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NON-ABELIAN STOKES THEOREMS IN YANG–MILLS AND GRAVITY THEORIES

D. I. Diakonov[a b *], ***V. Yu. Petrov***[b **]

[a] *NORDITA, DK-2100 Copenhagen Ø, Denmark*

[b] *St. Petersburg Nuclear Physics Institute*
188350, Gatchina, Leningrad region, Russia

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We discuss the interpretation of the non-Abelian Stokes theorem for the Wilson loop in the Yang–Mills theory. For the «gravitational Wilson loops», i. e., holonomies in curved $d = 2, 3, 4$ spaces, we then derive «non-Abelian Stokes theorems» that are similar to our formula in the Yang–Mills theory. In particular, we derive an elegant formula for the holonomy in the case of a constant-curvature background in three dimensions and a formula for small-area loops in any number of dimensions.

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1. INTRODUCTION

One of the main objects in the Yang–Mills theory and in gravity is the parallel transporter along closed contours, or holonomy. In Yang–Mills theory, it is conventionally called the Wilson loop; it can be written as a path-ordered exponential

$$W_r = \frac{1}{d(r)}\,\mathrm{Tr}\ \mathrm{P}\ \exp\left(i\oint d\tau\,\frac{dx^\mu}{d\tau}\,A^a_\mu\,T^a\right), \qquad (1)$$

where $x^\mu(\tau)$ with $0 \le \tau \le 1$ parameterizes the closed contour, A^a_μ is the Yang–Mills field (or connection) and T^a are the gauge group generators in a given representation r whose dimension is $d(r)$. For d-dimensional vectors in curved Riemannian spaces, the «gravitational Wilson loop», or holonomy, can also be written as a trace of the path-ordered exponential of the connection given by the Christoffel symbol,

$$W^G_{vector} = \frac{1}{d}\left[\mathrm{P}\ \exp\left(-\oint d\tau\,\frac{dx^\mu}{d\tau}\,\Gamma_\mu\right)\right]^\kappa_\kappa. \qquad (2)$$

One can also consider parallel transporters of spinors in a curved background: the holonomy is then defined not by the Christoffel symbols, but by the spin connection that is not uniquely determined by the metric tensor (see the precise definitions below).

The Yang–Mills Wilson loop is invariant under gauge transformations of the background field A_μ; the gravitational Wilson loop is invariant under general coordinate transformations, or diffeomorphisms, provided the contour is transformed as well.

It is generally believed that in three and four dimensions, the average of the Wilson loop in a pure Yang–Mills quantum theory exhibits the area-law behaviour for large and simple (e. g., flat rectangular) contours. This must be true not for all representations, but only those with a nonzero «N-ality»; in the simplest case of the $SU(2)$ gauge group, these are the representations with a half-integer spin J.

One of the difficulties in proving the area law for the Wilson loop is that it is a complicated object by itself: it is impossible to compute it analytically in a general non-Abelian background field, not to mention averaging it over an ensemble of configurations.

A decade ago, we suggested a formula for the Wilson loop in a given background belonging to any gauge group and any representation [1]. In this formula, the path ordering along the loop is removed at the price of an additional integration over all gauge transformations of the given non-Abelian background field, or more precisely, over a coset depending on the particular representation in which the Wilson loop is considered. Furthermore, the Wilson loop can be presented in the form of a surface integral [2], see the next

*E-mail: diakonov@nordita.dk
**E-mail: victorp@thd.pnpi.spb.ru

section. We call this representation the non-Abelian Stokes theorem. It is quite different from previous interesting statements [3–6] that were also referred to as «non-Abelian Stokes theorem» but which involved surface ordering. Our formula has no surface ordering. A classification of «non-Abelian Stokes theorems» for arbitrary groups and their representations was recently given by Kondo et al. [7] who used the naturally arising techniques of flag manifolds.

Although these formulas do not usually facilitate finding Wilson loops in particular backgrounds, they can be used in averaging Wilson loops over ensembles of Yang–Mills configurations or over different metrics, and in more general settings, see, e.g., [7–11].

The main aim of this paper is to present new formulas for the gravitational holonomies in curved $d = 2, 3, 4$ spaces; these formulas are similar to our non-Abelian Stokes theorem for the Yang–Mills case. We eliminate the path ordering in Eq. (2) and write the holonomies as exponentials of surface integrals. Instead of path-ordering, we must integrate over certain covariantly unit vectors (for $d = 3$) or covariantly unit (anti)self-dual tensors (for $d = 4$). Remarkably, these formulas put parallel transporters of different spins on the same footing. In particular, holonomies for half-integer spins are presented in terms of the metric tensor (and its derivatives) only, but not in terms of the vielbein or the spin connection.

In addition to a purely theoretical interest, we have a practical motivation in mind. Recently, it was shown, both in the continuum and on the lattice, that the $SU(2)$ Yang–Mills partition function in $d = 3$ can be exactly rewritten in terms of local gauge-invariant quantities given by the six components of the dual space metric tensor. This rewriting can be useful in directly investigating the spectrum and the correlation functions of the theory in a gauge-invariant way, but it is insufficient to study the interactions of external sources because these couple to the Yang–Mills potential and not to gauge-invariant quantities. The present paper demonstrates, however, that a typical source, i.e., the Yang-Mills Wilson loop, can be expressed not only through the potential (or connection) but also through the metric tensor, which is gauge-invariant. Thus, not only the partition function, but also the Wilson loops in the $d = 3$ Yang–Mills theory can be expressed through local gauge-invariant quantities. A detailed formulation of the resulting theory is given elsewhere.

Although the main content of the paper is the non-Abelian Stokes theorems for holonomies in 3 and 4 dimensions, we add three short sections with relevant material. For completeness, we add the Stokes theorem in two dimensions, compute the holonomy in the special case of a constant curvature with a cylinder topology in three dimensions, and give a general formula for the «gravitational Wilson loop» for small loops in any number of dimensions.

2. NON-ABELIAN STOKES THEOREM IN THE YANG–MILLS THEORY

We let τ parameterize the loop defined by the trajectory $x^\mu(\tau)$ and let $A(\tau)$ be the tangent component of the Yang–Mills field along the loop in the fundamental representation of the gauge group,

$$A(\tau) = A^a_\mu t^a \frac{dx^\mu}{d\tau}, \quad \mathrm{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab}.$$

Gauge transformations of $A(\tau)$ are given by

$$A(\tau) \to S(\tau)A(\tau)S^{-1}(\tau) + iS(\tau)\frac{d}{d\tau}S^{-1}(\tau). \qquad (3)$$

Let H_i be the Cartan subalgebra generators ($i = 1, \ldots, r$, where r is the rank of the gauge group) and the r-dimensional vector $\mathbf{m}$ be the highest weight of the representation r in which the Wilson loop is considered. The formula for the Wilson loop derived in Ref. [1] is a path integral over all gauge transformations $S(\tau)$ that are periodic along the contour:

$$W_r = \int DS(\tau) \times \times \exp\left(i\int d\tau \ \mathrm{Tr}\left[m_i H_i (SAS^{-1} + iS\dot{S}^{-1})\right]\right). \qquad (4)$$

We stress that Eq. (4) is manifestly gauge invariant, as is the Wilson loop itself. For example, in the simple case of the $SU(2)$ group, Eq. (4) becomes

$$W_J = \int DS(\tau) \times \times \exp\left(i\,J\int d\tau \ \mathrm{Tr}\left[\tau_3(SAS^\dagger + iS\dot{S}^\dagger)\right]\right), \qquad (5)$$

where τ_3 is the third Pauli matrix and $J = 1/2$, 1, $3/2, \ldots$ is the «spin» of the representation of the Wilson loop considered.

The path integrals over all gauge rotations in Eqs. (4) and (5) are not of the Feynman type: they do not contain terms quadratic in the derivatives in τ. A certain regularization of these equations is therefore implied ensuring that $S(\tau)$ is sufficiently smooth. For example, one can introduce quadratic terms in the angular velocities $iS\dot{S}^\dagger$ with small coefficients eventually set equal to zero;

see Ref. [1] for details. Equation (5) was derived in Ref. [1] in two independent ways: i) by a direct discretization and ii) by using the standard Feynman representation of path integrals as a sum over all intermediate states, in this case for the axial top supplemented by an action of the «Wess–Zumino» type. Another discretization leading to the same result was recently used by Kondo [7]. A similar formula has been used by Alekseev, Faddeev, and Shatashvili [16] in deriving a formula for group characters to which the Wilson loop is reduced for a constant A field (which is the case actually considered in [16]). In Ref. [17], Eq. (4) was rederived in an independent way specifically for the fundamental representation of the $SU(N)$ gauge group. Finally, another derivation of a variant of Eq. (5) using lattice regularization was recently given in Ref. [18].

The second term in the exponent in Eqs. (4) and (5) is in fact a «Wess–Zumino»-type action, and it can be rewritten not as a line but as a surface integral associated with a closed contour. For simplicity, we consider the $SU(2)$ gauge group and parameterize the $SU(2)$ matrix S in Eq. (5) by Euler's angles,

$$S = \exp\left(i\frac{\gamma\tau_3}{2}\right)\exp\left(i\frac{\beta\tau_2}{2}\right)\exp\left(i\frac{\alpha\tau_3}{2}\right) = \begin{pmatrix} \cos\frac{\beta}{2}\exp\left(i\frac{\alpha+\gamma}{2}\right) & \sin\frac{\beta}{2}\exp\left(-i\frac{\alpha-\gamma}{2}\right) \\ -\sin\frac{\beta}{2}\exp\left(i\frac{\alpha-\gamma}{2}\right) & \cos\frac{\beta}{2}\exp\left(-i\frac{\alpha+\gamma}{2}\right) \end{pmatrix}. \quad (6)$$

The derivation of Eq. (5) implies that $S(\tau)$ is a periodic matrix. This means that $\alpha \pm \gamma$ and β are periodic functions of τ with the period 4π.

The second term in the exponent in Eq. (5), which we denote by Φ, is then

$$\Phi = \int d\tau\ \mathrm{Tr}(\tau_3 iS\dot{S}^\dagger) = \int d\tau\,(\dot\alpha\cos\beta + \dot\gamma) = \int d\tau\,[\dot\alpha(\cos\beta - 1) + (\dot\alpha + \dot\gamma)] = \int d\tau\,\dot\alpha(\cos\beta - 1). \quad (7)$$

The last term is a total derivative and can be actually dropped because $\alpha + \gamma$ is 4π-periodic, and therefore, does not contribute to Eq. (5) even for half-integer representations J. We note that α can be 2π-periodic if γ (which drops from Eq. (7)) is 2π-, 6π-, ...-periodic. If $\alpha(1) = \alpha(0) + 2\pi k$, $\alpha(\tau)$ makes k windings. The integration over all possible $\alpha(\tau)$ implied in Eq. (5) can be divided into distinct sectors with different winding numbers k.

Introducing a unit 3-vector

$$n^a = \frac{1}{2}\,\mathrm{Tr}\,(S\tau^a S^\dagger\tau_3) = (\sin\beta\cos\alpha, \sin\beta\sin\alpha, \cos\beta), \quad (8)$$

we can rewrite Φ as

$$\Phi = \frac{1}{2}\int d\tau d\sigma\,\epsilon^{abc}\,\epsilon_{ij}n^a\partial_i n^b\partial_j n^c, \quad i,j = \tau,\sigma, \quad (9)$$

where we integrate over any spanning surface for the contour (we call it a «disk»), and $\mathbf{n}$ or α and β are continued to the interior of the disk without singularities. We denote the second coordinate by σ such that $\sigma = 1$ corresponds to the edge of the disk coinciding with the contour and $\sigma = 0$ corresponds to the center of the disk. See Ref. [18] for the details on the continuation to the interior of the disk.

We note that if the surface is closed or infinite, the right-hand side of Eq. (9) is the integer topological charge of the $\mathbf{n}$ field on the surface,

$$Q = \frac{1}{8\pi}\int d\sigma d\tau\,\epsilon^{abc}\epsilon_{ij}n^a\partial_i n^b\partial_j n^c. \quad (10)$$

Equation (9) can also be rewritten in the form that is invariant under surface reparameterizations. Introducing the invariant surface element

$$d^2S^{\mu\nu} = d\sigma\,d\tau\left(\frac{\partial x^\mu}{\partial\tau}\frac{\partial x^\nu}{\partial\sigma} - \frac{\partial x^\nu}{\partial\tau}\frac{\partial x^\mu}{\partial\sigma}\right) = \epsilon^{\mu\nu}d(\mathrm{Area}), \quad (11)$$

we can rewrite Eq. (9) as

$$\Phi = \frac{1}{2}\int d^2S^{\mu\nu}\epsilon^{abc}n^a\partial_\mu n^b\partial_\nu n^c. \tag{12}$$

For the Wilson loop, we then obtain [1]

$$W_J = \int D\mathbf{n}(\sigma,\tau)\exp\left[iJ\int d\tau(A^a n^a) + \frac{iJ}{2}\int d^2S^{\mu\nu}\epsilon^{abc}n^a\partial_\mu n^b\partial_\nu n^c\right]. \tag{13}$$

The interpretation of this formula is obvious: the unit vector $\mathbf{n}$ plays the role of the instant direction of the colour «spin» in the color space. However, multiplying its length by J does not guarantee that we deal with a true quantum state from the representation labelled by J; this is achieved only by introducing the «Wess–Zumino» term in Eq. (13) that fixes the representation to which the probe quark of the Wilson loop belongs to be exactly J.

Finally, we can rewrite the exponent in Eq. (13) such that both terms appear to be surface integrals [2],

$$W = \int D\mathbf{n}(\sigma,\tau)\exp\left[\frac{iJ}{2}\int d^2S^{\mu\nu}\left(-F^a_{\mu\nu}n^a + \epsilon^{abc}n^a\left(D_\mu n\right)^b\left(D_\nu n\right)^c\right)\right], \tag{14}$$

where

$$D^{ab}_\mu = \partial_\mu\delta^{ab} + \epsilon^{acb}A^c_\alpha$$

is the covariant derivative and

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc}A^b_\mu A^b_\nu$$

is the field strength. Indeed, expanding the exponent in Eq. (14) in powers of A_μ, we observe that the quadratic term cancels while the linear term is a total derivative reproducing the $A^a n^a$ term in Eq. (13); the zero-order term is «Wess–Zumino» term (9) or (7). We note that both terms in Eq. (14) are explicitly gauge invariant. We call Eq. (14) the non-Abelian Stokes theorem. We stress that it is different from the previously proposed Stokes-like representations of the Wilson loop, based on ordering elementary surfaces inside the loop [3–6]. For a further discussion of Eq. (14), see [18].

We now briefly discuss gauge groups higher than $SU(2)$: for that purpose, we must return to Eq. (4). Although it is valid for any group and any representation, its surface form depends explicitly on the group representation in which the Wilson loop is considered. Equation (4) says that one can in fact integrate not over all gauge transformations S but only over those that do not commute with the combination of Cartan generators m_iH_i where $\mathbf{m}$ is the highest weight of a given representation. In the $SU(2)$ case, one has

$$m_iH_i = J\tau_3, \quad J = 1/2, 1, 3/2, \ldots,$$

because $SU(2)$ has the rank 1 and there is only one Cartan generator. In the $SU(2)$ case, one therefore integrates over the coset $SU(2)/U(1)$ for any representation; this coset can be parameterized by the $\mathbf{n}$ field as described above.

For higher groups, there are several possibilities of taking cosets: a particular coset depends on the representation of the Wilson loop. For example, in the case where the Wilson loop is in the fundamental representation of the $SU(N)$ group, the combination m_iH_i is proportional to one particular generator of the Cartan subalgebra that commutes with the $SU(N-1)\times U(1)$ subgroup. (For $SU(3)$, this generator is the Gell-Mann λ_8 matrix or a permutation of its elements.) For the fundamental representation of the $SU(N)$ group, the appropriate coset is therefore given by

$$SU(N)/SU(N-1)/U(1) = CP^{N-1}.$$

A possible parameterization of this coset is given by a complex N-vector u^α of the unit length, $u^\dagger_\alpha u^\alpha = 1$. To be specific, the Cartan combination in the fundamental representation can always be set equal to

$$m_iH_i = \mathrm{diag}(1,0,\ldots,0)$$

by rotating the axes and subtracting the unit matrix. In this basis, u^α is just the first column of the unitary matrix $S^\dagger$ and $u^\dagger_\alpha$ is the first row of S. Unitarity of S implies that

$$u^\dagger_\alpha u^\alpha = 1.$$

In this parameterization, Eq. (4) can be written as

$$\begin{gathered} W^{SU(N)}_{fund} = \int Du\,Du^\dagger\,\delta(u^\dagger_\alpha u^\alpha - 1)\times \\ \times\exp i\int d\tau\frac{dx^\mu}{d\tau}u^\dagger_\alpha\left(i\nabla_\mu\right)^\alpha_\beta u^\beta, \\ \left(\nabla_\mu\right)^\alpha_\beta = \partial_\mu\delta^\alpha_\beta - iA^a_\mu\left(t^a\right)^\alpha_\beta. \end{gathered} \tag{15}$$

Using the identity

$$\epsilon_{ij}\partial_i\left(u^\dagger\nabla_i u\right) = \epsilon_{ij}\left[(\nabla_i u)^\dagger(\nabla_j u) + u^\dagger\nabla_i\nabla_j u\right] = \\ = \epsilon_{ij}\left[-\frac{i}{2}(u^\dagger F_{ij}u) + (\nabla_i u)^\dagger(\nabla_j u)\right], \quad (16)$$

we can present Eq. (15) in a surface form,

$$W^{SU(N)}_{fund} = \int Du\,Du^\dagger\delta(|u|^2-1)\times \\ \times\exp\left(i\int dS^{\mu\nu}\left[\frac{1}{2}(u^\dagger F_{\mu\nu}u) + i(\nabla_\mu u)^\dagger(\nabla_\nu u)\right]\right), \quad (17)$$

where $F_{\mu\nu}$ is the field strength in the fundamental representation. Equation (17) was first published in Ref. [17], however with an unexpected overall coefficient 2 in the exponent. Equation (17) presents the non-Abelian Stokes theorem for the Wilson loop in the fundamental representation of $SU(N)$. In the particular case of the $SU(2)$ group, transition to Eq. (14) is achieved by identifying the unit 3-vector

$$n^a = u^\dagger_\alpha(\tau^a)^\alpha_\beta u^\beta,$$

where

$$u^\alpha = \begin{pmatrix} \cos\frac{\beta}{2}\exp\left(-i\frac{\alpha+\gamma}{2}\right) \\ \sin\frac{\beta}{2}\exp\left(i\frac{\alpha-\gamma}{2}\right)\end{pmatrix}, \\ 2i\,u^\dagger\partial_\tau u = \dot\alpha(\cos\beta-1) + (\dot\alpha+\dot\gamma). \quad (18)$$

It must be mentioned that the quantity

$$\int d\sigma d\tau\epsilon_{ij}\,i\partial_i u^\dagger_\alpha\partial_j u^\alpha = 2\pi Q \quad (19)$$

appearing in Eq. (17) is the topological charge of the 2-dimensional CP^{N-1} model. For closed or infinite surfaces, Q is an integer.

In the case where the Wilson loop is taken in the adjoint represention of the $SU(N)$ gauge group, the combination $m_i H_i$ in Eq. (4) is the highest root. Only group elements of the form $\exp(i\alpha_i H_i)$ commute with this combination (these elements belong to the maximum torus subgroup $U(1)^{N-1}$). In the case of the adjoint representation, one therefore integrates over the flag manifold [19, 7]

$$SU(N)/U(1)^{N-1} = F^{N-1}.$$

3. «GRAVITATIONAL WILSON LOOPS»

An object similar to the Wilson loop of the Yang–Mills theory also exists in gravity theory. It is the parallel transporter of a vector on a Riemannian manifold along a closed contour, also called a holonomy. The holonomy is trivial if the space is flat but becomes a non-trivial functional of the curvature if it is nonzero. In the remaining sections, we present new formulas for the parallel transporters on $d = 2, 3, 4$ Riemannian manifolds.

We first recall some notation from differential geometry. We use [20] as a general reference book. Let $g_{\mu\nu} = g_{\nu\mu}$ $(\mu,\nu = 1,\ldots d)$ be the covariant metric tensor, with the contravariant tensor $g^{\mu\nu}$ being its inverse, $g_{\mu\nu}g^{\nu\kappa} = \delta^\kappa_\mu$. The determinant of the covariant metric tensor is denoted by g. The Christoffel symbol is defined by

$$\Gamma^\mu_{\nu\kappa} = g^{\mu\lambda}\Gamma_{\lambda,\nu\kappa} = \frac{g^{\mu\lambda}}{2}(\partial_\nu g_{\lambda\kappa} + \partial_\kappa g_{\lambda\nu} - \partial_\lambda g_{\nu\kappa}), \\ \Gamma^\kappa_{\nu\kappa} = \frac{\partial_\nu g}{2g}. \quad (20)$$

The action of the covariant derivative on a contravariant vector is defined as

$$(\nabla_\rho)^\kappa_\lambda v^\lambda = (\partial_\rho\delta^\kappa_\lambda + \Gamma^\kappa_{\rho\lambda})v^\lambda. \quad (21)$$

The commutator of two covariant derivatives determines the Riemann tensor,

$$[\nabla_\rho\nabla_\sigma]^\kappa_\lambda = R^\kappa{}_{\lambda\rho\sigma} = g^{\kappa\kappa'}R_{\kappa'\lambda\rho\sigma} = \\ = \partial_\rho\Gamma^\kappa_{\sigma\lambda} - \partial_\sigma\Gamma^\kappa_{\rho\lambda} + \Gamma^\kappa_{\rho\tau}\Gamma^\tau_{\sigma\lambda} - \Gamma^\kappa_{\sigma\tau}\Gamma^\tau_{\rho\lambda}. \quad (22)$$

A contraction of the Riemann tensor gives the symmetric Ricci tensor,

$$R_{\lambda\sigma} = R^\kappa{}_{\lambda\kappa\sigma}, \quad R^\kappa_\rho = R^\kappa{}_{\lambda\rho\sigma}g^{\lambda\sigma}. \quad (23)$$

Its full contraction is the scalar curvature

$$R = R_{\lambda\sigma}g^{\lambda\sigma} = R^\kappa_\kappa. \quad (24)$$

The parallel transporter of a contravariant vector along a curve $x^\mu(\tau)$ is determined by solving the equation

$$\frac{dx^\mu}{d\tau}(\nabla_\mu)^\kappa_\lambda v^\lambda(\tau) = 0. \quad (25)$$

The solution can be written using the evolution operator

$$v^\kappa(\tau) = \left[W^G(\tau)\right]^\kappa_\lambda v^\lambda(0), \quad (26)$$

where $v^\lambda(0)$ is the vector at the starting point of the contour and $v^\lambda(\tau)$ is the parallel-transported vector at the point labelled by τ. The evolution operator can be symbolically written as a path-ordered exponential of the Christoffel symbol,

$$\left[W^G(\tau)\right]^\kappa_\lambda = \left[\mathrm{P}\ \exp\left(-\int_0^\tau d\tau \frac{dx^\mu}{d\tau}\Gamma_\mu\right)\right]^\kappa_\lambda. \quad (27)$$

We define the «gravitational Wilson loop» as the trace of the parallel transporting evolution operator along the closed curve $x^\mu(\tau)$ with $x^\mu(1) = x^\mu(0)$,

$$W^G_{vector} = \frac{1}{d}\left[W^G(1)\right]^\kappa_\kappa. \quad (28)$$

This quantity is diffeomorphism-invariant: the metric tensor is transformed under coordinate changes $x^\mu \to x'^\mu(x)$, but if the contour is changed as

$$x^\mu(\tau) \to x'^\mu(x(\tau)),$$

the gravitational Wilson loop or the holonomy remains the same. In this respect, the gravitational holonomy is different from the Yang–Mills Wilson loop that is invariant under gauge transformations without changing the contour.

The parallel transporter of a covariant vector is given by the transposed matrix; its trace coincides with that of the matrix used in transporting contravariant vectors.

4. RELATION OF GRAVITY QUANTITIES TO THOSE OF THE YANG–MILLS THEORY

We now show that the «gravitational Wilson loop» is not only analogous to but directly expressible through the Yang–Mills Wilson loops of the $SU(2)$ group. For this purpose, we introduce the standard vielbein e^A_μ and its inverse $e^{A\mu}$ such that

$$\begin{aligned} e^A_\mu e^A_\nu = g_{\mu\nu}, \quad e^A_\mu e^{B\mu} = \delta^{AB}, \\ e^{A\mu}e^{A\nu} = g^{\mu\nu}, \quad \det e^A_\mu = \sqrt{g}. \end{aligned} \quad (29)$$

We decompose the vector experiencing the parallel transport in vielbeins, $v^\lambda = c^A e^{A\lambda}$, with the reciprocal decomposition

$$c^A = e^A_\kappa v^\kappa, \quad (30)$$

and insert this in Eq. (25) defining the parallel transport. We then have

$$\begin{aligned} 0 &= \frac{dx^\mu}{d\tau}(\nabla_\mu)^\kappa_\lambda c^A e^{A\lambda} = \\ &= \frac{dx^\mu}{d\tau}\left[e^{A\kappa}\partial_\mu c^A + c^A(\partial_\mu e^{A\kappa} + \Gamma^\kappa_{\mu\lambda}e^{A\lambda})\right] = \\ &= \frac{dx^\mu}{d\tau}e^{B\kappa}(\partial_\mu \delta^{BA} + \omega^{BA}_\mu)c^A, \end{aligned} \quad (31)$$

where we introduced the spin connection

$$\begin{aligned} \omega^{AB}_\mu = -\omega^{BA}_\mu = \frac{1}{2}e^{A\kappa}(\partial_\mu e^B_\kappa - \partial_\kappa e^B_\mu) - \\ - \frac{1}{2}e^{B\kappa}(\partial_\mu e^A_\kappa - \partial_\kappa e^A_\mu) - \\ - \frac{1}{2}e^{A\kappa}e^{B\lambda}e^C_\mu(\partial_\kappa e^C_\lambda - \partial_\lambda e^C_\kappa) \end{aligned} \quad (32)$$

and used the fundamental relations

$$\partial_\mu e^{A\kappa} + \Gamma^\kappa_{\mu\lambda}e^{A\lambda} = -\omega^{AB}_\mu e^{B\kappa}, \quad (33)$$

$$\partial_\mu e^A_\kappa - \Gamma^\lambda_{\mu\kappa}e^A_\lambda = -\omega^{AB}_\mu e^B_\kappa. \quad (34)$$

One can introduce the $SO(d)$ «field strength»

$$\begin{aligned} \mathcal{F}^{AB}_{\mu\nu} &= [\partial_\mu + \omega_\mu, \partial_\nu + \omega_\nu]^{AB} = \\ &= \partial_\mu\omega^{AB}_\nu - \partial_\nu\omega^{AB}_\mu + \omega^{AC}_\mu\omega^{CB}_\nu - \omega^{AC}_\nu\omega^{CB}_\mu \end{aligned} \quad (35)$$

related to the Riemann tensor as

$$\begin{aligned} \mathcal{F}^{AB}_{\mu\nu}e^A_\kappa e^B_\lambda &= -R_{\kappa\lambda\mu\nu}, \\ \mathcal{F}^{AB}_{\mu\nu} &= -R_{\kappa\lambda\mu\nu}e^{A\kappa}e^{B\lambda}, \\ \mathcal{F}^{AB}_{\mu\nu}e^{A\mu}e^{B\nu} &= R. \end{aligned} \quad (36)$$

The above material is common for any number of dimensions. To proceed further, we consider the cases where $d = 3$ and $d = 4$ separately. The case where $d = 2$ is considered in Sec. 6.

4.1. $d = 3$

In three dimensions, one can immediately identify the spin connection with the $su(2)$-valued Yang–Mills field as

$$A^c_i = -\frac{1}{2}\epsilon^{abc}\,\omega^{ab}_i. \quad (37)$$

Working in three dimensions, we denote the Lorentz indices by $i, j, \ldots = 1, 2, 3$ and the flat triade indices by $a, b, \ldots = 1, 2, 3$. Recalling the generators in the $J = 1$ representation,

$$(T^c)^{ab} = -i\epsilon^{cab}, \quad [T^cT^d] = i\epsilon^{cdf}T^f, \quad (38)$$

we can rewrite the last parenthesis in Eq. (31) as

$$\partial_i\delta^{ab} + \omega^{ab}_i = \partial_i\delta^{ab} - iA^c_i(T^c)^{ab} \equiv (D_i)^{ab}, \quad (39)$$

which is the standard Yang–Mills covariant derivative in the adjoint representation. In the fundamental (spinor) representation, the Yang–Mills covariant derivative is

$$\begin{aligned} (\nabla_i)^\alpha_\beta &= \partial_i\delta^\alpha_\beta - iA^c_i\left(\frac{\sigma^c}{2}\right)^\alpha_\beta = \\ &= \partial_i\delta^\alpha_\beta + \frac{1}{8}\omega^{ab}_i\left[\sigma^a\sigma^b\right]^\alpha_\beta, \quad \alpha, \beta = 1, 2, \end{aligned} \quad (40)$$

which coincides with the known expression for the covariant derivative in the spinor representation in a curved space.

The standard Yang–Mills field strength is directly related to that in Eq. (35),

$$F^a_{ij} = \partial_i A^a_j - \partial_j A^a_i + \epsilon^{abc} A^b_i A^c_j = -\frac{1}{2}\epsilon^{abc}\mathcal{F}^{bc}_{ij}. \quad (41)$$

It then follows from Eq. (36) that

$$\epsilon^{abc} F^a_{ij} e^b_k e^c_l = R_{ijkl}. \quad (42)$$

We next consider the parallel transporter of a 3-vector in a curved space, as defined by Eq. (25). In accordance with Eqs. (31) and (39), solving Eq. (25) is equivalent to solving the Yang–Mills equation for the parallel transporter,

$$\frac{dx^i}{d\tau}(D_i)^{ab} c^b = 0, \quad (43)$$

whose solution is

$$c^a(\tau) = \left[W_1^{YM}(\tau)\right]^{ab} c^b(0),$$
$$\left[W_1^{YM}(\tau)\right]^{ab} = \left[\mathrm{P}\,\exp\left(i\int d\tau \frac{dx^i}{d\tau} A^c_i T^c\right)\right]^{ab}, \quad (44)$$

where the subscript «1» indicates that the path-ordered exponential is taken in the $J = 1$ representation. The parallel transport of a contravariant vector is therefore given by

$$v^k(\tau) = c^a(\tau) e^{ak}(\tau) = = e^{ak}(\tau)\left[W_1^{YM}(\tau)\right]^{ab} e^b_l(0) v^l(0), \quad (45)$$

which immediately implies the sought relation between the «gravitational» and Yang–Mills parallel transporters,

$$\left[W_1^G(\tau)\right]^k_l = e^a_k(\tau)\left[W_1^{YM}(\tau)\right]^{ab} e^{bl}(0). \quad (46)$$

The relation becomes especially neat for the Wilson loops, i.e., for the traces of parallel transporters along closed contours. Because the vielbeins take identical values at the end points of a closed contour, $e^a_k(1) = e^a_k(0)$, we obtain

$$W^G_{vector} = \frac{1}{3}\left[W_1^G\right]^k_k = \frac{1}{3}\left[W_1^{YM}\right]^{aa} = W_1^{YM}. \quad (47)$$

In a similar way, one can show that the same equation is valid for the gravitational parallel transporter of covariant vectors and, more generally, for parallel transporters of any integer spin J. In this case, the Yang–Mills Wilson loop must be taken in the same representation as the gravitational one,

$$W^G_J = W^{YM}_J. \quad (48)$$

It is understood that the right-hand side of Eq. (48) is expressed through the Yang–Mills field equal to the spin connection in accordance with Eq. (37), while the left-hand side is expressed through the Christoffel symbols, that is, through the metric. It must be stressed that the spin connection is defined via the vielbein, which is not uniquely determined by the metric tensor. The Wilson loop, being a gauge-invariant quantity, is nevertheless uniquely determined by the metric tensor and its derivatives. This is the meaning of Eq. (48).

For a half-integer J, there is no way to define the parallel transporter other than through the spin connection. Nevertheless, as we show in Sec. 8, where we present the holonomy for any spin in a surface form, the «gravitational Wilson loop» is also expressible through the metric tensor and its derivatives, even for half-integer spins.

4.2. $d = 4$

In four Euclidean dimensions, the rotation group is $SO(4)$, with its algebra isomorphic to that of $SU(2)\times SU(2)$, and therefore, all irreducible representations of $SO(4)$ can be classified by (J_1, J_2), where $J_{1,2} = 0, 1/2, 1, \ldots$ label the representations of the two $SU(2)$ subgroups. For example, the 4-vector representation whose parallel transporter was considered in the beginning of this section, transforms in the $(1/2, 1/2)$ representation of $SU(2)\times SU(2)$. Because of this, it is convenient to decompose the spin connection ω^{AB}_μ into self-dual and anti-self-dual parts using 't Hooft's η and $\bar\eta$ symbols

$$\eta^{aAB} = \frac{1}{2i}\,\mathrm{Tr}\,\sigma^a(\sigma^{A+}\sigma^{B-} - \sigma^{B+}\sigma^{A-}),$$
$$\sigma^{A\pm} = (\pm i\sigma, 1), \quad (49)$$

$$\bar\eta^{aAB} = \frac{1}{2i}\,\mathrm{Tr}\,\sigma^a(\sigma^{A-}\sigma^{B+} - \sigma^{B-}\sigma^{A+}). \quad (50)$$

We use the capital Latin characters to denote flat 4-dimensional vierbein indices, $A, B, \ldots = 1, 2, 3, 4$, while $a, b, \ldots = 1, 2, 3$; σ^a are the three Pauli matrices. The spin connection ω^{AB}_μ transforms in the 6-dimensional representation of $SO(4)$, which can be decomposed into the sum $(1, 0) + (0, 1)$ of the adjoint representations of the two $SU(2)$ subgroups. We write

$$\omega^{AB}_\mu = -\frac{1}{2}\pi^a_\mu\,\eta^{aAB} - \frac{1}{2}\rho^a_\mu\,\bar\eta^{aAB}. \quad (51)$$

The $SO(4)$ «field strength» in Eq. (35) is then decomposed as

$$\mathcal{F}^{AB}_{\mu\nu} = -\frac{1}{2} F^a_{\mu\nu}(\pi)\,\eta^{aAB} - \frac{1}{2} F^a_{\mu\nu}(\rho)\,\bar{\eta}^{aAB}, \qquad (52)$$

where

$$F^a_{\mu\nu}(\pi) = \partial_\mu \pi^a_\nu - \partial_\nu \pi^a_\mu + \epsilon^{abc}\pi^b_\mu \pi^c_\nu, \qquad (53)$$

$$F^a_{\mu\nu}(\rho) = \partial_\mu \rho^a_\nu - \partial_\nu \rho^a_\mu + \epsilon^{abc}\rho^b_\mu \rho^c_\nu \qquad (54)$$

are the usual Yang–Mills field strengths of the $SU(2)$ Yang–Mills potentials π^a_μ and ρ^a_μ. We stress that $6 \cdot 4 = 24$ variables ω^{AB}_μ equivalent to $2 \cdot 3 \cdot 4 = 24$ variables π^a_μ, and ρ^a_μ are defined by only $4 \cdot 4 = 16$ tetrades e^A_μ via Eq. (32), and therefore not all of them are independent.

Contracting Eq. (36) with the η and $\bar{\eta}$ symbols, we obtain

$$F^a_{\mu\nu}(\pi) = \frac{1}{2}\eta^{aAB} e^{A\kappa} e^{B\lambda} R_{\kappa\lambda\mu\nu}, \qquad (55)$$

$$F^a_{\mu\nu}(\rho) = \frac{1}{2}\bar{\eta}^{aAB} e^{A\kappa} e^{B\lambda} R_{\kappa\lambda\mu\nu}. \qquad (56)$$

We now return to the parallel transporter of a 4-vector. As shown in the beginning of this section, finding this parallel transporter is equivalent to solving the equation

$$\frac{dx^\mu}{d\tau}\left(\partial_\mu\,\delta^{AB} + \omega^{AB}_\mu\right) c^B = 0. \qquad (57)$$

We represent the 4-vector c^A as a combination of two spinors,

$$c^A = \chi^\dagger_\alpha \left(\sigma^{A+}\right)^\alpha_\beta \psi^\beta, \quad \chi^\dagger_\alpha \psi^\beta = \frac{1}{2} c^A \left(\sigma^{A-}\right)^\beta_\alpha, \quad \alpha, \beta = 1, 2. \qquad (58)$$

Inserting this in Eq. (57) and decomposing ω^{AB}_μ as in Eq. (51), we obtain

$$\frac{dx^\mu}{d\tau}\left\{\partial_\mu\left[\chi^\dagger \sigma^{A+}\psi\right] - \frac{1}{2}\left(\pi^a_\mu\,\eta^{aAB} + \rho^a_\mu\,\bar{\eta}^{aAB}\right)\left[\chi^\dagger \sigma^{B+}\psi\right]\right\} = 0. \qquad (59)$$

Using the definition of the η-symbols in Eqs. (49) and (50), it is easy to verify that this equation is satisfied provided the spinors χ and ψ satisfy

$$\frac{dx^\mu}{d\tau}\left[\partial_\mu\,\delta^\alpha_\beta - i\,\pi^a_\mu \left(\frac{\sigma^a}{2}\right)^\alpha_\beta\right]\chi^\beta = 0 \quad \text{or} \quad \frac{dx^\mu}{d\tau}\chi^\dagger_\alpha\left[\overleftarrow{\partial}_\mu\,\delta^\alpha_\beta + i\,\pi^a_\mu\left(\frac{\sigma^a}{2}\right)^\alpha_\beta\right] = 0, \qquad (60)$$

$$\frac{dx^\mu}{d\tau}\left[\partial_\mu\,\delta^\alpha_\beta - i\,\rho^a_\mu\left(\frac{\sigma^a}{2}\right)^\alpha_\beta\right]\psi^\beta = 0. \qquad (61)$$

The expressions in square brackets are identical to the Yang–Mills covariant derivatives, with the role of the Yang–Mills potentials played by π^a_μ and ρ^a_μ, respectively. Equations (60) and (61) define the Yang–Mills parallel transporters in the fundamental representation. Their solution can be written as evolution operators,

$$\chi^\alpha(\tau) = \left[W^\pi(\tau)\right]^\alpha_\gamma \chi^\gamma(0) \quad \text{or} \quad \chi^\dagger_\alpha(\tau) = \chi^\dagger_\gamma(0)\left[W^{\pi\dagger}(\tau)\right]^\gamma_\alpha, \qquad (62)$$

$$\psi^\beta(\tau) = \left[W^\rho(\tau)\right]^\beta_\delta \psi^\delta(0), \qquad (63)$$

$$\left[W^\pi(\tau)\right]^\alpha_\gamma = \left[\mathrm{P}\,\exp\left(i\int d\tau\,\frac{dx^\mu}{d\tau}\,\pi^a_\mu\,\frac{\sigma^a}{2}\right)\right]^\alpha_\gamma, \qquad (64)$$

$$\left[W^\rho(\tau)\right]^\alpha_\gamma = \left[\mathrm{P}\,\exp\left(i\int d\tau\,\frac{dx^\mu}{d\tau}\,\rho^a_\mu\,\frac{\sigma^a}{2}\right)\right]^\alpha_\gamma. \qquad (65)$$

Returning to the 4-vector c^A in Eq. (58), we see that its evolution is determined by

$$c^A(\tau) = \left[W_{vector}(\tau)\right]^{AB} c^B(0),$$

$$\left[W_{vector}(\tau)\right]^{AB} = \frac{1}{2}\,\mathrm{Tr}\left[W^{\pi\dagger}(\tau)\sigma^{A+} W^\rho(\tau)\sigma^{B-}\right]. \qquad (66)$$

We now choose a closed contour and take the trace of the evolution operator. The «gravitational Wilson loop» for a 4-vector is then given by

$$W^G_{(\frac{1}{2},\frac{1}{2})} = \frac{1}{4} e^{A\kappa}(1)\left[W_{vector}(1)\right]^{AB} e^B_\kappa(0) = \frac{1}{4}\left[W_{vector}(1)\right]^{AA} = \frac{1}{2}\,\mathrm{Tr}\,W^\pi \cdot \frac{1}{2}\,\mathrm{Tr}\,W^\rho. \qquad (67)$$

Its generalization to the holonomy in an arbitrary representation (J_1, J_2) is obvious,

$$W^G_{(J_1,J_2)} = W^\pi_{J_1} \cdot W^\rho_{J_2}, \quad W^{\pi,\rho}_J = \frac{1}{2J+1}\,\mathrm{Tr}_{(2J+1)}\,W^{\pi,\rho}. \qquad (68)$$

Thus, the holonomy in the (J_1, J_2) representation in a curved $d = 4$ space is equal to the product of two Yang–Mills Wilson loops, with the role of the Yang–Mills potentials played by the self-dual π^a_μ and anti-self-dual ρ^a_μ parts of the spin connection. In Sec. 9, we show that both W^π and W^ρ can be written in terms of the metric tensor.

5. SMALL WILSON LOOPS

For small-area contours, the «gravitational Wilson loop» can be expanded in powers of the area. The most straightforward way to do this is to use the path-ordered form of W^G in Eq. (27). We take a square contour of the size $a \times a$ lying in the 12 plane and expand the path-ordered exponential in powers of a. After some simple algebra, we obtain the first nontrivial term of this expansion, which happens to be $O(a^4)$,

$$W^G_{vector} = \frac{1}{d}\left[W^G_{vector}\right]^{\kappa}_{\kappa} = 1 + \frac{a^4}{d} R^{\kappa}{}_{\lambda 12} R^{\lambda}{}_{\kappa 12} = \\ = 1 - \frac{2(\Delta S)^{\mu\nu}(\Delta S)^{\mu'\nu'}}{4d} R_{\kappa\lambda\mu\nu} R_{\rho\sigma\mu'\nu'} g^{\kappa\rho} g^{\lambda\sigma}, \quad (69)$$

where $(\Delta S)^{\mu\nu}$ is the surface element lying in the $\mu\nu$ plane. We note that the first correction to the holonomy is negative-definite. We emphasize that the first-order term in ΔS is in general present in the expansion of the parallel transporter, however it vanishes after taking the trace owing to the identity $R^{\kappa}{}_{\kappa\mu\nu} \equiv 0$, and therefore, the expansion of the trace starts with the $(\Delta S)^2$ term.

In three dimensions, Eq. (69) can be further simplified because the Riemann tensor is expressed through the Ricci tensor via

$$R_{ijkl} = R_{ik} g_{jl} - R_{il} g_{jk} + R_{jl} g_{ik} - R_{jk} g_{il} + \\ + \frac{R}{2}(g_{il} g_{jk} - g_{ik} g_{jl}). \quad (70)$$

Because the Riemann tensor is antisymmetric with respect to each pair of subscripts, we can replace

$$g^{km} g^{ln} \to \frac{1}{2}(g^{km} g^{ln} - g^{kn} g^{lm}) = \\ = \frac{1}{2g} \epsilon^{kli} \epsilon^{mnj} g_{ij}. \quad (71)$$

Introducing the dual surface element

$$\Delta S^{pq} = \epsilon^{pqr} \Delta S_r, \quad (72)$$

we have

$$\epsilon^{kli} \epsilon^{pqr} R_{klpq} = -4 \left(R^{ir} - \frac{1}{2} R\, g^{ir}\right), \quad (73)$$

which as a matter of fact is the Einstein tensor. For the parallel transporter of an arbitrary spin J, the factor 2 in the numerator of Eq. (69) must be replaced by $J(J+1)$.

Combining all the factors, we obtain

$$W^G_J = 1 - \frac{2J(J+1)}{3g} \left(R^{ir} - \frac{1}{2} R\, g^{ir}\right) \times \\ \times g_{ij} \left(R^{js} - \frac{1}{2} R\, g^{js}\right) \Delta S_r \, \Delta S_s \, . \quad (74)$$

This is our final expression for the trace of the spin-J parallel transporter for small loops in a curved $d = 3$ space. We note that Eq. (74) is invariant under diffeomorphisms.

6. GRAVITATIONAL WILSON LOOP IN TWO DIMENSIONS

In a curved $d = 2$ space, the trace of the parallel transporter along a closed loop can be computed exactly for any metric and can be presented in the form of a «Stokes theorem». The result is related to the Gauss–Bonnet theorem and is generally known: we present it here for the sake of completeness.

The key observation is that in two dimensions, spin connection (32) has only one component,

$$\omega^{ab}_i = \epsilon^{ab}\, \omega_i. \quad (75)$$

In this section, all indices take only two values 1, 2. In accordance with Eq. (31), the parallel transporter of a vector is determined by the equation

$$\frac{dc^a}{d\tau} - \frac{dx^i}{d\tau} \omega_i\, \epsilon^{ab}\, c^b = 0, \quad (76)$$

which is solved by

$$c^a(\tau) = W^{ab}(\tau)\, c^b(0), \\ W^{ab}(\tau) = \begin{pmatrix} \cos\gamma(\tau) & \sin\gamma(\tau) \\ -\sin\gamma(\tau) & \cos\gamma(\tau) \end{pmatrix}, \\ \gamma(\tau) = \int_0^{\tau} d\tau\, \frac{dx^i}{d\tau} \omega_i. \quad (77)$$

According to the general theorem in Sec. 4, the gravitational Wilson loop is equal to the Yang–Mills one, and we obtain

$$W^G_1 = \frac{1}{2} W^{aa}(1) = \cos\Phi, \quad (78)$$

where

$$\Phi = \gamma(1) = \int_0^1 d\tau \frac{dx^i}{d\tau}\, \omega_i = \frac{1}{2} \oint dx^i\, \epsilon_{ab}\, \omega^{ab}_i. \quad (79)$$

This formula is not fully satisfactory because the holonomy is expressed through the spin connection and not through the metric. Expressing it through the metric can be achieved if we apply the Stokes theorem and write Eq. (79) in a surface form. We have

$$\Phi = \frac{1}{2}\int dS\, \epsilon_{ab}\, \epsilon^{ij}\, \partial_i \omega_j^{ab}, \tag{80}$$

where dS is the element of the spanning surface for the contour. Introducing the field strength related to the Riemann tensor,

$$\begin{aligned} F_{ij}^{ab} = \partial_i\omega_j^{ab} - \partial_j\omega_j^{ab} + \omega_i^{ac}\omega_j^{cb} - \omega_j^{ac}\omega_i^{cb} = \\ = R^{kl}{}_{ij}\, e_k^a\, e_l^b, \\ \epsilon_{ab}\, e_k^a\, e_l^b = \epsilon_{kl}\sqrt{g}, \end{aligned} \tag{81}$$

and noticing that the commutator term is zero in two dimensions, we rewrite Eq. (80) as

$$\Phi = \frac{1}{2}\int dS\sqrt{g}\, R, \quad W_1^G = \cos\Phi, \tag{82}$$

where

$$R = (1/2)\, \epsilon^{ij}\, \epsilon_{kl}\, R_{ij}^{kl}$$

is the scalar curvature. It is gratifying that the holonomy is expressed through the Einstein–Hilbert action, which is known to be a total derivative in two dimensions. Needless to explain, Eq. (82) is diffeomorphism-invariant.

In two dimensions, there is essentially only one component of the Riemann tensor,

$$R_{1212} = \frac{1}{2} R\, g \tag{83}$$

(see [20]). Taking this into account, it is easy to verify that for small areas, the expansion of Eq. (82) gives the same result as Eq. (69) written for small loops.

7. AN EXAMPLE OF BIG LOOPS: A CONSTANT-CURVATURE BACKGROUND IN THREE DIMENSIONS

In three dimensions, the Riemann tensor is expressible through the Ricci tensor, see Eq. (70). Therefore, the diffeomorphism-invariant information about curved spaces is fully contained in the three eigenvalues of the symmetric Ricci tensor,

$$R_j^i = \lambda\, \delta_j^i, \tag{84}$$

with the scalar curvature being the sum of the three,

$$R = \lambda_1 + \lambda_2 + \lambda_3.$$

For example, the de Sitter S^3 space corresponds to

$$\lambda_1 = \lambda_2 = \lambda_3 = R/3 = \text{const}.$$

In this section, we consider another constant-curvature case, namely the cylinder space $S^2 \times R$ characterized by

$$\lambda_1 = \lambda_2 = R/2 = \text{const}, \quad \lambda_3 = 0.$$

We show that the parallel transporter in these spaces can be computed for any form of the contour and any metric and that the gravitational Wilson loop is given by an elegant formula.

A general metric can be considered as the one induced by 6 external coordinates $w^A(x_1, x_2, x_3)$,

$$g_{ij} = \partial_i w^A \partial_j w^A, \quad A = 1, \ldots, 6. \tag{85}$$

In the special case of the cylinder space $S^2 \times R$, it is sufficient to use only four external coordinates w^a ($a = 1, 2, 3$) and w^4 subject to the constraint

$$\sum_{a=1}^{3} (w^a)^2 = \frac{2}{R}. \tag{86}$$

An example of such external coordinates is given by

$$w^{1,2,3}(x) = \sqrt{\frac{2}{R}}\,\frac{x^{1,2,3}}{r}, \quad w^4(x) = \sqrt{\frac{2}{R}}\ln r, \tag{87}$$

leading to the metric tensor

$$g_{ij} = \frac{2}{R}\frac{1}{r^2}\delta_{ij}, \quad \sqrt{g} = \left(\frac{2}{R}\right)^{3/2}\frac{1}{r^3}. \tag{88}$$

A simple calculation using formulas in Sec. 3 shows that this metric indeed gives a zero eigenvalue of the Ricci tensor with the other two eigenvalues equal to the constant $R/2$. Because the eigenvalues of the Ricci tensor are diffeomorphism-invariant, a general change of coordinates $x^i \to y^i(x)$ in Eq. (87) results in the same eigenvalues. Therefore, the most general description of the cylinder spaces $S^2 \times R$ is given by

$$\begin{aligned} w^a(x) = \sqrt{\frac{2}{R}}\,\frac{y^a(x)}{|y(x)|}, \quad w^4(x) = \sqrt{\frac{2}{R}}\ln|y(x)|, \\ g_{ij} = \frac{2}{R}\,\frac{\partial_i y^a \partial_i y^a}{y^2}, \end{aligned} \tag{89}$$

$$\begin{aligned} \sqrt{g} = \left(\frac{2}{R}\right)^{\frac{3}{2}}\frac{1}{3!}\,\epsilon^{ijk}\,\epsilon_{abc}\,\frac{\partial_i y^a \partial_j y^b \partial_k y^c}{|y|^3} = \\ = \frac{1}{2}\sqrt{\frac{R}{2}}\,\epsilon^{ijk}\,\epsilon_{abc}\,\partial_i w^a\,\partial_j w^b\, w^c\,\partial_k w^4, \end{aligned} \tag{90}$$

where $y^a(x)$ are three arbitrary functions of the coordinates x^i. We note that g_{ij} is given by the product of two matrices

$$M_i^a = \partial_i y^a/|y|,$$

and hence, $\sqrt{g}$ is itself a determinant (of the matrix M).

Our aim is to calculate the Wilson loop for any contour in any metric (89) corresponding to the cylinder spaces. We use the diffeomorphism invariance of the Wilson loop. If we compute it for a general contour in some metric representing cylinder spaces, the most general case is recovered by diffeomorphisms of both the contour and the metric. We start with the specific metric given by Eqs. (87) and (88).

Given metric tensor (88), we construct a vielbein corresponding to it. This is, of course, not unique but any choice of the vielbein suits us. We choose

$$e_i^a = \sqrt{\frac{2}{R}}\,\frac{1}{r}\,\delta_i^a, \quad e_i^a e_j^a = g_{ij}. \tag{91}$$

Given the vielbein, we construct the spin connection (or the Yang–Mills field) from its definition (32) and obtain

$$A_i^a = -\frac{1}{2}\epsilon^{abc}\omega_i^{bc} = \epsilon^{aij}\,\frac{x^j}{r^2}, \tag{92}$$

which happens to be the field of the Wu–Yang monopole; the scalar curvature R has dropped from the spin connection. According to the theorem in Sec. 4, the gravitational Wilson loop is equal to the Yang–Mills Wilson loop, provided the Yang–Mills potential A_i^a is the spin connection of the metric under consideration. Therefore, all we have to do is to compute the Wilson loop for a general contour in the field of the Wu–Yang monopole.

This task is easily solvable if we use another invariance, the gauge invariance of the Wilson loop. It is well known that the Wu–Yang monopole in hedgehog gauge (92) can be transformed to the string gauge where the potential has only one nonzero component along the third color axis (plus a Dirac string). In this gauge, the Yang–Mills potential is basically Abelian, and the Wilson loop in any representation J is therefore given by

$$W_J^G = W_J^{YM} = \frac{1}{2J+1}\sum_{m=-J}^{J}\exp(im\Phi), \qquad \Phi = \oint dx^i A_i^3 = \int dS_i\,\frac{x^i}{r^3}. \tag{93}$$

In the last equation, we used the normal Stokes theorem for the circulation and also used the fact that in the string gauge, the magnetic field of the monopole is the Coulomb field of a point charge; dS_i is the element of the spanning surface for the contour and is orthogonal to the surface.

Equation (93) is the gravitational Wilson loop for arbitrary contours but in a specific metric given by Eq. (88). To generalize it to the general metric given by (89), it only remains to perform the general coordinate transformation of Eq. (93). To this end, it is convenient to use, instead of dS_i, its dual dS^{ij} such that $dS_i = \epsilon_{ijk}\,dS^{jk}$. We recall that under a general coordinate transformation $x^i \to y^i(x)$, the contravariant vector transforms as

$$V^i \to V^k\partial_k y^i,$$

and the antisymmetric contravariant tensor transforms as

$$dS^{ij} \to dS^{mn}\,\partial_m y^i\,\partial_n y^j.$$

The flux in Eq. (93) is therefore given by

$$\Phi = \int dS_i\,\frac{x^i}{r^3} = \int dS^{ij}\,\epsilon_{ijk}\,\frac{x^k}{r^3} \to \to \int dS^{mn}\,\frac{\epsilon_{ijk}\,\partial_m y^i\,\partial_n y^j\,y^k}{|y|^3}. \tag{94}$$

This equation takes a more symmetric form in terms of external coordinates (89),

$$\Phi = \left(\frac{2}{R}\right)^{\frac{3}{2}}\frac{1}{2}\int dS_k\,\epsilon_{abc}\,\epsilon^{ijk}\,\partial_i w^a\,\partial_j w^b\,w^c, \qquad \sum_{a=1}^{3} w^{a2} = \frac{2}{R}. \tag{95}$$

Equations (93) and (95) are our final result for the gravitational Wilson loop in the cylinder $S^2 \times R$ space of the constant curvature R. The Wilson loop implicitly depends on the metric through Eq. (89). We now make several comments.

1) The parallel transporter must depend on the metric along the contour but not on the spanning surface for the contour, because this surface can be drawn arbitrarily. This is indeed so despite the surface form of the result, because

$$\partial_k\left(\epsilon_{abc}\,\epsilon^{ijk}\,\partial_i w^a\,\partial_j w^b\,w^c\right) = 0. \tag{96}$$

Therefore, the flux in Eq. (95) can be presented as a circulation of a certain vector.

2) The flux in Eq. (95) has the form of a well-known expression for the winding number of a mapping

$S^2 \mapsto S^2$. For a closed or infinite surface, the winding number is normalized as

$$\frac{1}{8\pi}\left(\frac{2}{R}\right)^{3/2} \times \\ \times \int dS_k \epsilon_{abc}\epsilon^{ijk}\partial_i w^a \partial_j w^b\, w^c = Q = \text{integer}. \quad (97)$$

3) For small contours, Eqs. (93) and (95) reproduce the result of the previous section. To check this, we rewrite the general small-loop expansion (69) for the specific metric in Eq. (87). We find

$$R_{klpq} = \frac{2}{R\,r^6}\epsilon_{klu}x^u\epsilon_{pqv}x^v, \quad g^{ij} = \frac{R}{2}r^2\delta^{ij}. \quad (98)$$

Inserting this in Eq. (69) and then performing a general coordinate transformation $x^i \to y^i(x)$, we obtain, after some simple algebra,

$$W_J^G = 1 - \frac{J(J+1)}{6}\left(\frac{\epsilon_{pqu}y^u\partial_i y^p\partial_j y^q\Delta S^{ij}}{|y|^3}\right)^2, \quad (99)$$

which exactly coincides with the expansion of Eq. (93) in the small loop area ΔS up to the second order.

8. THE NON-ABELIAN STOKES THEOREM IN $d = 3$ GRAVITY

In Sec. 4, we have shown that the gravitational Wilson loop viewed as a functional of the metric is equal to the Yang–Mills Wilson loop viewed as a functional of the Yang–Mills potential, provided this potential is set equal to the spin connection corresponding to the metric in question.

We now present the Yang–Mills Wilson loop in terms of our non-Abelian Stokes formula, see Eq. (14):

$$W_J^G[\text{metric}] = W_J^{YM}[\text{spin connection}] = \\ = \int D\mathbf{n}\,\delta(\mathbf{n}^2-1)\exp\frac{iJ}{2}\times \\ \times \int d^2S^{ij}\left[-F_{ij}^a n^a + \epsilon^{abc}n^a\,(D_i n)^b\,(D_j n)^c\right]. \quad (100)$$

We next replace the surface element by its dual $dS^{ij} = \epsilon^{ijp}\,dS_p$ with the aim to rewrite this representation for the Wilson loop in terms of the metric of the curved three-dimensional space. To this end, we first decompose the integration unit vector $\mathbf{n}$ in the dreibein:

$$n^a = m^i\,e_i^a, \quad n^a n^a = m^i m^j e_i^a e_j^a = \\ = m^i m^j g_{ij} = 1. \quad (101)$$

The new 3-vector $\mathbf{m}$ is a covariant unit vector. Because the background metric g_{ij} is fixed, we only change the integration variables from $\mathbf{n}$ to $\mathbf{m}$ as

$$\int D\mathbf{n}\,\delta(\mathbf{n}^2-1)\ldots = \\ = \int D\mathbf{m}\,\sqrt{g}\,\delta(m^i m^j\,g_{ij}-1)\ldots \quad (102)$$

We next use relation (42) of the field strength F_{ij}^a computed from the spin connection

$$A_i^a = (1/2)\epsilon^{abc}\omega_i^{bc}$$

to the Riemann tensor. The first term in the exponent of Eq. (100) becomes

$$\text{first term} = \\ = -dS_p\epsilon^{ijp}\left(-\frac{1}{2}\right)\epsilon^{abc}\,m^n e_n^a\,R^l{}_{kij}\,e_l^b\,e^{ck}. \quad (103)$$

Using

$$\epsilon^{abc}e^{bl}e^{ck} = \frac{1}{\sqrt{g}}\epsilon^{lkm}e_m^a, \quad \sqrt{g} = \det\,e_i^a, \quad (104)$$

equation (103) can be continued as

$$\text{first term} = dS_p\epsilon^{ijp}\frac{1}{2\sqrt{g}}\,R_{ijkl}\epsilon^{klm}g_{mn}m^n. \quad (105)$$

The combination of the covariant Riemann tensor and two antisymmetric epsilon symbols has been encountered in Sec. 5: in three dimensions, it gives the Einstein tensor, see Eq. (73). We thus obtain

$$\text{first term} = dS_p\,\sqrt{g}\,(R\delta_n^p - 2R_n^p)\;m^n, \quad (106)$$

where R_n^p is the Ricci tensor and $R = R_k^k$ is the scalar curvature.

We now turn to the second term in the exponent in Eq. (100) and again use decomposition (101). We exploit fundamental relation (33) that can be presented as

$$D_j^{bb'}n^{b'} = e_k^b\,(\nabla_j)_l^k\,m^l, \quad (107)$$

where

$$D_j^{bb'} = \partial_j\delta^{bb'} + \epsilon^{bcb'}A_j^c$$

is the Yang–Mills covariant derivative and

$$(\nabla_j)_l^k = \partial_j\delta_l^k + \Gamma_{jl}^k$$

is the gravitational covariant derivative. The second term is therefore given by

$$\text{second term} = dS_p\,\epsilon^{abc}e_k^a e_l^b e_n^c\,\epsilon^{ijp}\,m^k(\nabla_i)_{l'}^l\times \\ \times m^{l'}(\nabla_j)_{n'}^n m^{n'} = \\ = dS_p\,\sqrt{g}\,\epsilon^{ijp}\,\epsilon_{kln}\,m^k(\nabla_i\,m)^l(\nabla_j\,m)^n. \quad (108)$$

Gathering Eqs. (102), (106), and (108) together, we finally obtain a non-Abelian Stokes theorem for the gravitational Wilson loop or the trace of the spin-J parallel transporter along a closed contour:

$$W_J^G = \int D\mathbf{m}\,\sqrt{g}\,\delta(m^i m^j\,g_{ij} - 1) \times \times \exp\, i\frac{J}{2}\int dS_k\,\sqrt{g}\left[\left(R\delta_p^k - 2R_p^k\right)\,m^p + + \epsilon^{ijk}\,\epsilon_{pqr}\,m^p(\nabla_i\,m)^q(\nabla_j\,m)^r\right]. \quad (109)$$

Several comments are in order here.

1) The holonomy, which was defined as a path-ordered exponential, is here expressed by a simple exponential of an integral over the spanning surface for the closed contour. That is why we call our formula a «Stokes theorem». The price to pay is the functional integration over the covariantly unit vector $\mathbf{m}$ defined on the surface.

2) Equation (109) is invariant under diffeomorphisms in the sense that the holonomy remains invariant under a general coordinate transformation

$$x^i \to x'^i(x^i)$$

and the appropriate change of the surface.

3) The parallel transporter depends only on the contour but must not depend on the spanning surface. The surface integral in Eq. (109) has the form

$$\int dS_k\,\sqrt{g}\,V^k, \quad (110)$$

and the condition that it does not depend on the form of the surface is

$$\partial_k\left(\sqrt{g}\,V^k\right) = 0, \quad (111)$$

or equivalently,

$$(\nabla_k)_l^k\,V^l = 0, \quad (112)$$

because

$$\Gamma_{kl}^k = \Gamma_{lk}^k = \partial_l \ln\sqrt{g}.$$

The verification of Eq. (112) is rather lengthy and we relegate it to the Appendix.

4) With condition (112) or equivalently (111) satisfied, the surface integral can be written as

$$\int dS_k\,\sqrt{g}\,V^k = \int dS_k \epsilon^{ijk}\partial_j B_k = -\oint dx^i\,B_i \quad (113)$$

proving that it depends only on the contour, as it should be. However, the vector field B_i cannot be uniquely determined from the metric tensor and the covariantly unit vector $\mathbf{m}$.

5) The following comment is closely related to the previous one. Parallel transporters of integer spins $1, 2, \ldots$ are defined via Christoffel's Γ symbols and hence by the metric tensor, while parallel transporters of half-integer spins $1/2$, $3/2, \ldots$ are not: they are defined by the spin connection that is not uniquely constructed from the metric. Nevertheless, it should be expected that the holonomy for half-integer spins, being a diffeomorphism-invariant quantity, can be expressed through the metric only. Equation (109) solves this non-trivial problem: only the metric and its derivatives are involved. The solution is possible only with the holonomy represented in the form of a surface integral, as in Eq. (109). One cannot solve this problem in a contour form because it is not uniquely expressible through the metric. If that were possible, one would be able to write a parallel transporter along an open contour in terms of the metric as well, but that is not so for half-integer spins.

6) Equation (109) solves another long-standing problem in the Yang–Mills theory. It was recently shown [12–14] that the $SU(2)$ Yang–Mills partition function in three dimensions can be exactly rewritten in terms of gauge-invariant quantities given by the six components of the dual space metric tensor. The usual argument why this rewriting is not very useful is that external sources couple to the Yang–Mills potential and not to gauge-invariant quantities. However, we now have demonstrated that a typical source—the Yang–Mills Wilson loop—can be expressed not only through the potential but also through the metric tensor, which is gauge-invariant. Thus, not only the partition function, but also the Wilson loops in the $d = 3$ Yang–Mills theory can be expressed through local gauge-invariant quantities.

9. THE NON-ABELIAN STOKES THEOREM IN $d = 4$ GRAVITY

The aim of this section is to express the holonomy $W^G_{(J_1,J_2)}$ in the representation (J_1, J_2) in a curved $d = 4$ space through the metric tensor and its derivatives. Equation (68) presents the holonomy in terms of the (anti)self-dual parts of the spin connection. The latter is not uniquely determined by the metric, which is not satisfactory. In addition, we would like to eliminate the path ordering in the Yang–Mills Wilson loops $W^{\pi,\rho}$ entering Eq. (68). Both goals are achieved via the non-Abelian Stokes theorem similar to that of the previous section, which we now derive.

We start by applying representation (14) to the Yang–Mills Wilson loop W^π,

$$W_J^\pi = \int D\mathbf{n}\delta(\mathbf{n}^2-1)\exp\left(i\frac{J}{2}\int dS^{\mu\nu}\left[-F^a_{\mu\nu}(\pi)n^a + \right.\right.$$
$$\left.\left.+\ \epsilon^{abc}n^a\left(D_\mu(\pi)n\right)^b\left(D_\nu(\pi)n\right)^c\right]\right), \quad (114)$$

where

$$D^{ab}_\mu(\pi) = \partial_\mu\,\delta^{ab} + \epsilon^{acb}\,\pi^c_\mu$$

is the covariant derivative with respect to the self-dual part of the spin connection and $F^a_{\mu\nu}(\pi)$ is the appropriate field strength (53); it is related to the Riemann tensor via Eq. (55). We next introduce the antisymmetric tensor

$$m^{\kappa\lambda} = \frac{1}{2}n^a\,\eta^{aAB}\,e^{A\kappa}e^{B\lambda}. \quad (115)$$

The first term in Eq. (114) can be written as $-R_{\kappa\lambda\mu\nu}m^{\kappa\lambda}$. The tensor $m^{\kappa\lambda}$ has actually only two independent components. To see this, we introduce two covariant projector operators

$$P^+_{\kappa\lambda\mu\nu} = \frac{1}{4}\eta^{aAB}\eta^{aCD}e^A_\kappa e^B_\lambda e^C_\mu e^D_\nu =$$
$$= \frac{1}{4}(g_{\kappa\mu}g_{\lambda\nu} - g_{\kappa\nu}g_{\lambda\mu} + \sqrt{g}\epsilon_{\kappa\lambda\mu\nu}), \quad (116)$$

$$P^-_{\kappa\lambda\mu\nu} = \frac{1}{4}\bar\eta^{aAB}\bar\eta^{aCD}e^A_\kappa e^B_\lambda e^C_\mu e^D_\nu =$$
$$= \frac{1}{4}(g_{\kappa\mu}g_{\lambda\nu} - g_{\kappa\nu}g_{\lambda\mu} - \sqrt{g}\epsilon_{\kappa\lambda\mu\nu}), \quad (117)$$

satisfying the projector conditions

$$P^\pm_{\kappa\lambda\mu\nu}\,g^{\mu\mu'}g^{\nu\nu'}P^\pm_{\mu'\nu'\rho\sigma} = P^\pm_{\kappa\lambda\rho\sigma}, \quad (118)$$

$$P^\pm_{\kappa\lambda\mu\nu}\,g^{\mu\mu'}g^{\nu\nu'}P^\mp_{\mu'\nu'\rho\sigma} = 0, \quad (119)$$

$$P^\pm_{\kappa\lambda\mu\nu}\,g^{\kappa\mu}g^{\lambda\nu} = 3. \quad (120)$$

$P^\pm_{\kappa\lambda\mu\nu}$ are (covariantly) orthogonal projectors, each having three zero and three nonzero eigenvalues. They project a general antisymmetric tensor into (covariantly) self-dual and anti-self-dual parts, respectively. It is easy to verify that the tensor $m^{\kappa\lambda}$ introduced in Eq. (115) is self-dual,

$$P^-_{\kappa\lambda\mu\nu}\,m^{\kappa\lambda} = 0, \quad (121)$$

and satisfies the normalization condition

$$m^{\kappa\lambda}\,m_{\kappa\lambda} = P^+_{\kappa\lambda\mu\nu}\,m^{\kappa\lambda}\,m^{\mu\nu} = 1, \quad (122)$$

which follows from the normalization $\mathbf{n}^2 = 1$. Therefore, $m^{\kappa\lambda}$ indeed has only two independent degrees of freedom in a given metric. We change the integration variables in Eq. (114) from $\mathbf{n}$ to $m^{\kappa\lambda}$,

$$\int D\mathbf{n}\,\delta(\mathbf{n}^2-1)\ldots = \int Dm^{\kappa\lambda}\sqrt{g}\ \times$$
$$\times\ \delta(P^-_{\kappa\lambda\mu\nu}\,m^{\mu\nu})\,\delta(m^{\kappa\lambda}m_{\kappa\lambda}-1)\ldots \quad (123)$$

We now compute the covariant derivative of $m^{\kappa\lambda}$ as

$$m^{\kappa\lambda}{}_{;\mu} = \partial_\mu m^{\kappa\lambda} + \Gamma^\kappa_{\mu\nu}m^{\nu\lambda} + \Gamma^\lambda_{\mu\nu}m^{\kappa\nu} = \frac{1}{2}\eta^{aAB}\ \times$$
$$\times\left[\partial_\mu n^a\,e^{A\kappa}e^{B\lambda} + n^a(\partial_\mu e^{A\kappa} + \Gamma^\kappa_{\mu\nu}e^{A\nu})e^{B\lambda} + \right.$$
$$\left. + n^a e^{A\kappa}(\partial_\mu e^{B\lambda} + \Gamma^\lambda_{\mu\nu}e^{B\nu})\right] =$$
$$= \frac{1}{2}\eta^{aAB}\left[\partial_\mu n^a\,e^{A\kappa}e^{B\lambda} - \right.$$
$$\left. -\ n^a\omega^{AC}_\mu e^{C\kappa}e^{B\lambda} - n^a e^{A\kappa}\omega^{BC}_\mu e^{C\lambda}\right], \quad (124)$$

where in the last equation, we have used fundamental relation (33). We now insert the decomposition of the spin connection ω^{AB}_μ into the self-dual and anti-self-dual parts, Eq. (51). Using the relations for the $\eta,\bar\eta$ symbols,

$$\eta^{aAB}\eta^{bAC} = \delta^{ab}\delta^{BC} + \epsilon^{abc}\eta^{cBC},$$
$$\bar\eta^{aAB}\bar\eta^{bAC} = \delta^{ab}\delta^{BC} + \epsilon^{abc}\bar\eta^{cBC}, \quad (125)$$

$$\eta^{aAB}\bar\eta^{bAC} = \eta^{aAC}\bar\eta^{bAB}, \quad (126)$$

it is easy to see that only the self-dual piece of ω^{AB}_μ survives in Eq. (124), giving

$$m^{\kappa\lambda}{}_{;\mu} = \frac{1}{2}\eta^{aAB}e^{A\kappa}e^{B\lambda}\left(\partial_\mu\,\delta^{ab} + \epsilon^{acb}\pi^c_\mu\right)n^b =$$
$$= \frac{1}{2}\eta^{aAB}e^{A\kappa}e^{B\lambda}\left(D_\mu(\pi)\,n\right)^a. \quad (127)$$

In other words, the gravitational covariant derivative of $m^{\kappa\lambda}$ is expressed through the Yang–Mills covariant derivative of the $\mathbf{n}$ field entering the second term in Eq. (114).

Using consecutively Eqs. (125) and (127), we finally rewrite Eq. (114) in terms of the metric:

$$W^\pi_{J_1} = \int Dm^{\kappa\lambda}\sqrt{g}\,\delta(P^-_{\kappa\lambda\mu\nu}\,m^{\mu\nu})\delta(m^{\kappa\lambda}m_{\kappa\lambda}-1)\ \times$$
$$\times\ \exp\left(i\frac{J_1}{2}\int dS^{\mu\nu}\left[-R_{\kappa\lambda\mu\nu}\,m^{\kappa\lambda} - \right.\right.$$
$$\left.\left. -\ \frac{1}{2}\sqrt{g}\,\epsilon_{\kappa\rho\sigma\tau}\,g_{\lambda\lambda'}\,m^{\kappa\lambda'}\,m^{\lambda\rho}{}_{;\mu}\,m^{\sigma\tau}{}_{;\nu}\right]\right). \quad (128)$$

Similarly, W^ρ is obtained by integrating over the anti-self-dual covariantly unit tensors:

$$W^\rho_{J_2} = \int Dm^{\kappa\lambda}\,\sqrt{g}\,\delta(P^+_{\kappa\lambda\mu\nu}\,m^{\mu\nu})\,\delta(m^{\kappa\lambda}m_{\kappa\lambda}-1)\times \times\exp\left(i\frac{J_2}{2}\int dS^{\mu\nu}\left[-R_{\kappa\lambda\mu\nu}\,m^{\kappa\lambda}+ +\frac{1}{2}\sqrt{g}\,\epsilon_{\kappa\rho\sigma\tau}\,g_{\lambda\lambda'}\,m^{\kappa\lambda'}\,m^{\lambda\rho}{}_{;\mu}\,m^{\sigma\tau}{}_{;\nu}\right]\right). \quad (129)$$

As derived in Sec. 4.2, the gravitational holonomy in the representation (J_1, J_2) is the product of the two components,

$$W^G_{(J_1,J_2)} = W^\pi_{J_1} W^\rho_{J_2}. \quad (130)$$

Equations (128), (129), and (130) constitute the «non-Abelian Stokes theorem» for the holonomy in a curved $d = 4$ space. It expresses the holonomy via surface integrals over spanning surfaces for the contour, and presents the holonomy in terms of the metric tensor and its derivatives only, without refering to the spin connection, even for half-integer representations (J_1, J_2).

10. CONCLUSIONS

The main results of this paper are the non-Abelian Stokes theorems for holonomies: the Yang–Mills Wilson loop (Eq. (14)) and the traces of parallel transporters in curved $d = 3$ (Eq. (109)) and $d = 4$ (Eqs. (128) and (129)) spaces. In all these cases, the path-ordered exponentials of the connections are replaced by ordinary exponentials of surface integrals, which, however, do not actually depend on the way the surface is spanned on the contour. The price to pay for the removal of path ordering is high: we obtain functional integrals instead. In the simplest case of the $SU(2)$ Yang–Mills theory, this is an integral over a unit 3-vector $\mathbf{n}$ «living» on the surface; for the $d = 3$ Riemannian manifold, this is an integral over a covariantly unit 3-vector $\mathbf{m}$, and for $d = 4$, one integrates over (anti)self-dual covariantly unit tensors.

In spite of the occurrence of functional integration, we believe that our formulas are aesthetically appealing. Compared to path-ordered exponentials, they are better suited to averaging over quantum ensembles of Yang–Mills fields or over various metrics. We hope that elegant formulas can also be used in more general settings.

In addition to the general non-Abelian Stokes formulas, we have presented holonomy as a surface integral for a specific background, namely for a constant-curvature $d = 3$ space with the cylinder topology $S^2 \times R$. The «gravitational Wilson loop» is given by a formula for the character whose argument is the winding number of external coordinates, see Sec. 8.

Parallel transporters of integer spins have a dual description: such a transporter can be defined either as a path-ordered exponential of Christoffel symbols or as a path-ordered exponential of the spin connection in the appropriate representation. In Sec. 4, we have shown that these representations are equivalent. Even though the spin connection is not uniquely determined by the metric tensor, this equivalence implies that the holonomy written in terms of the spin connection can in fact be expressed through the metric only.

For half-integer spins, the situation is far less trivial because the only way to define the holonomy is via the spin connection, and it is not at all clear beforehand that the holonomy can be uniquely written through the metric tensor and its derivatives. The non-Abelian Stokes theorem proved in this paper demonstrates that this rewriting can be achieved, but only with the holonomy presented in the surface form. Although the surface integral does not depend on the way one draws the surface and can actually be written as an integral along the contour, the contour form is not uniquely defined by the surface one, which reflects the ambiguity in determining the spin connection from the metric.

This finding has an interesting implication for the Yang–Mills theory in three dimensions, which can be identically reformulated as a quantum gravity theory with the partition function written as a functional integral over the metric tensor of the dual space [12–14]. This metric tensor is local and gauge invariant (in the Yang–Mills sense). However, one might wish to calculate the average of the Wilson loop, which is originally defined by the Yang–Mills potential, but not by the metric tensor. In the «quantum gravity» formulation, the Yang–Mills Wilson loop becomes a parallel transporter in the gravitational sense. It is therefore very important that the Yang–Mills Wilson loop in any representation can be expressed through the gauge-invariant metric tensor. Thus, not only the partition function but also the Wilson loop can be presented in terms of local and gauge-invariant quantities. This subject is described in more detail elsewhere [15].

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APPENDIX

Proof that Eq. (109) does not depend on the surface

The path-integral representation for the «gravitational Wilson loop» in Eq. (109) must not depend on the choice of the spanning surface for a given contour, but only on the contour itself. To prove that this is so, we verify Eq. (112),

$$(\nabla_k)^k_l\, V^l = 0, \tag{131}$$

where

$$\begin{aligned} V^k &= \left(R\delta^k_p - 2R^k_p\right) m^p + \\ &\quad + \epsilon^{ijk}\epsilon_{pqr} m^p (\nabla_i m)^q (\nabla_j m)^r, \\ m^i m^j g_{ij} &= 1. \end{aligned} \tag{132}$$

To simplify the notation, we denote covariant derivatives by «;» (see [20]). Explicitly, the covariant derivatives of a scalar, a vector, and a tensor are given by

$$\begin{aligned} &S_{;k} = \partial_k\, S, \\ &V^i_{;k} = \partial_k\, V^i + \Gamma^i_{kl}\, V^l, \quad V_{i;k} = \partial_k\, V_i - \Gamma^l_{ik}\, V_l, \\ &T^{ij}_{;k} = \partial_k\, T^{ij} + \Gamma^i_{kl}\, T^{lj} + \Gamma^j_{kl}\, T^{il}, \\ &T_{ij;k} = \partial_k\, T_{ij} - \Gamma^l_{ik}\, T_{lj} - \Gamma^l_{jk}\, T_{il}, \quad \text{etc.} \end{aligned} \tag{133}$$

The ordinary derivative of a convolution of two tensors can be written as the sum of covariant derivatives,

$$\begin{aligned} \partial_k \left(T^{(1)\ldots i}_{\ldots}\, T^{(2)\ldots}_{\ldots i}\right) &= T^{(1)\ldots i}_{\ldots;k}\, T^{(2)\ldots}_{\ldots i} + \\ &\quad + T^{(1)\ldots i}_{\ldots}\, T^{(2)\ldots}_{\ldots i;k}\,. \end{aligned} \tag{134}$$

We apply the covariant derivative to the first term of the vector V^k,

$$\begin{aligned} &\nabla_k \left[\left(R\delta^k_p - 2R^k_p\right) m^p\right] = \\ &\quad = \left(R\delta^k_p - 2R^k_p\right)_{;k} m^p + \left(R\delta^k_p - 2R^k_p\right) m^p_{;k}. \end{aligned} \tag{135}$$

The covariant derivative of the Einstein tensor is known to be zero [20, Eq. (92.10)]. Therefore, only the second term survives in Eq. (135).

We next apply the covariant derivative to the second term of V^k as

$$\begin{aligned} &\nabla_k \left[\epsilon^{ijk}\, \epsilon_{pqr}\, m^p (\nabla_i\, m)^q (\nabla_j\, m)^r\right] = \\ &\quad = \epsilon^{ijk}\, \epsilon_{pqr}\, (\nabla_k\, m)^p (\nabla_i\, m)^q (\nabla_j\, m)^r + \\ &\quad\quad + 2\,\epsilon^{ijk}\, \epsilon_{pqr}\, m^p (\nabla_i\, m)^q (\nabla_k \nabla_j\, m)^r. \end{aligned} \tag{136}$$

The first term here vanishes, for the following reasons. Differentiating the condition that m^i is a covariantly unit vector, we obtain

$$\begin{aligned} 0 = \partial_k \left(m^i m^j g_{ij}\right) &= 2g_{ij} \left(\nabla_k\, m\right)^i m^j = \\ &= 2\left(\nabla_k\, m\right)^i m_i, \end{aligned} \tag{137}$$

because the covariant derivative of the metric tensor is zero. This implies that the three vectors $(\nabla_{1,2,3}\, m)^i$ are not linearly independent, because three linearly independent vectors cannot be orthogonal to a given vector (in this case, m_i) in three dimensions. The first term in Eq. (136) is the antisymmetrized product of these three linearly dependent vectors and is therefore zero.

The second term in Eq. (136) contains the commutator of covariant derivatives, equal to

$$\begin{aligned} \epsilon^{ijk} (\nabla_k \nabla_j\, m)^r &= \frac{1}{2}\epsilon^{ijk} \left[\nabla_k \nabla_j\right]^r_s\, m^s = \\ &= \frac{1}{2}\epsilon^{ijk}\, g^{rt}\, R_{tskj}\, m^s \end{aligned} \tag{138}$$

where R_{tskj} is the Riemann tensor. Therefore, the second (and the only nonzero) term in Eq. (136) can be written as

$$\epsilon^{ijk}\, \epsilon_{pqr}\, g^{rt}\, R_{tskj}\, m^p m^s \left(\nabla_i\, m\right)^q. \tag{139}$$

We next use Eq. (70) to express the Riemann tensor through the Ricci and metric tensors and write the product of two epsilon symbols as a determinant made of Kronecker deltas. Performing all convolutions, we obtain that Eq. (139) can be identically rewritten as

$$\begin{aligned} &\left[g_{qs} \left(R\delta^i_p - 2R^i_p\right) - g_{ps} \left(R\delta^i_q - 2R^i_q\right)\right] \times \\ &\quad \times m^p m^s \left(\nabla_i\, m\right)^q. \end{aligned} \tag{140}$$

Here, the first term is zero because of Eq. (137) and in the second term, we use

$$g_{ps} m^p m^s = 1.$$

This gives

$$-\left(R\,\delta^i_q - 2R^i_q\right) \left(\nabla_i\, m\right)^q, \tag{141}$$

which cancels exactly with Eq. (135). Thus, $(\nabla_k)^k_l\, V^l = 0$, q.e.d.

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