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VECTOR BOSON IN THE CONSTANT ELECTROMAGNETIC FIELD

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The propagator and the complete sets of in- and out-solutions of the wave equation, together with the Bogoliubov coefficients relating these solutions are obtained for the vector W boson (with the gyromagnetic ratio $g = 2$) in the constant electromagnetic field. When only the electric field is present, the Bogoliubov coefficients are independent of the boson polarization and are the same as for the scalar boson. For the collinear electric and magnetic fields, the Bogoliubov coefficients for states with the boson spin perpendicular to the field are again the same as in the scalar case. For the W^- spin parallel (antiparallel) to the magnetic field, the Bogoliubov coefficients and the one-loop contributions to the imaginary part of the Lagrange function are obtained from the corresponding expressions for the scalar case by the substitution $m^2 \to m^2 + 2eH$ ($m^2 \to m^2 - 2eH$). For the gyromagnetic ratio $g = 2$, the vector boson interaction with the constant electromagnetic field is described by the functions that can be expected by comparing the scalar and Dirac particle wave functions in the constant electromagnetic field.

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1. INTRODUCTION

Vector bosons occupy an intermediate place between low-spin particles (with the spins 0 and 1/2) and higher-spin particles. They can therefore share some of the problems encountered in considering higher-spin particle interactions with a strong electromagnetic field. The most conspicuous feature of the vector boson interaction in the case of $g = 2$ is the appearance of tachyonic modes in the overcritical magnetic field. The ways to deal with this problem in the framework of non-abelian theories are analyzed in [1]. But are there any others? According to [2], problems in treating the pair production by the electric field by diagonalizing the Hamiltonian precisely occur for $g = 2$. This is surprising in view of a successful calculation of the Lagrange function of the constant field in the one-loop approximation [3]. We calculate the pair production by the constant field using the Bogoliubov coefficients (which contain all the information about this process); as expected, the results obtained are in agreement with those in [3] and [4].

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2. VECTOR BOSON IN THE CONSTANT ELECTRIC FIELD

We assume $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ and set $e = |e|$. The wave function of the W^- boson (with $g = 2$) in a source-free space (where $\partial_\mu F^{\mu\nu} = 0$) satisfies the equation [1, 5]

$$(-D_\sigma D^\sigma + m^2)\psi_\mu - 2ieF_{\mu\nu}\psi^\nu = 0 \tag{1}$$

and the constraint

$$D_\mu \psi^\mu = 0, \quad D_\mu = \partial_\mu + ieA_\mu. \tag{2}$$

With the vector potential chosen such that $A_3 = -Et$ and $A_1 = A_2 = A_0 = 0$, it follows from (1) that ψ^1 and ψ^2 satisfy the same equation as in scalar case,

$$(-D^2 + m^2)\psi^{1,2} = 0. \tag{3}$$

For ψ^3 and ψ^0, it follows from (1) that

$$\begin{aligned}(-D^2 + m^2)\psi^3 - 2ieE\psi^0 = 0,\\ (-D^2 + m^2)\psi^0 - 2ieE\psi^3 = 0.\end{aligned} \tag{4}$$

Introducing $\psi^\pm = \psi^0 \pm \psi^3$, we rewrite Eqs. (4) as

$$(-D^2 + m^2 \mp 2ieE)\psi^\pm = 0, \tag{5}$$

which can be obtained from (3) by the substitution $m^2 \to m^2 \mp 2ieE$. We see that the vector boson wave function can be obtained from the corresponding scalar boson wave function by simple rules.

We now do this. We let ${}_+\psi_{\mathbf{p}}$ denote the positive-frequency in-solution for the (negatively charged) scalar boson. The subscript $\mathbf{p} = (p_1, p_2, p_3)$ is dropped in what follows. Then [6]

$$ {}_+\psi \propto D_\nu(\tau) \exp(i\mathbf{p}\cdot\mathbf{x}), \tag{6} $$

where $D_\nu(\tau)$ is the parabolic cylinder function [7] and

$$ \begin{aligned} \nu &= \frac{i\lambda}{2} - \frac{1}{2}, \\ \tau &= -\sqrt{2eE}\exp\left(-i\frac{\pi}{4}\right)\left(t - \frac{p_3}{eE}\right), \\ \lambda &= \frac{m^2 + p_1^2 + p_2^2}{eE}. \end{aligned} \tag{7} $$

For the vector boson, we obtain

$$ {}_+\psi = \begin{bmatrix} \psi^0 \\ \psi^1 \\ \psi^2 \\ \psi^3 \end{bmatrix} = $$

$$ = \begin{bmatrix} c_+ D_{\nu+1}(\tau) + c_- D_{\nu-1}(\tau) \\ c_1 D_\nu(\tau) \\ c_2 D_\nu(\tau) \\ c_+ D_{\nu+1}(\tau) - c_- D_{\nu-1}(\tau) \end{bmatrix} \exp(i\mathbf{p}\cdot\mathbf{x}), \tag{8} $$

were

$$ \psi^1 = {}_+\psi^1, \quad \psi^2 = {}_+\psi^2, $$

and

$$ \begin{aligned} \psi^0 &\equiv {}_+\psi^0 = \frac{1}{2}({}_+\psi^+ + {}_+\psi^-), \\ \psi^3 &\equiv {}_+\psi^3 = \frac{1}{2}({}_+\psi^+ - {}_+\psi^-), \\ {}_+\psi^\pm &= 2c_\pm D_{\nu\pm1} \exp(i\mathbf{p}\cdot\mathbf{x}). \end{aligned} \tag{9} $$

The function $D_{\nu\pm1}(\tau)$ is obtained from $D_\nu(\tau)$ in Eqs. (6) and (7) by the substitution

$$ m^2 \to m^2 \mp 2ieE. $$

Arbitrary coefficients c_1, c_2, and $c_\pm \equiv {}_+c_\pm$ determine the polarization of the vector boson. They are not independent because of constraint (2),

$$ c_1p_1 + c_2p_2 + \sqrt{2eE}e^{i\pi/4}[(1+\nu){}_+c_+ - {}_+c_-] = 0. \tag{10} $$

For the negative-frequency in-solution (for the scalar boson), we have

$$ {}_-\psi \propto [D_\nu(\tau)]^* \exp(i\mathbf{p}\cdot\mathbf{x}) \tag{11} $$

instead of (6). The star denotes the complex conjugation. Similarly to (8), the parabolic cylinder functions entering ${}_-\psi^\pm$ are obtained from $[D_\nu(\tau)]^*$ in (11) by the substitution

$$ m^2 \to m^2 \mp 2ieE, $$

and therefore,

$$ {}_-\psi = $$

$$ = \begin{bmatrix} c_+ D_{\nu^*-1}(\tau^*) + c_- D_{\nu^*+1}(\tau^*) \\ c_1 D_{\nu^*}(\tau^*) \\ c_2 D_{\nu^*}(\tau^*) \\ c_+ D_{\nu^*-1}(\tau^*) - c_- D_{\nu^*+1}(\tau^*) \end{bmatrix} \exp(i\mathbf{p}\cdot\mathbf{x}). \tag{12} $$

(We have $c_\pm = {}_-c_\pm$ in Eq. (12) and similarly in other cases.) The constraint takes the form

$$ c_1p_1 + c_2p_2 + \sqrt{2eE}e^{-i\pi/4}[{}_-c_+ + \nu{}_-c_-] = 0. \tag{13} $$

Nothing prevents us from assuming that c_1 and c_2 in (12) are the same as in (8).

The negative-frequency out-solution is obtained from the positive-frequency in-solution by changing the sign of τ in the parabolic cylinder functions in Eq. (8),

$$ {}^-\psi = \begin{bmatrix} c_+ D_{\nu+1}(-\tau) + c_- D_{\nu-1}(-\tau) \\ c_1 D_\nu(-\tau) \\ c_2 D_\nu(-\tau) \\ c_+ D_{\nu+1}(-\tau) - c_- D_{\nu-1}(-\tau) \end{bmatrix} \times $$

$$ \times \exp(i\mathbf{p}\cdot\mathbf{x}), \quad {}^-c_\pm = -{}_+c_\pm, \tag{14} $$

see (112a). The constraint is given by

$$ c_1p_1 + c_2p_2 + \sqrt{2eE}e^{i\pi/4}[{}^-c_- - (1+\nu){}^-c_+] = 0. \tag{15} $$

Similarly, the positive-frequency out-solution can be found from ${}_-\psi$ in Eq. (12) by changing the sign of τ^*,

$$ {}^+\psi = \begin{bmatrix} c_+ D_{\nu^*-1}(-\tau^*) + c_- D_{\nu^*+1}(-\tau^*) \\ c_1 D_{\nu^*}(-\tau^*) \\ c_2 D_{\nu^*}(-\tau^*) \\ c_+ D_{\nu^*-1}(-\tau^*) - c_- D_{\nu^*+1}(-\tau^*) \end{bmatrix} \times $$

$$ \times \exp(i\mathbf{p}\cdot\mathbf{x}). \tag{16} $$

The corresponding constraint is

$$ c_1p_1 + c_2p_2 - \sqrt{2eE}e^{-i\pi/4}[\nu{}^+c_- + {}^+c_+] = 0. \tag{17} $$

For the scalar boson, the in- and out-solutions are related by [6]

$$
\begin{gathered}
{}_+\psi_n = c_{1n}{}^+\psi_n + c_{2n}{}^-\psi_n, \\
{}_-\psi_n = c_{2n}^*{}^+\psi_n + c_{1n}^*{}^-\psi_n, \\
c_{1n} = \frac{\sqrt{2\pi}}{\Gamma((1-i\lambda)/2)}\exp\left[-\frac{\pi}{4}(\lambda-i)\right], \\
c_{2n} = \exp\left[-\frac{\pi}{2}(\lambda+i)\right], \\
|c_{1n}|^2 - |c_{2n}|^2 = 1.
\end{gathered} \tag{18}
$$

The subscript n indicates a set of quantum numbers; here, $n = \mathbf{p}$. By a straightforward calculation similar to the one in the scalar case, we find that Eqs. (18) also hold for the vector boson and that

$$
\begin{gathered}
{}^+c_- = \frac{i}{\nu}{}_+c_- = -{}_-c_- = -\frac{i}{\nu}{}^-c_-, \\
{}^+c_+ = -i(1+\nu){}_+c_+ = -{}_-c_+ = i(1+\nu){}^-c_+.
\end{gathered} \tag{19}
$$

These relations guarantee that the wave functions ${}_\pm\psi$ and ${}^\pm\psi$ are normalized in the same manner and that any constraint can be obtained from any other using Eqs. (19).

As seen from (18), the Bogoliubov coefficients c_{1n} and c_{2n} do not depend on the boson polarization in the constant electric field. The imaginary part of the Lagrange function is therefore given by $3\,\mathrm{Im}\,\mathrm{L}_{spin0}$ in agreement with [3,4].

3. VECTOR BOSON IN THE CONSTANT ELECTROMAGNETIC FIELD

We now add a collinear constant magnetic field to the constant electric field. For $A_2 = Hx_1$, we obtain from Eq. (1) that

$$
\begin{gathered}
(-D^2+m^2)\psi_1 - 2ieH\psi_2 = 0, \\
(-D^2+m^2)\psi_2 + 2ieH\psi_1 = 0.
\end{gathered} \tag{20}
$$

Introducing

$$
\begin{gathered}
\tilde{\psi}_1 = \psi_1 - i\psi_2, \quad \tilde{\psi}_2 = \psi_1 + i\psi_2, \\
\psi_1 = \frac{1}{2}(\tilde{\psi}_1 + \tilde{\psi}_2), \quad \psi_2 = \frac{i}{2}(\tilde{\psi}_1 - \tilde{\psi}_2),
\end{gathered} \tag{21}
$$

we rewrite Eqs. (20) as

$$
\begin{gathered}
(-D^2+m^2+2eH)\tilde{\psi}_1 = 0, \\
(-D^2+m^2-2eH)\tilde{\psi}_2 = 0,
\end{gathered} \tag{22}
$$

and therefore, $\tilde{\psi}_{1,2}$ can be obtained from the scalar boson wave function by the substitutions

$$
m^2 \to m^2 \pm 2eH.
$$

We can therefore write

$$
\begin{gathered}
\tilde{\psi}_1 \propto 2c_1 D_{n-1}(\zeta), \quad \tilde{\psi}_2 \propto 2c_2 D_{n+1}(\zeta), \\
\zeta = \sqrt{2eH}\left(x_1 + \frac{p_2}{eH}\right).
\end{gathered} \tag{23}
$$

Instead of (8), we thus have

$$
{}_+\psi_{p_2p_3n} = \begin{bmatrix} [c_+D_{\nu+1}(\tau) + c_-D_{\nu-1}(\tau)]D_n(\zeta) \\ [c_1D_{n-1}(\zeta) + c_2D_{n+1}(\zeta)]D_\nu(\tau) \\ i[c_1D_{n-1}(\zeta) - c_2D_{n+1}(\zeta)]D_\nu(\tau) \\ [c_+D_{\nu+1}(\tau) - c_-D_{\nu-1}(\tau)]D_n(\zeta) \end{bmatrix} \times \exp(i(p_2x_2 + p_3x_3)), \tag{24}
$$

and similarly for the other ψ functions. Here,

$$
\nu = \frac{i\lambda}{2} - \frac{1}{2}, \quad \lambda = \frac{m^2 + eH(2n+1)}{eE}. \tag{25}
$$

The constraints can be obtained from the previous ones by the substitution

$$
c_1p_1 + c_2p_2 \to -i\sqrt{2eH}[(1+n)c_2 - c_1]. \tag{26}
$$

We note that $D_\mu\psi^\mu$ is proportional to the scalar wave function

$$
D_n(\zeta)D_\nu(\tau)\exp[i(p_2x_2 + p_3x_3)]
$$

(which is dropped in the expressions similar to (10) with modification (26), or in (116)). Equations (67) and (98) were used to obtain the constraints. It follows from the derivation that the presence of c_1 in the right-hand side of (26) is due to the assumption that $D_{n-1}(\zeta)$ is not zero in Eq. (24), i.e., $n \geq 1$.

Using (24) and (26), we can build three polarization states $\psi(i,x)$, $i = 1,2,3$, see Sec. 7. For these states, the respective minimum values of n in Eq. (25) are $-1, 0, 1$. Thus the Bogoliubov coefficients depend on all the four quantum numbers ($n = p_2, p_3, n, i$) through the minimum value of n.

Because

$$
2\,\mathrm{Im}\,\mathrm{L} = \sum_n \ln(1 + |c_{2n}|^2),
$$

it is easy to show that in agreement with [4],

$$
\begin{aligned}
\mathrm{Im}\,2\mathrm{L}_{spin1} = 2\cdot 3\,\mathrm{Im}\,\mathrm{L}_{spin0} + \\
+ \left\{\ln\left[1 + \exp\left(-\pi\frac{m^2 - eH}{eE}\right)\right] - \right. \\
\left. - \ln\left[1 + \exp\left(-\pi\frac{m^2 + eH}{eE}\right)\right]\right\}\frac{\alpha}{\pi}EHVT.
\end{aligned} \tag{27}
$$

The factors outside the braces give the statistical weight of the «correcting» states, see Eqs. (3.6) and (3.7) in [6].

The Bogoliubov coefficients allow finding the transition probability from any initial to any final state (with arbitrary occupation numbers) [6]. For example, if the initial state is the vacuum, we have

$$|c_{1n}|^{-2}\{1+w_n+w_n^2+w_n^3+\cdots\}=1,\quad w_n=\frac{|c_{2n}|^2}{|c_{1n}|^2}, \tag{28}$$

for the cell with the set of quantum numbers $n = p_2, p_3, n, i$. The term $|c_{1n}|^{-2}w_n^k$ gives the probability for the production of k pairs, $k = 0, 1, 2, \ldots$. The events occurring in cells with different quantum numbers are independent.

4. THE FREE VECTOR BOSON PROPAGATOR

The wave functions of a free vector boson with the momentum $p^\mu = (p^0, 0, 0, p^3)$ can be written as

$$\psi^\mu(i,x)=\frac{u^\mu(i)}{\sqrt{2|p^0|}}\exp(ip\cdot x),\quad \eta^{\mu\nu}=\mathrm{diag}(-1,1,1,1),\quad \mu=0,1,2,3, \tag{29}$$

$$u(1)=\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\quad u(2)=\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\quad u(3)=\frac{1}{m}\begin{bmatrix}p_3\\0\\0\\p^0\end{bmatrix}.$$

These solutions satisfy wave equation (1) and constraint (2) with the external field switched off. Summing $\psi^\mu(i,x)\psi^{\nu*}(i,x')$ over polarizations, we find

$$\sum_{i=1}^{3}\psi^\mu(i,x)\psi^{\nu*}(i,x')=\frac{1}{2|p^0|}\times \begin{bmatrix}\frac{p_3^2}{m^2} & 0 & 0 & \frac{p_3p^0}{m^2}\\ 0&1&0&0\\ 0&0&1&0\\ \frac{p_3p^0}{m^2}&0&0&\frac{(p^0)^2}{m^2}\end{bmatrix}\exp\left[ip\cdot(x-x')\right]. \tag{30}$$

If we use helicity states instead of linear polarization states (29) (cf. § 16 in [8]), we obtain the same result (30). In general, we must replace the matrix in the right-hand side of (30) by $\eta^{\mu\nu}+p^\mu p^\nu/m^2$. This case differs from the scalar particle case only by the presence of this matrix in the expression for the propagator (which is similar to (51)). The vector boson propagator can therefore be obtained from the scalar one,

$$G_{spin0}(x,x')=\frac{1}{(2\pi)^4}\int d^4p\frac{\exp[ip\cdot(x-x')]}{p^2+m^2-i\varepsilon}=\frac{1}{(4\pi)^2}\int_0^\infty\frac{ds}{s^2}\exp\left[-ism^2+\frac{i(x-x')^2}{4s}\right], \tag{31}$$

considered as a unit matrix over discrete indices, by acting on it with the differential operator

$$G^{\mu\nu}(x,x')=\left(\eta^{\mu\nu}-\frac{1}{m^2}\frac{\partial^2}{\partial x_\mu\partial x_\nu}\right)G_{spin0}(x,x'). \tag{32}$$

Because the scalar boson propagator satisfies the equation

$$(-\partial_\mu\partial^\mu+m^2)G_{spin0}(x,x')=\delta^4(x-x'), \tag{33}$$

we have

$$(-\partial_\sigma\partial^\sigma+m^2)G^{\mu\nu}(x,x')=\left(\eta^{\mu\nu}-\frac{1}{m^2}\frac{\partial^2}{\partial x_\mu\partial x_\nu}\right)\delta^4(x-x') \tag{34}$$

for the vector boson. We note that the right-hand side is not simply given by $\delta^4(x-x')$. The complication is due to the existence of constraints. This prevents us from using the well-known methods of constructing propagators of scalar and spinor particles in an external field [9, 10]. An elegant way to circumvent this difficulty was given by Vanyashin and Terentyev [3].

5. THE VECTOR BOSON PROPAGATOR IN THE CONSTANT MAGNETIC FIELD

To write the propagator, we need the complete set of orthonormalized solutions. The orthonormalization is performed by expressing the vector current as [5]

$$J_\mu=-i\{\psi^{\nu*}(D_\mu\psi_\nu-D_\nu\psi_\mu)-(D^*_\mu\psi^*_\nu-D^*_\nu\psi^*_\mu)\psi^\nu\},\quad D_\mu=\partial_\mu+ieA_\mu. \tag{35}$$

We note that our expression for D_μ in Eq. (35) coincides with that in [5]; although our $\eta_{\mu\nu}$ has a different sign, we also replace $e\to -e$, using that $e=|e|$ and assuming that W^- is a particle by analogy with the electron.

In the space without a field, the expression for J_μ can be written similarly to the scalar case up to divergence terms, (see § 15 in [8]). It is remarkable that with constraint (2), the same is true in the presence of a field. Indeed,

$$-\psi^{\nu *} D_\nu \psi_\mu = -\partial_\nu(\psi^{\nu *}\psi_\mu) + \psi_\mu D^*_\nu \psi^{\nu *}. \quad (36)$$

The last term in the right-hand side vanishes because of Eq. (2) for the boson with $g = 2$. Similarly,

$$(D^*_\nu \psi^*_\mu)\psi^\nu = \partial_\nu(\psi^\nu \psi^*_\mu) - \psi^*_\mu D_\nu \psi^\nu = \partial_\nu(\psi^\nu\psi^*_\mu), \quad (37)$$

and therefore,

$$J_\mu = -i\{\psi^{\nu *} D_\mu \psi_\nu - (D^*_\mu \psi^*_\nu)\psi^\nu - \partial_\nu[\psi^{\nu *}\psi_\mu - \psi^\nu \psi^*_\mu]\}. \quad (38)$$

To normalize the wave functions, we need only J_0. Straightforward calculations show that the divergence terms do not contribute to J_0 for the fields considered here. We then have

$$J^0 = -J_0 = i\{\psi^{\nu *} D_0 \psi_\nu - (D^*_0 \psi^*_\nu)\psi^\nu\}. \quad (39)$$

For orthonormalization, we use the expression

$$J^0(\psi', \psi) = i\{\psi'^{\nu *} D_0 \psi_\nu - (D^*_0 \psi'^*_\nu)\psi^\nu\}. \quad (40)$$

Our vector potentials are such that $A_0(x) = 0$. It then follows that $D_0 = \partial/\partial t$ and

$$J^0(\psi', \psi) = i\{\psi'^*_k \overleftrightarrow{\partial}_t \psi_k - \psi'^{0*} \overleftrightarrow{\partial}_t \psi^0\}, \quad (41)$$

where the sum over k runs from 1 to 3.

The positive-frequency solution of wave equation (1) with $A_\mu(x) = \delta_{\mu 2} H x_1$ is given by

$$\psi^\mu_{p_2,p_3,n} = \begin{bmatrix} c^0 D_n(\zeta) \\ c_1 D_{n-1}(\zeta) + c_2 D_{n+1}(\zeta) \\ i[c_1 D_{n-1}(\zeta) - c_2 D_{n+1}(\zeta)] \\ c_3 D_n(\zeta) \end{bmatrix} \times \exp\left[i(p_2 x_2 + p_3 x_3 - p^0 t)\right]. \quad (42)$$

The elements of this column correspond to $\mu = 0, 1, 2, 3$,

$$\zeta = \sqrt{2eH}\left(x_1 + \frac{p_2}{eH}\right),$$

$$p^0 = \sqrt{m^2 + p_3^2 + eH(2n+1)}.$$

The coefficients c determining the boson polarization satisfy the constraint

$$-ip^0 c^0 + ip_3 c_3 + \sqrt{2eH}[(n+1)c_2 - c_1] = 0. \quad (43)$$

For states with the polarizations c' and c, Eqs. (41) and (42) imply that

$$J^0(\psi', \psi) = 2p^0\{2c'^*_1 c_1 D^2_{n-1}(\zeta) + 2c'^*_2 c_2 D^2_{n+1}(\zeta) + (c'^*_3 c_3 - c'^{0*} c^0) D^2_n(\zeta)\}. \quad (44)$$

Integrating over x_1, we find

$$\int_{-\infty}^{\infty} dx_1 J^0(\psi', \psi) = 2p^0 n! \sqrt{\frac{\pi}{eH}} \times \left\{\frac{2}{n} c'^*_1 c_1 + 2(n+1) c'^*_2 c_2 + c'^*_3 c_3 - c'^{0*} c^0\right\}, \quad (45)$$

$$\int_{-\infty}^{\infty} dx_1 D^2_n(\zeta) = n! \sqrt{\frac{\pi}{eH}}.$$

Using the orthonormalization conditions

$$\int dx_1 J^0({}_\pm\psi(i,x), {}_\pm\psi(j,x)) = \pm\,\delta_{ij}, \quad i,j = 1,2,3, \quad (46)$$

and constraint (43), we find the positive-frequency polarization states

$$\psi^\mu(1,x) = N(1) \begin{bmatrix} (n+1)\sqrt{2eH} p^0 D_n(\zeta) \\ i m^2_\perp D_{n+1}(\zeta) \\ m^2_\perp D_{n+1}(\zeta) \\ (n+1)\sqrt{2eH} p_3 D_n(\zeta) \end{bmatrix} \times \exp\left[i(p_2 x_2 + p_3 x_3 - p^0 t)\right], \quad (47)$$

where

$$\mu = 0,1,2,3, \quad m^2_\perp = m^2 + eH(2n+1),$$

$$p^0 = \sqrt{m^2 + p_3^2 + eH(2n+1)},$$

$$N(1) = n_1 N_0, \quad N_0 = \left(\frac{eH}{\pi}\right)^{1/4} \frac{1}{\sqrt{2|p^0| n!}}, \quad n_1 = \frac{1}{\sqrt{2(n+1) m^2_\perp (m^2 + eHn)}}, \quad (48)$$

$$\psi(2,x) = n_2 N_0 \begin{bmatrix} p_3 D_n(\zeta) \\ 0 \\ 0 \\ p^0 D_n(\zeta) \end{bmatrix} \times \exp\left[i(p_2 x_2 + p_3 x_3 - p^0 t)\right], \quad n_2 = \frac{1}{\sqrt{m^2_\perp}}, \quad (49)$$

$$\psi(3,x) = n_3 N_0 \times \\ \times \begin{bmatrix} \sqrt{2eH}p^0 D_n(\zeta) \\ i[-(m^2+eHn)D_{n-1}(\zeta)+eHD_{n+1}(\zeta)] \\ (m^2+eHn)D_{n-1}(\zeta)+eHD_{n+1}(\zeta) \\ \sqrt{2eH}p_3 D_n(\zeta) \end{bmatrix} \times \\ \times \exp\left[i(p_2x_2+p_3x_3-p^0t)\right],$$

$$n_3 = \sqrt{\frac{n}{2m^2(m^2+eHn)}}. \qquad (50)$$

We separate the scalar wave function normalization factor N_0 from the normalization factors $N(i)$, because we concentrate our attention on the differences from the scalar case. We also note that

$$N(3) \propto \Gamma^{-1/2}(n),$$

which vanishes for $n=0$. For the state $\psi(3,x)$, only the values $n=1,2,3,\ldots$ are therefore possible. The same follows from the fact that constraint (43) cannot be satisfied because it does not involve c_1 for $n=0$.

We now construct the vector boson propagator. We start from the expression (which is a special case of a more general result derived in Sec. 6, see Eqs. (80) and (81))

$$G^{\mu\nu}(x,x') = i\int_{-\infty}^{\infty}\frac{dp_2}{2\pi}\int_{-\infty}^{\infty}\frac{dp_3}{2\pi}\times \\ \times \sum_{n=-1}^{\infty}\sum_{i=1}^{3}\begin{cases} {}_+\psi^\mu(i,x){}_+\psi^{*\nu}(i,x'), & t>t', \\ {}_-\psi^\mu(i,x){}_-\psi^{*\nu}(i,x'), & t<t'. \end{cases} \qquad (51)$$

In what follows, we let E_n denote the previous quantity p^0 and use the relations

$$-\frac{1}{2\pi i}\int_{-\infty}^{\infty}dp^0\frac{\exp[-ip^0(t-t')]}{(p^0-E_n+i\epsilon)(p^0+E_n-i\epsilon)} = \\ = \frac{1}{2E_n}\begin{cases}\exp[-iE_n(t-t')], & t>t', \\ \exp[iE_n(t-t')], & t<t', \end{cases} \qquad (52)$$

$$\frac{1}{i(E_n^2-(p^0)^2)} = \int_0^\infty ds\exp[-is(E_n^2-p_0^2)]$$

to rewrite (51) as (with $p^0=-p_0$)

$$G^{\mu\nu}(x,x') = i\sqrt{\frac{eH}{\pi}}\sum_{n=-1}^{\infty}\int_{-\infty}^{\infty}\frac{dp_2}{2\pi}\int_{-\infty}^{\infty}\frac{dp_3}{2\pi}\times \\ \times\int_{-\infty}^{\infty}\frac{dp^0}{2\pi}\int_0^\infty ds a^{\mu\nu}(x,x')\frac{1}{n!}\times \\ \times\exp\left[-is(m_\perp^2+p_3^2-p_0^2)+\right. \\ \left.+i[p_2(x_2-x_2')+p_3(x_3-x_3')-p^0(t-t')]\right], \\ m_\perp^2 = m^2+eH(2n+1). \qquad (53)$$

We note that the lower line in the right-hand side of (52) is obtained from the upper line by the substitution $t \leftrightarrow t'$, which does not change anything, because the right-hand side can be written as

$$(2E_n)^{-1}\exp[-iE_n|t-t'|].$$

The form of the left-hand side that is explicitly symmetric in t and t' is

$$\int_0^\infty ds\exp[-isE_n^2]\int_{-\infty}^{\infty}\frac{dp^0}{2\pi}\exp[isp_0^2-ip^0(t-t')] = \\ = \frac{e^{i\pi/4}}{2\sqrt{\pi}}\int_0^\infty\frac{ds}{\sqrt{s}}\exp\left[-isE_n^2-i\frac{(t-t')^2}{4s}\right]. \qquad (54)$$

We first obtain the scalar particle propagator in the proper-time representation [10]. We replace $a^{\mu\nu}(x,x')$ by $D_n(\zeta)D_n(\zeta')$ in (53). Using the formula

$$D_n(\zeta) = \sqrt{\frac{2}{\pi}}e^{\zeta^2/4}\times \\ \times\int_0^\infty dy y^n e^{-y^2/2}\cos\left(\zeta y-\frac{n\pi}{2}\right), \qquad (55)$$

we then find

$$\sum_{n=0}^{\infty}\frac{D_n(\zeta)D_n(\zeta')}{n!}\exp[-i\tau(2n+1)] = (2\sin 2\tau)^{-1/2}\times \\ \times\exp\left[-i\frac{\pi}{4}+i\frac{(\zeta-\zeta')^2}{8\,\mathrm{tg}\,\tau}-i\frac{(\zeta+\zeta')^2}{8\,\mathrm{ctg}\,\tau}\right],$$

$$\tau = eHs, \quad \zeta' = \sqrt{2eH}\left(x_1'+\frac{p_2}{eH}\right). \qquad (56)$$

Subsequent integration over p_2 gives

$$\int_{-\infty}^{\infty}\frac{dp_2}{2\pi}\sum_{n=0}^{\infty}\frac{D_n(\zeta)D_n(\zeta')}{n!}\times \\ \times\exp[-i\tau(2n+1)+ip_2z_2]= \\ =-i\sqrt{\frac{eH}{\pi}}(4\sin\tau)^{-1}\times \\ \times\exp\left[-i\frac{eHz_2(x_1+x_1')}{2}+i\frac{eH(z_1^2+z_2^2)}{4\operatorname{tg}\tau}\right], \\ z_\mu=x_\mu-x_\mu'. \quad (57)$$

Using

$$\int_{-\infty}^{\infty}\frac{dp_3}{2\pi}\int_{-\infty}^{\infty}\frac{dp^0}{2\pi}\exp[i(p_3z_3-p^0z^0)-is(p_3^2-p_0^2)]= \\ =\frac{1}{4\pi s}\exp[i\frac{z_3^2-z_0^2}{4s}], \quad (58)$$

we find [6, 9–11]

$$G_{spin0}(x,x')=\frac{eH}{(4\pi)^2}\int_0^{\infty}\frac{ds}{s\sin(eHs)}\times \\ \times\exp\left[-i\frac{eHz_2(x_1+x_1')}{2}\right]\times \\ \times\exp\left[-ism^2+i\frac{z_3^2-z_0^2}{4s}+i\frac{(z_1^2+z_2^2)eH}{4\operatorname{tg}(eHs)}\right]. \quad (59)$$

We now show how to obtain $a^{\mu\nu}(x,x')$ in Eq. (53) and how to turn it into a differential matrix that gives the vector boson propagator when inserted in the integrand in (59). As a preliminary step, we write two formulas directly related to (56):

$$\sum_{n=-1}^{\infty}\frac{D_{n+1}(\zeta)D_{n+1}(\zeta')}{(n+1)!}\exp[-i\tau(2n+1)]= \\ =\exp(2i\tau)\sum_{n=0}^{\infty}\frac{D_n(\zeta)D_n(\zeta')}{n!}\exp[-i\tau(2n+1)], \quad (60)$$

$$\sum_{n=1}^{\infty}\frac{D_{n-1}(\zeta)D_{n-1}(\zeta')}{(n-1)!}\exp[-i\tau(2n+1)]= \\ =\exp(-2i\tau)\sum_{n=0}^{\infty}\frac{D_n(\zeta)D_n(\zeta')}{n!}\times \\ \times\exp[-i\tau(2n+1)]. \quad (61)$$

We see that the expressions in Eqs. (60) and (61) differ from the scalar case only by the factors $e^{2i\tau}$ and $e^{-2i\tau}$.

We now return to $a^{\mu\nu}(x,x')$. As seen from (51) and (53),

$$a^{\mu\nu}(x,x')\propto\sum_{i=1}^{3}\psi^\mu(i,x)\psi^{\nu*}(i,x'). \quad (62)$$

Taking, e.g., $a^{11}(x,x')$, we see from (49) that $\psi^1(2,x)=0$, i.e., the term with $i=2$ does not contribute to $a^{11}(x,x')$. In accordance with (47), the contribution of the term with $i=1$ is

$$n_1^2m_\perp^4D_{n+1}(\zeta)D_{n+1}(\zeta'), \\ n_1^2=\frac{1}{2(n+1)m_\perp^2(m^2+eHn)}. \quad (63)$$

The term with $i=3$ gives

$$n_3^2[-(m^2+eHn)D_{n-1}(\zeta)+eHD_{n+1}(\zeta)]\times \\ \times[-(m^2+eHn)D_{n-1}(\zeta')+eHD_{n+1}(\zeta')], \\ n_3^2=\frac{n}{2m^2(m^2+eHn)}. \quad (64)$$

We now have $a^{11}(x,x')$ as the sum of Eqs. (63) and (64):

$$a^{11}(x,x')=\frac{1}{2(m^2+eHn)}\times \\ \times\left(\frac{m_\perp^2}{n+1}+\frac{(eH)^2n}{m^2}\right)D_{n+1}(\zeta)D_{n+1}(\zeta')+ \\ +\frac{n(m^2+eHn)}{2m^2}D_{n-1}(\zeta)D_{n-1}(\zeta')- \\ -\frac{eHn}{2m^2}[D_{n+1}(\zeta)D_{n-1}(\zeta')+D_{n-1}(\zeta)D_{n+1}(\zeta']. \quad (65)$$

Next, we note that

$$\frac{1}{m^2+eHn}\left(\frac{m_\perp^2}{n+1}+\frac{(eH)^2n}{m^2}\right)=\frac{1}{n+1}+\frac{eH}{m^2}, \quad (66)$$

i.e., the undesirable factor $1/(m^2+eHn)$ contained in n_1^2 and n_3^2 in Eqs. (63) and (64) disappears in the sum in Eq. (65).

In what follows, we use the relations

$$\left(\frac{d}{d\zeta}+\frac{\zeta}{2}\right)D_n(\zeta)=nD_{n-1}(\zeta), \\ \left(\frac{d}{d\zeta}-\frac{\zeta}{2}\right)D_n(\zeta)=-D_{n+1}(\zeta), \quad (67)$$

see Eqs. (8.2.15–16) in [7]. We also write the sum and the difference of these expressions:

$$2\frac{d}{d\zeta}D_n(\zeta)=nD_{n-1}(\zeta)-D_{n+1}(\zeta), \\ \zeta D_n(\zeta)=nD_{n-1}(\zeta)+D_{n+1}(\zeta). \quad (68)$$

It is then easy to verify that

$$a^{11}(x,x') = \frac{D_{n+1}(\zeta)D_{n+1}(\zeta')}{2(n+1)} + \\ + \frac{n}{2}D_{n-1}(\zeta)D_{n-1}(\zeta') + \\ + \frac{2eH}{m^2}\frac{\partial^2}{\partial\zeta\partial\zeta'}D_n(\zeta)D_n(\zeta'). \quad (69)$$

The first term in the right-hand side of (69) is involved in Eq. (60) and the second term is used in (61); the necessary factor $n!$ comes from N_0, see Eq. (48). The third term can be written as

$$\frac{1}{m^2}\frac{\partial^2}{\partial x_1 \partial x'_1}D_n(\zeta)D_n(\zeta'). \quad (70)$$

In a similar manner, we find the other components

$$a^{\mu\nu}(x,x') = a^{\nu\mu *}(x',x).$$

It is easy to verify that the differential operator $A^{\mu\nu}(x,x')$ corresponding to $a^{\mu\nu}(x,x')$ is given by

$$A^{\mu\nu} = B^{\mu\nu} + C^{\mu\nu}, \quad C^{\mu\nu} = \frac{1}{m^2}\Pi^\mu(x)\Pi^{\nu *}(x'), \\ \Pi_\mu(x) = -i\frac{\partial}{\partial x^\mu} + eA_\mu(x), \\ \Pi^*_\mu(x') = i\frac{\partial}{\partial x'^\mu} + eA_\mu(x'). \quad (71)$$

In our case,

$$A_\mu(x) = \delta_{\mu 2}Hx_1, \quad \Pi^0(x) = i\frac{\partial}{\partial t}, \\ \Pi^{0*}(x') = -i\frac{\partial}{\partial t'}. \quad (72)$$

The nonzero components $B^{\mu\nu}$ are

$$B^{11} = B^{22} = \cos\tau, \quad B^{21} = -B^{12} = \sin\tau, \\ B^{33} = -B^{00} = 1. \quad (73)$$

The difference of $B^{\mu\nu}$ from $\eta^{\mu\nu}$ is due to the interaction of the boson magnetic moment with the magnetic field. We can say that $B^{\mu\nu}$ with $\mu,\nu = 1,2$ are «dressed» by the magnetic field.

Thus,

$$G^{\mu\nu}(x,x') = \frac{eH}{(4\pi)^2}\int_0^\infty \frac{ds}{s\sin(eHs)}\exp(-ism^2)A^{\mu\nu}\times \\ \times\exp\left[-\frac{ieHz_2(x_1+x'_1)}{2}\right]\times \\ \times\exp\left[\frac{i(z_3^2-z_0^2)}{4s}+\frac{i}{4}(z_1^2+z_2^2)eH\operatorname{ctg}(eHs)\right], \\ z_\mu = x_\mu - x'_\mu. \quad (74)$$

It is somewhat surprizing that this representation does not coincide with the Vanyashin–Terentyev representation [3] with the electric field switched off. Possibly, these are two different representations for the same propagator, and it would be interesting to verify this hypothesis.

6. THE VECTOR BOSON PROPAGATOR IN THE CONSTANT ELECTRIC FIELD

We first give the generalization of Eq. (51) for the case where the external field can create pairs [12, 6]. For this purpose, we write

$$G(x,x')_{abs} = i\langle 0_{out}|T(\Psi(x)\Psi^\dagger(x'))|0_{in}\rangle = \\ = \langle 0_{out}|0_{in}\rangle G(x,x'), \quad (75)$$

where T is the chronological ordering operator and

$$\Psi(x) = \sum_n [a_{n\,out}\,{}^+\psi_n(x) + b^+_{n\,out}\,{}^-\psi_n(x)], \\ \Psi^\dagger(x) = \sum_n [a^\dagger_{n\,in}\,{}_+\psi^*_n(x) + b_{n\,in}\,{}_-\psi^*_n(x)]. \quad (76)$$

As usual, a_n and b_n are the particle and antiparticle destruction operators in a state with the quantum numbers n:

$$\Psi^\dagger(x')|0_{in}\rangle = \sum_k {}_+\psi^*_k(x')a^\dagger_{k\,in}|0_{in}\rangle, \\ \langle 0_{out}|\Psi(x) = \langle 0_{out}|\sum_n a_{n\,out}\,{}^+\psi_n(x). \quad (77)$$

For $t > t'$, it follows from (75) and (77) that

$$G(x,x')_{abs} = i\sum_{n,k}{}^+\psi_n(x)\,{}_+\psi^*_k(x')\times \\ \times\langle 0_{out}|a_{n\,out}a^\dagger_{k\,in}|0_{in}\rangle, \quad t > t'. \quad (78)$$

In our case, the Bogoliubov transformations have the form (cf. Eq. (18)) [6]

$$a^\dagger_{n\,out} = c^*_{1\,n}a^\dagger_{n\,in} + c_{2\,n}b_{n\,in}, \\ b_{n\,out} = c^*_{2\,n}a^\dagger_{n\,in} + c_{1\,n}b_{n\,in}. \quad (79)$$

The first equation in (79) implies that

$$a^\dagger_{k\,out}|0_{in}\rangle = c^*_{1\,k}a^\dagger_{k\,in}|0_{in}\rangle.$$

We insert $a^\dagger_{k\,in}|0_{in}\rangle$ obtained from here in (78) and use the commutation relation

$$[a_{k\,out}, a^\dagger_{n\,out}] = \delta_{kn}.$$

We then obtain

$$G(x,x')_{abs} = \langle 0_{out}|0_{in}\rangle i \sum_n {}^{+}\psi_n(x)\ {}_{+}\psi^*_n(x')\frac{1}{c^*_{1\,n}}, \quad t > t'. \tag{80}$$

Similarly, for $t < t'$, we find

$$G(x,x')_{abs} = \langle 0_{out}|0_{in}\rangle i \sum_n {}_{-}\psi_n(x)\ {}^{-}\psi^*_n(x')\frac{1}{c^*_{1\,n}}, \quad t < t'. \tag{81}$$

If the external field does not create pairs, the expressions obtained become those in (51).

In terms of the states ${}_{+}\psi'$ and ${}_{+}\psi$ in (8), transition current (41) becomes

$$J^0({}_{+}\psi', {}_{+}\psi) = \sqrt{2eE}e^{\pi\lambda/4}\left[c'^*_1 c_1 + c'^*_2 c_2 + 2i({}_{+}c'^*_{-}\ {}_{+}c_{+} - {}_{+}c'^*_{+}\ {}_{+}c_{-})\right], \tag{82}$$

where we used Eq. (8.2.11) in [7] (and its complex conjugate):

$$D_{\nu^*+1}(\tau^*)\ \overset{\leftrightarrow}{\frac{d}{dt}}\ D_{\nu-1}(\tau) = \sqrt{2eE}\exp\left(\frac{\pi\lambda}{4}\right) = \tag{83}$$

$$= -D_{\nu^*-1}(\tau^*)\ \overset{\leftrightarrow}{\frac{d}{dt}}\ D_{\nu+1}(\tau) = D_{\nu^*}(\tau^*)i\ \overset{\leftrightarrow}{\frac{d}{dt}}\ D_{\nu}(\tau). \tag{84}$$

The constraint is given in (10). Using Eqs. (82), (8), and (10), we find the ${}_{+}\psi$ polarization states

$${}_{+}\psi(1,x) = N(1)\begin{bmatrix} p_2\sqrt{\dfrac{eE}{2}}e^{i\pi/4}[D_{\nu+1}(\tau) - \nu D_{\nu-1}(\tau)] \\ 0 \\ m^2_{\perp}D_{\nu}(\tau) \\ p_2\sqrt{\dfrac{eE}{2}}e^{i\pi/4}[D_{\nu+1}(\tau) + \nu D_{\nu-1}(\tau)] \end{bmatrix}\exp(i\mathbf{p}\cdot\mathbf{x}), \quad N(i) = n_i N_0, \tag{85}$$

$$N(1) = n_1 N_0, \quad n_1 = \sqrt{\frac{1}{m^2_{\perp}(m^2+p_1^2)}}, \quad N_0 = (2eE)^{-1/4}\exp\left(-\frac{\pi\lambda}{8}\right),$$

$${}_{+}\psi(2,x) = N(2)\begin{bmatrix} D_{\nu+1}(\tau) + (1+\nu)D_{\nu-1}(\tau) \\ 0 \\ 0 \\ D_{\nu+1}(\tau) - (1+\nu)D_{\nu-1}(\tau) \end{bmatrix}\exp(i\mathbf{p}\cdot\mathbf{x}), \tag{86}$$

$$n_2 = \sqrt{\frac{eE}{2m^2_{\perp}}}, \quad m^2_{\perp} = m^2 + p_1^2 + p_2^2,$$

$${}_{+}\psi(3,x) = N(3)\begin{bmatrix} p_1\sqrt{\dfrac{eE}{2}}e^{i\pi/4}[D_{\nu+1}(\tau) - \nu D_{\nu-1}(\tau)] \\ (m^2+p_1^2)D_{\nu}(\tau) \\ p_1 p_2 D_{\nu}(\tau) \\ p_1\sqrt{\dfrac{eE}{2}}e^{i\pi/4}[D_{\nu+1}(\tau) + \nu D_{\nu-1}(\tau)] \end{bmatrix}\exp(i\mathbf{p}\cdot\mathbf{x}), \tag{87}$$

$$n_3 = \frac{1}{\sqrt{m^2(m^2+p_1^2)}}.$$

The $^+\psi$ polarization states can be obtained from these ones using Eqs. (19) (see also (16)):

$$^+\psi(1,x) = N(1)\begin{bmatrix} p_2\sqrt{\dfrac{eE}{2}}e^{-i\pi/4}[(1+\nu)D_{\nu^*-1}(-\tau^*)+D_{\nu^*+1}(-\tau^*)] \\ 0 \\ m_\perp^2 D_{\nu^*}(-\tau^*) \\ p_2\sqrt{\dfrac{eE}{2}}e^{-i\pi/4}[(1+\nu)D_{\nu^*-1}(-\tau^*)-D_{\nu^*+1}(-\tau^*)] \end{bmatrix}\exp(i\mathbf{p}\cdot\mathbf{x}), \tag{88}$$

$$^+\psi(2,x) = N(2)\begin{bmatrix} i(1+\nu)[-D_{\nu^*-1}(-\tau^*)+\dfrac{1}{\nu}D_{\nu^*+1}(-\tau^*)] \\ 0 \\ 0 \\ i(1+\nu)[-D_{\nu^*-1}(-\tau^*)-\dfrac{1}{\nu}D_{\nu^*+1}(-\tau^*)] \end{bmatrix}\exp(i\mathbf{p}\cdot\mathbf{x}), \tag{89}$$

$$^+\psi(3,x) = N(3)\begin{bmatrix} p_1\sqrt{\dfrac{eE}{2}}e^{-i\pi/4}[(1+\nu)D_{\nu^*-1}(-\tau^*)+D_{\nu^*+1}(-\tau^*)] \\ (m^2+p_1^2)D_{\nu^*}(-\tau^*) \\ p_1p_2D_{\nu^*}(-\tau^*) \\ p_1\sqrt{\dfrac{eE}{2}}e^{-i\pi/4}[(1+\nu)D_{\nu^*-1}(-\tau^*)-D_{\nu^*+1}(-\tau^*)] \end{bmatrix}\exp(i\mathbf{p}\cdot\mathbf{x}). \tag{90}$$

In Eqs. (85)–(90), the states $\psi(i,x)$ are characterized by p_1, p_2, p_3, and i; ν and λ are given in (7).

We note that the transition current $J^0(^+\psi', {}^+\psi)$ expressed in terms of ^+c has the same form as $J^0({}_+\psi', {}_+\psi)$ expressed in terms of $_+c$, see Eq. (82). A similar statement is true for the negative-frequency states. Because $\nu+1 = -\nu^*$ in accordance with Eq. (7), it follows from (19) that

$$^+c_-'^{\,*}\,{}^+c_+ = {}_+c_-'^{\,*}\,{}_+c_+ = \\ = {}^-c_-'^{\,*}\,{}^-c_+ = {}_-c_-'^{\,*}\,{}_-c_+. \tag{90a}$$

Therefore,

$$J^0({}_+\psi(i,x), {}_+\psi(j,x)) = \\ = J^0(^+\psi(i,x), {}^+\psi(j,x)) \propto \delta_{i,j}, \tag{91}$$

and

$$J^0({}_-\psi(i,x), {}_-\psi(j,x)) = J^0(^-\psi(i,x), {}^-\psi(j,x)) = \\ = -J^0({}_+\psi(i,x), {}_+\psi(j,x)). \tag{91a}$$

As previously, we focus our attention on the differences from the scalar case in expressions similar to (53). The proper-time representation of the scalar particle propagator is given by [12]

$$G(x,x')_{spin\,0} = \frac{eE}{(4\pi)^2}\exp\left[\frac{i}{2}eE(t+t')z_3\right]\times \\ \times\int_0^\infty \frac{ds}{s\,\mathrm{sh}(eEs)}\exp\left[-ism^2+\frac{i}{4s}(z_1^2+z_2^2)+ \\ +\frac{i}{4}eE(z_3^2-z_0^2)\,\mathrm{cth}(eEs)\right]. \tag{92}$$

This can be derived similarly to the magnetic case, but with the role of Eq. (52) played by the relation [12, 6]

$$\sqrt{2}\int_0^\infty \frac{d\theta}{\sqrt{\mathrm{sh}\,2\theta}}\times \\ \times\exp\left\{-i2\varkappa\theta-\frac{i}{8}\left[\frac{(T+T')^2}{\mathrm{cth}\,\theta}+\frac{(T-T')^2}{\mathrm{th}\,\theta}\right]\right\} = \\ = \Gamma\left(i\varkappa+\frac{1}{2}\right)\times \\ \times\begin{cases} D_{-i\varkappa-(1/2)}(\chi)D_{-i\varkappa-(1/2)}(-\chi'), & T>T', \\ D_{-i\varkappa-(1/2)}(-\chi)D_{-i\varkappa-(1/2)}(\chi'), & T<T', \end{cases} \tag{93}$$

where

$$\begin{aligned} &\theta = eEs, \quad T = \sqrt{2eE}\left(t - \frac{p_3}{eE}\right), \quad T' = \sqrt{2eE}\left(t' - \frac{p_3}{eE}\right), \\ &\chi = -\tau^* = e^{i\pi/4}T, \quad \chi' = e^{i\pi/4}T', \quad \varkappa = \frac{\lambda}{2} = \frac{m_\perp^2}{2eE}. \end{aligned} \tag{94}$$

The lower line in the right-hand side of (93) can be obtained from the upper line by the substitution $T \leftrightarrow T'$. As seen from the left-hand side of (93), this does not change the value of (93), cf. the remark after Eq. (53).

By analogy with the magnetic case, we expect the appearance of the factors $e^{\pm 2\theta}$ in the integrand of (93), cf. Eqs. (73) and (60), (61). To make the insertion possible, we must rotate the integration contour clockwise by a certain angle. This is in line with the Vanyashin–Terentyev approach [3]. After the substitution $\varkappa \to \varkappa + i$, it then follows from (93) that

$$\begin{aligned} &\sqrt{2}\int\limits_C \frac{d\theta}{\sqrt{\operatorname{sh} 2\theta}} \exp\left\{-i2\varkappa\theta + 2\theta - \frac{i}{8}\left[\frac{(T+T')^2}{\operatorname{cth}\theta} + \frac{(T-T')^2}{\operatorname{th}\theta}\right]\right\} = \\ &= \Gamma\left(i\varkappa - \frac{1}{2}\right)\left\{\begin{array}{ll} D_{-i\varkappa+(1/2)}(\chi)D_{-i\varkappa+(1/2)}(-\chi'), & T > T',\\ D_{-i\varkappa+(1/2)}(-\chi)D_{-i\varkappa+(1/2)}(\chi'), & T < T'. \end{array}\right. \end{aligned} \tag{95}$$

Similarly, substituting $\varkappa \to \varkappa - i$ in (93), we obtain

$$\begin{aligned} &\sqrt{2}\int\limits_0^\infty \frac{d\theta}{\sqrt{\operatorname{sh} 2\theta}} \exp\left\{-i2\varkappa\theta - 2\theta - \frac{i}{8}\left[\frac{(T+T')^2}{\operatorname{cth}\theta} + \frac{(T-T')^2}{\operatorname{th}\theta}\right]\right\} = \\ &= \Gamma\left(i\varkappa + \frac{3}{2}\right)\left\{\begin{array}{ll} D_{-i\varkappa-(3/2)}(\chi)D_{-i\varkappa-(3/2)}(-\chi'), & T > T',\\ D_{-i\varkappa-(3/2)}(-\chi)D_{-i\varkappa-(3/2)}(\chi'), & T < T'. \end{array}\right. \end{aligned} \tag{96}$$

The integration over p_3 contained in the sum over n in Eqs. (80) and (81) gives

$$\int\limits_{-\infty}^{\infty} \frac{dp_3}{2\pi} \exp\left[ip_3z_3 - \frac{i}{8}(T+T')^2 \operatorname{th}\theta\right] = \frac{1}{2}e^{-i\pi/4}\sqrt{\frac{eE\operatorname{cth}\theta}{\pi}} \exp\left\{\frac{iz_3^2 eE}{4\operatorname{th}\theta} + \frac{ieEz_3(t+t')}{2}\right\}, \quad z_3 = x_3 - x_3', \tag{97}$$

where T and T' are functions of p_3, see (94). Further calculations leading to (92) are similar to those in the magnetic case.

We now consider the differences from the scalar case. We first rewrite relations (67) and (68) between the parabolic cylinder functions for the present case as

$$\begin{aligned} &\left(\frac{d}{d\tau'^*} + \frac{\tau'^*}{2}\right) D_{\nu^*}(\tau'^*) = \nu^* D_{\nu^*-1}(\tau'^*), \\ &\left(\frac{d}{d\tau'^*} - \frac{\tau'^*}{2}\right) D_{\nu^*}(\tau'^*) = -D_{\nu^*+1}(\tau'^*), \end{aligned} \tag{98}$$

$$\begin{aligned} &2\frac{d}{d\tau'^*} D_{\nu^*}(\tau'^*) = \nu^* D_{\nu^*-1}(\tau'^*) - D_{\nu^*+1}(\tau'^*), \\ &\tau'^* D_{\nu^*}(\tau'^*) = \nu^* D_{\nu^*-1}(\tau'^*) + D_{\nu^*+1}(\tau'^*). \end{aligned} \tag{99}$$

The other necessary relations are obtained from these by the substitution $\tau'^* \to -\tau^*$.

Because

$$\begin{aligned} &c_{1n} = \frac{\sqrt{2\pi}}{\Gamma(-i\varkappa + 1/2)} \exp\left[-\frac{\pi\varkappa}{2} + \frac{i\pi}{4}\right], \\ &\frac{i}{c_{1n}^*} N_0^2 = \frac{\exp[3i\pi/4]}{2\sqrt{\pi eE}}\Gamma\left(i\varkappa + \frac{1}{2}\right), \end{aligned} \tag{100}$$

we can write the propagator as

$$G^{\mu\nu}(x,x') = \frac{\exp[3i\pi/4]}{2\sqrt{\pi eE}} \times \\ \times \int \frac{d^3p}{(2\pi)^3} a^{\mu\nu}(x,x') \exp[i\mathbf{p}\cdot(\mathbf{x} - \mathbf{x}')]. \tag{101}$$

The scalar particle propagator can be obtained from the right-hand side of (101) if we replace $a^{\mu\nu}(x,x')$ with expression (93). As an example, we now calculate $a^{33}(x,x')$. For $t>t'$, we have

$$a^{33}(x,x') \propto \sum_{i=1}^{3} {}^{+}\psi^3(i,x)\,{}_{+}\psi^{3*}(i,x'). \tag{102}$$

The first term in the sum is

$$\begin{gathered} {}^{+}\psi^3(1,x)\,{}_{+}\psi^{3*}(1,x') \propto -\frac{ieE}{2}\tau^* \times \\ \times D_{\nu^*}(-\tau^*)\tau'^* D_{\nu^*}(\tau'^*)\frac{p_2^2}{m_\perp^2(m^2+p_1^2)}, \end{gathered} \tag{103}$$

where we used the second equation in (99) and the one obtained from it by the substitution $\tau'^* \to -\tau^*$. Similarly,

$$\begin{gathered} {}^{+}\psi^3(3,x)\,{}_{+}\psi^{3*}(3,x') \propto \\ \propto -\frac{ieE}{2}\tau^* D_{\nu^*}(-\tau^*)\tau'^* D_{\nu^*}(\tau'^*)\frac{p_1^2}{m^2(m^2+p_1^2)}. \end{gathered} \tag{104}$$

Adding Eqs. (103) and (104), we obtain

$$\begin{gathered} -\frac{ieE}{2}\tau^* D_{\nu^*}(-\tau^*)\tau'^* D_{\nu^*}(\tau'^*) \times \\ \times \left[\frac{p_2^2}{m_\perp^2(m^2+p_1^2)} + \frac{p_1^2}{m^2(m^2+p_1^2)}\right]. \end{gathered} \tag{105}$$

The expression in the square brackets can be simplified as

$$\frac{1}{m^2+p_1^2}\left(\frac{p_2^2}{m_\perp^2}+\frac{p_1^2}{m^2}\right) = \frac{1}{m^2}-\frac{1}{m_\perp^2}. \tag{106}$$

The undesirable factor $(m^2+p_1^2)^{-1}$ involved in (103) and (104) disappears in sum (105). The first term in the right-hand side of (106) gives the following contribution to (105):

$$\begin{gathered} -\frac{ieE}{2m^2}\tau^* D_{\nu^*}(-\tau^*)\tau'^* D_{\nu^*}(\tau'^*) = \frac{1}{m^2} \times \\ \times (p_3-eEt)(p_3-eEt')D_{\nu^*}(-\tau^*)D_{\nu^*}(\tau'^*). \end{gathered} \tag{107}$$

This already has the desired form. We now rewrite the contribution of the second term in the right-hand side of (106) to (105) in the initial form (i.e., before using the second equation in (99)),

$$\begin{gathered} \frac{ieE}{2m_\perp^2}\left[-(1+\nu)^2 D_{\nu^*-1}(-\tau^*)D_{\nu^*-1}(\tau'^*) + \right.\\ +(1+\nu)D_{\nu^*+1}(-\tau^*)D_{\nu^*-1}(\tau'^*)+ \\ +(1+\nu)D_{\nu^*-1}(-\tau^*)D_{\nu^*+1}(\tau'^*) - \\ \left. -D_{\nu+1}(-\tau^*)D_{\nu^*+1}(\tau'^*)\right]. \end{gathered} \tag{108}$$

This expression still contains the undesirable factor $1/m_\perp^2$. But we must take the contribution from the term with $i=2$ in (102) into account:

$$\begin{gathered} {}^{+}\psi^3(2,x)\,{}_{+}\psi^{3*}(2,x') \propto \frac{ieE}{2m_\perp^2}(1+\nu) \times \\ \times \left[-D_{\nu^*-1}(-\tau^*)D_{\nu^*+1}(\tau'^*) - \right.\\ -\frac{1}{\nu}D_{\nu^*+1}(-\tau^*)D_{\nu^*+1}(\tau'^*) - \\ -\nu D_{\nu^*-1}(-\tau^*)D_{\nu^*-1}(\tau'^*) - \\ \left. -D_{\nu^*+1}(-\tau^*)D_{\nu^*-1}(\tau'^*)\right]. \end{gathered} \tag{109}$$

It is easy to see that in the sum of (108) and (109), the undesirable terms are cancelled and the unpleasant denominator

$$m_\perp^2 = -ieE(1+2\nu)$$

disappears:

$$\begin{gathered} (108)+(109) = \frac{1}{2} \times \\ \times \left[(1+\nu)D_{\nu^*-1}(-\tau^*)D_{\nu^*-1}(\tau'^*) + \right.\\ \left. +\frac{1}{\nu}D_{\nu^*+1}(-\tau^*)D_{\nu^*+1}(\tau'^*)\right]. \end{gathered} \tag{110}$$

Thus, $a^{33}(x,x')$ is given by the sum of expressions (107) and (110). The first term in the right-hand side of (110) is used in (96) and the second term in (95). In the same manner, we find all the other $a^{\mu\nu}(x,x')$ components. Similarly to the magnetic case, we have

$$\begin{gathered} G^{\mu\nu}(x,x') = \frac{eE}{(4\pi)^2}\int_C \frac{ds}{s\,\mathrm{sh}(eEs)}A^{\mu\nu} \times \\ \times \exp\left[\frac{ieE}{2}z_3(t+t')\right] \times \\ \times \exp\left[-ism^2 + \frac{i}{4s}(z_1^2+z_2^2) + \right.\\ \left. +\frac{i}{4s}(z_3^2-z_0^2)eE\,\mathrm{cth}(eEs)\right], \end{gathered} \tag{111}$$

where $A^{\mu\nu}$ is given by (71), but the vector potential is

$$A_\mu(x) = -\delta_{\mu 3}Et.$$

The nonzero $B^{\mu\nu}$ components are

$$\begin{gathered} B^{11} = B^{22} = 1, \quad B^{33} = -B^{00} = \mathrm{ch}(2eEs), \\ B^{30} = -B^{03} = \mathrm{sh}(2eEs). \end{gathered} \tag{112}$$

We see that the electric field dresses $B^{\mu\nu}$ with $\mu, \nu = 3, 0$.

Proceeding to the case where $t < t'$, we note that in accordance with (19),

$$_{-}c_{\pm} = -^{+}c_{\pm}, \quad ^{-}c_{\pm} = -_{+}c_{\pm}. \tag{112a}$$

This implies that $_{-}\psi$ ($^{-}\psi$) is obtained from $^{+}\psi$ ($_{+}\psi$) by changing the sign of the arguments in the parabolic cylinder functions and the sign of ψ^0 and ψ^3. The overall change of sign of $\psi(2,x)$ does not affect the corresponding term in (102). In $\psi(1,x)$ and $\psi(3,x)$, changing the sign of ψ^0 and ψ^3 and of the arguments τ^* and τ'^* is equivalent to changing the sign of only the D-function arguments τ^* and τ'^* if ψ^0 and ψ^3 are expressed through the left-hand sides of (99). As expected, it now follows from (93)–(96) that $G^{\mu\nu}(x,x')$ retains the same form (111) for $t < t'$.

7. THE VECTOR BOSON PROPAGATOR IN THE CONSTANT ELECTROMAGNETIC FIELD

After we have considered the magnetic and electric fields separately, the construction of the vector boson propagator in both fields meets no new problems. We take the vector potential in the form

$$A_\mu(x) = \delta_{\mu 2} H x_1 - \delta_{\mu 3} E t. \tag{113}$$

The transition current between the states $_{+}\psi'$ and $_{+}\psi$ is

$$J^0(_{+}\psi', _{+}\psi) = 2\left\{ [c_1'^* c_1 D_{n-1}^2(\zeta) + c_2'^* c_2 D_{n+1}^2(\zeta)] D_{\nu^*}(\tau^*) i \overset{\leftrightarrow}{\frac{d}{dt}} D_\nu(\tau) - \left[c_-'^* c_+ D_{\nu^*-1}(\tau^*) i \overset{\leftrightarrow}{\frac{d}{dt}} D_{\nu+1}(\tau) + \right.\right.$$
$$\left.\left. + c_+'^* c_- D_{\nu^*+1}(\tau^*) i \overset{\leftrightarrow}{\frac{d}{dt}} D_{\nu-1}(\tau) \right] D_n^2(\zeta) \right\}. \tag{114}$$

Taking Eq. (84) into account and integrating over x_1, we obtain

$$\int\limits_{-\infty}^{\infty} dx_1 J^0(_{+}\psi', _{+}\psi) = n! \sqrt{\frac{2\pi E}{H}} 2e^{\pi\varkappa/2} \left[\frac{1}{n} c_1'^* c_1 + (1+n) c_2'^* c_2 + i(c_-'^* c_+ - c_+'^* c_-) \right]. \tag{115}$$

The constraint is given by

$$\sqrt{2eH}[(1+n)c_2 - c_1] + \sqrt{2eE} e^{-i\pi/4} [_{+}c_- - (1+\nu)\, _{+}c_+] = 0. \tag{116}$$

Using (115) and (116), we find the $_{+}\psi$ polarization states (in what follows, the factor $\exp[i(p_2 x_2 + p_3 x_3)]$ is dropped for brevity):

$$_{+}\psi(1,x) = N(1) \begin{bmatrix} (1+n)\sqrt{e^2 EH} e^{i\pi/4} [D_{\nu+1}(\tau) - \nu D_{\nu-1}(\tau)] D_n(\zeta) \\ i m_\perp^2 D_\nu(\tau) D_{n+1}(\zeta) \\ m_\perp^2 D_\nu(\tau) D_{n+1}(\zeta) \\ (1+n)\sqrt{e^2 EH} e^{i\pi/4} [D_{\nu+1}(\tau) + \nu D_{\nu-1}(\tau)] D_n(\zeta) \end{bmatrix}, \tag{117}$$

$$N(i) = n_i N_0, \quad N_0 = \left(\frac{H}{2\pi E}\right)^{1/4} \frac{e^{-\pi\varkappa/4}}{\sqrt{n!}},$$
$$n_1 = \frac{1}{\sqrt{2m_\perp^2 (m^2 + eHn)(1+n)}}, \tag{118}$$

$$_{+}\psi(2,x) = N(2) \begin{bmatrix} [D_{\nu+1}(\tau) + (1+\nu) D_{\nu-1}(\tau)] D_n(\zeta) \\ 0 \\ 0 \\ [D_{\nu+1}(\tau) - (1+\nu) D_{\nu-1}(\tau)] D_n(\zeta) \end{bmatrix}, \quad n_2 = \sqrt{\frac{eE}{2m_\perp^2}}, \tag{119}$$

$$
{}_{+}\psi(3,x) = N(3)\begin{bmatrix} \sqrt{e^2EH}[D_{\nu+1}(\tau) - \nu D_{\nu-1}(\tau)]D_n(\zeta) \\ e^{i\pi/4}D_\nu(\tau)[-(m^2+eHn)D_{n-1}(\zeta) + eHD_{n+1}(\zeta)] \\ e^{-i\pi/4}D_\nu(\tau)[(m^2+eHn)D_{n-1}(\zeta) + eHD_{n+1}(\zeta)] \\ \sqrt{e^2EH}[D_{\nu+1}(\tau) + \nu D_{\nu-1}(\tau)]D_n(\zeta) \end{bmatrix}, \tag{120}
$$

$$
n_3 = \sqrt{\frac{n}{2m^2(m^2+eHn)}}. \tag{121}
$$

To obtain the polarization states of $^{+}\psi$ (or of $_{-}\psi$ and $^{-}\psi$), we again use Eq. (19) (cf. Eqs. (88)–(90) and (47)–(50)). We then obtain

$$
{}^{+}\psi(1,x) = N(1)\begin{bmatrix} (1+n)\sqrt{e^2EH}e^{-i\pi/4}[(1+\nu)D_{\nu^*-1}(-\tau^*) + D_{\nu^*+1}(-\tau^*)]D_n(\zeta) \\ im_\perp^2 D_{\nu^*}(-\tau^*)D_{n+1}(\zeta) \\ m_\perp^2 D_{\nu^*}(-\tau^*)D_{n+1}(\zeta) \\ (1+n)\sqrt{e^2EH}e^{-i\pi/4}[(1+\nu)D_{\nu^*-1}(-\tau^*) - D_{\nu^*+1}(-\tau^*)]D_n(\zeta) \end{bmatrix}, \tag{122}
$$

$$
{}^{+}\psi(2,x) = N(2)\begin{bmatrix} i(1+\nu)\left[-D_{\nu^*-1}(-\tau^*) + \dfrac{1}{\nu}D_{\nu^*+1}(-\tau^*)\right]D_n(\zeta) \\ 0 \\ 0 \\ i(1+\nu)\left[-D_{\nu^*-1}(-\tau^*) - \dfrac{1}{\nu}D_{\nu^*+1}(-\tau^*)\right]D_n(\zeta) \end{bmatrix}, \tag{123}
$$

$$
{}^{+}\psi(3,x) = N(3)\begin{bmatrix} -i\sqrt{e^2EH}[-\nu^* D_{\nu^*-1}(-\tau^*) + D_{\nu^*+1}(-\tau^*)]D_n(\zeta) \\ e^{i\pi/4}D_{\nu^*}(-\tau^*)[-(m^2+eHn)D_{n-1}(\zeta) + eHD_{n+1}(\zeta)] \\ e^{-i\pi/4}D_{\nu^*}(-\tau^*)[(m^2+eHn)D_{n-1}(\zeta) + eHD_{n+1}(\zeta)] \\ -i\sqrt{e^2EH}[-\nu^* D_{\nu^*-1}(-\tau^*) - D_{\nu^*+1}(-\tau^*)]D_n(\zeta) \end{bmatrix}. \tag{124}
$$

The first and the fourth lines in the right-hand sides of (122) and (124) can be written in a more compact form using relations that can be obtained from (99) by the substitution $\tau'^* \to -\tau^*$.

Further calculations are quite similar to those in Secs. 5 and 6. The result was of course evident in advance: $A^{\mu\nu}$ is now given by (71) with the vector potential (113) and all the nonzero $B^{\mu\nu}$ are «dressed», see Eqs. (73) and (112). The scalar particle propagator is given by

$$
G_{spin\,0}(x,x') = \frac{e^2EH}{(4\pi)^2}\int\limits_0^\infty \frac{ds}{\operatorname{sh}(eEs)\sin(eHs)} \times
$$
$$
\times \exp\left\{-ism^2 + \frac{i}{4}\left[(z_1^2+z_2^2)eH\operatorname{ctg}(eHs) + \right.\right.
$$
$$
\left. + (z_3^2 - z_0^2)eE\operatorname{cth}(eEs)\right] +
$$
$$
\left. + \frac{i}{2}\left[eEz_3(t+t') - eHz_2(x_1+x_1')\right]\right\}, \tag{125}
$$

$$
z_\mu = x_\mu - x'_\mu.
$$

This expression agrees with the calculations by Ritus [10, 11]. The overall phase factor $e^{-i\pi/2}$ is involved in his formulas because of a different definition of the propagator. We also note that Eq. (125) is symmetric in t and t' and that

$$
G_{spin\,0}(x,x',e) = G_{spin\,0}(x',x,-e).
$$

Therefore,

$$
G^{\mu\nu}(x,x') = \frac{e^2EH}{(4\pi)^2}\int\limits_0^\infty \frac{ds}{\operatorname{sh}(eEs)\sin(eHs)}A^{\mu\nu} \times
$$
$$
\times \exp\left\{-ism^2 + \frac{i}{4}\left[(z_1^2+z_2^2)eH\operatorname{ctg}(eHs) + \right.\right.
$$
$$
\left. + (z_3^2 - z_0^2)eE\operatorname{cth}(eEs)\right] +
$$
$$
\left. + \frac{i}{2}\left[eEz_3(t+t') - eHz_2(x_1+x_1')\right]\right\}. \tag{126}
$$

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REFERENCES

1. V. V. Skalozub, Particles and Nuclei, **16**, 1005 (1985).
2. V. M. Mostepanenko, V. M. Frolov, and V. A. Sheluto, Yad. Fiz. **37**, 1261 (1983).
3. V. S. Vanyashin and M. V. Terentyev, Zh. Eksp. Teor. Fiz. **48**, 565 (1965).
4. M. S. Marinov and V. S. Popov, Yad. Fiz. **15**, 1271 (1972).
5. A. A. Grib, S. G. Mamaev, and V. M. Mostepanenko, *Vacuum Quantum Effects in Strong Fields*, Moscow (1988).
6. A. I. Nikishov, Trudy FIAN **111**, 152 (1979); J. Sov. Laser Res. **6**, 619 (1985).
7. A. Erdélyi et al., *Higher Transcendental Functions*, Vol. 2, Mc Grow-Hill Book Co. (1953).
8. V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, *Quantum Electrodynamics*, Nauka, Moscow (1980).
9. J. Schwinger, Phys. Rev. **82**, 661 (1951).
10. V. I. Ritus, Zh. Eksp. Teor. Fiz. **75**, 1560 (1978); Trudy FIAN **111**, 134 (1979); J. Sov. Laser Res. **6**, 497 (1986).
11. V. I. Ritus, Trudy FIAN **168**, 52 (1986); A. I. Nikishov and N. B. Narozhny, Trudy FIAN **168**, 175 (1986); in *Issues in Intense-Field Quantum Electrodynamics*, ed. by V. L. Ginzburg, Nova Science Publ. Inc., Commack (1987).
12. A. I. Nikishov, Zh. Eksp. Teor. Fiz. **57**, 1210 (1969).