensitive, they have every ground for being included and can be improved using various numerical procedures similar to the «contour surgery» proposed in [2].

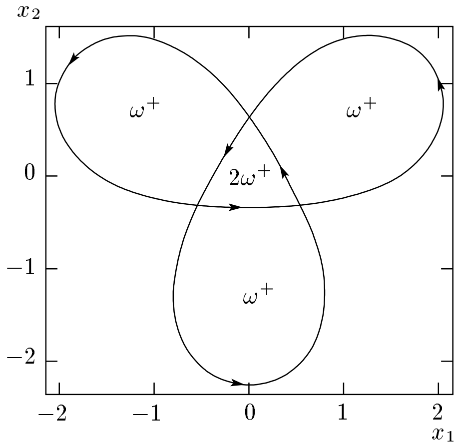

Fig. 1. The geometry of a three-petal vortex region of piecewise-constant vorticity with a selfintersecting contour. The vorticity distribution is ω^+ in petals and $2\omega^+$ in the core, so the jump in vorticity is the invariant ω^+ in tracing the contour

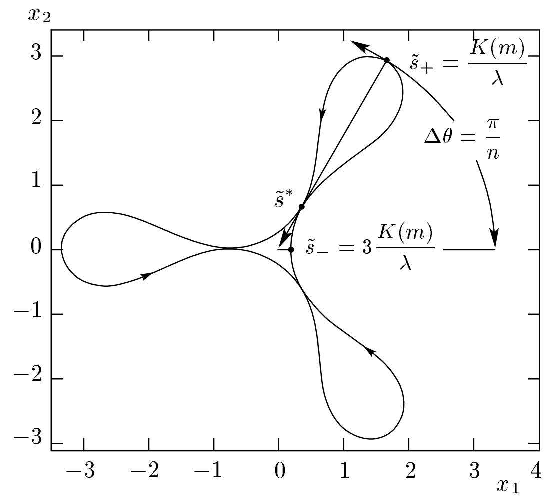

Fig. 2. Three-petal vortex structure. The point $\tilde{s}_+ = K(m)/\lambda$ lies in the petal tip and $\tilde{s}_- = 3K(m)/\lambda$ lies between the petals. $\tilde{s}^*$ is the selfcontacting point of the contour

Because the contour is closed and its curvature is a periodic function of $\tilde{s}$, the boundary shape of the vortices must have an n-petal structure. An example of this structure is given in Fig. 2. From this figure and the analysis of (4.16), it is clear that the contour curvature of the n-petal vortex structure, being an oscillatory function with the period $4K(m)/\lambda$, has extrema at the

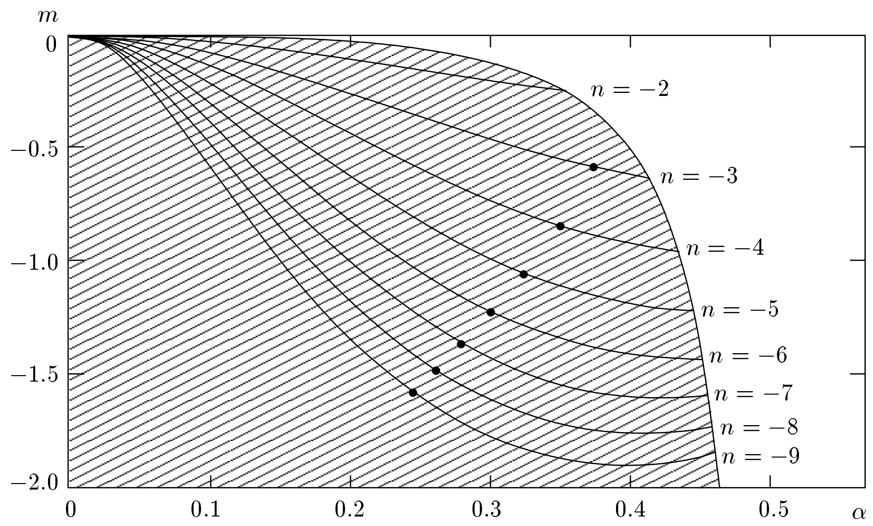

Fig. 3. The family of n-petal vortex regimes in the plane αm. The characteristic curves assign the dependence $m_n(\alpha)$ for $n = -1, -2, \ldots, -9$. The limit points where the corresponding vortex structure has the contour with a self-contact are marked as •

points

$$\tilde{s}_- = (4j-1)\frac{K(m)}{\lambda}, \quad \tilde{s}_+ = (4j-3)\frac{K(m)}{\lambda},$$

$$j = 1, 2, \ldots, n,$$

where $K(m)$ is the complete elliptic integral of the first kind. At these points, the contour curvature takes the extreme values

$$\begin{aligned} \tilde{\kappa}_+ &= b + \frac{a-b}{1-\alpha} = \\ &= 2^{-1/3}\frac{m(\alpha+2) - \alpha(1+2\alpha)}{[(1-m)^2\alpha(m-\alpha^4)]^{1/3}}, \\ \tilde{\kappa}_- &= b + \frac{a-b}{1+\alpha} = \\ &= 2^{-1/3}\frac{m(\alpha-2) - \alpha(1-2\alpha)}{[(1-m)^2\alpha(m-\alpha^4)]^{1/3}}. \end{aligned} \tag{4.19}$$

The subscript notation $\mp$ means that $f_\mp = f(\tilde{s}_\mp)$. The relative position of the turning points $\tilde{s}_-$ and $\tilde{s}_+$ depends on the parameters α and m. To establish which of them is at the tip of the petal and which is in the trough between the petals, it is necessary to compute the distances between these points and the symmetry center (the coordinate origin). For this purpose, we introduce ρ and θ as the polar coordinates,

$$\hat{z}(\tilde{s}) = \rho e^{i\theta}.$$

In accordance with (4.15) and (4.12), the variables ρ and θ are then given by

$$\rho^2 = 4\left(c_1^2 + c_2 + \tilde{\kappa}\right), \tag{4.20}$$

$$\theta = \operatorname{arcctg}\left(\frac{\partial\tilde{\kappa}/\partial\tilde{s}}{c_1 - \tilde{\kappa}^2/2}\right) + \tilde{\varphi}. \tag{4.21}$$

Expressing c_1 and c_2 in terms of α and m and using (4.19), we find from (4.20) that

$$\begin{aligned} \rho_+^2 &= 2^{2/3}\frac{\left[m(1+2\alpha) - \alpha^3(\alpha+2)\right]^2}{\alpha(m-\alpha^4)\left[(1-m)^2\alpha(m-\alpha^4)\right]^{1/3}}, \\ \rho_-^2 &= 2^{-1/3}\frac{\left[m(1-2\alpha) - \alpha^3(\alpha-2)\right]^2}{\alpha(m-\alpha^4)\left[(1-m)^2\alpha(m-\alpha^4)\right]^{1/3}}. \end{aligned}$$

The relative position of the turning points depends on whether 1 is greater or less than the ratio

$$\left(\frac{\rho_-}{\rho_+}\right)^2 = 1 + \frac{8\alpha(\alpha^2-m)(m-\alpha^4)}{[m(1+2\alpha) - \alpha^3(\alpha+2)]^2}.$$

It is easy to see that the inequality $\alpha^4 \le m \le \alpha^2$ entails the inequality $\rho_- \ge \rho_+$; in this interval of the parameters, therefore the tops of the petals lie at the points $\tilde{s}_-$. In the event that $m \le \alpha^4$ (and consequently, the reverse inequality $\rho_- \le \rho_+$ holds), the tips of the petals lie at the points $\tilde{s}_+$.

It is amply clear that in the region of the permissible parameters (4.17), not all solutions (4.16) correspond

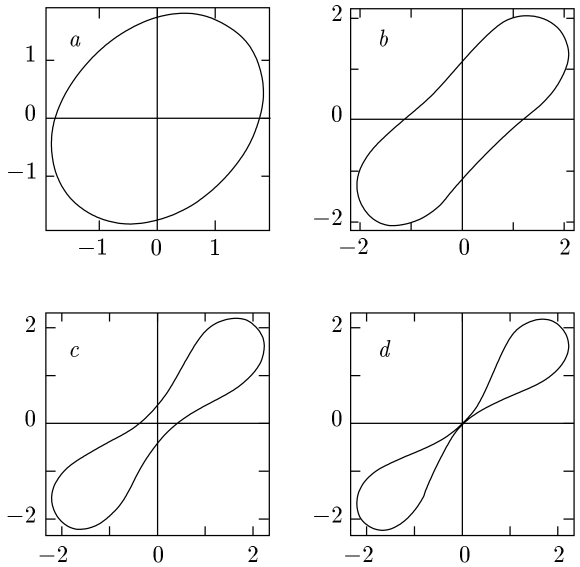

Fig. 4. Shapes of boundaries for double-petal vortex structures: $\alpha = 0.050$ (a), 0.200 (b), 0.300 (c), 0.353 (d)

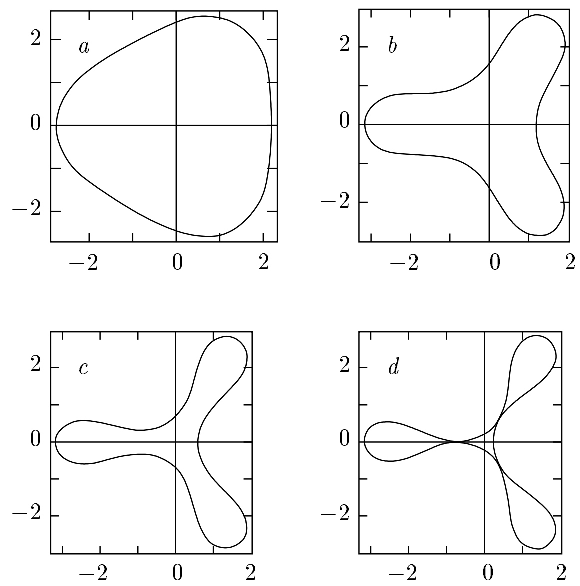

Fig. 5. Shapes of boundaries for three-petal vortex structures: $\alpha = 0.050$ (a), 0.200 (b), 0.300 (c), 0.371 (d)

to vortex structures with closed contours. The condition under which periodic solution (4.16) corresponds to a closed contour can be formulated as

$$\Delta\theta = \theta_- - \theta_+ = \frac{\pi}{n}. \qquad (4.22)$$

This condition has a simple geometrical interpretation shown in Fig. 2. From this figure, it is easy to see that $2\Delta\theta$ is merely the angular distance between neighboring petals. To evaluate its value, it suffices to note that the position vector and the tangent one are mutually orthogonal at the turning points. It thus follows from (4.21) that

$$\theta_\pm = \tilde{\varphi}_\pm + \frac{\pi}{2}\Delta_\pm,$$

where the sign function $\Delta_\pm$ is defined as

$$\Delta_\pm = \operatorname{sign}\left[m(1 \pm 2\alpha) - \alpha^3(\alpha \pm 2)\right]. \qquad (4.23)$$

The expression for $\tilde{\varphi}_\pm$ can be easily found from (4.18) as

$$\tilde{\varphi}_\pm = \frac{4j - 2 \mp 1}{\lambda} \times \left[bK(m) + (a-b)\,\Pi\left(\alpha^2|m\right)\right] - \pi, \qquad (4.24)$$

where

$$\Pi(u|m) = \Pi\left(u; \frac{\pi}{2}|m\right)$$

is the complete elliptic integral of the third kind.

Equations (4.23) and (4.24) allow us to rewrite (4.22) as

$$bK(m) + (a-b)\,\Pi\left(\alpha^2|m\right) = \frac{\pi}{2}\lambda\left(\frac{1}{n} - \Delta\right), \qquad (4.25)$$

where

$$\Delta = \frac{1}{2}(\Delta_- - \Delta_+) = \frac{1}{2}\left\{\operatorname{sign}\left[m(1-2\alpha) - \alpha^3(\alpha-2)\right] - \operatorname{sign}\left[m(1+2\alpha) - \alpha^3(\alpha+2)\right]\right\}.$$

The analysis shows that Eq. (4.25) has solutions in the form of n-petal structures in the region

$$m \le \alpha^3\frac{\alpha - 2}{1 - 2\alpha},$$

where $\Delta = 0$, for $n \le -2$. In Fig. 3, this region is marked by a shaded background. The solutions are presented by the characteristic curves that determine the dependence $m_n(\alpha)$ for every n. For a fixed n, the multipetal structure can therefore be described by a single parameter α. The vortex shapes for $n = 2, 3$ depending on α are shown in Figs. 4 and 5. For every n-petal regime, the characteristic curve has a limit point where the corresponding vortex structure has a self-contacting contour. Solutions without intersections of contours are on the left of the point and those with self-intersections are on the right.

A prerequisite to the formation of a self-contact in a contour can be formulated on the basis of geometrical

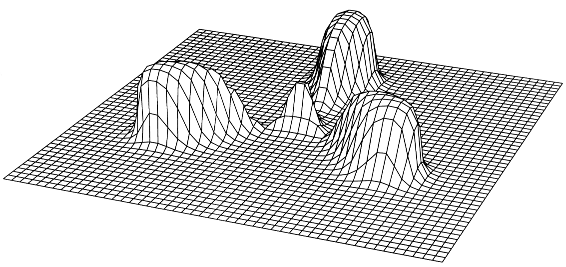

Fig. 6. Surface plot of the streamfunction field for the limiting three-petal vortex structure

considerations following from Fig. 2. At the tangency point $\tilde{s}^*$, the angles θ and $\tilde{\varphi}$ are related by

$$\theta(\tilde{s}^*) = \tilde{\varphi}(\tilde{s}^*).$$

Equation (4.21) now implies the condition

$$\tilde{\kappa}^2 (\tilde{s}^*) = 2c_1.$$

One more condition is obtained by taking into account that in tracing the contour from the point $\tilde{s}_+$ to the tangency point $\tilde{s}^*$, the tangent vector is rotated through $\pi/2$, and therefore

$$\tilde{\varphi}(\tilde{s}^*) - \tilde{\varphi}_+ = \pi/2.$$

Using the relation

$$\tilde{\varphi}_+ = \frac{\pi}{2}\left(\frac{1}{n} - \Delta\right) - \pi,$$

which follows from (4.24) and (4.25) with $j = 1$, we obtain the conditions

$$\tilde{\varphi}(s^*) = \frac{\pi}{2}\left(\frac{1}{n} - \Delta - 1\right),$$

$$\tilde{\kappa}^2 (s^*) = 2^{-2/3} \times \times \frac{2\alpha^4(1+m) + \alpha^2(1+m(m-10)) + 2m(m+1)}{[(1-m)^2\alpha(m-\alpha^4)]^{2/3}}.$$

Together with (4.25), these conditions fix all the parameters of the limiting regimes presented in the Table.

In the quasigeostrophic barotropic model, the physical interpretation of ψ is the pressure deviation, and in the plasma model based on the Hasegawa–Mima equation, this quantity characterizes the electric potential. To illustrate the spatially-temporal character of distributions of ψ, we assume for simplicity that the background vorticity is absent, i.e., $\omega^- = 0$. Using the results obtained in Sec. 4.1, we can then establish the formula

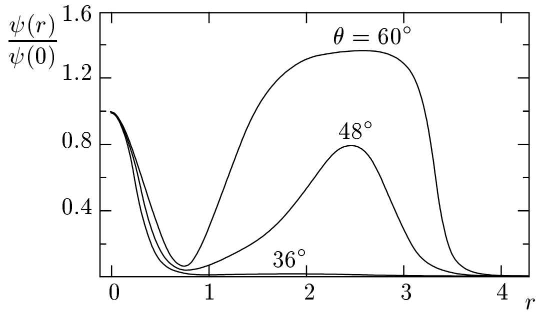

Fig. 7. The radial profile of the streamfunction for the limiting three-petal vortex structure given in Fig. 6. The profiles correspond to the directions $\theta = 60, 48, 36$

$$\psi(\mathbf{x}) = \omega^+ \int \theta^{+\prime} G(\mathbf{x}, \mathbf{x}')\, d\mathbf{x}' = = \omega^+ R^2 \, \mathrm{Im} \int_C \left[\frac{1}{|\hat{z} - z|} - \varepsilon^{-1} K_1\left(\frac{|\hat{z} - z|}{\varepsilon}\right)\right] \times \times \frac{\hat{z}_s(\hat{\bar{z}} - \bar{z})}{|\hat{z} - z|} ds, \quad (4.26)$$

where

$$z = (x_1 + ix_2)/R \quad \text{and} \quad \varepsilon = |r/R|.$$

The distribution $\psi(\mathbf{x})/\psi(0)$ associated with the presence of the three-petal vortex of limiting type is calculated in accordance with (4.26) and is shown in Fig. 6. The radial profiles corresponding to this vortex are presented in Fig. 7.

5. CONCLUDING REMARKS

To gain greater insight into the physical significance of the results and decide in which range of parameters these results can be applicable, we make some estimates for the Hasegawa–Mima model and for the axial model of electronic vortices, in parallel. We note that for these models, the values of the r_L and r_S parameters cover a broad range. According to factual evidence [15], the Larmor ion radius r_L measures 10^3 cm for the interplanetary gas and 10^{-2} cm for the solar corona. Depending on the type of plasma, the skin layer width r_S varies between $5 \cdot 10^5$ and $5 \cdot 10^{-3}$ cm.

To illustrate the obtained results in more detail, we consider the Hasegawa–Mima model of plasma with the parameters $T_e = 10^4$ K, $n_0 = 10^{14}$ cm^{-3}, $B_0 = 10^4$ G,

Table. Values of parameters characterizing the limiting regimes

n	α	m	$\tilde{\kappa}_-$	$\tilde{\kappa}_+$	ρ_-	ρ_+
−2	0.352823	−0.245778	−0.456761	1.79081	0.	2.12018
−3	0.371469	−0.580662	−0.820287	1.95339	0.193635	3.3365
−4	0.348897	−0.844407	−1.01623	2.03108	0.42446	3.51701
−5	0.323504	−1.0545	−1.15832	2.08942	0.635998	3.65998
−6	0.300157	−1.22456	−1.27263	2.13903	0.83048	3.78634
−10	0.231285	−1.66566	−1.60011	2.29932	1.49709	4.22362

and $m_i = 1.67 \cdot 10^{-24}$ g, which are typical for a low-pressure gas discharge. In accordance with (4.2), we find $r_L \approx 10^{-2}$ cm. Because the theory of limiting vortex structures has only two control parameters (the angular rotation velocity ω_0 and the vorticity jump $\nu = \omega^+ - \omega^-$), we put $\omega_0 = 10$ s^{-1}, $\omega^- = 0$, and $\omega^+ = 10^6$ s^{-1} in order to calculate some characteristics of a three-petal drift vortex. In this case, Eq. (4.11) gives $R \approx 10r_L = 10^{-1}$ cm, and therefore, each petal of the vortex structure has the radial length $\rho_+ R \approx 3.3 \cdot 10^{-1}$ cm. Next, upon numerical integration with $\varepsilon = r_L/R \approx 10^{-1}$, we obtain from (4.26) that $\psi(0) \approx 5.07R^2\omega^+$. Thus, we can estimate the magnitudes of the electric potential Φ and the electron number density n_e at the center of the three-petal drift vortex. It follows from (4.3) that $\Phi(0) \approx 4.4 \cdot 10^2$ V and $n_e(0) \approx 1.5 \cdot 10^{16}$ cm^{-3}.

We note, in closing, some possible generalizations of the Hamiltonian versions of $2D$ contour dynamics. The technique that we have described can also be used for $3D$ vortex objects, for example, in quasigeostrophic baroclinic models of geophysical fluid dynamics. The Hamiltonian versions of $2D$ contour dynamics can be successfully applied to the study of nonplanar models in all the cases where the velocity field is invariant along the vorticity field direction. Typical examples are flows on the sphere and also flows with the rotational and helical spatial symmetry of the vortex field.

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