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DOUBLE POTTS CHAIN AND EXACT RESULTS FOR SOME TWO-DIMENSIONAL SPIN MODELS

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An exact analytical solution for the q-state Potts model on a $2 \times \infty$ ladder with arbitrary two-, three-, and four-site interactions in a unit cell is presented in a closed form. This solution is used to show that the finite-size internal energy equation [6] yields an accurate value of the critical temperature for the triangular Potts lattice with three-site interactions in alternate triangular faces. It is argued that the above equation is exact at least for self-dual models on isotropic strips.

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1. INTRODUCTION

The methods that allow extracting information about a multidimensional system from solutions of its lower-dimensional counterparts play an important role in statistical physics. One of the most well-known examples of this kind is the finite-size scaling approach [1, 2].

There are cases that evoke particular interest when the critical properties of a system experiencing a phase transition can be exactly determined from the data pertaining to its subsystems. For instance, for the Ising strips, the intersection point of the partition function zero locus in a complex temperature plane with the real positive axis yields the exact value of the critical temperature for the two-dimensional Ising model [3]. Exact critical temperatures for the $S = 1/2$ Ising models on square, triangular, honeycomb, and centered square (Union Jack) anisotropic lattices are obtained by using strip clusters when an effective field is applied to one side of the strip only [4]. Another exotic way of estimating the critical point of the square-lattice Ising model was proposed in [5]. The authors of this paper showed that in the quasidiagonal form of a transfer matrix of a finite-width strip, all coefficients of the characteristic equation for the sub-block containing the largest eigenvalue have an extremum located precisely at the exact value of the phase transition temperature of the infinite lattice.

In the present paper, we concentrate our attention on the method to calculate the critical temperature proposed by Wosiek [6] (see also [7–12]). The author of [6] introduced a maximum criterion for the ratio of moments of the transfer matrix and obtained the following equation for determining the critical point position in a d-dimensional system:

$$u_1(K_c) = u_2(K_c). \tag{1}$$

Here, u_1 and u_2 are the respective internal energies of $(d-1)$-dimensional and two coupled $(d-1)$-dimensional subsystems and K_c is the critical coupling (the normalized inverse critical temperature) of the d-dimensional system.

It is remarkable that at $d = 2$, Eq. (1) (see [6]) yields the exact value of K_c for the isotropic square and triangular Ising lattices, as well as for the three-site Potts model on the square lattice with isotropic interactions. Subsequently, several other models were added to the list, which now includes another isotropic Baxter model (two square Ising lattices coupled by four-particle interactions), the Baxter–Wu model (triangular lattice with three-site interactions of Ising spins) [10], and the q-state Potts model on an isotropic square lattice with an arbitrary value of q [12]. The physical nature of

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Eq. (1) can be elucidated when it either yields an exact solution or admits an approximate estimate or does not give any solution at all for a given model.

For a two-dimensional system, Eq. (1) connects the internal energies of infinitely long linear and double chains. Therefore, in order to test Eq. (1) rigorously, it is necessary to have analytical solutions for such subsystems.

In Sec. 2, we give an exact analytical solution for the two-chain Potts strip with a large number of independent parameters. As a special case, it contains a solution for the linear Potts chain.

Our solution for the double Potts chain enables us to cover all the previously known cases where Eq. (1) exactly reproduces the critical temperatures for the two-dimensional Ising, Baxter–Wu, and Potts models. In addition, we discover (Sec. 3) a new model for which Eq. (1) yields the exact result. This is the q-state Potts model on the triangular lattice with purely three-site interactions in a half of the triangular faces [13].

In Sec. 4, we discuss the results. In particular, we show that duality is a sufficient condition for the validity of Eq. (1) for isotropic spin lattices. In Sec. 5, we summarize the results obtained in the work.

2. SOLUTION OF THE DOUBLE q-STATE POTTS CHAIN WITH THE $\mathbf{S}_q$ SYMMETRY

We consider a two-chain (ladder) lattice with spin variables σ_l^i attached to its sites ($i = 1, 2$ is the chain index and $l = 1, 2, 3, \ldots$ labels the sites in the longitudinal direction of the ladder); the spin variables take the values $1, 2, \ldots, q$.

We write the Hamiltonian of the system as

$$\mathcal{H} = -\sum_l H(\sigma_l^1, \sigma_l^2; \sigma_{l+1}^1, \sigma_{l+1}^2). \tag{2}$$

The locality of interactions in this Hamiltonian allows us to introduce the transfer matrix V with the elements

$$\langle \sigma_1, \sigma_2 | V | \sigma_1', \sigma_2' \rangle = \exp \frac{H(\sigma_1, \sigma_2; \sigma_1', \sigma_2')}{k_B T} \tag{3}$$

(where T is the temperature and k_B is the Boltzmann constant) and reduce the problem of calculating the free energy density f of an infinitely long strip to finding the largest eigenvalue λ_1 of the matrix V:

$$f = \frac{1}{2} \ln \lambda_1. \tag{4}$$

Transfer matrix (3) has the size $q^2 \times q^2$. It is real and all its elements are positive, but the matrix is not symmetric in general ($V_{ij} \neq V_{ji}$).

To solve the eigenvalue problem for the transfer matrix, we use the group-theoretical approach (see, e.g., Ref. [14], where this approach was applied to a quasidiagonalization of the Ising model transfer matrix on parallelepipeds $L \times L \times \infty$). In order to obtain a solution for the two-leg spin ladder (in which we are particularly interested) in the most general form, we procee in the reversed order. Namely, we first select a symmetry group in the space $|\sigma_1, \sigma_2\rangle$, which enables us to quasidiagonalize the transfer matrix up to sub-block secular equations that can be solved analytically; only then we expand the Hamiltonian density H into a series in the invariants of the symmetry group.

We take a model that is invariant, e.g., under transformations of the symmetric group $\mathbf{S}_q$ of the degree q. For the Potts model, this means that we are dealing with a system in the zero external field. Fortunately, the field is not required to test Eq. (1).

It is known (see, e.g., [15]) that the largest eigenvalue of the transfer matrix is located in the sub-block of the identity irreducible representation. In accordance with group theory, the basis vectors ψ_i of the identity irreducible representation can be obtained by successively acting with the permutation operators of the $\mathbf{S}_q$ group on the orths $|1,1\rangle$, $|1,2\rangle$, ... , $|q,q\rangle$. Acting by elements of the symmetric group first on the orth $|1,1\rangle$ and then on $|1,2\rangle$, we find that the two linear combinations obtained involve all the orths. The normalized basis vectors are given by

$$\psi_1 = \frac{1}{\sqrt{q}} \sum_{i=1}^{q} |i,i\rangle, \quad \psi_2 = \frac{1}{\sqrt{q(q-1)}} \sum_{i,j=1}^{q}{}' |i,j\rangle \tag{5}$$

(the prime at the second sum indicates that the terms with $i = j$ are omitted). Hence, the sub-block of the identity irreducible representation has the size 2 by 2, and therefore, its eigenvalues (one of which is λ_1) can be easily obtained by solving an algebraic equation of only the second degree. We note that if we take the group $\mathbf{S}_q \times \mathbf{C}_s$ (where $\mathbf{C}_s$ is the group of mirror reflections in the plane placed between the chains of the two-leg ladder), the sub-block corresponding to the identity irreducible representation again has the size 2×2, and therefore, this symmetry (which only reduces the number of independent parameters in the Hamiltonian) does not justify itself in the given case.

We now represent Hamiltonian (2) as a sum of terms that are invariant under transformations of the

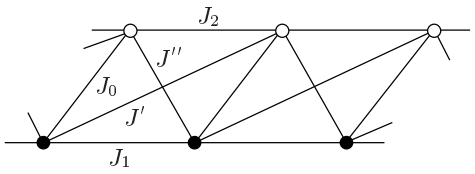

Fig. 1. Geometry of two-site couplings in the double q-state Potts chain with the $\mathbf{S}_q$ symmetry

group $\mathbf{S}_q$:

$$\mathcal{H} = -\sum_l [J_1\delta_{\sigma_l^1\sigma_{l+1}^1} + J_2\delta_{\sigma_l^2\sigma_{l+1}^2} + J_0\delta_{\sigma_l^1\sigma_l^2} + \\ + J'\delta_{\sigma_l^1\sigma_{l+1}^2} + J''\delta_{\sigma_l^2\sigma_{l+1}^1} + J_3\delta_{\sigma_l^1\sigma_l^2\sigma_{l+1}^1} + J_3'\delta_{\sigma_l^1\sigma_{l+1}^1\sigma_{l+1}^2} + \\ + \tilde{J}_3\delta_{\sigma_l^1\sigma_l^2\sigma_{l+1}^2} + \tilde{J}_3'\delta_{\sigma_l^2\sigma_{l+1}^1\sigma_{l+1}^2} + J_4\delta_{\sigma_l^1\sigma_l^2\sigma_{l+1}^1\sigma_{l+1}^2}]. \quad (6)$$

The Kronecker symbols entering here are defined as

$$\delta_{\sigma_1\cdots\sigma_k} = \begin{cases} 1, & \text{if } \sigma_1 = \ldots = \sigma_k, \\ 0, & \text{otherwise.} \end{cases} \quad (7)$$

The structure of the two-site couplings in Hamiltonian (6) is shown in Fig. 1. Matrix elements of the original transfer matrix are written as

$$\langle\sigma_1,\sigma_2|V|\sigma_1',\sigma_2'\rangle = \exp\Big[K_1\delta_{\sigma_1\sigma_1'} + K_2\delta_{\sigma_2\sigma_2'} + \\ + \frac{1}{2}K_0(\delta_{\sigma_1\sigma_2} + \delta_{\sigma_1'\sigma_2'}) + K'\delta_{\sigma_1\sigma_2'} + K''\delta_{\sigma_2\sigma_1'} + \\ + K_3\delta_{\sigma_1\sigma_2}\delta_{\sigma_1\sigma_1'} + K_3'\delta_{\sigma_1\sigma_1'}\delta_{\sigma_1\sigma_2'} + \tilde{K}\delta_{\sigma_1\sigma_2}\delta_{\sigma_1\sigma_2'} + \\ + \tilde{K}'\delta_{\sigma_2\sigma_2'}\delta_{\sigma_2\sigma_1'} + K_4\delta_{\sigma_1\sigma_1'}\delta_{\sigma_1\sigma_2}\delta_{\sigma_1\sigma_2'}\Big], \quad (8)$$

where

$$K_0 = J_0/k_BT, \quad K_1 = J_1/k_BT, \quad K_2 = J_2/k_BT,$$

$$K' = J'/k_BT, \quad K'' = J''/k_BT, \quad K_3 = J_3/k_BT,$$

$$K_3' = J_3'/k_BT, \quad \tilde{K}_3 = \tilde{J}_3/k_BT, \quad \tilde{K}_3' = \tilde{J}_3'/k_BT,$$

$$K_4 = J_4/k_BT.$$

Using Eqs. (5) and (8), we calculate the matrix elements

$$Q_{ij} = \psi_i^+ V\psi_j$$

of the sub-block corresponding to the identity irreducible representation:

$$Q_{11} = [q - 1 + \exp(K_1 + K_2 + K' + K'' + K_3 + \\ + K_3' + \tilde{K}_3 + \tilde{K}_3' + K_4)]\exp K_0, \\ Q_{12} = (q-1)^{1/2}[q - 2 + \exp(K_1 + K'' + K_3) + \\ + \exp(K_2 + K' + \tilde{K}_3)]\exp(K_0/2), \\ Q_{21} = (q-1)^{1/2}[q - 2 + \exp(K_1 + K' + K_3') + \\ + \exp(K_2 + K'' + \tilde{K}_3')]\exp(K_0/2), \\ Q_{22} = (q-2)(q - 3 + e^{K_1} + e^{K_2} + e^{K'} + e^{K''}) + \\ + \exp(K_1 + K_2) + \exp(K' + K''). \quad (9)$$

As a result, we find that the largest eigenvalue of the transfer matrix of the double q-state Potts chain with Hamiltonian (6) is given by

$$\lambda_1^{(2)} = \frac{1}{2}(Q_{11} + Q_{22}) + \\ + \left[\frac{1}{4}(Q_{11} - Q_{22})^2 + (q-1)A\exp K_0\right]^{1/2}, \quad (10)$$

where

$$A = [q-2+\exp(K_1+K''+K_3)+\exp(K_2+K'+\tilde{K}_3)]\times \\ \times[q-2+\exp(K_1+K'+K_3')+\exp(K_2+K''+\tilde{K}_3')]. \quad (11)$$

The versions of the double Potts chains solved previously [3, 12, 16–18] correspond to a particular choice of the interaction constants. Setting

$$J_0 = J'(= J)$$

with all the other interaction constants vanishing, we arrive at the solution for the linear Potts chain [19],

$$\lambda_1^{(1)}(K) = e^K + q - 1. \quad (12)$$

3. THE TRIANGULAR POTTS LATTICE WITH THREE-SITE INTERACTIONS ON ALTERNATE TRIANGLE FACES

A large number of independent parameters in the model solved in the previous section enables us to test Eq. (1) for a wide class of two-dimensional spin systems.

In addition to the cases listed in the Introduction, in which Eq. (1) is satisfied exactly, we consider the Potts model on a triangular lattice with three-site interactions in each up-triangle (Fig. 2). The position of the critical point in this model was found with both three- and two-site interactions [13]. However, it is known [12]

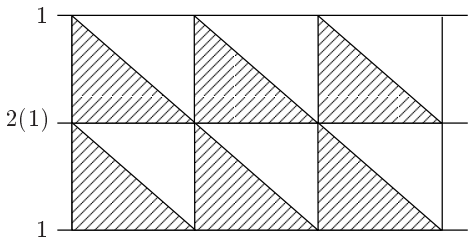

Fig. 2. Fragment of the Potts lattice with three-site interactions in alternate triangular faces (shaded)

that for the triangle lattice with pair couplings, Eq. (1) yields the exact result only for the Ising case ($q = 2$). We therefore discuss the model with purely three-site interactions. In this case,

$$K_c = \ln(1+q). \tag{13}$$

We now show that this value satisfies Eq. (1) by subsystems in the shape of strips with the periodic boundary condition in the transverse direction.

The internal energy of the one-dimensional subsystem is

$$u_1(K) \equiv \frac{\partial f_1}{\partial K} = [(q-1)e^{-K}+1]^{-1}. \tag{14}$$

Substituting $K_3 = \tilde{K}_3$ with all the other interaction constants vanishing, we obtain from (4), (10), and (11) the free energy density of the double Potts chain:

$$f_2(K) = \frac{1}{2}\ln\left[\frac{1}{2}(e^{2K}+q^2-1)+ \right. \\ + \left[\frac{1}{4}(e^{2K}-(q-1)^2)^2 + \right. \\ \left.\left. + q(q-1)(2e^K+q-2)\right]^{1/2}\right]. \tag{15}$$

The internal energy is given by

$$u_2(K) = \frac{\partial f_2}{\partial K}.$$

Differentiating Eq. (15) with respect to K, we find the expression for $u_2(K)$.

The analysis shows that the dependences $u_1(K)$ and $u_2(K)$ have a crossing point that lies exactly at

$$K = K_c = \ln(1+q)$$

both for integer and non-integer q. The internal energy of the system at the critical point is given by

$$u_\infty(K_c) = u_1(K_c) = u_2(K_c) = \frac{1}{2}(1+q^{-1}). \tag{16}$$

Thus, using solutions for only the linear and double Potts chains, Eq. (1) has enabled us to extract the exact value of K_c for the bulk two-dimensional Potts model on a triangular lattice with alternating faces that interact by three-site forces.

4. DISCUSSION

In Ref. [8], Eq. (1) was extended to

$$u_L(K_c) = u_{L'}(K_c), \quad L, L' = 1, 2, 3, \ldots, \tag{17}$$

where u_L is the internal energy per site of L coupled $(d-1)$-dimensional subsystems. In the two-dimensional case, L denotes the width of the strip.

The validity of condition (17) for arbitrary L and L' means the absence of a «singular» (i.e., L-dependent) part of the internal energy density at the critical point,

$$u_L(K_c) = \text{const} \quad \text{on} \quad L. \tag{18}$$

In other words, the amplitudes of all finite-size corrections to the critical internal energy of the system $u_\infty(K_c)$ are equal to zero.

For the square isotropic Ising lattice, the derivative of the inverse correlation length $\kappa_L(K)$ with respect to the temperature-like variable K has a similar property [20, 21],

$$\left.\frac{\partial \kappa_L}{\partial K}\right|_{K=K_c} = \left.\frac{\partial \kappa_{L'}}{\partial K}\right|_{K=K_c}, \tag{19}$$

i.e., $\partial\kappa_L/\partial K|_c$ does not depend on L. This property has enabled us to exactly determine the value of the thermal critical exponent $y_t\,(=1)$ for this model using only the finite-size data [20, 21].

Equations (1) and (17) are valid for the ferromagnetic isotropic square Potts lattices. These models are self-dual and their critical coupling (in the anisotropic case) is determined from the condition

$$(\exp K_x - 1)(\exp K_y - 1) = q. \tag{20}$$

For the antiferromagnetic square-lattice Potts model, the criticality condition is [22]

$$(\exp K_x + 1)(\exp K_y + 1) = 4 - q, \tag{21}$$

where $K_x < 0$ and $K_y < 0$. We performed a verification and found that in the antiferromagnetic case, the curves $u_1(K)$ and $u_2(K)$ do not have any self-crossing point, and therefore, Eq. (1) does not lead to the exact value that follows from Eq. (21), nor to any approximate estimate for the critical point.

It is not difficult to show that if the model is self-dual and the dual point therefore coincides with the original one, Eqs. (1) and (17) are valid.

Indeed, we consider for instance the Ising model on the isotropic square lattice $L \times N$ with toroidal boundary conditions. The partition function of this system has a fundamental property: it is invariant (up to a multiplicative factor exponentially depending on L) under the duality transformation (see [23]),

$$Z_{L,N}(K^*) = (\operatorname{sh} 2K)^{-LM} Z_{L,M}(K), \tag{22}$$

where K and K^* are related by

$$\operatorname{th} K^* = \exp(-2K). \tag{23}$$

(We here used another normalization of the exchange constant in the Ising model, namely $J_{\text{Potts}} = 2J_{\text{Ising}}$.) In the limit of an infinitely long strip ($N \to \infty$), Eq. (22) transforms to the duality condition for the largest eigenvalue,

$$\lambda_1^{(L)}(K^*) = (\operatorname{sh} 2K)^{-L} \lambda_1^{(L)}(K). \tag{24}$$

This implies that the values of the normalized internal energy in dually conjugated points (K and K^*) are related by

$$u_L(K^*) \frac{\partial K^*}{\partial K} = u_L(K) - 2u_0(K), \tag{25}$$

where the additive term u_0 ($= \operatorname{cth} 2K$) does not depend on L. Another important feature related to the isotropy of the lattice is that the dually conjugated points K and K^* merge into one point at criticality,

$$K^* = K = K_c. \tag{26}$$

Using Eq. (23), we find that at the critical point

$$K_c = \frac{1}{2} \ln\left(1 + \sqrt{2}\right),$$

the derivative

$$\left.\frac{\partial K^*}{\partial K}\right|_c = -1.$$

Consequently,

$$u_L(K_c) = u_0(K_c) = \sqrt{2}. \tag{27}$$

Thus, the critical internal energy per site $u_L(K_c)$ of an Ising cylinder with isotropic square cells satisfies condition (18) for all $L = 1, 2, \ldots$ This, in turn, leads to the validity of Eqs. (1) and (17).

Similarly, Eqs. (1) and (17) can be derived for other isotropic spin model partition functions that satisfy a functional equation like

$$Z_L(K^*) = [g(K)]^L Z_L(K). \tag{28}$$

In the cases where the model is self-dual but the critical manifold is a line or a surface (as, e.g., for anisotropic lattices), Eqs. (1) and (17) no longer hold. This is not difficult to prove if we again consider the two-dimensional Ising model. For the anisotropic square lattice, the duality condition becomes

$$\lambda_1^{(L)}(K_x^*, K_y^*) = \\ = [\operatorname{sh}(2K_x) \operatorname{sh}(2K_y)]^{-L/2} \lambda_1^{(L)}(K_x, K_y) \tag{29}$$

with

$$\operatorname{th} K_x^* = \exp(-2K_y) \text{ and } \operatorname{th} K_y^* = \exp(-2K_x). \tag{30}$$

It then follows that on the critical line

$$\operatorname{sh}(2K_x) \operatorname{sh}(2K_y) = 1, \tag{31}$$

condition (29) relates the values of the free energy at distinct (dually conjugated) points (K_x, K_y) and (K_y, K_x),

$$f_L(K_x, K_y) = f_L(K_y, K_x) + \\ + \frac{1}{2} \ln\left[\operatorname{sh}(2K_x) \operatorname{sh}(2K_y)\right]. \tag{32}$$

This violates Eqs. (1) and (17), which identify the internal energies at the same point.

The critical internal energy density of the strip $L \times \infty$ cut out from an anisotropic lattice depends on the size L. This is easy to verify using the results of Sec. 2 if one calculates the values $u_1(K_c)$ and $u_2(K_c)$ for the anisotropic Ising and Potts lattices.

On the other hand, we can establish the same property if we take the Onsager solution [24] for the two-dimensional Ising model. The dominant eigenvalue of the transfer matrix of the cylinder $L \times \infty$ with spatially anisotropic interactions is given by

$$\lambda_1^{(L)}(K; \alpha) = [2 \operatorname{sh}(2K)]^{L/2} \times \\ \times \exp\left[\frac{\gamma_1 + \gamma_3 + \ldots + \gamma_{2L-1}}{2}\right], \tag{33}$$

where $\alpha = J_y/J_x$ is the lattice anisotropy parameter and γ_r are positive solutions of the equations

$$\operatorname{ch} \gamma_r = \operatorname{ch}(2\alpha K) \operatorname{cth}(2K) - \frac{\operatorname{sh}(2\alpha K)}{\operatorname{sh}(2K)} \cos\left(\frac{\pi r}{L}\right). \tag{34}$$

From this, we obtain the internal energy per site

$$u_L(K; \alpha) = \operatorname{cth}(2K) + \\ + \frac{1}{2L}\left(\frac{\partial \gamma_1}{\partial K} + \frac{\partial \gamma_3}{\partial K} + \ldots + \frac{\partial \gamma_{2L-1}}{\partial K}\right). \tag{35}$$

The functions $\gamma_r(K)$ have a smooth extremum (minimum) that in the isotropic case ($\alpha = 1$) lies exactly at $K = K_c$, and therefore,

$$\left.\frac{\partial \gamma_r}{\partial K}\right|_{K=K_c} = 0 \quad (r \neq 0). \tag{36}$$

As a result, the second term in Eq. (35) disappears and the critical internal energy ceases to depend on L. When $\alpha \neq 1$, Eq. (36) is not valid and $u_L(K;\alpha)$ depends on the strip width in a complicated way. This explains the failure of the exact calculations of K_c from Eq. (1) in the anisotropic Ising lattice [10].

In closing this section, we note that in spite of Eqs. (1) and (17), Eq. (19) cannot be deduced from the dual invariance of the system.

5. CONCLUSIONS

Using the group-theoretical approach, we obtained the exact analytical solution for the double Potts chain with Hamiltonian (6). The solution allows examining Eq. (1) for a large number of models with Ising ($q = 2$) and arbitrary Potts spins (including non-integer q). The validity of Eq. (1) for the triangular Potts lattice with purely three-site interactions in alternate triangular faces was established.

We have also shown that Eqs. (1) and (17) are a consequence of the duality symmetry of models for which the critical point coincides with its dual image.

As far as the author knows, the inverse theorem has not been proved. Duality plus isotropy or, more precisely, self-conjugation of the critical point are not necessary conditions for Eq. (1). In general, therefore, there can exist systems that are not invariant under the duality transformation or a combination of the dual and star–triangle transformations, but for which all amplitudes of finite-size corrections to the critical internal energy (or to some other quantity) are equal to zero.

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