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A TURBULENCE MODEL IN UNBOUNDED SMOOTH SHEAR FLOWS. THE WEAK TURBULENCE APPROACH

G. D. Chagelishvili[a,b]*, *R. G. Chanishvili*[a], *T. S. Hristov*[c], *J. G. Lominadze*[a]

[a] *Department of Theoretical Astrophysics, Abastumani Astrophysical Observatory 380060, Tbilisi, Georgia*

[b] *Space Research Institute of Russian Academy of Sciences 117810, Moscow, Russia*

[c] *University of California, Irvine Irvine CA 92697-3975, USA*

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We discuss a new concept of the subcritical transition to turbulence in unbounded smooth (noninflectional) spectrally stable shear flows. This concept (the so-called «bypass» transition) follows from considering the non-normality of the linear dynamics of vortex disturbances in shear flows and is most easily interpreted by tracing the evolution of spatial Fourier harmonics (SFHs) of the disturbances. The key features of the concept are as follows: the transition of the flow by only finite amplitude vortex disturbances despite the fact that the phenomenon is energetically supported by a linear process (the transient growth of SFHs); the anisotropy of processes in the $\mathbf{k}$-space; the onset of chaos due to the dynamical (not stochastic) process — nonlinear processes that close the transition feedback loop by the angular redistribution of SFHs in the $\mathbf{k}$-space. The evolution of two-dimensional small-scale vortex disturbances in the parallel flow with a uniform shear is analyzed within the weak turbulence approach. This numerical test analysis is carried out to prove the most problematic statement of the concept, the existence of a positive feedback caused by the nonlinear process. Numerical calculations also show the existence of a threshold: if the amplitude of the initial disturbance exceeds the threshold value, the self-maintenance of disturbances becomes realistic. The latter is a characteristic feature of the flow transition to the turbulent state and its maintenance.

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1. INTRODUCTION

Shear flows are permanently interesting because they are widely spread both in the terrestrial and astrophysical environment (galaxies, stars, jets, planet atmospheres, oceans, etc.) and in the laboratory and industry (tokamaks, MHD facilities, etc.). Some simple and important hydrodynamic shear flows (e.g., the Couette flow) remain insensitive to infinitesimal disturbances at any Reynolds numbers but become turbulent at finite disturbances even at moderate (subcritical) Reynolds numbers. Moreover, the transition to turbulence occurring in such flows strongly depends not only on the amplitude of the initial disturbances but also on their type and spectrum. Physics of these facts was not explained even one decade ago [1–6].

Specific features of shear flows rigrorously established recently [7] led to difficulties in studying linear phenomena in the framework of the canonical modal analysis, i.e., the technique where all the disturbed quantities are expanded in Fourier integrals in time. The point is that the operators arising in this approach are not self-adjoint [8]. Their eigenfunctions are not orthogonal to each other, which yields a strong interference among them. As a result, even if all the imaginary parts of all eigenfrequencies are negative and the eigenfunctions monotonically decay with time (i.e., the flow is spectrally stable), a particular solution can reveal a large relative growth over a finite time interval. The analysis of separate eigenfunctions and eigenfrequencies is therefore not sufficient to arrive at definite

*E-mail: georgech123@yahoo.com

conclusions on the linear evolution of disturbances. In addition, taking the interference into account usually leads to insurmountable complications. This has given impetus to the so-called nonmodal analysis as a tool for describing the evolution of disturbances in smooth shear flows (i.e., those without the inflection point), primarily in the parallel flow with the uniform shear of velocity. Within this approach, the temporal behavior of the spatial Fourier harmonics (SFHs) of disturbances is studied without any spectral expansion in time. Being an optimal tool, the nonmodal analysis considerably simplifies the mathematical description of the processes and is capable of revealing the key phenomena that escape perception in the modal approach (in particular, the phenomena caused by the non-normality of the linear dynamics). Many new unexpected results on time evolution of both the vortex mode [9–17] and acoustic wave [18, 19] disturbances have already been obtained within this approach; it was also successfully applied to the study of the MHD waves [20–22]. New linear mechanisms of the mutual transformation of wave modes [23–25] and conversion of vortices to waves [26–29] have been discovered. A new concept of the subcritical transition to turbulence in smooth shear flows (those without the inflection point) has been formulated [30–37]. The latter, named the «bypass» transition, is the subject of our analysis.

According to the concept, the subcritical transition to turbulence that occurs in spectrally stable shear flows is caused by the interplay among four (linear and nonlinear) basic phenomena. The transition scenario based on this concept is presented in detail (in qualitative terms) in Sec. 2. In Sec. 2, we also consider the «philosophical» problem of turbulence, i.e., how a completely deterministic and causal system can have chaotic solutions. In Sec. 3, we give numerical test calculations to prove the most problematic statement of the concept — the existence of a positive nonlinear feedback. The subsequent results of numerical calculations are also presented in Sec. 3. We have restricted ourselves to the investigation of the action of nonlinearity for the two-dimensional symmetric disturbance (which is quite simple and most suitable for testing) in the weak turbulence approximation. In reality, the shear flow turbulence has a three-dimensional ($3D$) nature. However, from the discussion presented in Sec. 4 it follows that nonlinear processes should easier cope with the «mission» of the positive feedback in the actual $3D$ case than in the $2D$ one. The weak turbulence equation for a $2D$ vortex mode disturbances in the parallel flow with a uniform shear is derived in the Appendix.

2. SCENARIO OF THE SUBCRITICAL TRANSITION TO TURBULENCE

Vortex mode (aperiodic/nonoscillating) disturbances are the creator of turbulence in the unbounded, parallel flow with a constant shear rate and a uniform density that we consider here. Therefore, the presented scenario involves disturbances of only this type. The nonmodal formalism allows revealing the following specific features in the evolution of SFHs:

(a) The wave number of a SFH along the axis orthogonal to the flow velocity (i.e., along the flow shear) varies in time; in the linear approximation, there is a «drift» of a SFH in the wave-number space, i.e., in the **k**-space.

Actually, (cf. [9–28]) in a parallel flow with uniform shear

$$\mathbf{U}_0 = (Ay, 0, 0) \qquad (1)$$

(where A is the shear parameter that is assumed to be positive), disturbances cannot have the form of a simple plane wave because of the effect of the shearing background on the wave crests. The SFH wave numbers are then time-dependent: if a SFH with the wave numbers k_x, $k_y(0)$, and k_z is initially disturbed,

$$v_x(0) = \tilde{v}_x(k_x, k_y(0), k_z, 0) \times \times \exp(ik_x x + ik_y(0)y + ik_z z), \qquad (2)$$

then the evolution of its phase for $t > 0$ is determined by the equations

$$v_x(t) \propto \exp(ik_x x + k_y(t)y + ik_z z), \qquad (3)$$

$$k_y(t) = k_y(0) - k_x At \qquad (4)$$

that describe the «linear drift» of the SFH in the wave-number space.

The values of the spatial characteristics (i.e., k_x, $k_y(t)$, and k_z) define the energy exchange intensity between SFHs and the background flow to a greater extent. Therefore, the linear drift leads to the variation of the intensity of this exchange.

(b) Not all SFHs can draw energy from the shear; only the SFHs that are located in a certain region of the **k**-space (called the «amplification region» below) are amplified. Moreover, each SFH is amplified during a limited time interval until it leaves the amplification region as a result of the linear drift. In addition, the presence of SFHs in this region imposes conditions mainly on the direction (and not the magnitude) of their wave vector. Therefore, the process of the energy exchange between vortex mode disturbances and the shear flow has a pronounced anisotropic character

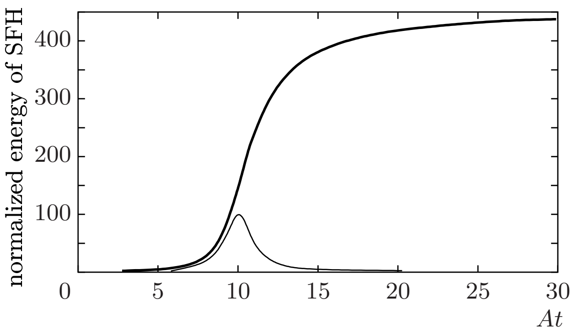

Fig. 1. Time evolution of the normalized energy of $2D$ and $3D$ SFHs defined in the linear stage and in the invisced case (i.e., with only processes (**a**) and (**b**) involved). Thin solid line corresponds to a $2D$ SFH with the parameters $k_y(0)/k_x = 10$, $k_z = 0$, $\tilde{v}_x(0)/\tilde{v}_y(0) = -10$, and $\tilde{v}_z(0) = 0$. The bold solid line corresponds to a $3D$ SFH with the parameters $k_y(0)/k_x = 10$, $k_z/k_x = 1$, $\tilde{v}_x(0)/\tilde{v}_y(0) = -5$, and $\tilde{v}_z(0)/\tilde{v}_y(0) = -5$. Here, $\tilde{v}_x(0)$, $\tilde{v}_y(0)$, and $\tilde{v}_z(0)$ are the components of the SFH velocity at $t = 0$. k_x, $k_y(0)$, and k_z are related to the wave numbers of the SFH (see Fig. 2)

in the **k**-space. Physics of this process is described in detail in [38].

Therefore, vortex mode disturbances at the linear stage of the evolution are pumped by the background shear flow and grow within a limited time interval, i.e., exhibit a transient growth. There is an essential difference between the transient growths of $2D$ and $3D$ SFHs [16–20], which can be seen by comparing the evolution of their energy, as in Fig. 1. This figure shows time evolution of the normalized energy of $2D$ and $3D$ SFHs. It corresponds to the linear dynamics of separate SFHs in the invisced case (i.e., when only processes (**a**) and (**b**) are at work).

The amplification region in the **k**-space is much wider for $3D$ SFHs than for $2D$ ones. Moreover, in contrast to $2D$ SFHs, the energy of $3D$ SFHs does not decrease after passing the amplification region ($3D$ SFHs do not return energy to the flow) but it saturates and approaches a value that may be much higher than their initial value. In reality, however, a viscous dissipation becomes efficient as $|k_y(t)| \to \infty$ and (if no new phenomena, e.g., nonlinear phenomena are involved) converts the energy of SFHs into heat. We list the viscous dissipation as item (**c**).

Thus, the nonmodal approach demonstrates not only the possibility of the algebraic/transient growth of SFHs of vortex mode disturbances in shear flows, but also the anisotropic properties of linear processes in the wave-number space. This anisotropy is also observed in nonlinear processes.

(**d**) Nonlinear processes, apart from the usual fragmentation of the disturbance scale, are also responsible for the angular redistribution of SFHs in the **k**-space, i.e., they could «supply» SFHs to the amplification region, closing a feedback loop of the transition to turbulence. In a forced shear flow, the nonlinear terms do not contribute to the energy transfer between the mean flow and disturbances.

Processes (**a**) and (**b**) are quantitatively analyzed and well-acknowledged in papers devoted to the nonmodal approach. The existence of a positive feedback (caused by the nonlinear processes) has been checked using model equations [34, 35]. In Sec. 3, we prove it using the Navier–Stokes equation in the weak turbulence approach.

It is plausible that the angular redistribution of SFHs in the **k**-space is the main process caused by the nonlinearity. The nonlinear processes then indirectly favor the energy extraction by SFHs from the shear flow (the SFH scale decrease to the dissipative scale should be ensured by the linear drift of SFHs in the **k**-space).

The scenario of the subcritical transition to turbulence (called the «bypass» transition) is based on the interplay of the linear and nonlinear basic phenomena itemized above. In presenting this scenario, we schematically describe these processes in the plane $k_z = \mathrm{const}$ (which is parallel to the plane $k_x k_y$). It is obvious that the boundaries of the **k**-space regions where phenomena (**b**) and (**c**) occur are vague. We fix the regions where these phenomena are operative for clarifying the analysis. The viscous dissipation becomes essential for harmonics with the wave numbers satisfying the inequality

$$\sqrt{k_x^2 + k_y^2} > k_\nu,$$

where the value of k_ν depends on the Reynolds number. As follows from Fig. 1, the real growth of the disturbance energy occurs when the ratio $|k_y(t)/k_x|$ reaches moderate values (the dashed region in Fig. 2). We can therefore separate three regions inside the circle

$$\sqrt{k_x^2 + k_y^2} < k_\nu :$$

I(I′), II(II′), and III(III′). We now discuss what happens to a SFH of the vortex mode disturbance injected in region I(I′), for instance at point *1* (see Fig. 2). The wave number of the SFH varies in time, thereby leading to a drift in the direction marked by the arrows.

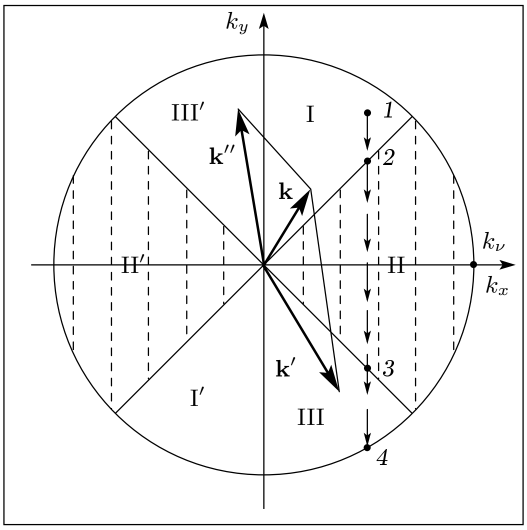

Fig. 2. A conventional separation of regions of the action of the basic physical processes that are responsible for the onset of turbulence/chaos in accordance with the bypass transition. The energy exchange between the disturbances and the background flow is essential (a transient growth takes place) in regions II(II′) dashed by vertical lines; nonlinear processes (e.g., of the type $\mathbf{k}' + \mathbf{k}'' \to \mathbf{k}$) and the «linear drift» are effective in all regions I(I′), II(II′), and III(III′) inside the circle $\sqrt{k_x^2 + k_y^2} < k_\nu$. The viscous dissipation of SFHs dominates outside the circle $\sqrt{k_x^2 + k_y^2} > k_\nu$

After a certain moment, when the harmonics passes point *2*, its energy starts to grow. This growth is transient and lasts until the SFH leaves the amplification region II(II′) (point *3* in Fig. 2). Continuing its drift, the harmonics then reaches point *4*, where the dissipative processes are switched on and convert the disturbance energy into the heat. Consequently, if the nonlinear phenomena are inefficient, nothing interesting can occur as regards the transition, and the disturbances eventually disappear. A permanent extraction of the shear energy by disturbances is necessary for their maintenance. This is possible in the case of the permanent existence of disturbances in regions I(I′) and II(II′) that can be provided by nonlinear processes, in particular, by the three-wave processes

$$\mathbf{k}' + \mathbf{k}'' \to \mathbf{k}$$

(see Fig. 2), four-wave processes

$$\mathbf{k}' + \mathbf{k}'' + \mathbf{k}''' \to \mathbf{k},$$

five-wave processes, etc. This means a predominant transfer of the disturbance energy by the nonlinear processes from region III(III′) to regions I(I′) and II(II′). However, there are no restrictions on the reverse transfer (from regions I(I′) and II(II′) to region III(III′)). But, as shown is Sec. 3, the nonlinear processes ensure a preferential transfer of the disturbance energy to the amplification region.

The reproduction of disturbances in region I(I′) depends on both the amplitude and the spectrum of the initial disturbances. The nonlinear decay processes are insignificant at low amplitudes and are not able to resist the linear drift of SFHs in the **k**-space. As a result, low-amplitude disturbances are damped without any trace, i.e., without inducing the transition to turbulence. The higher is the initial disturbance amplitude, the more noticeable nonlinear effects occur. At a certain amplitude (which evidently depends on the initial disturbance spectrum and the Reynolds number), nonlinear processes can compensate the action of the linear drift, thereby ensuring the permanent return of SFHs to the amplification region (this is justified by simulations in Sec. 3). This eventually ensures a permanent extraction of energy from the background flow and the maintenance of disturbances. Therefore, a certain threshold must occur in accordance with the scenario discussed here.

Any theory aiming at explaining the transition to turbulence must distinctly answer the problem of how a completely deterministic and causal system can have chaotic solutions. In accordance with the above scenario, the onset of turbulence/chaos occurs because of dynamical (not stochastic) processes and can be explained as follows.

We assume that we initially have a spatially localized vortical disturbance with sufficiently regular features: a package of spatial Fourier harmonics. In general, a disturbance of some physical variable, e.g., velocity can be represented as

$$\mathbf{v}(\mathbf{r},t) = \int d\mathbf{k}|\tilde{\mathbf{v}}(\mathbf{k},t)|\exp[i\varphi(\mathbf{k},t) + i\mathbf{k}\cdot\mathbf{r}], \qquad (5)$$

where $|\tilde{\mathbf{v}}(\mathbf{k},t)|$ and $\varphi(\mathbf{k},t)$ are real functions of $\mathbf{k}$ and t. We assume that the initial phase $\varphi(\mathbf{k},0)$ is a weakly varying function of $\mathbf{k}$. In this case, the initial disturbance $\mathbf{v}(\mathbf{r},0)$ is regular and sufficiently smooth in space.

What kind of processes govern the phase evolution at any point of the **k**-space?

We consider processes in at arbitrarily chosen point in the **k**-space inside the package. Following the scenario, the SFH that happens to be at the point at the initial moment of time, leaves this point because of the linear drift. But this «loss» is compensated by the linear and nonlinear processes: a portion of energy «arrives» as the result of the linear drift; portions of energy are transferred from numerous points of the $\mathbf{k}$-space as a result of the nonlinear decay processes (three-wave, four-wave, etc.) described above. The total energy of the SFH at the chosen point is composed of these portions. Naturally, all these portions have their own phases. It is clear that the Fourier harmonic phase at the point must be a certain sum of these phases. It is evident that the phase $\varphi(\mathbf{k},t)$ becomes a strongly varying function of $\mathbf{k}$ with the passage of time, because the phases of SFHs at neighboring points of the $\mathbf{k}$-space can differ from each other by any value. Consequently, an initially regular disturbance becomes more and more irregular, thereby tending to the chaotic behavior.

3. THE WEAK TURBULENCE APPROACH

In accordance with the above scenario, nonlinear processes do not contribute to the energy transfer between the mean flow and perturbations. They result in (i) the fragmentation of the disturbance scale, i.e., the energy transfer from large scales to smaller ones and finally to the dissipative ones; (ii) the angular redistribution of SFHs in the $\mathbf{k}$-space. It must be noted that the energy transfer to the small dissipative scales also occurs because of the linear drift of SFHs (process (**a**)), which could be even more operative than the nonlinear fragmentation of the disturbance scale. We again emphasize that the main role of the nonlinear processes in the presented scenario consists in (ii) rather than (i), because in doing so, they could «supply» SFHs to the amplification region, closing the feedback loop of the transition to turbulence. The existence of a positive nonlinear feedback is the most problematic statement of the concept. It has been verified using model equations [34, 35]. In this section, we attempt to prove it using the Navier–Stokes equation. We performed numerical calculations for a $2D$ symmetric vortex mode disturbance in the weak turbulence approximation. As we see in what follows, the $2D$ symmetric disturbance is most suitable for testing the existence of the positive nonlinear feedback.

The weak turbulence equation describing the evolution of the energy spectral density of a $2D$ disturbance is derived in the Appendix,

$$\frac{\partial E_{\mathbf{k}}}{\partial t} + \nabla_{\mathbf{k}}(\mathbf{V}E_{\mathbf{k}}) - \frac{2Ak_xk_y}{k_x^2+k_y^2}E_{\mathbf{k}} + \nu(k_x^2+k_y^2)E_{\mathbf{k}} = \hat{N}E_{\mathbf{k}}, \quad (6)$$

where

$$\nabla_{\mathbf{k}} = (\partial/\partial k_x, \partial/\partial k_y),$$

$$\mathbf{V} = (-Ak_x, 0),$$

and $E_{\mathbf{k}}$ is the energy density of the $2D$ vortex mode disturbances at a fixed point of the $\mathbf{k}$-space. (In other words, $E_{\mathbf{k}}$ is the spectral density of energy.) The term $\hat{N}E_{\mathbf{k}}$ is defined by Eq. (A.36). As can be seen from Eq. (6) (and as described in Sec. 2), the energy spectral density ($\partial E_{\mathbf{k}}/\partial t$) changes because of the following reasons.

1) The linear «drift» of SFHs in the wave-number space (the second term in the left-hand-side). This term does not cause a variation of the total disturbance energy,

$$\int d\mathbf{k}\nabla_{\mathbf{k}}(\mathbf{V}E_{\mathbf{k}}) = 0,$$

but results in a transfer of SFHs from the amplification region to the attenuation one.

2) The energy exchange between disturbances and the background flow (the third term in the left-hand side). Assuming that $A > 0$, we can state that the $2D$ SFHs for which $k_y(t)/k_x > 0$ gain energy from the background flow and their amplitude increases, whereas the amplitudes of SFHs for which $k_y(t)/k_x < 0$ decrease.

3) The viscosity (the last term in the left-hand side), which transforms the disturbance energy into heat and which is significant for large wave numbers.

4) The nonlinear three-wave processes (the term in the right-hand side), leading to the energy exchange between different SFHs [39-41]. It is easy to show that

$$\int d\mathbf{k}\hat{N}E_{\mathbf{k}} = 0,$$

i.e., the nonlinear term leads only to the energy redistribution in the $\mathbf{k}$-space (not to a change of the total disturbance energy).

The conditions for wave vectors ($\mathbf{k}' + \mathbf{k}'' = \mathbf{k}$) and frequencies ($\omega_1 + \omega_2 = \omega$) are usually imposed on three-wave processes in the weak turbulence equations [39–41]. Because both conditions cannot be simultaneously satisfied for waves with certain wave vectors, the restriction of three-wave processes arises. Moreover, these conditions cause the existence of some completely non-decaying spectra. The vortex mode disturbances considered here are aperiodic ($\omega_1, \omega_2, \omega = 0$) and therefore automatically satisfy the second condition ($\omega_1 + \omega_2 = \omega$). Hence, there are no forbidden three-wave processes for SFHs in our case. However, they have different probabilities. For example, the probability of the processes $\mathbf{k}' + \mathbf{k}' = \mathbf{k}$ is equal to zero, although it is not forbidden in principle. Therefore, the nonlinear term in Eq. (6) is equal to zero if a single SFH mode is disturbed. This explains the following well-known fact: a single SFH mode is an exact solution of the complete incompressible Navier–Stokes equation, while a superposition of modes is usually not.

The net effect of all the three-wave processes depends on two factors: the probability with which different decay acts occur (the coefficients of $E_{\mathbf{k}'}E_{\mathbf{k}''}$ and $E_{\mathbf{k}}E_{\mathbf{k}''}$ in Eq. (A.36)) and the distribution of SFHs in the **k**-space (the values of $E_{\mathbf{k}'}E_{\mathbf{k}''}$ and $E_{\mathbf{k}}E_{\mathbf{k}''}$). If the spectral density of energy is increased in the first and third quarters of the **k**-space at the cost of the second and fourth ones, we can say that the three-wave processes lead to the preferential transfer of SFHs to the amplification region, i.e., lead to the regeneration of SFHs, which can gain shear energy (lead to the positive feedback). This trend of nonlinear processes can be revealed by showing their asymmetry in the **k**-space with respect to the K_x axis. To proceed, we consider the initial $2D$ disturbance with the highest possible symmetry with respect to the K_x axis (see Fig. 3). In this case, processes (**a**) and (**c**) are symmetric with respect to K_x and process (**b**) is asymmetric because it results in removal of SFHs from the first and third quarters of the **k**-space to the second and fourth ones; process (**b**) is therefore asymmetric in the opposite direction to nonlinear process (**d**). That is why the symmetric $2D$ disturbance presented in Fig. 3 is most suitable for determining the trend of the nonlinear transfer of SFHs.

3.1. Results of the numerical calculation of the weak turbulence equation

We consider the $2D$ initial disturbance with the spectral density of energy that is symmetric in the **k**-space (see also Fig. 3),

$$E_{\mathbf{k}}(t=0) = \\ = \begin{cases} B\left\{\operatorname{arctg}\left(\beta_1(k_\nu^2-k_x^2-k_y^2)\right)\operatorname{arctg}\left(\beta_2(k_x^2+k_y^2-k_0^2)\right)\right\}^2 & \text{for } k_\nu^2>k_x^2+k_y^2 \text{ and } k_x^2+k_y^2>k_0^2,\\ 0 & \text{for } k_\nu^2<k_x^2+k_y^2 \text{ and } k_x^2+k_y^2<k_0^2,\end{cases} \tag{7}$$

where B defines the value of the initial disturbance energy, k_0 and $k_\nu = 1/\sqrt{A\nu}$ are the minimum and maximum values of the disturbance wave vectors respectively, and β_1 and β_2 denote the sharpness of the disturbance boundaries in the **k**-space. The calculations are carried out at $\beta_1 = 0.07$, $\beta_2 = 0.8$, $k_0 = 0.3$, and $k_\nu = 10$ (i.e., $A = 1$ and $\nu = 0.01$). The absence of SFHs with large wave numbers is justified by the action of viscosity. SFHs with small wave numbers are also absent, because we consider small-scale disturbances. The evolution of $E_{\mathbf{k}}$ was numerically investigated for a short time interval ($At \le 1$) because of two reasons. First, Eq. (6) is obtained in the weak turbulence approximation and it is therefore correct only for a relatively short time interval ($t \lesssim 1/A$). Second, the trend of nonlinear processes is revealed even for such short time intervals.

Initially, we tried to answer the question what the redistributing action of the nonlinear term $\hat{N}E_{\mathbf{k}}$ is in the **k**-space. Specifically, whether the term $\hat{N}E_{\mathbf{k}}$ transfers disturbance energy to the amplification region. For $2D$ disturbances, the amplification region covers the first and third quarters of the k_xk_y plane (where $k_xk_y > 0$) and the attenuation region covers its second and fourth quarters (where $k_xk_y < 0$). Introducing polar coordinates $\varphi = \operatorname{arctg}(k_y/k_x)$ and $k = \sqrt{k_x^2+k_y^2}$, we can say that the angle φ between 0 and $\pi/2$ corresponds to the amplification region, and between $-\pi/2$ and 0 to the attenuation one.

Obviously, the value and sign of $\hat{N}E_{\mathbf{k}}$ depend on φ and k. Taking the integral over k, we obtain the function that describes the nonlinear redistribution of energy only in φ,

$$\Psi(\varphi,t) \equiv \int dk\; k\hat{N}E_{\mathbf{k}}. \tag{8}$$

It is easy to see that if the conditions

$$\Psi(\varphi,t)|_{0<\varphi<\pi/2} > 0, \quad \Psi(\varphi,t)|_{-\pi/2<\varphi<0} < 0 \tag{9}$$

are satisfied, we can unambiguously state that the nonlinear processes transfer the disturbance energy to the amplification region, thereby realizing the positive feedback.

We thus determine the dependence of Ψ on φ. The result of our calculations at the time instance $At = 0.1$ is shown in Fig. 4. It is seen that conditions (9) are satisfied, i.e., the nonlinear three-wave processes lead to the preferential energy transfer to the amplification region. Because we used a symmetric initial disturbance (with SFHs having the same «weight» in the amplification and attenuation regions), we can conclude that the nonlinear three-wave processes do have the tendency to transfer SFHs to the amplification region. This conclusion can be considered as a numerical confirmation (in the weak turbulence approximation) of the suggestion given in 4).

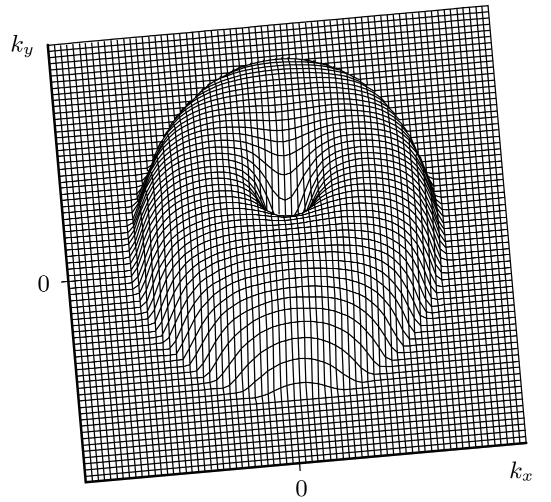

Fig. 3. The spectral density distribution of the disturbance energy in the $k_x k_y$ plane for the time instance $t = 0$, i.e., initial conditions for the numerical solution of Eq. (6). The absence of SFHs with large wave numbers is related to the action of viscosity. SFHs with small wave numbers are also absent, because we consider small-scale disturbances

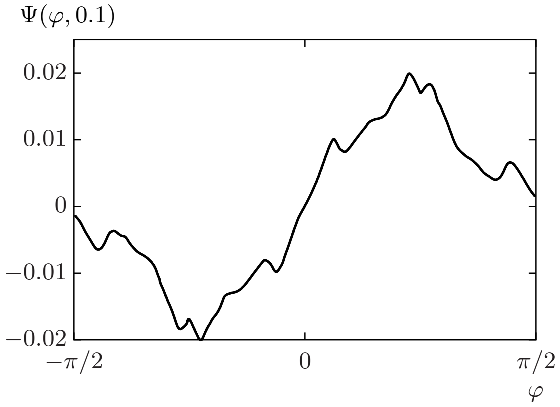

Fig. 4. The phase factor $\Psi(\varphi, t)$ (see Eq. (8)) for different $\varphi = \text{arctg}(k_y/k_x)$ at $At = 0.1$. It can be seen that the nonlinear term $\hat{N}E_{\mathbf{k}}$ results in a transfer of the $2D$ SFH energy from the attenuation region $-\pi/2 < \varphi < 0$ to the amplification one $0 < \varphi < \pi/2$

Figure 5 presents the disturbance normalized total energy $E_{tot}(t)/E_{tot}(0)$ vs time, where

$$E_{tot}(t) = \int d\mathbf{k} E_{\mathbf{k}}.$$

The three curves correspond to different values of the parameter B (see Eq. (7)), i.e., to different values of the initial disturbance energy $\int d\mathbf{k}E_{\mathbf{k}}$. The first curve pertains to low values of $B = B_1$ at which the effect of

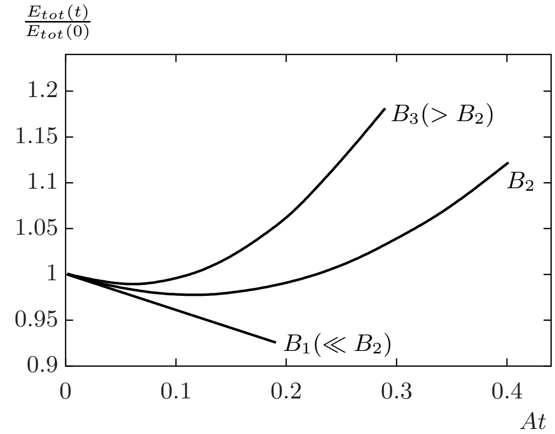

Fig. 5. The disturbance normalized total energy vs time. Each curve corresponds to a different amplitude of the initial disturbance (i.e., to a different value of the initial disturbance energy), $B_1 \ll B_2 < B_3$ (see Eq. (7)). The first curve suits well to low values of $B = B_1$ at which the effect of nonlinear processes can be ignored. In this case, the total energy of the disturbance is gradually decreasing. For the other two values $B = B_2$ and B_3 (with $B_1 \ll B_2 < B_3$), the effect of nonlinear processes is significant and the initial decrease of the total energy of disturbances is replaced by its growth

nonlinear processes can be ignored. As seen from Fig. 5, the total disturbance energy gradually decreases if nonlinear processes are negligible. For the other two values $B = B_2$ and B_3 (with $B_1 \ll B_2 < B_3$), at which the effect of nonlinear processes is significant, the initial decrease of the total energy is replaced by its growth. The higher is the initial energy, the sooner the growth begins. The results shown in Fig. 5 can be explained only by the nonlinear transfer of energy of the disturbances to the amplification region. The following arguments may prove this conclusion.

Only the unstable and dissipation processes ((**b**) and (**c**)) lead to changing the total disturbance energy. Viscosity (process (**c**)) always causes a decrease of the disturbance energy. As for process (**b**), its net effect depends on the distribution of the energy spectral density in the amplification and attenuation regions. If the «weight» of SFHs in the amplification region is «heavier» than that in the attenuation region, the net effect of process (**b**) causes an increase of the total energy of the $2D$ disturbance. Vice versa, if the «weight» of SFHs is «heavier» in the attenuation region, process (**c**) causes a decrease of the total energy. It follows from the above argument that in accordance with Eq. (6), the total energy of $2D$ disturbances can become higher only if the «weight» of SFHs in the amplification region is «heavier» than that in the attenuation region. In addition, the «weight» must be so much «heavier» that the net effect of the third term in Eq. (6) dominate over that of the viscous term.

Initially, the SFHs of the $2D$ disturbance considered here (see Eq. (7) and Fig. 3) have the same «weight» in the amplification and attenuation regions. If we assume that the effect of nonlinear processes is negligible, the disturbance is transferred to the attenuation region with time by the linear drift. This causes an increase of the «weight» of SFHs in the attenuation region toward higher values than in the amplification region, and the total energy of the disturbance under study must therefore begin to decrease. It is the temporal history that can explain the $B = B_1$ curve run in Fig. 5. The behavior of the curves with $B = B_2$ and B_3, namely the fact that the initial decrease of the total disturbance energy is replaced by its growth, thus unambiguously indicates that beginning with a certain time instance (which occurs the earlier the larger the disturbance amplitude is), the «weight» of SFHs in the amplification region dominates over the «weight» of SFHs in the attenuation one. This fact can be explained only by the preferential transfer of SFHs to the amplification region caused by the nonlinear processes. It also follows from Fig. 5 that there exists some threshold B_{th} for the initial disturbances. If $B > B_{th}$ (e.g., B_2, $B_3 > B_{th}$), the initial decrease of the total disturbance energy is replaced by its growth, which must eventually lead to the self-maintenance of disturbances. We did not calculate the threshold because of the following simple reasons. In our calculations, the threshold must appear at large times $At \gg 1$, where the weak turbulence approach becomes invalid. In addition, we made calculations for a definite disturbance and the calculation of the threshold in the particular case would not enrich the theory; much more important is the establishment of the threshold existence.

4. DISCUSSION

The aim of this paper was to prove the existence of the positive nonlinear feedback, the most problematic statement of the «bypass» transition to turbulence. We performed numerical calculations for the $2D$ case in the weak turbulence approximation. The results of calculations shown in Fig. 4 describe the preferential nonlinear transfer of the disturbance energy to the amplification region and the results in Fig. 5 evidence for the preferential transfer that can crucially change the temporal history: the total disturbance energy decrease can be replaced by its growth at certain amplitudes. This behavior makes the self-maintenance of the disturbance realistic. This is in turn the characteristic feature of the flow transition to the turbulent state and its maintenance.

We can therefore conclude that our numerical test calculations prove the existence of the positive nonlinear feedback in the $2D$ case. In reality, the shear flow turbulence has a $3D$ nature (cf. Ref. [6]). However, the qualitative analysis in Sec. 2 implies that nonlinear processes easier cope with the positive feedback in the actual $3D$ case than in $2D$ one. Indeed, we refer to the case discussed in Sec. 2, where SFHs of incompressible vortical $3D$ disturbances are initially in region I(I′) (see Fig. 2) and then drift along the k_y axis thus falling in the amplification region II(II′). They are amplified and reach region III(III′) because of the drift. In distinction to $2D$ SFHs (see Fig. 1), $3D$ SFHs do not become weaker after leaving the amplification region. The spectral energy density of $3D$ disturbances must therefore be higher in region III(III′) than in region I(I′). Combining this fact with the preferential nonlinear transfer of the SFH energy to the amplification region, we conclude that the positive nonlinear feedback must be easier realized in the $3D$ case than in the $2D$ one.

In accordance with the «bypass» transition to turbulence, the transient growth of disturbances is a key element of the subcritical transition. (The flow is spectrally stable.) At the same time, triggering the nonlinear positive feedback — nonlinear regeneration of the SFH that can draw the mean flow energy — is a necessary step to the transition. These facts require the existence of a sufficiently high level of initial disturbances in the system for the subcritical transition. It is obvious that finite disturbances can be produced by external forces. For instance, a pair of oblique waves with small, but finite amplitudes were used in Refs. [42, 43] as the initial condition in numerical simulations of the transition. However, finite disturbances must also have the intrinsic fluctuation origin according to Refs. [44, 45]. (These results shed new light on the fluctuation background of the vortex mode fluctuations in the laminar Couette flow.) Namely, according to Refs. [44, 45], the background of the vortex mode fluctuations in a certain subspace of the wave-number space is sufficiently strong at high Reynolds numbers and the level of its spectral energy density by far exceeds the level of the white noise. This must in turn trigger a nonlinear positive feedback and lead to the transition. The reality of this time history should be proved by direct numerical simulation.

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APPENDIX

Derivation of the weak turbulence equation

We let the x axis of a Cartesian coordinate system lie along the velocity of the mean flow and the y axis along the flow velocity shear, $\mathbf{U}_0(Ay,0,0)$. The fluid is assumed to be incompressible. Considering that the disturbed variables are independent of the z coordinate, the continuity equation and the equations of motion for the disturbances are given by

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0, \qquad (A.1)$$

$$\left(\frac{\partial}{\partial t}+Ay\frac{\partial}{\partial x}\right)v_x+Av_y+v_x\frac{\partial v_x}{\partial x}+ + v_y\frac{\partial v_x}{\partial y}=-\frac{\partial P}{\partial x}, \qquad (A.2)$$

$$\left(\frac{\partial}{\partial t}+Ay\frac{\partial}{\partial x}\right)v_y+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}=-\frac{\partial P}{\partial y}, \qquad (A.3)$$

where v_x and v_y are the respective disturbance velocities in the Cartesian coordinate system along the x and y axes and P is the pressure disturbance normalized by the undisturbed density of the fluid ρ_0. The action of viscosity in the weak turbulence equation is taken into account in the end. It is significant that we consider disturbances with the characteristic length scale much less than the distance between the flow boundaries. This allows us to neglect the boundary effects.

To simplify subsequent transformations, we introduce a coordinate system x_1y_1, with its origin and the x_1 axis coinciding with those of xy and the y axis convecting with the mean flow. This is equivalent to changing the variables as

$$x_1=x-Ayt,\quad y_1=y,\quad t_1=t, \qquad (A.4)$$

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial x_1},\quad \frac{\partial}{\partial y}=\frac{\partial}{\partial y_1}-At_1\frac{\partial}{\partial x_1},\quad \frac{\partial}{\partial t}=\frac{\partial}{\partial t_1}-Ay_1\frac{\partial}{\partial x_1}. \qquad (A.5)$$

In terms of the new variables, Eqs. (A.1)–(A.3) can be rewritten as

$$\frac{\partial}{\partial x_1}v_x+\left(\frac{\partial}{\partial y_1}-At_1\frac{\partial}{\partial x_1}\right)v_y=0, \qquad (A.6)$$

$$\frac{\partial v_x}{\partial t}+Av_y+v_x\frac{\partial v_x}{\partial x_1}+ + v_y\left(\frac{\partial}{\partial y_1}-At_1\frac{\partial}{\partial x_1}\right)v_x=-\frac{\partial P}{\partial x_1}, \qquad (A.7)$$

$$\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_x}{\partial y_1}+v_y\left(\frac{\partial}{\partial y_1}-At_1\frac{\partial}{\partial x_1}\right)v_y= =-\left(\frac{\partial}{\partial y_1}-At_1\frac{\partial}{\partial x_1}\right)P. \qquad (A.8)$$

Substitution (A.4) is not a physical transition to a new coordinate system, because in Eqs. (A.6)–(A.8) (as well as in Eqs. (A.1)–(A.3)), the quantities v_x and v_y are components of the disturbance velocity in the Cartesian coordinate system xy. The coefficients of the original set of linear equations (A.1)–(A.3) depend on the spatial coordinate y. As a result of the transformation, this spatial inhomogeneity is changed to the temporal one (Eqs. (A.7) and (A.8)).

The disturbed variables can be Fourier decomposed with respect to the Eulerian (laboratory) coordinates (x,y) and the Lagrangian (convected) coordinates (x_1,y_1),

$$\left\{\begin{array}{c} v_x \\ v_y \\ P \end{array}\right\}=\int\limits_{-\infty}^{\infty}dk_xdk_y\left\{\begin{array}{c} \hat{v}_x(k_x,k_y,t) \\ \hat{v}_y(k_x,k_y,t) \\ \hat{P}(k_x,k_y,t) \end{array}\right\}\times \times\exp(ik_xx+ik_yy), \qquad (A.9)$$

$$\left\{\begin{array}{c} v_x \\ v_y \\ P \end{array}\right\}=\int\limits_{-\infty}^{\infty}dk_{1x}dk_{1y}\left\{\begin{array}{c} \tilde{v}_x(k_{1x},k_{1y},t_1) \\ \tilde{v}_y(k_{1x},k_{1y},t_1) \\ \tilde{P}(k_{1x},k_{1y},t_1) \end{array}\right\}\times \times\exp(ik_{1x}x_1+ik_{1y}y_1). \qquad (A.10)$$

The two Fourier representations in Eqs. (A.9) and (A.10) are different, although they coincide at the initial moment ($t=0$) because $x\equiv x_1$ and $y\equiv y_1$. This difference is manifested in the dynamics of SFHs in the wave-number space. The wave vector $\mathbf{k}_1$ of a particular SFH is constant in time in the convected coordinates, while it varies in laboratory coordinates. Each of these two methods has its advantages. In the linear theory, Eq. (A.10) is convenient in studying the real spatial Fourier harmonics moving with them. However, in analyzing the weak turbulence equation (thus assuming the excitation of many degrees of freedom), it is impossible to follow the evolution of each Fourier harmonics. In the latter case, it is more convenient to study what occurs to the energy at a fixed point of the $\mathbf{k}$-space, i.e., to describe the variation of the spectral density of the disturbance energy at a fixed point of the **k**-space. In spite of this, expansion (A.10) is also useful for intermediate transformations.

To derive the weak turbulence equation, we insert expansion (A.10) in (A.6)–(A.8),

$$k_{1x}\tilde{v}_x(k_{1x},k_{1y},t) + (k_{1y} - k_{1x}At)\tilde{v}_y(k_{1x},k_{1y},t) = 0, \quad (A.11)$$

$$-\left(\frac{k_{1y}}{k_{1x}} - At\right)\frac{\partial \tilde{v}_y(\mathbf{k}_1,t)}{\partial t} - 2A\tilde{v}_y(\mathbf{k}_1,t) + \int d\mathbf{k}'_1 d\mathbf{k}''_1 \left\{\delta(\mathbf{k}'_1 + \mathbf{k}''_1 - \mathbf{k}_1)\left(\frac{k'_{1y}}{k'_{1x}} - At\right) \times \tilde{v}_y(\mathbf{k}'_1,t) i k''_{1x}\left(\frac{k''_{1y}}{k''_{1x}} - At\right)\tilde{v}_y(\mathbf{k}''_1,t)\right\} + \int d\mathbf{k}'_1 d\mathbf{k}''_1 \left\{\delta(\mathbf{k}'_1 + \mathbf{k}''_1 - \mathbf{k}_1)\tilde{v}_y(\mathbf{k}'_1,t) i k''_{1x} \times \left(\frac{k''_{1y}}{k''_{1x}} - At\right) \times \left[-\left(\frac{k''_{1y}}{k''_{1x}} - At\right)\right]\tilde{v}_y(\mathbf{k}''_1,t)\right\} = -ik_{1x}\tilde{P}, \quad (A.12)$$

$$\frac{\partial \tilde{v}_y(\mathbf{k}_1,t)}{\partial t} + \int d\mathbf{k}'_1 d\mathbf{k}''_1 \times \left\{\delta(\mathbf{k}'_1 + \mathbf{k}''_1 - \mathbf{k}_1)\left[-\left(\frac{k'_{1y}}{k'_{1x}} - At\right)\tilde{v}_y(\mathbf{k}'_1,t)\right] \times i k''_{1x}\tilde{v}_y(\mathbf{k}''_1,t)\right\} + \int d\mathbf{k}'_1 d\mathbf{k}''_1 \left\{\delta(\mathbf{k}'_1 + \mathbf{k}''_1 - \mathbf{k}_1)\tilde{v}_y(\mathbf{k}'_1,t) \times i k''_{1x}\tilde{v}_y(\mathbf{k}''_1,t)\left(\frac{k''_{1y}}{k''_{1x}} - At\right)\right\} = -ik_{1x}\left(\frac{k_{1y}}{k_{1x}} - At\right)\tilde{P}. \quad (A.13)$$

Eliminating $\tilde{P}$ from these equations gives a symmetric equation for v_y,

$$(k_{1x}^2 + k_{1y}^2)\frac{\partial \tilde{v}_y(k_{1x},k_{1y},t)}{\partial t} - 2Ak_{1x}k_{1y}(t)\tilde{v}_y(k_{1x},k_{1y},t) + \frac{i}{2}\int d\mathbf{k}'_1 d\mathbf{k}''_1 \left\{\delta(k'_{1x} + k''_{1x} - k_{1x})\delta(k'_{1y} + k''_{1y} - k_{1y}) \times k_{1x}[k''^2_1(t) - k'^2_1(t)]\left(\frac{k''_{1y}}{k''_{1x}} - \frac{k'_{1y}}{k'_{1x}}\right) \times \tilde{v}_y(k'_{1x},k'_{1y},t)\tilde{v}_y(k''_{1x},k''_{1y},t)\right\} = 0, \quad (A.14)$$

where

$$k_{1y}(t) = k_{1y} - k_{1x}At, \quad k_1^2(t) = k_{1x}^2 + k_{1y}^2(t).$$

Introducing the function

$$C_{\mathbf{k}} = [k_{1x}^2 + k_{1y}^2(t)]\tilde{v}_y(\mathbf{k}_1,t), \quad (A.15)$$

we rewrite Eq. (A.14) in a more convenient form (cf. Refs. [39–41])

$$i\frac{\partial C_{\mathbf{k}_1}}{\partial t} = \int d\mathbf{k}'_1 d\mathbf{k}''_1 V_{\mathbf{k}_1\mathbf{k}'_1\mathbf{k}''_1} C_{\mathbf{k}'_1} C_{\mathbf{k}''_1}, \quad (A.16)$$

where

$$V_{\mathbf{k}_1\mathbf{k}'_1\mathbf{k}''_1} = \delta(\mathbf{k}'_1 + \mathbf{k}''_1 - \mathbf{k}_1)\frac{k'^2_1(t) - k''^2_1(t)}{k'^2_1(t)k''^2_1(t)} \times (k'_{1x} + k''_{1x})\left(\frac{k'_{1y}}{k'_{1x}} - \frac{k''_{1y}}{k''_{1x}}\right). \quad (A.17)$$

We note that $C_{\mathbf{k}}$ is related to the vorticity of the spatial Fourier harmonics.

Assuming that many degrees of freedom (modes) are excited, we use the random phase approximation (cf. Ref. [39]), which can be expressed by

$$\left\langle C_{\mathbf{k}_1} C_{\mathbf{k}'_1}\right\rangle = n_{\mathbf{k}_1}(t)\delta(\mathbf{k}_1 + \mathbf{k}'_1) \equiv n(k_{1x},k_{1y},t)\delta(\mathbf{k}_1 + \mathbf{k}'_1), \quad (A.18)$$

where $\langle\ldots\rangle$ denotes the phase average.

To use the methods of the weak turbulence theory, we expand $C_{\mathbf{k}}$ as

$$C_{\mathbf{k}_1} = C^{(0)}_{\mathbf{k}_1} + C^{(1)}_{\mathbf{k}_1} + \ldots, \quad (A.19)$$

where

$$C^{(0)}_{\mathbf{k}_1} \gg C^{(1)}_{\mathbf{k}_1} \quad (A.20)$$

and

$$\frac{\partial C^{(0)}_{\mathbf{k}_1}}{\partial t} = 0, \quad (A.21)$$

which means that the nonlinearity is taken into account within the perturbation theory. Using Eqs. (A.19)–(A.21), it follows from (A.16) that

$$C^{(1)}_{\mathbf{k}_1} = -i\int d\mathbf{k}'_1 d\mathbf{k}''_1 \left\{C^{(0)}_{\mathbf{k}'_1}C^{(0)}_{\mathbf{k}''_1}\int_0^t dt' V_{\mathbf{k}_1\mathbf{k}'_1\mathbf{k}''_1}\right\}. \quad (A.22)$$

We next use the relations between higher correlations accepted in the weak turbulence theory,

$$\left\langle C^{(0)}_{\mathbf{k}_1} C^{(0)}_{\mathbf{k}'_1} C^{(0)}_{\mathbf{k}''_1}\right\rangle = 0, \quad (A.23)$$

$$\left\langle C^{(0)}_{\mathbf{k}_1}C^{(0)}_{\mathbf{k}'_1}C^{(0)}_{\mathbf{k}''_1}C^{(0)}_{\mathbf{k}'''_1}\right\rangle = \left\langle C^{(0)}_{\mathbf{k}_1}C^{(0)}_{\mathbf{k}'_1}\right\rangle\left\langle C^{(0)}_{\mathbf{k}''_1}C^{(0)}_{\mathbf{k}'''_1}\right\rangle + \left\langle C^{(0)}_{\mathbf{k}_1}C^{(0)}_{\mathbf{k}''_1}\right\rangle\left\langle C^{(0)}_{\mathbf{k}'_1}C^{(0)}_{\mathbf{k}'''_1}\right\rangle + \left\langle C^{(0)}_{\mathbf{k}_1}C^{(0)}_{\mathbf{k}'''_1}\right\rangle\left\langle C^{(0)}_{\mathbf{k}'_1}C^{(0)}_{\mathbf{k}''_1}\right\rangle = n_{\mathbf{k}_1}n_{\mathbf{k}''_1}\delta(\mathbf{k}_1+\mathbf{k}'_1)\delta(\mathbf{k}''_1+\mathbf{k}'''_1) + n_{\mathbf{k}_1}n_{\mathbf{k}'_1}\delta(\mathbf{k}_1+\mathbf{k}''_1)\delta(\mathbf{k}'_1+\mathbf{k}'''_1) + n_{\mathbf{k}_1}n_{\mathbf{k}'_1}\delta(\mathbf{k}_1+\mathbf{k}'''_1)\delta(\mathbf{k}'_1+\mathbf{k}''_1). \quad (A.24)$$

As can be easily seen, we then have

$$\frac{\partial n_{\mathbf{k}_1}}{\partial t}\delta(\mathbf{k}_1+\mathbf{k}'''_1) = -i\int d\mathbf{k}'_1 d\mathbf{k}''_1 \left(V_{\mathbf{k}_1\mathbf{k}'_1\mathbf{k}''_1}\left\langle C_{\mathbf{k}'_1}C_{\mathbf{k}''_1}C_{\mathbf{k}'''_1}\right\rangle + V_{\mathbf{k}'''_1\mathbf{k}'_1\mathbf{k}''_1}\left\langle C_{\mathbf{k}_1}C_{\mathbf{k}''_1}C_{\mathbf{k}'''_1}\right\rangle\right) \approx$$
$$\approx -i\int d\mathbf{k}'_1 d\mathbf{k}''_1 \left\{V_{\mathbf{k}_1\mathbf{k}'_1\mathbf{k}''_1}\left(\left\langle C^{(0)}_{\mathbf{k}'_1}C^{(0)}_{\mathbf{k}''_1}C^{(1)}_{\mathbf{k}'''_1}\right\rangle + \left\langle C^{(1)}_{\mathbf{k}'_1}C^{(0)}_{\mathbf{k}''_1}C^{(0)}_{\mathbf{k}'''_1}\right\rangle + \left\langle C^{(0)}_{\mathbf{k}'_1}C^{(1)}_{\mathbf{k}''_1}C^{(0)}_{\mathbf{k}'''_1}\right\rangle\right) + V_{\mathbf{k}'''_1\mathbf{k}'_1\mathbf{k}''_1}\left(\left\langle C^{(0)}_{\mathbf{k}_1}C^{(0)}_{\mathbf{k}'_1}C^{(1)}_{\mathbf{k}''_1}\right\rangle + \left\langle C^{(1)}_{\mathbf{k}_1}C^{(0)}_{\mathbf{k}'_1}C^{(0)}_{\mathbf{k}''_1}\right\rangle + \left\langle C^{(0)}_{\mathbf{k}_1}C^{(1)}_{\mathbf{k}'_1}C^{(0)}_{\mathbf{k}''_1}\right\rangle\right)\right\}. \quad (A.25)$$

In view of (A.22) and (A.24), this becomes

$$= 2\int d\mathbf{k}'_1 d\mathbf{k}''_1 \psi_{\mathbf{k}'_1\mathbf{k}''_1} \times \left\{\delta(\mathbf{k}_1-\mathbf{k}'_1-\mathbf{k}''_1)\delta(\mathbf{k}_1+\mathbf{k}'''_1) \times \left(n_{\mathbf{k}_1}n_{\mathbf{k}''_1}\int_0^t dt'\psi_{\mathbf{k}'_1\mathbf{k}''_1} + 2n_{\mathbf{k}_1}n_{\mathbf{k}''_1}\int_0^t dt'\psi_{-\mathbf{k}'_1\mathbf{k}''_1}\right) + \delta(\mathbf{k}_1+\mathbf{k}'_1+\mathbf{k}''_1)\delta(\mathbf{k}_1+\mathbf{k}'''_1) \times \left(n_{\mathbf{k}'_1}n_{\mathbf{k}''_1}\int_0^t dt'\psi_{\mathbf{k}'_1\mathbf{k}''_1} + 2n_{\mathbf{k}_1}n_{\mathbf{k}''_1}\int_0^t dt'\psi_{\mathbf{k}_1\mathbf{k}''_1}\right)\right\}, \quad (A.26)$$

where

$$\psi_{\mathbf{k}'_1\mathbf{k}''_1}(t) \equiv \left(\frac{1}{\mathbf{k}''^2_1(t)} - \frac{1}{\mathbf{k}'^2_1(t)}\right)(k'_{1x}+k''_{1x})\left(\frac{k'_{1y}}{k'_{1x}} - \frac{k''_{1y}}{k''_{1x}}\right).$$

Changing the variables in the second part of the integral as

$$\mathbf{k}'_1 \to -\mathbf{k}'_1 \quad \text{and} \quad \mathbf{k}''_1 \to -\mathbf{k}''_1$$

and taking into account that

$$\psi_{-\mathbf{k}'_1,-\mathbf{k}''_1} = -\psi_{\mathbf{k}'_1\mathbf{k}''_1}, \quad (A.27)$$

we continue the transformations as

$$\frac{\partial n_{\mathbf{k}_1}}{\partial t} = 4\int d\mathbf{k}'_1 d\mathbf{k}''_1 \delta(\mathbf{k}_1-\mathbf{k}'_1-\mathbf{k}''_1)\psi_{\mathbf{k}'_1\mathbf{k}''_1} \times \left(n_{\mathbf{k}'_1}n_{\mathbf{k}''_1}\int_0^t dt'\psi_{\mathbf{k}'_1\mathbf{k}''_1} - 2n_{\mathbf{k}_1}n_{\mathbf{k}''_1}\int_0^t dt'\psi_{\mathbf{k}_1\ -\mathbf{k}''_1}\right). \quad (A.28)$$

Inserting the expressions for $\psi_{\mathbf{k}'_1\mathbf{k}''_1}$ and $\psi_{\mathbf{k}_1\ -\mathbf{k}''_1}$ (see Eq. (A.26)) in the time integrals and integrating, we obtain

$$\int_0^t dt'\psi_{\mathbf{k}'_1\mathbf{k}''_1} = \frac{1}{A}(k'_{1x}+k''_{1x})\left(\frac{k'_{1y}}{k'_{1x}} - \frac{k''_{1y}}{k''_{1x}}\right) \times \left\{\frac{1}{k''^2_{1x}}\operatorname{arctg}\frac{At}{1+(k''_{1y}/k''_{1x})\left((k''_{1y}/k''_{1x})-At\right)} - \frac{1}{k'^2_{1x}}\operatorname{arctg}\frac{At}{1+(k'_{1y}/k'_{1x})\left((k'_{1y}/k'_{1x})-At\right)}\right\}, \quad (A.29)$$

$$\int_0^t dt'\psi_{\mathbf{k}'_1-\mathbf{k}''_1} = \frac{1}{A}(k_{1x}-k''_{1x})\left(\frac{k_{1y}}{k_{1x}} - \frac{k''_{1y}}{k''_{1x}}\right) \times \left\{\frac{1}{k''^2_{1x}}\operatorname{arctg}\frac{At}{1+(k''_{1y}/k''_{1x})\left((k''_{1y}/k''_{1x})-At\right)} - \frac{1}{k^2_{1x}}\operatorname{arctg}\frac{At}{1+(k_{1y}/k_{1x})\left((k_{1y}/k_{1x})-At\right)}\right\}. \quad (A.30)$$

As mentioned above, it is convenient to obtain the equation for the energy density at a fixed point of the **k**-space in order to construct the weak turbulence theory. For this, we use Eqs. (A.4)–(A.5) to transform Eqs. (A.28)–(A.30) to the new variables k_x and k_y,

$$\langle v^2\rangle = \langle v_x^2+v_y^2\rangle = \int d\mathbf{k}_1(\tilde v_x^2+\tilde v_y^2) = \int d\mathbf{k}_1 \tilde v_y^2(k_{1x},k_{1y},t)\left(1+\frac{k^2_{1y}(t)}{k^2_{1x}(t)}\right) = \int d\mathbf{k}_1\frac{n(k_{1x},k_{1y},t)}{k^2_{1x}[k^2_{1x}+k^2_{1y}(t)]} = \int d\mathbf{k}\frac{n(k_x,k_y+k_xAt,t)}{k_y^2(k_x^2+k_y^2)}, \quad (A.31)$$

where

$$k_x \equiv k_{1x}, \quad k_y \equiv k_{1y}(t) = k_{1y} - k_{1x}At. \quad (A.32)$$

On the other hand,

$$\langle v^2 \rangle = \int d\mathbf{k} \left(\hat{v}_x^2(k_x, k_y, t) + \hat{v}_y^2(k_x, k_y, t)\right) = \\ = \int d\mathbf{k} E(\mathbf{k}, t) \equiv \int d\mathbf{k} E_{\mathbf{k}}, \quad (A.33)$$

whence

$$E_{\mathbf{k}} = \frac{n(k_x, k_y + k_x A t, t)}{k_x^2 (k_x^2 + k_y^2)}. \quad (A.34)$$

$E_{\mathbf{k}}$ is the energy density of the $2D$ vortex mode disturbances at a fixed point of the $\mathbf{k}$-space. In other words, this is the spectral density of energy.

Inserting integrals (A.29) and (A.30) in Eq. (A.28), changing the variables in accordance with (A.4), and using (A.34), we obtain the equation for the spectral density of the disturbance energy at a fixed point in the $\mathbf{k}$-space,

$$\frac{\partial E_{\mathbf{k}}}{\partial t} + \nabla_{\mathbf{k}}(\mathbf{V} E_{\mathbf{k}}) - \frac{2 A k_x k_y}{k_x^2 + k_y^2} E_{\mathbf{k}} + \\ + \nu (k_x^2 + k_y^2) E_{\mathbf{k}} = \hat{N} E_{\mathbf{k}}, \quad (A.35)$$

where

$$\nabla_{\mathbf{k}} = (\partial/\partial k_x, \partial/\partial k_y), \quad \mathbf{V} = (-A k_x, 0),$$

and

$$\hat{N} E_{\mathbf{k}} = \frac{4}{A} \int d\mathbf{k}' d\mathbf{k}'' \delta(\mathbf{k} - \mathbf{k}' - \mathbf{k}'') \times \\ \times (k'^2 - k''^2)(k'_y k''_x - k''_y k'_x)^2 \times \\ \times \left\{ \frac{1}{k^2} \left(\frac{1}{k''^2_x} \operatorname{arctg} \frac{At}{1 + (k''_y/k''_x)\left((k''_y/k''_x) + At\right)} - \right.\right. \\ \left. - \frac{1}{k'^2_x} \operatorname{arctg} \frac{At}{1 + (k'_y/k'_x)\left((k'_y/k'_x) + At\right)} \right) E_{\mathbf{k}'} E_{\mathbf{k}''} + \\ + 2 \frac{1}{k'^2} \left(\frac{1}{k''^2_x} \operatorname{arctg} \frac{At}{1 + (k''_y/k''_x)\left((k''_y/k''_x) + At\right)} - \right. \\ \left. - \frac{1}{k_x^2} \operatorname{arctg} \frac{At}{1 + (k_y/k_x)\left((k_y/k_x) + At\right)} \right) \times \\ \left. \times E_{\mathbf{k}} E_{\mathbf{k}''} \right\}. \quad (A.36)$$

In the derivation of Eqs. (A.35) and (A.36), the viscosity term was omitted. It was then added in Eq. (A.35) (the fourth term in the left-hand side).

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