 

# APPROXIMATE ANALYTICAL SOLUTIONS OF THE BABY SKYRME MODEL

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We show that many properties of the baby skyrmions, which have been determined numerically, can be understood in terms of an analytic approximation. In particular, we show that the approximation captures properties of the multiskyrmion solutions (derived numerically) such as their stability towards decay into various channels, and that it is more accurate for the «new baby Skyrme model» describing anisotropic physical systems in terms of multiskyrmion fields with axial symmetry. Some universal characteristics of configurations of this kind are demonstrated that are independent of their topological number.

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## 1. INTRODUCTION

It is known that the two-dimensional $O(3)$ $\sigma$-model [1] possesses metastable states that can shrink or spread out under perturbation because of the conformal (scale) invariance of the model [2–4]. This implies that the metastable states can be of any size, and therefore, a term of the fourth order in derivatives, the so-called Skyrme term, must be added to break the scale invariance of the model. But the resulting energy functional has no minima, and a further extra term is needed to stabilize the size of the corresponding solitons; this term contains no derivatives of the field and is often called the potential (or mass) term. The field can then be viewed as the magnetization vector of a two-dimensional ferromagnetic substance [1], and the potential term describes the coupling of the magnetization vector to a constant external magnetic field. Because the extra terms contribute to the masses of the solitons, their dependence deviates from a simple law in which the skyrmion mass is proportional to the skyrmion (topological) number and the two-skyrmion configuration becomes stable, showing that the model possesses bound states [5].

In this paper, we demonstrate that the simple analytical method used for the description of the three-dimensional Skyrme model presented in [6] can also be used to study various properties of the low-energy states of the corresponding two-dimensional $\sigma$-model when the parameters that determine the contributions of the Skyrme and the potential terms are not large. More precisely, it was possible to describe the basic properties of the three-dimensional skyrmions for large baryon numbers analytically [6], and it is therefore worthwhile to derive such a description for the two-dimensional $O(3)$ $\sigma$-model as well. In general, such analytical discussions of soliton models are useful because they lead to a better understanding of the soliton properties. The two-dimensional $O(3)$ $\sigma$-model is widely used to describe ferromagnetic systems, high-temperature superconductivity, etc., and the results obtained here can therefore be useful for the understanding of these phenomena.

Our method is based on the ansatz introduced in [6] and is accurate for the so-called «new baby Skyrme

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model» [7] that describes anisotropic physical systems. Its accuracy actually increases as the skyrmion number $n$ increases, and this method allows predicting some universal properties of the ring-like configurations for large $n$, independently of its particular value. Although such models are not integrable, the «new baby Skyrme model» appears to have the properties of an integrable system in the case where $n$ is large.

## 2. NEAR THE NONLINEAR $O(3)$ $\sigma$-MODEL

The Lagrangian density of the $O(3)$ $\sigma$-model with the additional terms introduced and discussed in [5, 7, 8] is[1)]

$$\mathcal{L} = \frac{g^2}{2}\left(\partial_\alpha \mathbf{n}\right)^2 - \frac{1}{4e^2}\left[\partial_\alpha \mathbf{n}, \partial_\beta \mathbf{n}\right]^2 - g^2 V. \qquad (1)$$

Here, $\partial_\alpha = \partial/\partial x^\alpha$; $x^\alpha$, $\alpha = 0,1,2$, refer to both time and spatial components of $(t,x,y)$, and the field $\mathbf{n}$ is a scalar field with three components $n_a$, $a = 1,2,3$, satisfying the condition

$$\mathbf{n}^2 = n_1^2 + n_2^2 + n_3^2 = 1.$$

The constants $g$ and $e$ are free parameters, with $g^2$ having the dimension of energy. It is useful to think of $g^2$ and $1/ge$ as natural units of energy and length respectively. The first term in (1) is familiar from $\sigma$-models, the second term, which is of the fourth order in derivatives, is the analogue of the Skyrme term, and the last term is the potential term. The respective potentials for the «old baby Skyrme model» (OBM) and the «new baby Skyrme model» (NBM) describing anisotropic systems are given by

$$\begin{aligned} V_{OBM} &= \mu^2\left(1 - n_3\right), \\ V_{NBM} &= \frac{1}{2}\mu^2\left(1 - n_3^2\right), \end{aligned} \qquad (2)$$

were $\mu$ has the dimension of energy, and $1/\mu$ therefore determines a second length scale in our model. Evidently, $V_{NBM} \le V_{OBM}$ at a fixed value of $\mu$.

In three spatial dimensions, the Skyrme term is necessary for the existence of soliton solutions, but the inclusion of a potential is optional from the mathematical standpoint. Physically, however, a potential of a certain form is required in order to give the pions a mass [9]. By contrast, in two dimensions, a potential term must be included in the above Lagrangian in order for soliton solutions to exist. As shown in [10], the different potential terms give quite different properties to the multiskyrmion configurations when the skyrmion number is large. Our analytical treatment here supports this conclusion, as shown in Sects. 3–5.

We are only interested in configurations with finite energy, and we therefore define the configuration space to be the space of all maps $\mathbf{n}$: $R^2 \to S^2$ that tend to the constant field $(0,0,1)$ (the so-called vacuum) at spatial infinity,

$$\lim_{|x|\to\infty} \mathbf{n}(\mathbf{x}) = (0,0,1). \qquad (3)$$

Every configuration $\mathbf{n}$ can thus be regarded as a representative of a homotopy class in $\pi_2(S^2) = \mathbf{Z}$ and has the corresponding integer degree given by

$$\deg\left[\mathbf{n}\right] = \frac{1}{8\pi}\int d^2x\, \epsilon^{bc}\, \mathbf{n}\left(\partial_b \mathbf{n} \times \partial_c \mathbf{n}\right). \qquad (4)$$

The vacuum field is invariant under the symmetry group

$$G = E_2 \times SO(2)_{iso} \times P,$$

where $E_2$ is the Euclidean group of two-dimensional translations and rotations, acting on fields via pull-back. The $SO(2)_{iso}$ subgroup of the three-dimensional rotation group acting on $S^2$ is the subgroup that leaves the vacuum invariant (we call its elements iso-rotations to distinguish them from rotations in physical space). Finally, $P$ is a combined reflection in both space and the target space $S^2$.

We are interested in stationary points of $\deg[\mathbf{n}] \ne 0$; the maximal subgroups of $G$ under which such fields can be invariant are labelled by a nonzero integer $n$ and consist of spatial rotations by some angle $\alpha \in [0, 2\pi]$ and simultaneous iso-rotation by $-n\alpha$. Fields that are invariant under such a group are of the form

$$\begin{aligned} n_1 = \sin f(\tilde r)\,\cos(n\phi), \quad n_2 = \sin f(\tilde r)\,\sin(n\phi), \\ n_3 = \cos f(\tilde r), \end{aligned} \qquad (5)$$

where $(\tilde r, \phi)$ are polar coordinates and $f(\tilde r)$ is the profile function. Such fields are the analogues and generalizations of the hedgehog fields in the Skyrme model. In this parametrization, which involves azimuthal symmetry of the fields, it is assumed that all the skyrmions sit on top of each other in forming the multiskyrmion configuration.

It is easy to show that the degree of field (5),

$$\deg\left[\mathbf{n}\right] = n, \qquad (6)$$

is equal to the azimuthal winding number $n$.

1) The first several paragraphs of this section follow Refs. [5, 8] very closely and are included to make the paper more selfcontained.

The respective static energy functionals related to Lagrangian (1) for the OBM and the NBM are given by

$$E_{cl}(n)_{OBM} = \frac{g^2}{2}\int r\,dr\left(f'^2 + \frac{n^2\sin^2 f}{r^2} + \right.$$
$$\left. + a\left[\frac{n^2 f'^2\sin f^2}{r^2} + 2\,(1-\cos f)\right]\right), \quad (7)$$

$$E_{cl}(n)_{NBM} = \frac{g^2}{2}\int r\,dr\left(f'^2 + \frac{n^2\sin^2 f}{r^2} + \right.$$
$$\left. + a\left[\frac{n^2 f'^2\sin f^2}{r^2} + \left(1-\cos^2 f\right)\right]\right). \quad (8)$$

In (7) and (8), the length $(\sqrt{ge\mu})^{-1}$ is absorbed such that the scale size of the localized structures is a function of the dimensionless spatial coordinate $r = \sqrt{ge\mu}\,\tilde{r}$ and the dimensionless parameter $a = \mu/ge$ becomes the only nontrivial parameter of the model. Finiteness of the energy functional requires that the profile function must satisfy the boundary conditions $f(0) = \pi$ and $f(\infty) = 0$.

Setting $\phi = \cos f$ in (7), we rewrite the energy functional as

$$E_{cl}(n)_{OBM} = \frac{g^2}{2}\int r\,dr\left(\frac{\phi'^2}{1-\phi^2} + \frac{n^2\left(1-\phi^2\right)}{r^2} + \right.$$
$$\left. + a\left[\frac{n^2\phi'^2}{r^2} + 2\,(1-\phi)\right]\right) \quad (9)$$

and similarly for $E_{cl}(n)_{NBM}$. We next parameterize the field $\phi$ using the ansatz introduced in [6] for the description of the three-dimensional skyrmions,

$$\phi = \cos f = \frac{(r/r_n)^p - 1}{(r/r_n)^p + 1}, \quad \phi' = \frac{p}{2r}(1-\phi^2). \quad (10)$$

After the integration with respect to $r$, this leads to the analytic expressions for the energy

$$E_{cl}(n)_{OBM} = \pi g^2\left(\frac{4n^2}{p} + p + \right.$$
$$\left. + \frac{4a\pi}{p\,\sin(2\pi/p)}\left[\frac{n^2(p^2-4)}{3r_n^2 p} + r_n^2\right]\right), \quad (11)$$

$$E_{cl}(n)_{NBM} = \pi g^2\left(\frac{4n^2}{p} + \right.$$
$$\left. + p + \frac{4a\pi}{p\,\sin(2\pi/p)}\left[\frac{n^2(p^2-4)}{3r_n^2 p} + \frac{2}{p}r_n^2\right]\right). \quad (12)$$

Here, $p$ and $r_n$ are parameters which still must be determined by minimizing the energy. In fact, $r_n$ corresponds to the radius of the $n$-soliton configuration. We remark that in deriving (11) and (12) we used the Euler-type integrals (see also [6])

$$\int_0^\infty \frac{2r\,dr}{1+(r/r_n)^p} = \frac{2\pi r_n^2}{p\,\sin(2\pi/p)}, \quad p > 2,$$
$$\int_0^\infty \frac{dr\,(r/r_n)^p}{r\left[1+(r/r_n)^p\right]^2} = \frac{1}{p}, \quad p > 0,$$
$$\int_0^\infty \frac{dr\,(r/r_n)^{2p}}{r^3\left[1+(r/r_n)^p\right]^4} = \frac{\pi\left(p^2-4\right)}{3r_n^2\,p^4\,\sin(2\pi/p)}, \quad p > 1, \quad (13)$$
$$\int_0^\infty \frac{2r\,dr}{[1+(r/r_n)^p]^2} = \left(1-\frac{2}{p}\right)\frac{2\pi r_n^2}{p\,\sin(2\pi/p)}, \quad p > 1.$$

It can be easily proved that the minimization of the energies in Eqs. (11) and (12) implies that

$$(r_n^{min})^2_{OBM} = \frac{n}{\sqrt{3}}\sqrt{\frac{p^2-4}{p}},$$
$$(r_n^{min})^2_{NBM} = n\sqrt{\frac{p^2-4}{6}}, \quad (14)$$

i.e., $(r_n^{min})^2_{NBM} = \sqrt{p/2}\,(r_n^{min})^2_{OBM}$, and the minimum energy values are therefore equal to

$$E_{cl}(n)_{OBM} = $$
$$= 4\pi g^2\left[\frac{n^2}{p} + \frac{p}{4} + \frac{2an\pi}{\sqrt{3}p\sin(2\pi/p)}\,\frac{\sqrt{p^2-4}}{\sqrt{p}}\right], \quad (15)$$

$$E_{cl}(n)_{NBM} = $$
$$= 4\pi g^2\left[\frac{n^2}{p} + \frac{p}{4} + \frac{2\sqrt{2}an\pi}{\sqrt{3}\sin(2\pi/p)}\,\frac{\sqrt{p^2-4}}{p^2}\right]. \quad (16)$$

It is obvious that the energy contributions of the Skyrme and the potential terms are equal due to (14), which is in agreement with the result obtained from Derrick's theorem. Equations (15) and (16) provide an upper bound for the energies of baby skyrmions for any value of $p$. To obtain the lowest upper bound, we must minimize the right-hand sides of (15) and (16) with respect to the parameter $p$. In what follows, we investigate various cases that correspond to different values of the only nontrivial parameter of the model, $a$.

We first consider the case where $a \ll 1$, i.e., the model parameter is very small. We observe that for $a = 0$, ansatz (10) is a solution of the model for $p = 2n$, which implies that $p \to 2n$ as $a \to 0$. In accordance with (14), the radius of the multiskyrmion configuration then increases with $n$,

$$(r_n^{min})^2_{OBM} \propto n^{3/2}, \quad (r_n^{min})^2_{NBM} \propto n^2.$$

Moreover, the configuration consists of a ring of the thickness $\delta \approx 4r_n/p$, and therefore

$$\delta_{OBM} \propto 2n^{-1/4}, \quad \delta_{NBM} \propto \text{const}.$$

We remark that the ring thickness is determined as the difference of the values of $\phi$ inside (which is equal to $-1$) and outside (which is equal to $+1$) the ring (i.e., $d\phi = 2$) divided by its derivative at $r = r_n$, where $\phi(r_n) = 0$; as a consequence of (10) $\phi'(r_n) = p/2r_n$.

Magnetic solitons of this type have been observed in [11, 12] as solutions of the Landau–Lifshitz equations defining the dynamics of ferromagnets. (We note that the static solutions of the baby Skyrme model and the Landau–Lifshitz equations are related.) In general, $\phi$ given by (10) for $p = 2n$ is a low-energy approximation of multiskyrmion configurations (for $n > 1$), because the corresponding energies given by (15) and (16) are infinite for $n = 1$. Indeed, it is a matter of simple algebra to show that

$$\begin{aligned}
E_{cl}(n=2)_{OBM} &= 4\pi g^2 (2 + a\pi), \\
E_{cl}(n=2)_{NBM} &= 4\pi g^2 \left(2 + \frac{a\pi}{\sqrt{2}}\right), \\
E_{cl}(n=3)_{OBM} &= 4\pi g^2 \left(3 + a\pi\frac{8}{3\sqrt{3}}\right), \\
E_{cl}(n=3)_{NBM} &= 4\pi g^2 \left(3 + a\pi\frac{8}{9}\right), \\
E_{cl}(n=4)_{OBM} &= 4\pi g^2 \left(4 + a\pi\sqrt{5}\right), \\
E_{cl}(n=4)_{NBM} &= 4\pi g^2 \left(4 + a\pi\frac{\sqrt{5}}{2}\right).
\end{aligned} \tag{17}$$

For large $n$, the energies take the asymptotic values

$$\begin{aligned}
E_{cl}(n)_{OBM} &= 4\pi n g^2 \left(1 + \sqrt{\frac{2n}{3}}a\right), \\
E_{cl}(n)_{NBM} &= 4\pi n g^2 \left(1 + \sqrt{\frac{2}{3}}a\right).
\end{aligned} \tag{18}$$

We note that the energy of the OBM per unit skyrmion number increases as $n$ increases, while the energy of the NBM per skyrmion decreases as $n$ increases and becomes constant for large $n$. In fact, the energies given by (17) are the upper bounds of the multiskyrmion energies because the exact profile function corresponding to the minimum of the energy differs from that given by (10).

## 3. PERTURBATION THEORY FOR THE MODEL PARAMETER

In this section, we obtain energy corrections up to the second or higher orders with respect to the model parameter $a$. The corresponding energies for the OMB and NBM can be written as

$$E_{cl}(n) = 4\pi g^2[f(p) + a\,h(p)], \tag{19}$$

where $f(p)$ and $h(p)$ can be evaluated from (15) and (16), respectively. Letting $p = 2n + \epsilon$ and expanding energies (15) and (16) up to the second order in $\epsilon$, we obtain $f(p) = n + \epsilon^2/8n$, $h(p) = h_0 + \epsilon h_1$, where

$$h_1 = (2n)^{-1}\beta h_0.$$

In fact, the corresponding functions for the OBM and the NBM are given by

$$\begin{aligned}
\frac{h_{0_{OBM}}}{n} &= \sqrt{\frac{2n}{3}}\,\frac{\pi}{n\,\sin(\pi/n)}\sqrt{1 - 1/n^2}, \\
\beta_{OBM} &= \frac{\pi}{n}\,\text{ctg}(\pi/n) - \frac{1}{2} + \frac{1}{n^2 - 1},
\end{aligned} \tag{20}$$

$$\begin{aligned}
\frac{h_{0_{NBM}}}{n} &= \sqrt{\frac{2}{3}}\,\frac{\pi}{n\,\sin(\pi/n)}\sqrt{1 - 1/n^2}, \\
\beta_{NBM} &= \frac{\pi}{n}\,\text{ctg}(\pi/n) - 1 + \frac{1}{n^2 - 1}.
\end{aligned} \tag{21}$$

Minimization of (19) with respect to $\epsilon$ implies that

$$\epsilon^{min} = -4anh_1 = -2a\beta h_0.$$

At large values of $n$, the parameters $\epsilon$ and $p = 2n + \epsilon$ take the values

$$\begin{aligned}
\epsilon(n)_{OBM} &\approx -an\sqrt{\frac{2n}{3}}, \\
\epsilon(n)_{NBM} &\approx 2a\sqrt{\frac{2}{3}}\,\frac{\pi^2/3 - 1}{n},
\end{aligned} \tag{22}$$

$$\begin{aligned}
p(n)_{OBM} &\approx 2n - an\sqrt{\frac{2n}{3}}, \\
p(n)_{NBM} &\approx 2n + 2a\sqrt{\frac{2}{3}}\,\frac{\pi^2/3 - 1}{n}.
\end{aligned} \tag{23}$$

For any $a$, the effective power $p(n)_{OBM}$ becomes negative as $n$ increases and the approach based on the assumption that $\epsilon_{OBM}$ is small is not self-consistent (also see the next section). On the contrary, for the NBM, $p(n)_{NBM} \approx 2n$ as $n$ increases, which implies that our consideration is self-consistent in this case. In terms of (19)–(21), the energy per skyrmion of the $n$-skyrmion configuration takes the value

$$\frac{E_{cl}(n)}{4\pi g^2 n} = 1 + a\frac{h_0}{n} - a^2\frac{h_0^2\beta^2}{2n^2}, \tag{24}$$

which gives

$$\begin{aligned}
\frac{E_{cl}(2)_{OBM}}{4\pi g^2 2} &= 1 + 1.5708a - 0.034\,a^2,\\
\frac{E_{cl}(2)_{NBM}}{4\pi g^2 2} &= 1 + 1.1107\,a - 0.2741\,a^2,\\
\frac{E_{cl}(3)_{OBM}}{4\pi g^2 3} &= 1 + 1.6120\,a - 0.068\,a^2,\\
\frac{E_{cl}(3)_{NBM}}{4\pi g^2 3} &= 1 + 0.9308\,a - 0.0317\,a^2,\\
\frac{E_{cl}(4)_{OBM}}{4\pi g^2 4} &= 1 + 1.7562\,a - 0.191\,a^2,\\
\frac{E_{cl}(4)_{NBM}}{4\pi g^2 4} &= 1 + 0.8781\,a - 0.0084\,a^2,\\
\frac{E_{cl}(5)_{OBM}}{4\pi g^2 5} &= 1 + 1.9122\,a - 0.302\,a^2,\\
\frac{E_{cl}(5)_{NBM}}{4\pi g^2 5} &= 1 + 0.8552\,a - 0.0032\,a^2,\\
\frac{E_{cl}(6)_{OBM}}{4\pi g^2 6} &= 1 + 2.0649\,a - 0.404\,a^2,\\
\frac{E_{cl}(6)_{NBM}}{4\pi g^2 6} &= 1 + 0.8430\,a - 0.0015\,a^2.
\end{aligned} \tag{25}$$

For large $n$, the energies in Eq. (24) take the asymptotic values

$$\begin{aligned}
\frac{E_{cl}(n)_{OBM}}{4\pi g^2 n} &= \left(1 + a\sqrt{\frac{2n}{3}} - a^2\frac{n}{12}\right),\\
\frac{E_{cl}(n)_{NBM}}{4\pi g^2 n} &= \left(1 + a\sqrt{\frac{2}{3}} - a^2\frac{(\pi^2/3-1)^2}{3n^4}\right).
\end{aligned} \tag{26}$$

We note that the energies of the two models behave differently when we consider terms of the second order in the model parameter, i.e., the terms $\sim a^2$. Indeed, for the OBM, the contribution to the energy is linearly proportional to the skyrmion number $n$, while for the NBM, the contribution decreases rapidly as the skyrmion number increases. This implies that the linear approximation in $a$ is accurate for the NBM because the quadratic term becomes negligible for large $n$. Numerical results obtained for different values of $a$ for the OBM and NBM are presented in Tables 1 and 2, respectively.

As we have noted previously, our method cannot describe the one-skyrmion configuration because the corresponding energies become infinite. But setting $p = 2 + \varepsilon$ in (15) and (16) and expanding all terms up to the third order in $\varepsilon \ll 1$, we obtain

$$E_{cl}(n=1) = 4\pi g^2\left(1+\frac{\varepsilon^2}{8}-\frac{\varepsilon^3}{16} + 2a\sqrt{\frac{2}{3\varepsilon}}(1-\gamma\varepsilon)\right), \tag{27}$$

where $\gamma$ takes different value for each of the two models,

$$\gamma_{OBM} = \frac{1}{8}, \quad \gamma_{NBM} = \frac{3}{8}. \tag{28}$$

We note that with the terms of only up to the second order in $\varepsilon$ considered, the corresponding energy in Eq. (27) simplifies to

$$E_{cl} = 4\pi g^2\left(1 + \frac{\varepsilon^2}{8} + 2a\sqrt{\frac{2}{3\varepsilon}}\right),$$

and the minimum occurs at

$$\varepsilon_1 = 2\left(\frac{a}{\sqrt{3}}\right)^{2/5}.$$

Finally, the minimum of (27) occurs at

$$\varepsilon^{min} = 2\left(\frac{a}{\sqrt{3}}\right)^{2/5}\left[1+\frac{4}{5}\left(\frac{a}{\sqrt{3}}\right)^{2/5}\left(\gamma+\frac{3}{4}\right)\right] \tag{29}$$

and corresponds to a shift of $\varepsilon_1$ because higher-order corrections in $\varepsilon$ were considered in (27). The energy of the one-skyrmion configuration is

$$\begin{aligned}
\frac{E_{cl}(n=1)}{4\pi g^2} &= \\
&= \left\{1 + \frac{5}{2}\left(\frac{a}{\sqrt{3}}\right)^{4/5}\left[1 - \frac{1}{5}\left(\frac{a}{\sqrt{3}}\right)^{2/5}(8\gamma+1)\right]\right\} \approx \\
&\approx \left[1 + 1.611\,a^{4/5}\left(1 - 0.1605a^{2/5}(8\gamma+1)\right)\right].
\end{aligned} \tag{30}$$

Equation (30) implies that for a single skyrmion, the energy expansion in $a$ is proportional to a power of $a$ instead of being linearly proportional to $a$ (which is the case for the multiskyrmion configurations with $n \geq 2$), while its convergence is worse than for multiskyrmions, especially for the NBM. In fact, for $a = 0.4213$, the first two terms in (30) are equal to 1.807, and the next-order term decreases this value to 1.44, which gives an error of 7 % compared to the exact value 1.564 obtained from numerical simulations. We note that our one-skyrmion parameterization gives the same energy for both models if only the expansions up to the lowest order in $a$ are considered: the difference appears only in the term $\sim a\gamma\sqrt{\varepsilon}$ in (27).

It is clear from the results in Tables 1 and 2 that our approximate method gives the energy values that are very close to the exact values obtained by numerical simulations, especially for the NBM. In particular, the difference between the exact and the approximate energies for $a = 0.4213$ is less than 0.5 % for $n \geq 6$. For smaller values of $a$, the agreement between analytical and numerical results is even better. In evident agreement with (2), the energies of the NBM skyrmions given in Table 2 are smaller than those of the OBM skyrmions (see Table 1) at the same values of the model parameters.

We note that for the OBM (when $a$ is small), the energy per skyrmion of a multiskyrmion configuration with $n \geq 2$ is smaller compared to the single skyrmion energy, and therefore, these configurations are bound states, stable with respect to the decay into $n$ individual skyrmions. On the contrary, the ring-like OBM multiskyrmions with even $n$ (where $n \geq 4$) are unstable with respect to the decay into two-skyrmion configurations, while configurations with odd $n$ (where $n \geq 5$) are unstable with respect to the breakup into two- and three-skyrmion configurations. In addition, Table 1 and Eq. (30) show that for any $n \neq 1$, there is an upper limit for the model parameter, $a \leq a_{cr}(n)$, above which the ring-like $n$-skyrmion configuration can decay into $n$ individual skyrmions.

We now consider the case where $n = 3$ in more detail. As can be observed from the energies in Eqs. (17) and (30), the ring-like three-skyrmion configuration is stable with respect to the decay into a single and a two-skyrmion configuartion for $a \leq 0.77$ because

$$E_1 + E_2 - E_3 \approx 1.611\, a^{4/5} - a\pi \left( \frac{8}{3\sqrt{3}} - 1 \right), \quad (31)$$

and its difference becomes positive if and only if

$$a \leq \left( \frac{3\sqrt{3}\, 1.611}{\pi\,(8 - 3\sqrt{3})} \right)^5 \approx 0.77. \quad (32)$$

For the skyrmion configurations with $n = 1, 2, 3$, corrections to the energy of the higher order in $a$ lead to smaller critical values $a_{cr}(n)$.

Because our fields with axial symmetry (5) and (10) correspond to ring-like solutions of the Euler–Lagrange equations [1] for $a = 0$, they must also be solutions of the corresponding equations as $a \to 0$, i.e., when $a$ takes values in a small region close to zero. (In fact, this region actually becomes narrower as $n$ increases because the expansion in $a$ becomes less convergent in this limit.) On the other hand, the lattice-like configurations (tripole for $n = 3$, quadrupole for $n = 4$, etc.) are solutions of the equations when $a \geq a_{cr}(n)$ for given $n$ [5, 10, 13]. But the transition from the ring-like configuration to any other minimum energy configuration is a phenomenon that has not been studied in much detail and deserves further investigation.

Finally it should be stressed that in contrast to the linear approximation, the quadratic approximation given by (25) does not provide an upper bound for the energy.

## 4. AWAY FROM THE NONLINEAR $O(3)$ $\sigma$-MODEL

In the general case, for arbitrary values of the parameter $a$ and the skyrmion number $n$, soliton solutions can be obtained by numerically minimizing the energy in Eqs. (15) and (16) with respect to the variable $p$. This leads to an upper bound for the corresponding energies because the profile function is given by (10).

For large $a$ at fixed $n$ (or for large $n$ at fixed $a$), expansion (20) is not self-consistent for the OBM. But some analytical results can also be obtained in this case because for large $a$, Eq. (15) can be approximated by

$$E_{cl}(n)_{OBM} \approx 4\pi g^2 \frac{2an\pi}{\sqrt{3}p\, \sin(2\pi/p)} \frac{\sqrt{p^2-4}}{\sqrt{p}}. \quad (33)$$

The expansion of (33) up to second-order terms with respect to $p$ gives

$$E_{cl}(n)_{OBM} \approx 4\pi g^2 \frac{an\sqrt{p}}{\sqrt{3}} \left( 1 + \frac{c_2}{p^2} \right), \quad (34)$$

where

$$c_2 = 2(\pi^2/3 - 1);$$

its minimization implies that

$$p_{min} \approx \sqrt{3c_2} = 3.71$$

and the corresponding energy is therefore given by

$$\frac{E_{cl}(n)_{OBM}}{4\pi g^2} \approx \frac{4}{3} an \left( \frac{c_2}{3} \right)^{1/4} = 1.48\, an. \quad (35)$$

We note that in contrast with the results obtained near the nonlinear $\sigma$-model, the parameter $p$ is independent of the skyrmion number $n$ for large $a$. For $a \gg n$, the skyrmion radius is proportional to the square root of the skyrmion number, $r_n \propto n^{1/2}$, the skyrmion thickness is given by

$$\delta \propto r_n/p \propto n^{1/2},$$

**Table 1.** Energy per unit skyrmion number (in $4\pi g^2$) for different values of the parameter $a$ for the OBM with second-order corrections in $a$ taken into account

| | $n=1$ | $n=2$ | $n=3$ | $n=4$ | $n=5$ | $n=6$ | $n=8$ |
|---|---|---|---|---|---|---|---|
| $a=0.001$ | 1.0063 | 1.00157 | 1.0016 | 1.0017 | 1.0019 | 1.0021 | 1.0023 |
| $a=0.01$ | 1.0384 | 1.0157 | 1.0161 | 1.0176 | 1.0191 | 1.0206 | 1.0234 |
| $a=0.0316$ | 1.0933 | 1.0496 | 1.0508 | 1.0553 | 1.0601 | 1.0649 | 1.0737 |
| $a=0.1$ | 1.2227 | 1.1567 | 1.1605 | 1.1737 | 1.1882 | 1.2025 | 1.2291 |
| $a=0.316$ | 1.5113 | 1.4930 | 1.5026 | 1.5358 | 1.5638 | 1.6126 | 1.6835 |
| $a_{hed}=0.316$ (num) | 1.5647 | 1.4681 | 1.4901 | 1.5284 | 1.5692 | 1.6092 | 1.6832 |
| $a=0.316$ (num) | 1.564 | 1.468 | 1.460 | 1.450 | 1.456 | 1.449 | – |

The last two lines contain the exact results obtained from the numerical simulations of the respective multi-skyrmions with ring-like shapes ($n \geq 2$) and with shapes other than ring-like ($n \geq 3$) [10]. In the first case, we have numerically solved the equations using the hedgehog ansatz (5).

**Table 2.** Energy per unit skyrmion number for different values of the parameter $a$ for the NBM

| | $n=1$ | $n=2$ | $n=3$ | $n=4$ | $n=5$ | $n=6$ | $n=8$ | $n=12$ | $n=16$ |
|---|---|---|---|---|---|---|---|---|---|
| $a=0.01$ | 1.0363 | 1.0111 | 1.0093 | 1.0088 | 1.0085 | 1.0084 | 1.0083 | 1.0082 | 1.0082 |
| $a=0.0316$ | 1.0851 | 1.0348 | 1.0294 | 1.0277 | 1.0270 | 1.0266 | 1.0262 | 1.0260 | 1.0259 |
| $a=0.1$ | 1.1887 | 1.1083 | 1.0928 | 1.0877 | 1.0855 | 1.0843 | 1.0831 | 1.0823 | 1.0820 |
| $a=0.316$ | 1.3814 | 1.3238 | 1.2912 | 1.2768 | 1.2699 | 1.2662 | 1.2626 | 1.2602 | 1.2593 |
| $a=0.4213$ | 1.44 | 1.4193 | 1.3865 | 1.3684 | 1.3597 | 1.3549 | 1.3501 | 1.3467 | 1.3455 |
| $a=0.4213$ (num) | 1.564 | 1.405 | 1.371 | 1.358 | 1.352 | 1.349 | 1.3447 | 1.3407 | 1.3385 |

The last line contains the exact results determined by the numerical simulations [10] of multiskyrmions with ring-like shapes for $a=0.4213$, which coincide with ours for $n \leq 6$.

and therefore, the ring-like structure of the configuration is not very pronounced. Direct numerical minimization of (33) with respect to $p$ gives $p_{min}=4.5$ and the corresponding value of the energy is

$$\frac{E_{cl}(n)_{OBM}}{4\pi g^2} = 1.55\,an. \tag{36}$$

The energy obtained by solving the Euler–Lagrange equation numerically is [8]

$$\frac{E_{cl}(n)}{4\pi g^2} = 1.333\,an.$$

The profile function corresponding to this solution is given by

$$\cos f = \frac{r^2}{8n^2}(r_n^2 - r^2) + 2\frac{r^2}{r_n^2} - 1 \quad \text{for} \quad r \leq r_n$$

and

$$f = 0 \quad \text{for} \quad r > r_n.$$

This solution is quite different from our parameterization (10), and the 16 % difference between the exact and the approximate solutions is therefore understandable.

To conclude, we recall that for the NBM, parameterization (10) works well for arbitrarily large $n$ and its accuracy increases with increasing $n$, as illustrated in Table 2.

## 5. PROPERTIES OF THE SKYRMIONS: MEAN SQUARE RADII, ENERGY DENSITY, AND MOMENT OF INERTIA

Many properties of multiskyrmions can be determined using ansatz (10). For example, the mean square radius of the $n$-skyrmion configuration takes the simple form

$$\langle r^2\rangle_n = \frac{1}{2}\int dr\, r^2\phi' = \frac{2\pi r_n^2}{p\sin(2\pi/p)}, \qquad (37)$$

where $r_n$ is given by (14) for the OBM and NBM. For small $a$, it was shown in Sec. 2 that $p = 2n$, implying that the mean square radius becomes

$$\begin{aligned}\langle r^2\rangle_{n_{OBM}} &\approx \frac{\pi\sqrt{2(n^2-1)}}{\sin(\pi/n)\sqrt{3n}},\\ \langle r^2\rangle_{n_{NBM}} &\approx \frac{\pi\sqrt{2(n^2-1)}}{n\sin(\pi/n)\sqrt{3}},\end{aligned} \qquad (38)$$

which takes the respective values $\pi$, $8\pi/3\sqrt{3}$, $\pi\sqrt{5},\ldots$ and $\sqrt{2}\pi$, $8\pi/3$, $2\pi\sqrt{5},\ldots$ for $n = 2,3,4,\ldots$

For the NBM, even for a sufficiently large value of the parameter $a$, analytical formula (14) with the power $p$ taken from (23) gives the values of $\langle r^2\rangle_{n_{NBM}}$ in a remarkably good agreement with those obtained in numerical calculations. For example, the analytical result for $n = 3$ is $\sqrt{\langle r^2\rangle_3} = 2.987$, in natural units of the model $1/ge\mu$, to be compared with 2.872 obtained numerically. This agreement improves with increasing $n$, and we have $\sqrt{\langle r^2\rangle_{12}} \sim 10.92$ for $n = 12$, to be compared with 10.85 determined numerically. A similar agreement between analytical and numerical results takes place for the mean square radius of the energy distribution of multiskyrmions (the $3D$ case was considered in detail in [6]).

We note that the one-skyrmion configuration is (still) a singular case because (37) is undefined for $n = 1$. But as we have shown earlier, by expressing $p = 2 + \varepsilon$ and expanding (14) in $\varepsilon$, we obtain $r_{n=1}^2 = \sqrt{2\varepsilon/3}$, which leads to

$$\langle r^2\rangle_1 = 2\sqrt{\frac{2}{3\varepsilon^{min}}} \qquad (39)$$

for $\varepsilon^{min}$ given by (29). Our approximate method therefore shows that as the model parameter tends to zero, the mean square radius of the one-skyrmion field tends to infinity because

$$\langle r^2\rangle_1 \sim a^{-1/5};$$

on the other hand, because

$$\langle r^2\rangle_{NBM}(n) = \sqrt{n}\langle r^2\rangle_{OBM}(n),$$

the mean square radius is given by (39) for both models in this case.

The average energy density per unit surface element is defined as

$$\rho_E = \frac{E_{cl}(n)}{2\pi r_n\delta} \qquad (40)$$

with $\delta \approx 2r_n/n$, see discussion after (16). For the NBM, when $n$ is large, (40) takes the constant value

$$\rho_{E_{NBM}} \approx e\mu\, g^3\left(\sqrt{\frac{3}{2}} + a\right), \qquad (41)$$

i.e., is independent of $n$. Equation (41) therefore represents the fundamental property of multiskyrmions of this type. On the contrary, for the OBM with ring-like configurations (which do not correspond to the minimum of the energy [5, 10]) taken into account, the energy density increases with $n$ as $\sqrt{n}$ for small values of $a$.

Another quantity of physical significance determining the quantum corrections to the energy of skyrmions is the moment of inertia; it has been considered for two-dimensional models in [13]. To obtain the energy quantum correction of the soliton, due to its rotation around the axis perpendicular to the plane in which the soliton is located, we must take the $t$-dependent ansatz of the form

$$\begin{aligned} n_1 &= \sin f(\tilde r)\,\cos[n(\phi-\omega t)],\\ n_2 &= \sin f(\tilde r)\,\sin[n(\phi-\omega t)],\\ n_3 &= \cos f(\tilde r).\end{aligned} \qquad (42)$$

The $\omega$ dependence of the energy is then given by the simple formula:

$$E^{rot} = \frac{\Theta_J}{2}\omega^2, \qquad (43)$$

where $\Theta_J$, the so-called moment of inertia, is given by [13]

$$\Theta_J(n) = g^2n^2\int d^2r\,\sin^2 f\,\left(1 + af'^2\right). \qquad (44)$$

Using (10) and the relations

$$\begin{aligned} &\frac{1}{4}\int(1-\phi^2)r\,dr = \int_0^\infty \frac{(r/r_n)^p\, r\,dr}{\left[1+(r/r_n)^p\right]^2} = \\ &= \frac{2\pi\, r_n^2}{p^2\sin(2\pi/p)}, \quad p>2,\\ &\frac{1}{16}\int(1-\phi^2)^2\,\frac{dr}{r} = \int_0^\infty \frac{(r/r_n)^{2p}\,dr}{\left[1+(r/r_n)^p\right]^4 r} = \\ &= \frac{1}{6p}, \quad p>0,\end{aligned} \qquad (45)$$

we find that at large values of $n$, the moment of inertia simplifies to

$$\Theta_J(n) \approx 4\pi g^2 n r_n^2\left(\frac{2n}{p} + \frac{anp}{3r_n^2}\right), \qquad (46)$$

which holds for any multiskyrmion configuration described by ansatz (10), for both models. For small values of $a$, letting $p = 2n$ and taking $r_n^2$ given by (14), we find that

$$\Theta_J(n)_{OBM} \approx 4\pi g^2 n\, r_n^2 \left(1 + a\sqrt{\frac{2n}{3}}\right),$$
$$\Theta_J(n)_{NBM} \approx 4\pi g^2 n\, r_n^2 \left(1 + a\sqrt{\frac{2}{3}}\right), \qquad (47)$$

which implies that for large $n$, the moment of inertia is

$$\Theta_J(n) \approx E_{cl}(n)\, r_n^2, \qquad (48)$$

in agreement with simple semiclassical arguments for the thin massive ring. Similar semiclassical formulas have been obtained for the three-dimensional skyrmions (see, e.g., Refs. [6, 14]) and the moment of inertia was shown to be given by

$$\Theta_J = 2M_B r_B^2/3$$

for large baryon numbers; this expression is valid for a classical spherical bubble with the mass concentrated in its shell.

## 6. CONCLUSIONS

We have presented an analytical approach for deriving approximate expressions of skyrmion solutions in the two-dimensional $O(3)$ $\sigma$-model. These approximations are very accurate for small values of the parameter $a$ that determines the weight of the Skyrme term and the potential term in the Lagrangian. For other values of the model parameter, we have performed some numerical calculations and then combined them with further analytical work to investigate the binding and other properties of multiskyrmion states.

Two models have been studied: the «old baby Skyrme model» and the «new baby Skyrme model», which differ from each other in the form of potentials (2). For both models, the $a$ dependence of the energy of a single skyrmion differs from the cases where topological number $n \geq 2$. For the OBM, when $a$ is small, the $n = 3$ skyrmion configuration is stable with respect to the decay into a single skyrmion and a two-skyrmion configuration, while the ring-like multiskyrmion configurations with $n \geq 4$ are unstable with respect to the breakup into two- and three-skyrmion configurations. For the NBM, on the other hand, the hedgehog multiskyrmion configurations considered in [10] and here describe bound states, because the energy per skyrmion decreases as the skyrmion number increases. We note that the results obtained for the NBM are similar to the ones obtained for the three-dimensional model studied in [6]. In both cases, the energy per skyrmion decreases as the skyrmion number increases. The three-dimensional skyrmions obtained using the rational map ansatz [15] for large $n$ have the form of a bubble with the energy and the baryon number concentrated in the shell. The thickness and the energy density of the shell (which is analogous to the thickness of the ring in the two-dimensional case) are independent of the skyrmion number [6]. Similarly, in this paper we have shown that for large $n$, the two-dimensional baby skyrmions of the NBM correspond to ring-like configurations with a constant thickness and a constant energy density per unit surface of the ring. The building material for these objects is a band of matter with a constant thickness and the average energy density per unit surface. The baby skyrmions can therefore be obtained as dimensional reductions of the three-dimensional skyrmions at large $n$; the three-dimensional skyrmions can be derived from the two-dimensional baby skyrmions as dimensional extensions.

It was concluded in [8] that the Casimir energy, or quantum loop corrections, can destroy the binding properties of the two-skyrmion bound states. The validity of this argument for the two- and three-skyrmion bound states of the NBM would be worth investigating. Another interesting problem is to determine to what extent the region of sufficiently small $a$ is of importance from the standpoint of physics. For large $a$, the method overestimates the skyrmion masses for the OBM but is accurate for the NBM, especially for large $n$.

The existence of bound states of the three-dimensional skyrmions has rich phenomenological consequences in elementary particles and nuclear physics. It suggests possible existence of multibaryons with nontrivial flavor, strangeness, charm, or beauty; more details are given in [14] and references therein. Similarly, the existence of bound states of two-dimensional baby skyrmions with universal properties in the NBM, which describes anisotropic systems, can also have some consequences for the condensed state physics, which would be worth investigating in detail.

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