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THE VACUUM–VACUUM AMPLITUDE AND BOGOLIUBOV COEFFICIENTS

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We consider the problem of fixing the phases of Bogoliubov coefficients in quantum electrodynamics such that the vacuum–vacuum amplitude can be expressed through them. For a constant electric field and particles with spins 0 and 1/2, this is done starting from the definition of these coefficients. Using the symmetry between electric and magnetic fields, we extend the result to a constant electromagnetic field. It turns out that for a constant magnetic field, it is necessary to distinguish the in- and out-states, although they differ only by a phase factor. For a spin-1 particle with the gyromagnetic ratio $g = 2$, this approach fails and we reconsider the problem using the proper-time method.

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1. INTRODUCTION

Even if the electromagnetic field does not create pairs, virtual pairs lead to the appearance of a phase in the vacuum–vacuum amplitude. This makes it necessary to distinguish the in- and out-solutions even when it is commonly assumed that there is only one complete set of solutions as, e.g., in the case of a constant magnetic field. The in- and out-solutions then differ only by a phase factor that is in essence the Bogoliubov coefficient. In terms of the in- and out-states, the propagator takes the same form as for pair-creating fields.

We use the solutions with conserved quantum numbers and do not consider radiation processes. Then the events in a cell with quantum numbers n are independent of the events in cells with different quantum numbers. In other words, we work in the diagonal representation. The knowledge of the Bogoliubov coefficients is sufficient for obtaining the probability of any process in the external field (disregarding the radiation processes) [1–3]. But the real part of the action integral W that defines the vacuum–vacuum amplitude,

$$\langle 0_{out}|0_{in}\rangle = e^{iW}, \quad W = \int d^4x\mathcal{L}, \tag{1}$$

is not directly expressed through the Bogoliubov coefficients. At the same time, some effects related to $\operatorname{Re} W$ are observable. Thus, the Lagrange function $\mathcal{L}$ of a slowly varying field determines the dielectric permittivity and magnetic permeability of the field [4, 5].

The Lagrange function of a constant electromagnetic field was obtained in [6–8] in the one-loop approximation and in [9] in the two-loop approximation. Studying a model of particle production, De Witt noted that $\operatorname{Re} W$ can be expressed through Bogoliubov coefficients with the natural choice of their phases [10]. Our purpose is to choose these phases such that $\operatorname{Re} W$ can be expressed through them. We show that for the constant electric field and particles with spins 0 and 1/2, the natural choice would be sufficient if it were not for the necessity to make renormalizations. For a vector boson with the gyromagnetic ratio $g = 2$, the situation is more complicated even for a constant electric field.

We note that the transition amplitude for an electron to go from an in-state to an out-state is equal to unity. To show this, we write the Bogoliubov transformations and the relation between $\langle 0_{n\ out}|$ and $\langle 0_{n\ in}|$ [2] (where n is the set of quantum numbers)

$$\begin{aligned} a_{n\ out} &= c_{1n}a_{n\ in} - c^*_{2n}b^+_{n\ in}, \\ b^+_{n\ out} &= c_{2n}a_{n\ in} + c^*_{1n}b^+_{n\ in}, \\ \langle 0_{n\ out}| &= \langle 0_{n\ in}|(c^*_{1n} - c_{2n}a_{n\ in}b_{n\ in}), \end{aligned} \tag{1$'$}$$

where

$$|c_{1n}|^2 + |c_{2n}|^2 = 1.$$

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Here, $a_{n\ in}$ ($b^+_{n\ in}$) is the particle (antiparticle) annihilation (creation) operator, $a_{n\ in}|0_{n\ in}\rangle = 0$, and similarly for the out-states; $|0_{n\ in}\rangle$ is the vacuum state in the cell with the quantum number n, c_{1n} and c_{2n} are the Bogoliubov coefficients, and the asterisk denotes complex conjugation.

The third relation in (1′) implies Eq. (28) below and the first relation implies that

$$a^+_{n\ in} = c^{*-1}_{1n}[a^+_{n\ out} + c_{2n} b_{n\ in}].$$

Using this relation and the anticommutator $\{a_{n'\ out}, a^+_{n\ in}\} = \delta_{n',n}$, we find [2]

$$\langle 0_{n\ out}|a_{n\ out} a^+_{n\ in}|0_{n\ in}\rangle = c^{*-1}_{1n}\langle 0_{n\ out}|0_{n\ in}\rangle = 1. \quad (1'')$$

The Pauli principle prohibits virtual pair creation in the state occupied by the electron. Therefore, even the phase of the scattering amplitude remains unchanged. In particular, $c_{2n} = 0$ for the constant magnetic field, but we cannot assume that $c_{1n} = 1$ without violating Eq. (1″) and Eqs. (28), (29) below because $W \neq 0$ [4, 5]. In other words, even if $c_{2n} = 0$, the in- and out-vacua are different. (This is in contrast to the remark after Eq. (15) in [10].) The Bogoliubov coefficient c_{1n} must therefore be coordinated with the vacuum–vacuum amplitude. For the constant electromagnetic field, we represent the action integral as a sum over the set of quantum numbers n,

$$W = \int d^4x \mathcal{L}(x) = \Sigma_n W_n.$$

Then W_n define the phase of the Bogoliubov coefficient (in general, complex).

In Secs. 2 and 3, starting from the definition of the Bogoliubov coefficients, we consider the phase fixing for particles with the respective spins 0 and 1/2. In Secs. 4–6, we reconsider the problem using a more general proper-time method for spins 0, 1/2, and 1.

2. SCALAR PARTICLE IN THE CONSTANT ELECTROMAGNETIC FIELD

For a set of wave functions with conserved quantum numbers n, the Bogoliubov transformation is given by

$$\begin{aligned} {}_+\psi_n &= c_{1n}\, {}^+\psi_n + c_{2n}\, {}^-\psi_n, \\ {}_-\psi_n &= c^*_{2n}\, {}^+\psi_n + c^*_{1n}\, {}^-\psi_n, \end{aligned} \quad (2)$$

where

$$|c_{1n}|^2 - |c_{2n}|^2 = 1$$

and ${}_+\psi_n$ (${}^+\psi_n$) is the positive-frequency in- (out-) solution, and similarly for the negative-frequency states.

We are free to choose the phase of c_{1n} by redefining ψ_n. Indeed, if we substitute

$${}_\pm\psi_n = e^{\pm if}\, {}_\pm\psi^{new}_n, \quad {}^\pm\psi_n = e^{\mp if}\, {}^\pm\psi^{new}_n, \quad c_{1n} = e^{i2f} c^{new}_{1n},$$

then Eq. (2) and the propagator [2, 11]

$$G_0(x,x') = i\Sigma_n c^{*-1}_{1n} \times \begin{cases} {}^+\psi_n(x)\, {}_+\psi^*_n(x'), & t > t', \\ {}_-\psi_n(x)\, {}^-\psi^*_n(x'), & t < t' \end{cases} \quad (2')$$

retain their form in terms of the redefined quantities.

For definiteness, we assume that the particle charge is $e' = -e, e = |e|$. For a constant electric field, we then have [2] ($n = (p_1, p_2, p_3)$, $A_\mu = -\delta_{\mu 3} Et$)

$$\begin{aligned} c_{1n} &= \frac{\sqrt{2\pi}}{\Gamma\left(\frac{1}{2} - i\varkappa\right)} \exp\left(-\frac{\pi\varkappa}{2} + i\frac{\pi}{4}\right), \\ c_{2n} &= \exp\left(-\pi\varkappa - i\frac{\pi}{2}\right), \quad \varkappa = \frac{m^2 + p_1^2 + p_2^2}{2eE}. \end{aligned} \quad (3)$$

We note that in a weak electric field, $|c_{2n}|$ is exponentially small and can be neglected. The in- and out-states then differ only by a phase factor. The same must be true for the magnetic field, where $c_{2n} = 0$ exactly and $\ln c^*_{1n}$ is to be determined.

The probability amplitude that the vacuum in the state n remains vacuum is [2]

$$\langle 0_{n\ out}|0_{n\ in}\rangle = c^{*\ -1}_{1n}. \quad (4)$$

The total vacuum–vacuum amplitude is

$$\begin{aligned} \langle 0_{out}|0_{in}\rangle &= \prod_n c^{*\ -1}_{1n} = e^{iW_0}, \\ W_0 &= \sum_n W_{0n}, \quad W_{0n} = i \ln c^*_{1n}. \end{aligned} \quad (5)$$

As we see below, c^*_{1n} must be replaced by C^{*ren}_{1n} in (4) and (5). This is the renormalization of c^*_{1n}. From (3), we have

$$\ln c^*_{1n} = \frac{1}{2}\ln 2\pi - \frac{\pi\varkappa}{2} - \frac{i\pi}{4} - \ln\Gamma\left(\frac{1}{2} + i\varkappa\right). \quad (6)$$

As shown in [2], the vacuum–vacuum probability $|\langle 0_{out}|0_{in}\rangle|^2$ obtained from (5) and (3) agrees with the Schwinger result [8]. This implies that $\mathrm{Im}W_0$ is correctly given by (5) and (3). To find $\mathrm{Re}W_0$, we first consider the asymptotic representation (see Eq. (1.3.12) in [12])

$$\ln\Gamma\left(\frac{1}{2} + i\varkappa\right) = i\varkappa[\ln(i\varkappa) - 1] + \frac{1}{2}\ln 2\pi + \sum_{k=1} \frac{B_{2k}(1/2)}{2k(2k-1)}(i\varkappa)^{1-2k}. \quad (7)$$

(Letting k range to ∞, we can say that the right-hand side of (7) represent the left-hand side in a certain sense exactly; the information encoded in the right-hand side can be decoded [13].) From (6) and (7), it follows that

$$\ln c^*_{1n} = = -i\left[\varkappa(\ln\varkappa-1)+\frac{\pi}{4}+\sum_{k=1}\frac{(-1)^k B_{2k}(1/2)}{2k(2k-1)\varkappa^{2k-1}}\right]. \quad (8)$$

This asymptotic expansion contains only the imaginary part of $\ln c^*_{1n}$ or only the real part of W_0. It can be seen from (8) that as the first step, we must pass from $\ln c^*_{1n}$ to

$$\ln C^*_{1n} = \ln c^*_{1n} + i\left[\varkappa(\ln\varkappa - 1) + \frac{\pi}{4}\right] \quad (9)$$

in order to have $\ln C^*_{1n} \to 0$ as $\varkappa \to \infty$ (i.e. as $E \to 0$). Because charge renormalization is necessary, we must make the second step and introduce

$$\ln C^{*ren}_{1n} = \ln c^*_{1n} + i\left[\varkappa(\ln\varkappa - 1) + \frac{\pi}{4} + \frac{1}{24\varkappa}\right]. \quad (10)$$

In other words, we also let $\ln C^{*ren}_{1n}$ contain the term with $k = 1$ in (8). We then have the asymptotic representation

$$\ln C^{*ren}_{1n} = -i\sum_{k=2}\frac{(-1)^k B_{2k}(1/2)}{2k(2k-1)\varkappa^{2k-1}}. \quad (11)$$

Summing (11) over n as

$$\sum_k \to \int\frac{d^3pL^3}{(2\pi)^3}, \quad \int dp_3 \to eET, \quad (12)$$

and making renormalization [8], we obtain the correct asymptotic representation for $\operatorname{Re}\mathcal{L}_0$,

$$\operatorname{Re}\mathcal{L}_0 = \frac{1}{2}E^2 + \frac{(eE)^2}{16\pi^2}\times \times\sum_{k=2}\frac{(-1)^k B_{2k}(1/2)}{k(k-1)(2k-1)\varkappa_0^{2k-2}}, \quad \varkappa_0 = \frac{m^2}{2eE}. \quad (13)$$

To simplify formulas and minimize confusion with T in Eq. (50), we often set $L = T = 1$ in the expressions like (12). In addition, we drop the Maxwell part of the Lagrangian in what follows ($E^2/2$ in this case).

We now show that expression (9) can be brought to the form suggested by the proper-time formalism,

$$\ln C^*_{1n} \equiv \ln\sqrt{2\pi}+\eta(\ln\eta-1)-\ln\Gamma\left(\frac{1}{2}+\eta\right) = -F(\eta),$$

$$F(\eta) = \frac{1}{2}\int_0^\infty\frac{d\theta}{\theta}e^{-2\eta\theta}\left(\frac{1}{\operatorname{sh}\theta} - \frac{1}{\theta}\right), \quad \eta = i\varkappa. \quad (14)$$

Differentiating (14) with respect to η and using Eq. (2.4.22.5) in [14], we see that the results in the left- and the right-hand sides coincide. In addition, both sides have the same asymptotic behavior as $\eta \to \infty$. We therefore have

$$\ln C^*_{1n} = -\frac{1}{2}\int_0^\infty\frac{ds}{s\operatorname{sh}\theta}\exp\left[-is(m^2+p_\perp^2)\right]\times \times\left[1 - \frac{\operatorname{sh}\theta}{\theta}\right], \quad \theta = eEs, \quad p_\perp^2 = p_1^2 + p_2^2. \quad (15)$$

Next, we note that the term $i/24\varkappa$ in (10) can be written as

$$\frac{i}{24\varkappa} = -\frac{1}{12}\int_0^\infty d\theta e^{-2i\varkappa\theta}, \quad (16)$$

and therefore,

$$\ln C^{*ren}_{1n} = -\frac{1}{2}\int_0^\infty\frac{ds}{s\operatorname{sh}\theta}\exp\left[-is(m^2+p_\perp^2)\right]R(\theta),$$
$$R(\theta) = 1 - \left(\frac{1}{\theta} - \frac{\theta}{6}\right)\operatorname{sh}\theta. \quad (17)$$

Here, $R(\theta)$ is a «regulator». It is independent of the quantum numbers n and is the same as in the proper-time representation of the Lagrange function [8].

We now consider the case where a constant magnetic field is collinear with a constant electric field. Then

$$\ln C^{*ren}_{1n}(E,H) = -\frac{1}{2}\int_0^\infty\frac{ds}{s\operatorname{sh}\theta}\times \times\exp\left\{-is[m^2+eH(2l+1)]\right\}R(\theta,\tau),$$
$$\tau = eHs, \quad l = 0, 1, \ldots, \quad (18)$$

and we assume that $R(\theta,\tau)$ can be obtained by the same reasoning as in [8] (or simply taken from [8]),

$$R(\theta,\tau) = 1 - \left(\frac{1}{\theta\tau} + \frac{1}{6}\frac{H^2-E^2}{EH}\right)\operatorname{sh}\theta\sin\tau,$$
$$\tau = eHs, \quad \theta = eEs. \quad (19)$$

Integrating over p_3, we obtain (see (12) with $T = 1$)

$$\int dp_3\ln C^{*ren}_{1n}(E,H) = -\frac{1}{2}eE\int_0^\infty\frac{ds}{s\operatorname{sh}\theta}\times \times\exp\left\{-is[m^2+eH(2l+1)]\right\}R(\theta,\tau). \quad (20)$$

In this expression, we can turn the electric field off,

$$\int dp_3 \ln C_{1n}^{*ren}(E=0,H) =$$

$$= -\frac{1}{2}\int_0^\infty \frac{ds}{s^2}\exp\left\{-is[m^2+eH(2l+1)]\right\} R(0,\tau), \quad (21)$$

$$R(0,\tau) = 1 - \left(\frac{1}{\tau}+\frac{\tau}{6}\right)\sin\tau.$$

To remove the integration over p_3, we write the factor s^{-2} as $s^{-3/2}s^{-1/2}$ and note that $1/\sqrt{s}$ must arise from the integration over p_3,

$$\frac{1}{\sqrt{s}} = \frac{e^{i\pi/4}}{\sqrt{\pi}}\int_{-\infty}^{\infty} dp_3 \exp(-isp_3^2). \quad (22)$$

Therefore,

$$\ln C_{1n}^{*ren}(E=0,H) = -\frac{e^{i\pi/4}}{2\sqrt{\pi}}\int_0^\infty \frac{ds}{s^{3/2}} \times$$
$$\times \exp\left\{-is[m^2+eH(2l+1)+p_3^2]\right\} R(0,\tau). \quad (23)$$

(Substituting $s \to -it$, we see that expression (23) is purely imaginary.) From here or from (21), we obtain

$$i\sum_n \ln C_{1n}^{*ren}(E=0,H) =$$
$$= i\int\frac{dp_2}{2\pi}\int\frac{dp_3}{2\pi}\sum_{l=0}^{\infty}\ln C_{1n}^{*ren}(E=0,H) = \mathcal{L}_0 =$$
$$= -\frac{eH}{16\pi^2}\int_0^\infty \frac{ds}{s^2\sin\tau}\times$$
$$\times \exp(-ism^2)R(0,\tau) \quad (L=T=1), \quad (24)$$

which agrees with [8, 9]. Relation (39) below was used here and the sum over l was performed with the help of the formula

$$\sum_{l=0}^{\infty}\exp\left[-iseH(2l+1)\right] = \frac{1}{2i\sin(eHs)}. \quad (25)$$

3. ELECTRON IN THE CONSTANT ELECTROMAGNETIC FIELD

The Bogoliubov transformation is given by

$$\begin{aligned} {}_+\psi_n &= c_{1n}\,{}^+\psi_n + c_{2n}\,{}^-\psi_n, \\ {}_-\psi_n &= -c_{2n}^*\,{}^+\psi_n + c_{1n}^*\,{}^-\psi_n, \end{aligned} \quad (26)$$

where

$$|c_{1n}|^2 + |c_{2n}|^2 = 1.$$

For the constant electric field, we have

$$\begin{aligned} c_{1n}^* &= -i\sqrt{\frac{2\pi}{\varkappa}}\frac{e^{-\pi\varkappa/2}}{\Gamma(i\varkappa)}, \\ c_{2n} &= e^{-\pi\varkappa}, \quad n=(p_1,p_2,p_3,r). \end{aligned} \quad (27)$$

These Bogoliubov coefficients are independent of the spin state index $r=1,2$.

As in the scalar case, we start with the relations [2]

$$\langle 0_{n\ out}|0_{n\ in}\rangle = c_{1n}^*, \quad (28)$$

and

$$\begin{aligned} \langle 0_{out}|0_{in}\rangle &= \prod_n c_{1n}^* = e^{iW_{1/2}}, \\ W_{1/2} &= \sum_n W_{1/2;n},\ W_{1/2;n} = -i\ln c_{1n}^*. \end{aligned} \quad (29)$$

It follows from (27) that

$$\ln c_{1n}^* = -\frac{i\pi}{2} + \frac{1}{2}\ln\frac{2\pi}{\varkappa} - \frac{\pi\varkappa}{2} - \ln\Gamma(i\varkappa). \quad (30)$$

The asymptotic expansion for $\Gamma(i\varkappa)$ is

$$\ln\Gamma(i\varkappa) = \left(i\varkappa-\frac{1}{2}\right)\ln(i\varkappa) - i\varkappa + \frac{1}{2}\ln 2\pi +$$
$$+ i\sum_{k=1}(-1)^k\frac{B_{2k}}{2k(2k-1)}(\varkappa)^{1-2k} \quad (31)$$

(see Eq. (8.344) in [15] or Eq. (6.1.40.) in [16]). From (30) and (31), we obtain

$$\ln C_{1n}^* \equiv \ln c_{1n}^* + i\left(\varkappa\ln\varkappa - \varkappa + \frac{\pi}{4}\right) =$$
$$= -i\sum_{k=1}(-1)^k\frac{B_{2k}}{2k(2k-1)}(\varkappa)^{1-2k}, \quad (32)$$

$$\ln C_{1n}^{*ren} \equiv \ln C_{1n}^* - \frac{i}{12\varkappa} =$$
$$= -i\sum_{k=2}(-1)^k\frac{B_{2k}}{2k(2k-1)}(\varkappa)^{1-2k}. \quad (33)$$

As in the scalar case, we find that

$$\ln C_{1n}^* = -\frac{1}{2}\int_0^\infty \frac{dx}{x}e^{-2i\varkappa x}\left(\operatorname{cth}x - \frac{1}{x}\right), \quad (34)$$

$$\ln C_{1n}^{*ren} =$$
$$= -\frac{1}{2}\int_0^\infty \frac{dx}{x}e^{-2i\varkappa x}\left[1-\left(\frac{1}{x}+\frac{x}{3}\right)\operatorname{th}x\right]\operatorname{cth}x. \quad (35)$$

Equation (2.4.22.6) in [14] was used to verify (34), cf. the text before Eq. (15).

The generalization of (35) to the presence of a constant magnetic field is straightforward. We rewrite it as ($x = \theta = eEs$)

$$\ln C_{1n}^{*ren}(E,H) = -\frac{1}{2}\int\limits_0^\infty \frac{d\theta}{\theta} \times \exp[-is(m^2+2eHl)]R(\theta,\tau)\,\mathrm{cth}\,\theta, \quad (36)$$

where $n = (p_1, p_2, p_3, r)$; $l = l_{min}, l_{min}+1, \ldots, l_{min} = 0$ for $r = 1$, $l_{min} = 1$ for $r = 2$, and $R(\theta,\tau)$ can be taken from the Lagrange function [8, 9] ($\tau = eHs$),

$$R(\theta,\tau) = 1 - \left(\frac{1}{\theta\tau} + \frac{E^2-H^2}{3EH}\right)\mathrm{tg}\,\tau\,\mathrm{th}\,\theta. \quad (37)$$

Integrating over p_3 using the second equation in (12), we find

$$\int \frac{dp_3}{2\pi}\ln C_{1n}^{*ren} = -\frac{eE}{4\pi}\int\limits_0^\infty \frac{d\theta}{\theta} \times \exp[-is(m^2+2eHl)]R(\theta,\tau)\,\mathrm{cth}\,\theta. \quad (38)$$

The subsequent integration over p_2 is performed using the formula similar to (12) [2],

$$\int dp_2 = eHL. \quad (39)$$

To sum over r and l in (36), we use the formula

$$\sum_{r=1}^{2}\sum_{l_{min}}^{\infty} e^{-2iseHl} = -i\,\mathrm{ctg}(eHs) \quad (40)$$

that follows from (25). In agreement with the Lagrange function for the constant electromagnetic field [8, 9], we therefore have

$$\sum_n \ln C_{1n}^{*ren} = i\frac{e^2EH}{8\pi^2}\int\limits_0^\infty \frac{d\theta}{\theta} \times \exp(-ism^2)R(\theta,\tau)\,\mathrm{cth}\,\theta\,\mathrm{ctg}\,\tau \quad (L=T=1). \quad (41)$$

Returning to (38), we can switch the electric field off,

$$\int \frac{dp_3}{2\pi}\ln C_{1n}^{*ren} = -\frac{1}{4\pi}\int\limits_0^\infty \frac{ds}{s^2} \times \exp[-is(m^2+2eHl)]R(0,\tau), \quad (42)$$

$$R(0,\tau) = 1 - \left(\frac{1}{\tau} - \frac{\tau}{3}\right)\mathrm{tg}\,\tau,$$

where l are given in (36). As in the scalar case, using (22), we obtain

$$\ln C_{1n}^{*ren}(E=0,H) = -\frac{e^{i\pi/4}}{2\sqrt{\pi}}\int\limits_0^\infty \frac{ds}{s^{3/2}} \times \exp[-is(m^2+p_3^2+2eHl)]R(0,\tau), \quad (43)$$

where $n = (p_1, p_2, p_3, r)$, $l = 0, 1, 2, \ldots$ for $r = 1$, and $l = 1, 2, \ldots$ for $r = 2$.

In the subsequent sections, we give a heuristic derivation of $\ln C_{1n}^{*ren}$ not resorting to c_{1n}^*, but using the proper-time method. The main problem occurring here is that renormalizations must be made. We know how to renormalize $\mathcal{L}$ as a whole, but we must renormalize the contribution to it from a particular state n. To do this, we assume, as before, that the regulator is independent of n.

4. SCALAR PARTICLE

We take the vector potential of a constant electromagnetic field in the form

$$A_\mu = \delta_{\mu 2}Hx_1 - \delta_{\mu 3}Et, \quad (44)$$

but start with the particle in a constant magnetic field, $E = 0$ in (44). The propagator with coinciding x and x' is given by (see, e.g., [11])

$$G_0(x,x|E=0,H) = i\sqrt{\frac{eH}{\pi}}\sum_{l=0}^{\infty}\int\limits_{-\infty}^{\infty}\frac{dp_2}{2\pi} \times \int\limits_{-\infty}^{\infty}\frac{dp_3}{2\pi}\int\limits_{-\infty}^{\infty}\frac{dp^0}{2\pi}\int\limits_0^\infty ds\frac{D_l^2(\zeta)}{l!} \times \exp\{-is[m^2+eH(2l+1)+p_3^2-p_0^2]\}, \quad (45)$$

$$\zeta = \sqrt{2eH}\left(x_1 + \frac{p_2}{eH}\right).$$

In accordance with (1), we must integrate $\mathcal{L}_0$ and hence $G_0(x,x)$ over d^4x. The integration over x_1 is done using the formula

$$\int\limits_{-\infty}^{\infty} d\zeta D_l^2(\zeta) = \sqrt{2\pi}\,l!, \quad \text{or} \int\limits_{-\infty}^{\infty} dx_1 D_l^2(\zeta) = \sqrt{\frac{\pi}{eH}}\,l! \quad (46)$$

Integrating over p^0 and x_1, we obtain

$$\int_{-\infty}^{\infty} dx_1 G_0(x,x) = \frac{\exp(3\pi i/4)}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{dp_2}{2\pi} \times \\ \times \int_{-\infty}^{\infty} \frac{dp_3}{2\pi} \sum_{l=0}^{\infty} \int_0^{\infty} \frac{ds}{\sqrt{s}} \times \\ \times \exp\{-is[m^2 + eH(2l+1) + p_3^2]\}. \quad (47)$$

As noted in [3] (see Eq. (2.12) therin), it follows from Schwinger results [8] that for a scalar particle (boson),

$$-i\frac{\partial W_b}{\partial m^2} = \int d^4x G_b(x,x), \\ \text{or} \quad W_b = -i \int_{m^2}^{\infty} d\tilde{m}^2 \int d^4x G_b(x,x|\tilde{m}^2). \quad (48)$$

This implies that $\mathcal{L}_0$ can be obtained from (47) by inserting $-1/s$ in the integrand. Also inserting the regulator from (21), we obtain

$$iW_0(E=0,H) = i\mathcal{L}_0(E=0,H) = \\ = \frac{\exp(\pi i/4)}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{dp_2}{2\pi} \int_{-\infty}^{\infty} \frac{dp_3}{2\pi} \sum_{l=0}^{\infty} \int_0^{\infty} \frac{ds}{s^{3/2}} \times \\ \times \exp\{-is[m^2 + eH(2l+1) + p_3^2]\} R(0,\tau) = \\ = -\int_{-\infty}^{\infty} \frac{dp_2}{2\pi} \int_{-\infty}^{\infty} \frac{dp_3}{2\pi} \sum_{l=0}^{\infty} \ln C_{1n}^{*ren} \quad (L=T=1). \quad (49)$$

For the constant electromagnetic field described by vector potential (44), we now insert the expressions for the wave functions in (2′) (see [2] with the modifications for $e' = -e = -|e|$) and use relation (93) in [11] (or a relation similar to (96) below). We then find

$$G_0(x,x|E,H) = \frac{e^{3\pi i/4}}{2\sqrt{\pi eE}} \int_{-\infty}^{\infty} \frac{dp_2}{2\pi} \int_{-\infty}^{\infty} \frac{dp_3}{2\pi} \times \\ \times \sum_{l=0}^{\infty} \sqrt{\frac{eH}{\pi}} \frac{D_l^2}{l!} \sqrt{2} \int_0^{\infty} \frac{d\theta}{\sqrt{\operatorname{sh} 2\theta}} \times \\ \times \exp\left(-2i\varkappa\theta - i\frac{T^2}{2\operatorname{cth}\theta}\right), \\ \theta = eEs, \quad T = \sqrt{2eE}\left(t - \frac{p_3}{eE}\right). \quad (50)$$

Integrating over x_1 (see (46)) and t, we obtain

$$\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dt G_0(x,x|E,H) = \\ = \frac{i}{2} \int_{-\infty}^{\infty} \frac{dp_2}{2\pi} \int_{-\infty}^{\infty} \frac{dp_3}{2\pi} \sum_{l=0}^{\infty} \int_0^{\infty} \frac{ds}{\operatorname{sh}(eEs)} \times \\ \times \exp\{-is[m^2 + eH(2l+1)]\}. \quad (51)$$

Passing from $G_0(x,x)$ to $\mathcal{L}_0$ is realized by inserting the factor $-1/s$ in the integrand in (51). Also inserting the regulator $R(\tau,\theta)$, see Eq. (19), we obtain

$$W_0(E,H) = i \int_{-\infty}^{\infty} \frac{dp_2}{2\pi} \int_{-\infty}^{\infty} \frac{dp_3}{2\pi} \sum_{l=0}^{\infty} \ln C_{1n}^{*ren} = \\ = -\frac{i}{2} \int_{-\infty}^{\infty} \frac{dp_2}{2\pi} \int_{-\infty}^{\infty} \frac{dp_3}{2\pi} \sum_{l=0}^{\infty} \int_0^{\infty} \frac{ds}{s \operatorname{sh}\theta} \times \\ \times \exp\{-is[m^2 + eH(2l+1)]\} R(\tau,\theta). \quad (52)$$

5. SPINOR PARTICLE

We first consider the electron in the constant magnetic field, $E = 0$ in (44). The squared Dirac equation can be brought to the form

$$\left\{\frac{d^2}{d\zeta^2} - \frac{\zeta^2}{4} + \frac{p_0^2 - p_3^2}{2eH} - \frac{1}{2}\Sigma_3\right\} Z = 0, \\ \Sigma_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \quad (53)$$

where ζ is the same as in (45). We see that Z can be written as

$$Z = \operatorname{diag}(f_1, f_2, f_1, f_2) \exp\left[i(p_2x_2 + p_3x_3 - p^0t)\right], \quad (54)$$

and f_1 and f_2 must satisfy the equation

$$\left\{\frac{d^2}{d\zeta^2} - \frac{\zeta^2}{4} + \frac{p_0^2 - p_3^2}{2eH} \mp \frac{1}{2}\right\} f_{1,2} = 0. \quad (55)$$

We choose $f_1 = D_{l-1}(\zeta)$ and $f_2 = D_l(\zeta)$ in order that $p_\perp^2 = 2eHl$ in both cases. The solutions of the Dirac equation are obtained as the columns of the matrix [2]

$$(m - i\hat{\Pi})Z, \quad (56)$$

where $\hat{\Pi} = \gamma_\mu \Pi_\mu$, $\Pi_\mu = -i\partial_\mu + eA_\mu$.

Using the γ-matrices in the standard representation [4], we have

$$m - i\hat{\Pi} = \begin{pmatrix} m+\Pi^0 & 0 & -\Pi_3 & -\Pi_1 + i\Pi_2 \\ 0 & m+\Pi^0 & -\Pi_1 - i\Pi_2 & \Pi_3 \\ \Pi_3 & \Pi_1 - i\Pi_2 & m-\Pi^0 & 0 \\ \Pi_1 + i\Pi_2 & -\Pi_3 & 0 & m-\Pi^0 \end{pmatrix}. \tag{57}$$

In terms of ζ, we obtain

$$\begin{aligned} \Pi_1 + i\Pi_2 &= -i\sqrt{2eH}\left(\frac{d}{d\zeta} - \frac{\zeta}{2}\right), \\ \Pi_1 - i\Pi_2 &= -i\sqrt{2eH}\left(\frac{d}{d\zeta} + \frac{\zeta}{2}\right). \end{aligned} \tag{58}$$

Also using the relations

$$\begin{aligned} \left(\frac{d}{d\zeta} + \frac{\zeta}{2}\right) D_l(\zeta) &= l D_{l-1}(\zeta), \\ \left(\frac{d}{d\zeta} - \frac{\zeta}{2}\right) D_l(\zeta) &= -D_{l+1}(\zeta), \end{aligned} \tag{59}$$

we find (with the exponetial factor in (54) omitted for brevity)

$$(m - i\hat{\Pi})Z = \begin{pmatrix} (m+p^0)D_{l-1}(\zeta) & 0 & -p_3 D_{l-1}(\zeta) & il\sqrt{2eH}D_{l-1}(\zeta) \\ 0 & (m+p^0)D_l(\zeta) & -i\sqrt{2eH}D_l(\zeta) & p_3 D_l(\zeta) \\ p_3 D_{l-1}(\zeta) & -il\sqrt{2eH}D_{l-1}(\zeta) & (m-p^0)D_{l-1}(\zeta) & 0 \\ i\sqrt{2eH}D_l(\zeta) & -p_3 D_l(\zeta) & 0 & (m-p^0)D_l(\zeta) \end{pmatrix}. \tag{60}$$

Choosing the second and the first columns as ψ_1 and ψ_2 (with the subscripts 1 and 2 indicating spin states) and normalizing them, we obtain

$$_{+}\psi_1 = N_n \begin{bmatrix} 0 \\ (m+p^0)D_l(\zeta) \\ -il\sqrt{2eH}D_{l-1}(\zeta) \\ -p_3 D_l(\zeta) \end{bmatrix} e^{iq\cdot x},$$

$$N_n = \left(\frac{eH}{\pi}\right)^{1/4} \sqrt{\frac{1}{2p^0(p^0+m)l!}},$$

$$p^0 = \sqrt{m^2 + 2eHl + p_3^2}, \quad q\cdot x = p_2 x_2 + p_3 x_3 - p^0 t,$$

$$n = (p_2, p_3, l, r), \quad \zeta = \sqrt{2eH}\left(x_1 + \frac{p_2}{eH}\right), \tag{61}$$

$$_{+}\psi_2 = N_n \sqrt{l} \begin{bmatrix} (m+p^0)D_{l-1}(\zeta) \\ 0 \\ p_3 D_{l-1}(\zeta) \\ i\sqrt{2eH}D_{l-1}(\zeta) \end{bmatrix} e^{iq\cdot x}, \tag{62}$$

$$l = 0, 1, 2, \ldots$$

As can be seen from (62), l actually begins with unity in this state. The negative-frequency solutions $_{-}\psi_n$ are obtained from (61) and (62) by the substitution $q \to -q$. We note that Eqs. (61) and (62) differ from Eq. (10.5.9) in [4] because the authors there assumed the charge of a spinor particle to be positive.

Having obtained the wave functions, we next find the contribution to $\mathcal{L}_{1/2}$ from each state ψ_n. For the field that does not create pairs, the propagator has the standard form

$$G_{1/2}(x, x') = = i\Sigma_n \begin{cases} {}_{+}\psi_n(x) {}_{+}\bar{\psi}_n(x'), & t > t', \\ -{}_{-}\psi_n(x) {}_{-}\bar{\psi}_n(x'), & t < t', \end{cases} \quad \bar{\psi}_n = \psi_n^* \beta. \tag{63}$$

In the standard representation, we have

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{64}$$

From (61) and (64), we find

$$\mathrm{tr}_+\psi_1(x)_+\bar{\psi}_1(x) = \\ = N_n^2\{[(m+p^0)^2 - p_3^2]D_l^2(\zeta) - 2eHl^2 D_{l-1}^2(\zeta)\}. \quad (65)$$

Integrating over x_1, we obtain, see (46),

$$\int_{-\infty}^{\infty} dx_1 \mathrm{tr}_+\psi_1(x)_+\bar{\psi}_1(x) = \frac{m}{p^0}, \\ p^0 = \sqrt{m^2 + 2eHl + p_3^2}, \quad l = 0, 1, \ldots \quad (66)$$

From (62), we obtain, similarly,

$$\int_{-\infty}^{\infty} dx_1 \mathrm{tr}_+\psi_2(x)_+\bar{\psi}_2(x) = \frac{m}{p^0}, \quad l = 1, 2, \ldots \quad (67)$$

For the negative-frequency states, we must substitute $p^0 \to -p^0$. We can then write

$$\frac{1}{|p^0|} = \frac{e^{i\pi/4}}{\sqrt{\pi}} \int_0^{\infty} \frac{ds}{\sqrt{s}} \exp[-is(m^2 + 2eHl + p_3^2)], \quad (68)$$

incorporating both lines in (63). It thus follows from (63) and (66)–(68) that

$$\int_{-\infty}^{\infty} dx_1 \mathrm{tr}\, G_{1/2}(x,x) = \\ \sum_n \frac{e^{3i\pi/4}}{\sqrt{\pi}} m \int_0^{\infty} \frac{ds}{\sqrt{s}} \exp[-is(m^2 + 2eHl + p_3^2)], \quad (69)$$

where $l = 0, 1, \ldots$ for $r = 1$ and $l = 1, 2, \ldots$ for $r = 2$. We next use the analogue of (48) for the electron,

$$W_{1/2} = i \int_m^{\infty} d\tilde{m} \mathrm{Tr}\, G_{1/2}(x, x|\tilde{m}), \quad (70)$$

where Tr means the integration over d^4x and the trace over spin indices; we set $VT = 1$ as above. Because

$$i \int_m^{\infty} d\tilde{m}\, \tilde{m} \exp(-is\tilde{m}^2) = \frac{e^{-ism^2}}{2s}, \quad (70')$$

we see that $W_{1/2}$ can be obtained from (69) by inserting the factor $1/2ms$ in the integrand. We therefore find

$$\mathcal{L}_{1/2} = \Sigma_n \frac{e^{3i\pi/4}}{2\sqrt{\pi}} \times \\ \times \int_0^{\infty} \frac{ds}{s^{3/2}} \exp[-i(m^2 + 2eHl + p_3^2)] R(0, \tau). \quad (71)$$

This is in agreement with (43) and (29). To check this result, we integrate over $dp_2/2\pi$ with the help of (39), over $dp_3/2\pi$ with the help of (22), and use (40). Then, as expected, we obtain

$$\mathcal{L}_{1/2}(E = 0, H) = \frac{eH}{8\pi^2} \int_0^{\infty} \frac{ds}{s^2} \times \\ \times \exp(-ism^2) R(0, \tau) \operatorname{ctg} \tau, \quad (72)$$

see Eq. (47) in Ch. 1 in the last Ref. in [9] for $E = 0$.

Passing over to the constant electromagnetic field described by vector potential (44), we use γ-matrices in the spinor representation because both α_3 and Σ_3 are then diagonal. The squared Dirac equation has the form

$$(\Pi^2 + m^2 + g)Z = 0, \\ g = e \begin{pmatrix} (H - iE)\sigma_3 & 0 \\ 0 & (H + iE)\sigma_3 \end{pmatrix}, \quad (73)$$

with Π_μ defined in (56). Hence,

$$Z = \mathrm{diag}(f_1, f_2, f_3, f_4) \exp[i(p_2x_2 + p_3x_3)]. \quad (74)$$

In terms of ζ and T (see (45) and (50)), we obtain the equation

$$\left\{2eH\left[-\frac{\partial^2}{\partial\zeta^2} + \frac{\zeta^2}{4} \pm \frac{1}{2}\right] + \\ + 2eE\left[\frac{\partial^2}{\partial T^2} + \frac{T^2}{4} \mp \frac{i}{2}\right] + m^2\right\} f_{1,2} = 0 \quad (75)$$

for f_1 and f_2 and similarly,

$$\left\{2eH\left[-\frac{\partial^2}{\partial\zeta^2} + \frac{\zeta^2}{4} \pm \frac{1}{2}\right] + \\ + 2eE\left[\frac{\partial^2}{\partial T^2} + \frac{T^2}{4} \pm \frac{i}{2}\right] + m^2\right\} f_{3,4} = 0 \quad (76)$$

for f_3 and f_4. From these equations, it follows that

$$^+Z = \mathrm{diag}\{D_{l-1}(\zeta) D_{-i\varkappa-1}(\chi), \\ D_l(\zeta) D_{-i\varkappa}(\chi), D_{l-1}(\zeta) D_{-i\varkappa}(\chi), \\ D_l(\zeta) D_{-i\varkappa-1}(\chi)\} \times \\ \times \exp[i(p_2x_2 + p_3x_3)], \quad \chi = e^{i\pi/4}T. \quad (77)$$

Solutions of the Dirac equation with γ-matrices in the spinor representation are obtained as the columns of the matrix

$$(m - i\hat{\Pi})Z = \begin{pmatrix} m & 0 & \Pi^0 + \Pi_3 & \Pi_1 - i\Pi_2 \\ 0 & m & \Pi_1 + i\Pi_2 & \Pi^0 - \Pi_3 \\ \Pi^0 - \Pi_3 & -\Pi_1 + i\Pi_2 & m & 0 \\ -\Pi_1 - i\Pi_2 & \Pi^0 + \Pi_3 & 0 & m \end{pmatrix} Z. \quad (78)$$

In terms of χ, we have

$$\Pi^0 \pm \Pi_3 = -e^{-i\pi/4}\sqrt{2eE}\left(\frac{\partial}{\partial\chi} \pm \frac{\chi}{2}\right), \quad \chi = e^{i\pi/4}T. \quad (79)$$

Also taking (58), (59), and the relations

$$(\Pi^0 + \Pi_3)D_\nu(\chi) = -e^{-i\pi/4}\nu\sqrt{2eE}D_{\nu-1}(\chi), \quad (\Pi^0 - \Pi_3)D_\nu(\chi) = e^{-i\pi/4}\sqrt{2eE}D_{\nu+1}(\chi) \quad (80)$$

into account, we find four columns of the matrix $(m - i\hat{\Pi})\,{}^+Z$,

$$\begin{bmatrix} mD_{l-1}(\zeta)D_{-i\varkappa-1}(\chi) \\ 0 \\ e^{-i\pi/4}\sqrt{2eE}D_{l-1}(\zeta)D_{-i\varkappa}(\chi) \\ -i\sqrt{2eH}D_l(\zeta)D_{-i\varkappa-1}(\chi) \end{bmatrix}, \quad \begin{bmatrix} 0 \\ mD_l(\zeta)D_{-i\varkappa}(\chi) \\ il\sqrt{2eH}D_{l-1}(\zeta)D_{-i\varkappa}(\chi) \\ e^{i\pi/4}\varkappa\sqrt{2eE}D_l(\zeta)D_{-i\varkappa-1}(\chi) \end{bmatrix},$$
$$\begin{bmatrix} e^{i\pi/4}\varkappa\sqrt{2eE}D_{l-1}(\zeta)D_{-i\varkappa-1}(\chi) \\ i\sqrt{2eH}D_l(\zeta)D_{-i\varkappa}(\chi) \\ mD_{l-1}(\zeta)D_{-i\varkappa}(\chi) \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -il\sqrt{2eH}D_{l-1}(\zeta)D_{-i\varkappa-1}(\chi) \\ e^{-i\pi/4}\sqrt{2eE}D_l(\zeta)D_{-i\varkappa}(\chi) \\ 0 \\ mD_l(\zeta)D_{-i\varkappa-1}(\chi) \end{bmatrix}. \quad (81)$$

Here and below, $\exp[i(p_2x_2 + p_3x_3)]$ is dropped for brevity. We let ${}^+\psi_1$ (${}^+\psi_2$) denote the fourth (first) column multiplied by the normalization factor ${}^+N_n$ (${}^+N_n\sqrt{l}\,$):

$${}^+N_n = \exp\left(-\frac{\pi\varkappa}{4}\right)(l!2eE)^{-1/2}\left(\frac{eH}{\pi}\right)^{1/4}. \quad (82)$$

We next consider the positive-frequency solution of (73) as $t \to -\infty$,

$${}_+Z = \mathrm{diag}\{D_{l-1}(\zeta)D_{i\varkappa}(\tau), D_l(\zeta)D_{i\varkappa-1}(\tau), D_{l-1}(\zeta)D_{i\varkappa-1}(\tau), D_l(\zeta)D_{i\varkappa}(\tau)\}, \quad (83)$$

where $\tau = -e^{-i\pi/4}T$. In terms of this variable, we have

$$\Pi^0 \pm \Pi_3 = -e^{i\pi/4}\sqrt{2eE}\left(\frac{\partial}{\partial\tau} \mp \frac{\tau}{2}\right). \quad (84)$$

Similarly to (80), we find

$$(\Pi^0 + \Pi_3)D_\nu(\tau) = e^{i\pi/4}\sqrt{2eE}D_{\nu+1}(\tau), \quad (\Pi^0 - \Pi_3)D_\nu(\tau) = -e^{i\pi/4}\nu\sqrt{2eE}D_{\nu-1}(\tau). \quad (85)$$

Using these relations, we obtain the four columns of the matrix $(m - i\hat{\Pi})_+Z$ in (78) and (83),

$$\begin{bmatrix} mD_{l-1}(\zeta)D_{i\varkappa}(\tau) \\ 0 \\ e^{-i\pi/4}\varkappa\sqrt{2eE}D_{l-1}(\zeta)D_{i\varkappa-1}(\tau) \\ -i\sqrt{2eH}D_l(\zeta)D_{i\varkappa}(\tau) \end{bmatrix}, \quad \begin{bmatrix} 0 \\ mD_l(\zeta)D_{i\varkappa-1}(\tau) \\ il\sqrt{2eH}D_{l-1}(\zeta)D_{i\varkappa-1}(\tau) \\ e^{i\pi/4}\sqrt{2eE}D_l(\zeta)D_{i\varkappa}(\tau) \end{bmatrix},$$
$$\begin{bmatrix} e^{i\pi/4}\sqrt{2eE}D_{l-1}(\zeta)D_{i\varkappa}(\tau) \\ i\sqrt{2eH}D_l(\zeta)D_{i\varkappa-1}(\tau) \\ mD_{l-1}(\zeta)D_{i\varkappa-1}(\tau) \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -il\sqrt{2eH}D_{l-1}(\zeta)D_{i\varkappa}(\tau) \\ e^{-i\pi/4}\varkappa\sqrt{2eE}D_l(\zeta)D_{i\varkappa-1}(\tau) \\ 0 \\ mD_l(\zeta)D_{i\varkappa}(\tau) \end{bmatrix}. \quad (86)$$

Using the fourth and the first columns again, we have

$${}_+\psi_1(x) = {}_+N_n \begin{bmatrix} -il\sqrt{2eH}D_{l-1}(\zeta)D_{i\varkappa}(\tau) \\ e^{-i\pi/4}\varkappa\sqrt{2eE}D_l(\zeta)D_{i\varkappa-1}(\tau) \\ 0 \\ mD_l(\zeta)D_{i\varkappa}(\tau) \end{bmatrix} \times \exp[i(p_2x_2 + p_3x_3)], \quad (87)$$

$$ {}_+\psi_2(x) = $$
$$ = {}_+N_n\sqrt{l}\begin{bmatrix} mD_{l-1}(\zeta)D_{i\varkappa}(\tau) \\ 0 \\ e^{-i\pi/4}\varkappa\sqrt{2eE}D_{l-1}(\zeta)D_{i\varkappa-1}(\tau) \\ -i\sqrt{2eH}D_l(\zeta)D_{i\varkappa}(\tau) \end{bmatrix} \times $$
$$ \times \exp[i(p_2x_2+p_3x_3)], \quad (88) $$

where ${}_+N_n = {}^+N_n/\sqrt{\varkappa}$, see (82).

We note that ${}_-Z$ (${}^-Z$) can be obtained from ${}^+Z$ (${}_+Z$) by the substitution $\chi \to -\chi$ ($\tau \to -\tau$). To obtain ${}_-\psi$-functions from the corresponding ${}^+\psi$-functions, we also change the sign of $\sqrt{2eE}$ in the columns in addition to these substitutions; this is because of the relations (see (79) and (80))

$$ \begin{aligned} (\Pi^0+\Pi_3)D_\nu(\pm\chi) &= \mp e^{-i\pi/4}\nu\sqrt{2eE}D_{\nu-1}(\pm\chi), \\ (\Pi^0-\Pi_3)D_\nu(\pm\chi) &= \pm e^{-i\pi/4}\sqrt{2eE}D_{\nu+1}(\pm\chi). \end{aligned} \quad (89) $$

Thus,

$$ {}_-\psi_1(x) = {}_-N_n\begin{bmatrix} -il\sqrt{2eH}D_{l-1}(\zeta)D_{-i\varkappa-1}(-\chi) \\ -e^{-i\pi/4}\sqrt{2eE}D_l(\zeta)D_{-i\varkappa}(-\chi) \\ 0 \\ mD_l(\zeta)D_{-i\varkappa-1}(-\chi) \end{bmatrix} \times $$
$$ \times \exp[i(p_2x_2+p_3x_3)], \quad (90) $$

$$ {}_-\psi_2(x) = $$
$$ = {}_-N_n\sqrt{l}\begin{bmatrix} mD_{l-1}(\zeta)D_{-i\varkappa-1}(-\chi) \\ 0 \\ -e^{-i\pi/4}\sqrt{2eE}D_{l-1}(\zeta)D_{-i\varkappa}(-\chi) \\ -i\sqrt{2eH}D_l(\zeta)D_{-i\varkappa-1}(-\chi) \end{bmatrix} \times $$
$$ \times \exp[i(p_2x_2+p_3x_3)], \quad {}_-N_n = {}^+N_n, \quad (91) $$

and similarly for ${}^-\psi_1$ and ${}^-\psi_2$.

We note in passing that the wave functions for the electron in a constant electric field were written in [2] using γ-matrices in the standard representation. Acting on these functions by the operator

$$ U = \frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ I & -I \end{pmatrix}, $$

we obtain the solutions in the spinor representation. Taking the magnetic field into account is realized by the substitutions

$$ \exp(ip_2x_2)\{1, p_1-ip_2, p_1+ip_2\} \to \left(\frac{eH}{\pi}\right)^{1/4} \times $$
$$ \times \frac{1}{\sqrt{l!}}\{D_l(\zeta), -il\sqrt{2eH}D_{l-1}(\zeta), i\sqrt{2eH}D_{l+1}(\zeta)\} $$

for $r = 1$. For $r = 2$, we must replace l by $l-1$ in these substitutions.

The electron propagator is given by

$$ G_{1/2}(x,x') = $$
$$ = i\sum_n c_{1n}^{*-1}\begin{cases} {}^+\psi_n(x){}_+\bar{\psi}_n(x'), & t > t', \\ -{}_-\psi_n(x){}^-\bar{\psi}_n(x'), & t < t', \end{cases} \quad (92) $$

where $\bar{\psi} = \psi^*\beta$, $n = (p_2, p_3, l, r)$ for the constant electromagnetic field, and c_{1n}^* is given in (27), where $p_\perp^2 = 2eHl$ in the expression for $\varkappa$, see (3) and (15). In the spinor representation,

$$ \beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad (93) $$

and therefore,

$$ (a_1, a_2, a_3, a_4)\beta = (a_3, a_4, a_1, a_2). $$

Using (81), (82), and (87), we now obtain

$$ \mathrm{tr}({}^+\psi_1(x)\,{}_+\bar{\psi}_1(x)) = \sqrt{\frac{eH}{\pi}}\frac{me^{-\pi\varkappa/2}}{l!\sqrt{2eE\varkappa}} \times $$
$$ \times D_l^2(\zeta)\{e^{-i\pi/4}D_{-i\varkappa}(\chi)D_{-i\varkappa}(-\chi) + $$
$$ + e^{i\pi/4}\varkappa D_{-i\varkappa-1}(\chi)D_{-i\varkappa-1}(-\chi)\}. \quad (94) $$

Integrating over x_1, we obtain, see (46),

$$ \int\limits_{-\infty}^{\infty} dx_1\,\mathrm{tr}({}^+\psi_1(x)\,{}_+\bar{\psi}_1(x)) = $$
$$ = \frac{me^{-\pi\varkappa/2}}{\sqrt{2eE\varkappa}}\{e^{-i\pi/4}D_{-i\varkappa}(\chi)D_{-i\varkappa}(-\chi) + \quad (95) $$
$$ + e^{i\pi/4}\varkappa D_{-i\varkappa-1}(\chi)D_{-i\varkappa-1}(-\chi)\}, $$
$$ \varkappa = \frac{m^2+2eHl}{2eE}, \quad l = 0, 1, \ldots $$

For $r = 2$, we obtain the same expression, but with $l = 1, 2, \ldots$

We next multiply (95) with i/c_{1n}^* according to (92) and use the relation (see Eq. (93) in [11] with $-i\varkappa \to -i\varkappa + 1/2$)

$$ \Gamma(i\varkappa)D_{-i\varkappa}(\chi)D_{-i\varkappa}(-\chi) = \sqrt{2}\int\limits_0^\infty \frac{d\theta}{\sqrt{\mathrm{sh}\,2\theta}} \times $$
$$ \times \exp(-2i\varkappa\theta + \theta - \frac{i}{2}T^2\,\mathrm{th}\,\theta), \quad \theta = eEs, \quad (96) $$

and the relation obtained from this by the substitution $i\varkappa \to i\varkappa + 1$.

We now obtain from (95) and (96) that

$$\int_{-\infty}^{\infty} dx_1 \frac{i}{c_{1n}^*} \operatorname{tr}({}^+\psi_1(x)\; {}_+\bar{\psi}_1(x)) = -\frac{me^{-i\pi/4}}{\sqrt{2\pi eE}} \times \\ \times \int_0^{\infty} \frac{d\theta}{\sqrt{\operatorname{sh} 2\theta}} 2\operatorname{ch}\theta \exp\left(-2i\varkappa\theta - \frac{i}{2}T^2 \operatorname{th}\theta\right), \\ T = \sqrt{2eE}\left(t - \frac{p_3}{eE}\right). \tag{97}$$

Integrating this expression over t, we obtain

$$\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx_1 \frac{i}{c_{1n}^*} \operatorname{tr}({}^+\psi_1(x)\; {}_+\bar{\psi}_1(x)) = \\ = im \int_0^{\infty} ds \operatorname{cth}(eEs) \exp[-is(m^2 + 2eHl)], \\ l = 0, 1, 2, \ldots \tag{98}$$

For $r = 2$, we have the same expression, but with $l = 1, 2, \ldots$

Taking the remarks after Eq. (70′) into account and inserting the regulator $R(\theta, \tau)$ in the integrand, we obtain the contribution to $\mathcal{L}_{1/2}$ from the state $n = (p_2, p_3, l, r)$. Summing over l and r (see (40)) and integrating over $dp_2/2\pi$ and $dp_3/2\pi$ (see (39) and (12)), we obtain, in agreement with (41), that

$$\mathcal{L}_{1/2} = \frac{e^2HE}{8\pi^2} \int_0^{\infty} \frac{ds}{s} \times \\ \times \exp(-ism^2) R(\theta, \tau) \operatorname{cth}\theta \operatorname{ctg}\tau. \tag{99}$$

We finally note that for $H = 0$, we have

$$\int_{-\infty}^{\infty} dt \frac{i}{c_{1n}^*} \operatorname{tr}({}_+\psi_1(x)\; {}^+\bar{\psi}_1(x)) = im \int_0^{\infty} ds \operatorname{cth}(eEs) \times \\ \times \exp[-is(m^2 + p_1^2 + p_2^2)], \quad l = 0, 1, 2, \ldots \tag{100}$$

instead of (98). Inserting $1/2ms$ and $R(\theta, 0)$, we see that this agrees with (35) and (29).

6. VECTOR BOSON

The propagator and the effective Lagrange function for the vector boson with the gyromagnetic ratio $g = 2$ in a constant electromagnetic field were obtained by Vanyashin and Terentyev [17]. In another form, the propagator was found by the author [11]. In the latter paper, there is a misprint in Eq. (73), where the argument of sin and cos should be 2τ, not τ. In addition, the statement that the divergence term in the expression for the current in Eq. (38) does not contribute is not true when the magnetic field is present; this, however, is of no consequence because the expression was used only for the normalization of wave functions.

The results of Vanyashin and Terentyev imply that relation (48) in the present paper also holds for the vector boson if we take $G_b = G^{\mu}{}_{\mu}$. Using (48), we can reproduce the expression for $\mathcal{L}_1$ in [17] starting from our propagator. Indeed, our result for

$$G^{\mu}{}_{\mu} = \frac{e^2EH}{16\pi^2} \int_C \frac{ds}{\operatorname{sh}\theta \sin\tau} \exp(-ism^2) \times \\ \times \left\{ 2\cos 2\tau + 2\operatorname{ch} 2\theta - \frac{i}{m^2}[eH \operatorname{ctg}\tau + eE \operatorname{cth}\theta] \right\} \tag{101}$$

can be written in a simpler form if we note that

$$\frac{d}{ds} \frac{1}{\operatorname{sh}\theta \sin\tau} = \\ = -\frac{1}{\operatorname{sh}\theta \sin\tau}[eH \operatorname{ctg}\tau + eE \operatorname{cth}\theta]\}, \\ \tau = eHs, \quad \theta = eEs. \tag{102}$$

We can then integrate the term in the square brackets in (101) by parts,

$$-\frac{ie^2EH}{16\pi^2m^2} \int_C \frac{ds}{\operatorname{sh}\theta \sin\tau} \times \\ \times \exp(-ism^2)[eH \operatorname{ctg}\tau + eE \operatorname{cth}\theta]\} \to \\ \to -\frac{e^2EH}{16\pi^2} \int_C \frac{ds}{\operatorname{sh}\theta \sin\tau} \exp(-ism^2), \tag{103}$$

where we discarded a divergent term independent of E and H. Expression (101) is therefore equivalent to

$$\frac{e^2EH}{16\pi^2} \int_C \frac{ds}{\operatorname{sh}\theta \sin\tau} \times \\ \times \exp(-ism^2)(2\cos 2\tau + 2\operatorname{ch} 2\theta - 1). \tag{104}$$

Inserting $-1/s$ in the integrand, we obtain Eq. (21) in [17]; we agree with the subsequent formulas in that paper.

Returning to our present problem, we note that for a constant electric field, c_{1n}^* is independent of the polarization state of the vector boson and is the same as in the scalar case [11]. Nevertheless, $\operatorname{Im}\mathcal{L}_1$ is not simply $3\operatorname{Im}\mathcal{L}_0$ [17]. The knowledge of c_{1n}^* is therefore not useful in obtaining $\ln C_{1n}^{*ren}$. Resorting to the proper-time method, we find that the problem is more difficult than in the previous cases. As seen already from (101), the dependence on m^2 is more complicated here and the contributions from the electric and magnetic fields are not factorized in the proper-time integrand. For these reasons, we here consider only the constant magnetic field.

It follows from [11] that for the spin states $r = 1, 2, 3$,

$$
{}_+\psi_1^\mu(x)\, {}_+\psi_{1\mu}^*(x) = \sqrt{\frac{eH}{\pi}}\frac{1}{2|p^0|l!}\frac{1}{(l+1)(m^2+eHl)}\times \\ \times\{-(l+1)^2 eHD_l^2(\zeta) + \\ + [m^2 + eH(2l+1)]D_{l+1}^2(\zeta)\}, \quad (105)
$$

$$
{}_+\psi_2^\mu(x)\, {}_+\psi_{2\mu}^*(x) = \sqrt{\frac{eH}{\pi}}\frac{1}{2|p^0|l!}D_l^2(\zeta), \quad (106)
$$

$$
{}_+\psi_3^\mu(x)\, {}_+\psi_{3\mu}^*(x) = \sqrt{\frac{eH}{\pi}}\frac{1}{2|p^0|l!}\frac{l}{2m^2(m^2+eHl)}\times \\ \times\{-2eH[m^2 + eH(2l+1)]D_l^2(\zeta) + [eHD_{l+1}(\zeta) - \\ - (m^2+eHl)D_{l-1}(\zeta)]^2 + \\ + [eHD_{l+1}(\zeta) + (m^2+eHl)D_{l-1}(\zeta)]^2\}. \quad (107)
$$

Integrating the expressions in (105)–(107) over x_1 with the help of (46), we obtain $1/2|p^0|$ in all three cases, but

$$
l = l_{min}, l_{min}+1, \ldots, \\ l_{min} = \begin{cases} -1, & r = 1, \\ 0, & r = 2, \\ 1, & r = 3. \end{cases} \quad (108)
$$

The vector boson propagator is given by [11]

$$
G_1^{\mu\nu}(x, x') = i\int_{-\infty}^{\infty}\frac{dp_2}{2\pi}\int_{-\infty}^{\infty}\frac{dp_3}{2\pi}\times \\ \times\sum_{r=1}^{3}\sum_{l_{min}}^{\infty}\begin{cases} {}_+\psi_n^\mu(x)\,{}_+\psi_n^{*\nu}(x'), & t > t', \\ {}_-\psi_n^\mu(x)\,{}_-\psi_n^{*\nu}(x'), & t < t'. \end{cases} \quad (109)
$$

We see from (68) and (48) and the above results that the contribution to $\mathcal{L}_1$ from the state with the quantum numbers $n = (p_2, p_3, l, r)$ is

$$
i\ln c_{1n}^* = -i\frac{e^{i\pi/4}}{2\sqrt{\pi}}\times \\ \times\int_0^\infty\frac{ds}{s^{3/2}}\exp\left\{-is[m^2 + eH(2l+1) + p_3^2]\right\}. \quad (110)
$$

The sum over r and l is performed using the formula

$$
\sum_{r=1}^{3}\sum_{l_{min}}^{\infty}\exp[-iseH(2l+1)] = \frac{1+2\cos 2eHs}{2i\sin eHs} \quad (111)
$$

that can be obtained from (25). To integrate over $dp_2/2\pi$ and $dp_3/2\pi$, we use (39) and (12). Inserting $R(\tau)$, we then obtain

$$
\sum_n W_{spin1,n} = -\frac{eH}{16\pi^2}\int_0^\infty\frac{ds}{s^2\sin\tau}\times \\ \times\exp(-ism^2)(3 - 4\sin^2\tau)R(\tau). \quad (112)
$$

In accordance with [17], $R(\tau)$ is defined as

$$
\frac{3 - 4\sin^2\tau}{\sin\tau} \to 3\left(\frac{1}{\sin\tau} - \frac{1}{\tau} - \frac{\tau}{6}\right) - 4(\sin\tau - \tau) = \\ = \frac{3 - 4\sin^2\tau}{\sin\tau}R(\tau). \quad (113)
$$

This implies that

$$
R(\tau) = 1 - \frac{\sin\tau}{3 - \sin^2\tau}\left(\frac{3}{\tau} - \frac{7}{2}\tau\right), \\ R(\tau)|_{\tau\ll 1} = \frac{29}{120}\tau^4. \quad (114)
$$

From (110), we therefore have

$$
i\ln C_{1n}^{*ren} = -i\frac{e^{i\pi/4}}{2\sqrt{\pi}}\int_0^\infty\frac{ds}{s^{3/2}}\times \\ \times\exp\left\{-is[m^2 + eH(2l+1) + p_3^2]\right\}R(\tau), \quad (115)
$$

where l is given in (108). Substituting $\tau \to -it$ and rotating the integration contour, we see that $\ln C_{1n}^{*ren}$ is real, as it should be for the magnetic field.

7. CONCLUSIONS

We have shown how the renormalized phase of the vacuum–vacuum amplitude in quantum electrodynamics can be expressed through the properly fixed phases of the Bogoliubov coefficients; a nonzero phase of the former indicates nonzero phases of the latter. In general, the knowledge of the Bogoliubov coefficients is insufficient for obtaining the phase of the vacuum–vacuum amplitude. Additional information is needed. Thus, in the case of constant magnetic and electromagnetic fields, we have used the symmetry between the electric and magnetic fields in the Lagrange function. In the case of a vector boson, the knowledge of the Bogoliubov coefficients is not useful in fixing their phases. Resorting to the proper-time method shows that the expressions for the phases are in general more complicated than that for lower-spin particles. For this reason, we have presented the results only for the constant magnetic field, where they turned out to be as simple as expected.

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