BOUND STATES OF THE JOSEPHSON DEGREES OF FREEDOM AND TRAP OSCILLATIONS

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It is shown that the interaction of the Josephson degrees of freedom with states of condensate motion can produce their equilibrium bound states. As a result of the appearance of these states, first, the tunneling splitting is significantly increased in double-well trapped condensates. Second, the bound states can realize an absolute minimum of the thermodynamic energy for a sufficiently strong interaction. Transition to the new ground state is a second-order phase transition. The existense of the bound state leads to an equilibrium distortion of the condensate shape. This implies that the Josephson states can be detected by observing the change in the condensate shape.

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1. INTRODUCTION

Since the discovery of two-component condensates [1] and condensates in a double-well potential [2], the phenomena caused by phase coherence of two condensate modes attract considerable attention, both experimental and theoretical (see, e.g., [3] and references therein). In [2], spatial quantum coherence was observed by means of an interference pattern in two overlapping condensates. This interference pattern was confirmed in [4] by numerical simulation of the Gross–Pitaevskii equation. In [5], coherent oscillations of the relative populations were observed in driven two-component condensates with different internal states. As is well known, a clear manifestation of phase coherence is the Josephson effect. In numerous studies devoted to the Josephson effect in systems of two condensates in different internal states [5, 6] or in a double-well potential [7, 8], coherent Josephson oscillations are considered for various dynamical regimes caused by the competition between tunneling and intracondensate interaction (nonlinearity). In [7], the Josephson coupling energy is calculated for small-amplitude oscillations in a double-well potential. Damping effects due to the normal currents at a finite temperature are estimated there. In [8], it is shown that for a relatively weak

interaction, the particle number oscillations between the condensates are complete. They are suppressed when the total number of atoms in the condensates exceeds a critical value and the behavior of the system is governed by nonlinearity. Nonlinear Josephson-type oscillations in the relative oscillations of driven two-component condensates are studied in [6]. Decoherence effects and quantum corrections to mean-field solutions have been considered in [11, 12]. In [13], the damping effects of the Josephson current (even at zero temperature) are derived within the functional integral approach. A detailed treatment of the nonlinear classical dynamic of the condensates in a double-well potential was given in [9, 10]. In [14], the quantum and thermal fluctuations of the phase are studed for condensates in the double-well potential.

We emphasize that experimental observation of the Josephson effect is difficult because the small energy splitting associated with the Josephson coupling implies that thermal and quantum fluctuations destroy the phase coherence between two condensates even at the lowest achievable temperatures [13, 14]. While the energy splitting can be increased, e.g., by lowering the barrier height, it then becomes comparable with that of motion states of the condensates.

But the problem of the interaction between the Josephson degrees of freedom and states of motion (oscillations) of the trapped condensate has yet to be ana-

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lyzed. The present paper focuses on mechanisms of increasing the tunneling splitting in a double-well potential and of formation of the bound states of the Josephson degrees of freedom with trap oscillations. The mechanisms are generated by a sufficiently strong interaction between the Josephson and oscillation states. These mechanisms may be important for experimental detection of the Josephson states. The consideration proposed in the present paper is suitable for the double-well trapped condensates and two-component condensates in the same trap.

The results obtained in this paper are as follows.

1. As is well known [3, 7, 14], the Gross–Pitaevskii equations for two condensates with a weak Josephson coupling have stationary solutions corresponding to the lowest states with the eigenenergies $\pm E_J$, where E_J is the Josephson coupling energy. This implies that the double-well condensates form a macroscopic two-level subsystem with the tunneling energy splitting $2E_J$ if the dynamical Josephson oscillations of the particle number are disregarded. In what follows, it is shown that a sufficiently strong interaction between the macroscopic two-level subsystem and the condensate oscillation results in their equilibrium bound state. Appearance of the bound state generates an essential increase of the tunneling splitting of the macroscopic two-level subsystem.

2. In Sec. 3–5, we consider the interaction between anharmonic trap oscillations and the Josephson degrees of freedom generated by the particle number transfer between two condensates. Nonlinear dynamic oscillations of the particle number between two condensates with the Josephson coupling are considered in [9, 10]. In the present paper, we derive the quantized spectrum of the particle number generated by the Josephson coupling in order to formulate the problem of the interaction between the Josephson and oscillation degrees of freedom in an adequate manner. The states of this spectrum represent a quantum analogue of the nonlinar coherent Josephson oscillations considered in [10]. In what follows, the states of the quantized spectrum are called the Josephson states. The spectrum is highly nonequidistant and has a logarithmic singularity in the density of states at the energy $2E_J$. We show that any Josephson state can be realized by means of a given initial disbalance of the particle number in two condensates.

3. We consider the interaction between trap oscillations and the excited Josephson states corresponding to a sufficiently large initial disbalance of the particle number. We show that this interaction is responsible for the formation of a bound state of $\bar{n}_m \gg 1$ oscil-

lation quanta with the Josephson state corresponding to the initial disbalance of the particle number. In the Thomas–Fermi approximation at $\mu \gg \omega_0$, where μ is the chemical potential of the condensate and ω_0 is the characteristic frequency of the trap, $h = 1$ and the bound state arises in the region of a sufficiently dense oscillation spectrum. In this region, the level separations are small compared with the harmonic oscillation frequency ω_0. The equilibrium values of the oscillation quanta and the initial particle number disbalance are coupled self-consistently and can realize an absolute minimum of the thermodynamic energy at a sufficiently strong interaction. The thermodynamic average $\bar{n}_m \neq 0$ generates an equilibrium distortion of the condensate shape. This allows detecting the Josephson states by observing a change in the condensate shape.

2. BOUND STATE OF THE MACROSCOPIC TWO-LEVEL SYSTEM AND TRAP OSCILLATIONS

The Josephson coupling is realized for condensates in a symmetric/asymmetric double-well potential formed by two different traps with a barrier between them [2]. The barrier is created by laser light, and its height is directly proportional to the laser power and can therefore be varied easily. The proposed mechanism is also suitable for condensates in different internal states in the same trap. Experimentally, this may be a superposition of two Rb^{87} condensates in the states [5, 6]

$$|F = 1, m_F = -1\rangle, |F = 2, m_F = 1\rangle.$$

For a weak Josephson coupling, the basis states are the self-consistent ground states in the two condensates separately. The wave function of a condensate with the Josephson coupling is given by a superposition of these states, namely,

$$\Psi(\mathbf{r}, t) = \psi_1(\mathbf{r})a_1(t) + \psi_2(\mathbf{r})a_2(t),$$

where $\psi_i(\mathbf{r})$ are normalized solutions of the Gross–Pitaevskii equation,

$$a_i(t) = N_i^{1/2}(t)e^{i\theta_i(t)}, \quad i = 1, 2,$$

with N_i and $\theta_i(t)$ being the particle numbers and phases of each condensate.

As is well known [3, 7, 14], the Hamiltonian of two condensates with a weak Josephson coupling is given by

$$H_J - E_0 = E_C(\Delta N)^2 - 2E_J \cos\phi, \tag{1}$$

where

$$\Delta N = N_1 - N_2, \quad E_C = \partial \mu / \partial N,$$

$\mu \equiv \mu_1 = \mu_2$ are the chemical potentials, N is the total particle number, and $\phi = \theta_1 - \theta_2$ is the relative phase of the condensates. The quantities E_C and E_J depend on the total particle number N. In (1), the energy origin is the mean-field total energy of the condensates, namely, $E_0 \equiv \mu N$. The variables ΔN and ϕ are canonical. The equations of motion can be written in the Hamiltonian form,

$$\frac{\partial \phi}{\partial t} = \frac{\partial H_J}{\partial (\Delta N)}, \quad \frac{\partial (\Delta N)}{\partial t} = -\frac{\partial H_J}{\partial \phi}.$$

The lowest stationary solutions of these equations have the eigenenergies $E_{s,a} = \pm E_J$ corresponding to symmetric $(\phi_s = 2\pi n, \ (\Delta N)_s = 0)$ and antisymmetric $(\phi_a = \pi(2n + 1), \ (\Delta N)_a = 0)$ eigenfunctions.

Thus, the double-well trapped condensates form a macroscopic two-level system with the tunneling splitting $2E_J$ if we disregard the dynamical Josephson oscillations of the particle number.

In what follows, we show that a sufficiently strong interaction between the macroscopic two-level system and the condensate oscillation is responsible for the formation of their equilibrium bound state. In this state, the definite equilibrium number of the oscillation quanta is coupled to the two-level system.

The interaction can be realized by the following mechanisms. First, the interaction can be implemented if we allow the Josephson coupling energy to depend on the atom displacement. The latter is generated by the condensate oscillation. Second, the interaction can be realized by applying a pair of traveling-wave laser beams with the same Rabi frequency Ω and the wave-vector difference q, for instance, in the x direction. The one-dimensional condensate is considered for simplicity. The pulse frequences are chosen to be resonant with the transition energy between the two stationary Josephson states. A similar mechanism is used for the detection of the motion states of a single trapped two-level atom (see [15] and references therein) and N two-level atoms in a trap [16].

In what follows, we assume that the interaction generates the transition between the states $|e, n'\rangle$ and $|g, n\rangle$. Here, $|e\rangle$ and $|g\rangle$ are two Josephson states, n and n' are the numbers of oscillation quanta. The transition matrix element can be written as

$$G\langle e, n'|\sigma^+ \exp\left(i\eta(\hat{a}^+ + \hat{a})\right) + \\ + \sigma^- \exp\left(-i\eta(\hat{a}^+ + \hat{a})\right)|g, n\rangle. \quad (2)$$

The operators $\hat{a}^+$ and $\hat{a}$ are the creation and annihilation operators associated with the oscillation state and $\sigma^+ = |e\rangle\langle g|$. Each of two values $N\hat{\sigma}^z = \pm N$, where $\sigma^z = |e\rangle\langle e| - |g\rangle\langle g|$, corresponds to one of the two stationary Josephson states. The quantity G is determined by the specific mechanism inducing the interaction. If the interaction is realized by applying laser beams, it follows that in the rotating frame, $G = \Omega$ and $\eta = q(2MN\omega_0)^{-1/2}$ is the Lamb–Dicke parameter caused, for example, by the center-of-mass motion of N atoms in a trap with the characteristic frequency ω_0.

We consider the classical states of motion of the condensate. These states can be described in terms of the complex amplitudes $a^*, a = n^{1/2} e^{\pm i\varphi_1}$, where

$$n = \langle a|\hat{a}^+ \hat{a}|a\rangle = |a|^2$$

is the average number of quanta in the coherent state $|a\rangle$. The variables n and φ_1 are canonical. By the classical state of motion, we mean that its number of quanta is very large, $n \gg 1$. It is convenient to specify the relation between the amplitudes a, a^* and the operators $\hat{a}, \hat{a}^+$ as $a = N^{-1/2}\hat{a}$. The commutator of a and a^* is then equal to zero with macroscopic accuracy,

$$[a, a^*] = 1/N \to 0.$$

The Hamiltonian of the motion states can be written as $N\epsilon(n)$.

The Hamiltonian of the interacting stationary Josephson and oscillation degrees of freedom therefore becomes

$$\hat{H} = N\epsilon(n) + N\frac{\varepsilon}{2}\sigma^z + H_{int}, \quad \varepsilon \equiv \frac{2E_J}{N}. \quad (3)$$

We are interested in the situation where the resonance condition

$$k \cdot \frac{d\epsilon}{dn} = \varepsilon \quad (4)$$

is satisfied with an integer k. In this case, the Hamiltonian H_{int} can be represented as

$$H_{int} = N[g_k \sigma^+ a^k + g_k^* a^{*k} \sigma^-], \\ g_k = G\left(\frac{i^k}{k!}\right)\left(\frac{q^2}{2M\omega_0}\right)^{k/2}. \quad (5)$$

Dynamical solutions generated by the Hamiltonian of type (3), (5) are studied in numerous papers both for a single trapped atom, $N = 1$, (see, e.g., [15, 17], and references therein) and for N atoms in a trap [16]. In the present paper, we focus on the equilibrium properties of the system described by Hamiltonian (3), (5).

We note that in order to determine the partition function $Z(N; T)$ in what follows, we must have a macroscopic two-level system. The phase transitions within the Dicke model that corresponds to the case where $k = 1$ in Eqs. (3) and (5) have been discussed in the context of superradiance [18] and recently, for exciton condensation [19].

Substituting Hamiltonian (3), (5) in the expression $Z(N, T) = \mathrm{Sp}\, e^{-\beta \hat{H}}$ for the partition function, we obtain in the case of the classical oscillation degrees of freedom that

$$
Z(N; T) = \int\limits_0^\infty dn\, e^{-\beta N \epsilon(n)} \left(\mathrm{Sp}\, e^{-\beta \hat{H}_\sigma} \right)^N,
$$

$$
\hat{H}_\sigma = \frac{\varepsilon}{2} \sigma^z + [g_k \sigma^+ a^k + g_k^* a^{+k} \sigma^-], \tag{6}
$$

where $\beta = 1/T$, T is the temperature. Using the eigenenergies

$$
E_{1,2} = \pm \frac{\varepsilon}{2} \left(1 + \frac{4|g_k|^2 n^k}{\varepsilon^2} \right)^{1/2}
$$

of the Hamiltonian $\hat{H}_\sigma$, we arrive at the following expression for the partition function:

$$
Z(N; T) = \int\limits_0^\infty dn\, e^{-\beta N \epsilon(n)} \left(e^{-\beta E_1} + e^{-\beta E_2} \right)^N =
$$

$$
= \int\limits_0^\infty dn \exp \left\{ N \left(-\beta \varepsilon(n) + \right. \right.
$$

$$
\left. \left. + \ln \left[2 \,\mathrm{ch}\, \frac{\beta \varepsilon}{2} \left(1 + \frac{4|g_k|^2 n^k}{\varepsilon^2} \right)^{1/2} \right] \right) \right\} \equiv
$$

$$
\equiv \int\limits_0^\infty dn\, e^{-N F(n)}. \tag{7}
$$

The partition function in Eq. (7) has a maximum at the value $|a|_m^2 \equiv \bar{n}_m$ realizing a minimum of the function $F(n)$. That is, the value $\bar{n}_m(T)$ is determined by the equation

$$
\left(\frac{d\epsilon}{dn} \right)_{n=\bar{n}_m} = \frac{|g_k|^2 k \bar{n}_m^{k-1}}{\varepsilon} \left(1 + \frac{4|g_k|^2 \bar{n}_m^k}{\varepsilon^2} \right)^{-1/2} \times
$$

$$
\times \,\mathrm{th} \left[\beta \frac{\varepsilon}{2} \left(1 + \frac{4|g_k|^2 \bar{n}_m^k}{\varepsilon^2} \right)^{1/2} \right] \tag{8}
$$

and is the number of oscillation quanta per atom of the two-level system. This quantity plays the part of the order parameter of the system for $T < T_c$, where T_c is the transition temperature to the state with $\bar{n}_m \neq 0$.

The new splitting $E_1 - E_2 = \Delta \varepsilon$ of the Josephson levels depends on the temperature and is defined by the expression

$$
\Delta \varepsilon(T) = \varepsilon \left(1 + \frac{4|g_k|^2 \bar{n}_m^k}{\varepsilon^2} \right)^{1/2}. \tag{9}
$$

At $T = 0$, Eq. (8) becomes

$$
\left(1 + \frac{4|g_k|^2 \bar{n}_m^k}{\varepsilon^2} \right) = \left(\frac{k^2 |g_k|^2}{\varepsilon^2} \right)^2 \bar{n}_m^{2(k-1)}, \tag{10}
$$

where we used that

$$
\frac{d\epsilon}{dn} = \frac{\varepsilon}{k}.
$$

For a simple resonance with $k = 1$, the solution for $\bar{n}_m(T = 0)$ is given by

$$
\bar{n}_m = \left(\frac{\varepsilon}{2G(q l_0)} \right)^2 \left[\left(\frac{G^2 (q l_0)^2}{\varepsilon^2} \right)^2 - 1 \right],
$$

$$
l_0^2 = \frac{1}{2 M \omega_0}. \tag{11}
$$

We used that $g_1 = G(q l_0)$ in accordance with Eq. (5). We note that

$$
\left(\frac{d^2 F(n)}{dn^2} \right)_{n=\bar{n}_m} > 0.
$$

The solution for $\bar{n}_m$ exists when the interaction is sufficiently strong, that is,

$$
\left(\frac{G}{\varepsilon} \right)^2 (q l_0)^2 > 1. \tag{12}
$$

This relation implies that the system in question undergoes a quantum phase transition at the critical value $G_c = \varepsilon(q l_0)$ of the interaction strength.

At $G(q l_0) \gg \varepsilon$, the tunneling splitting $\Delta \varepsilon(T = 0)$ and thermodynamic energy E_m of the system discribed by Hamiltonian (3), (5) are determined by the expressions

$$
\Delta \varepsilon \approx \frac{G^2 (q l_0)^2}{\varepsilon} \gg \varepsilon, \quad E_m \approx -N \frac{G^2 (q l_0)^2}{4\varepsilon}. \tag{13}
$$

Therefore, first, the splitting caused by the interaction is much greater than the «unperturbed» splitting ε. Second, the bound state realizes the absolute minimum of the thermodynamic energy within the range $g_1 \gg \varepsilon$.

It is worth noting that the consideration proposed above applies in the case where $q r_0 \ll 1$, where r_0 is the (typical) mean interatom spacing. The l_0 value is

the characteristic size of the condensate. As a result, we arrive at the relation

$$ql_0 \gtrsim 1.$$

Transition to the state with $\bar{n}_m \neq 0$ is second-order at $k = 1$. Imposing that $\bar{n}_m \to 0$ at the transition temperature T_c, we obtain from (8) at $k = 1$ and $|g_1| \gg \varepsilon$ that

$$T_c = \frac{|g_1|^2}{2\varepsilon} = \frac{G^2(ql_0)^2}{2\varepsilon} = \frac{\Delta\varepsilon(T=0)}{2}. \quad (14)$$

It is interesting to discuss the solutions of Eq. (10) for multiple resonances with $k > 1$. As can be easily seen, the character of the ground state changes drastically at $k \neq 1$. Already at $k = 2$, solutions of Eq. (10) with $\bar{n}_m \gg 1$ are absent.

At $k \gg 1$, Eq. (10) implies

$$\bar{n}_m^k \approx \frac{2\varepsilon^2}{\pi k^2 |g_k|^2}, \quad \Delta\varepsilon \approx \varepsilon \left(1 + \frac{4}{\pi k^2}\right). \quad (15)$$

We see that $\bar{n}_m \sim 1$ and $\Delta\varepsilon \approx \varepsilon$ at $k \gg 1$, and therefore, the effect of the interaction is negligible, unlike the solution for a simple resonance with $k = 1$ obtained above.

For a one-dimensional condensate, the appearance of a state with $\bar{n}_m \neq 0$ corresponds to the center-of-mass oscillation of N atoms with a displaced zero point. The «capture» of oscillation quanta by the macroscopic two-level system can therefore be considered as a mechanism of the formation of the equilibrium coherent state of motion.

In the general case, the existence of a nonzero value of $\bar{n}_m$ leads to an equilibrium distortion of the condensate shape. The distortion is self-consistently coupled to the tunneling splitting. In particular, the stationary Josephson states can be detected by observing a change in the condensate shape, and vice versa.

A remark is in order. The quantities $d\epsilon/dn \equiv \omega(n)$ specify the level separations of the oscillation spectrum. They are independent of n for harmonic oscillations, where $\omega(n) = \omega_0$ and for the states in the region of a «dense» (semiclassical) oscillation spectrum. In this region, the level separations $\omega(n)$ are small compared with ω_0 and, in addition, are slowly-varying function of n, $\omega(n) \approx \omega \ll \omega_0$.

3. QUANTUM SPECTRUM OF THE PARTICLE NUMBER

The quantization of Hamiltonian (1) produces the particle number spectrum in the Josephson potential $E_J \cos\phi$. As is shown in what follows, any Josephson state can be realized by means of a given initial disbalance of the particle number. It is therefore interesting to obtain the complete spectrum generated by Hamiltonian (1) and to represent it as a function of the initial disbalance.

The Schrödinger equation for Hamiltonian (1) is derived by the quantization rule

$$(\Delta N) \to -i\partial/\partial\phi.$$

As a result, we obtain the Mathieu equation

$$\left[-E_C \frac{d^2}{d\phi^2} - 2E_J \cos\phi_r\right]\Psi = \varepsilon\Psi,$$
$$\varepsilon \equiv H_J - E_0. \quad (16)$$

For $\varepsilon > 2E_J$, this equation has a continuum spectrum. The states of this spectrum correspond to classical states with an unlimited phase change, $-\infty < \phi < +\infty$; they are called the self-trapping states in [9, 10]. In the region $-2E_J < \varepsilon < 2E_J$, Eq. (16) has a discrete spectrum. It corresponds to the finite-motion region of Hamiltonian (1), where the relative phase changes within

$$-\arccos(\varepsilon/2E_J) < \phi < \arccos(\varepsilon/2E_J)$$

for each ε. In the Josephson regime [3] at $E_J \gg E_C$, the number of levels in a well is large and the discrete spectrum is determined by the Bohr–Sommerfeld formula

$$\nu(\varepsilon_\nu) = \oint \frac{d\phi_r}{\pi} \Delta N(\phi_r; \varepsilon_\nu) =$$
$$= \oint \frac{d\phi_r}{\pi} \left[\frac{1}{E_C}(\varepsilon_\nu + 2E_J \cos\phi)\right]^{1/2} =$$
$$= \nu_c \left[E(\kappa) - (1 - \kappa^2)K(\kappa)\right], \quad (17)$$

where

$$\nu_c = \frac{8}{\pi}\left(\frac{E_J}{E_C}\right)^{1/2}, \quad \kappa^2 = \frac{\varepsilon_\nu + 2E_J}{4E_J}, \quad \nu_c \gg 1, \quad (18)$$

and $K(\kappa)$ and $E(\kappa)$ are the complete elliptic integrals of the first and second kind. The density of states $\rho_d(\varepsilon_\nu)$ follows from Eq. (17) and is equal to

$$\rho_d(\varepsilon_\nu) = \frac{1}{2\pi}\frac{d\nu}{d\varepsilon_\nu} = \frac{K(\kappa)}{\pi^2 \omega_m}. \quad (19)$$

The level separations in (17) are given by

$$\omega(\varepsilon_\nu) = \frac{d\varepsilon}{d\nu} = \frac{\pi\omega_m}{2K(\kappa)}, \quad \omega_m = 2(E_C E_J)^{1/2}. \quad (20)$$

At $(E_J/E_C)^{1/2} \gg 1$, we have the relation $\omega_m \ll 2E_J$. The quantity ω_m determines the maximum splitting of the levels in the Josephson well. In what follows, the states with $\nu < \nu_c$ are called the «libration» states.

In the region $\nu \geq \nu_c$, the $\nu(\varepsilon)$ dependence and density of states are determined by

$$
\begin{aligned}
\nu(\varepsilon) &= \frac{4}{\pi^2} \left(\frac{E_J}{E_C} \right)^{1/2} \kappa E(\kappa^{-1}), \\
\rho_c(\varepsilon) &= \frac{1}{2\pi^2} \frac{\kappa^{-1} K(\kappa^{-1})}{\omega_m}.
\end{aligned}
\tag{21}
$$

States (21) with $\nu > \nu_c$ are called the self-trapping states.

Equations (17) and (21) imply that

$$
\begin{aligned}
\varepsilon(\nu) &\approx -2E_J + \omega_m \nu, \quad 1 \ll \nu \ll \nu_c, \\
\varepsilon(\nu) &\approx 4\pi^2 E_C \nu^2, \quad \nu \gg \nu_c.
\end{aligned}
\tag{22}
$$

At the same time, it is easy to show that

$$
\frac{d^2 \varepsilon(\nu)}{d\nu^2} < 0 \quad \text{for} \quad \nu < \nu_c
$$

and

$$
\frac{d^2 \varepsilon(\nu)}{d\nu^2} > 0 \quad \text{for} \quad \nu > \nu_c.
$$

At $\nu = \nu_c$, the curve $\varepsilon(\nu)$ has an inflection point.

Because the energy is conserved, the state with a given value of ν can be realized by defining the initial values of $(\Delta N)_0$ and $\phi(0)$ as

$$
\varepsilon(\nu) = E_C (\Delta N)_0^2 - 2E_J \cos \phi(0).
$$

Supposing that $\phi(0) = 0$, we obtain the following relation between ν and $(\Delta N)_0$:

$$
\varepsilon(\nu) = -2E_J + E_C (\Delta N)_0^2.
\tag{23}
$$

Using Eqs. (22) and (23), we arrive at the expressions

$$
\nu = \frac{1}{2\pi} |(\Delta N)_0|
$$

for the self-trapping (sf) states and

$$
\nu = \left(\frac{E_C}{\omega_m} \right) (\Delta N)_0^2
\tag{24}
$$

for the «libration» (l) states.

Combining (19) and (21), we obtain the dependence

$$
\rho_{d,c}(\varepsilon) \propto \omega_m^{-1} \ln \left| 1 - \frac{\varepsilon}{2E_J} \right|^{-1}, \quad \varepsilon \to 2E_J^{\pm}.
\tag{25}
$$

Therefore, a new logarithmic singularity appears at the boundary separating the libration and self-trapping spectra.

4. INTERACTION OF THE EXCITED JOSEPHSON STATES AND ANHARMONIC CONDENSATE OSCILLATIONS

In this and the next sections, we show that the spectrum of the system can change drastically due to the interaction between excited Josephson states (22)–(24) with sufficiently large values of ν and oscillations of the condensate.

As in Sec. 2, two mechanisms can be proposed for the realization of the interaction. First, the interaction can be produced by a dependence of E_C in Eq. (23) on the atom displacements. The latter are generated by the condensate oscillation. Second, the interaction can be realized by applying a two-photon traveling-wave laser pulse with the Rabi frequency Ω. The pulse creates the condensates with different particle numbers and induces the interaction of atom displacements with the excited Josephson states corresponding to the particle number disbalance created by the pulse. The general description proposed in what follows is independent of the specific mechanism producing the interaction.

The states of motion of the condensate are classical. These states are defined in Sec. 2. For semiclassical Josephson states with $\nu \gg 1$, the c_ν, c_ν^* amplitudes can be written as

$$
c_\nu = \nu^{1/2} e^{i\varphi_2}.
$$

It is convenient to rewrite $\varepsilon(\nu)$ and c_ν in terms of the variable

$$
x = \frac{|(\Delta N)_0|}{N^{1/2}} \gg 1.
$$

Combining this inequality with the requirement that

$$
x = \frac{|(\Delta N)_0|}{N} \ll 1,
$$

we arrive at the conditions for the x values,

$$
1 \ll x \ll N^{1/2}.
\tag{26}
$$

Using Eqs. (22) and (23), we find that

$$
\nu = \nu(x),
$$

$$
\varepsilon(\nu) = N(-E_J/N + E_C x^2).
$$

In the general case, the $\nu(x)$ dependence is implicit. It is determined by Eqs. (17), (21), and (23). But in the particular cases of the «libration» ($\varepsilon(\nu) \ll E_J$) and self-trapping ($\varepsilon(\nu) \gg E_J$) states, the relations between ν and $(\Delta N)_0$ can be represented in a simple form, as can be seen from (24). Using Eqs. (23) and (24), we arrive at the following expressions:

$$
H_0 \equiv N\varepsilon_0(n, x) = N \left[-\frac{E_J}{N} + \epsilon(n) + E_C x^2 \right], \tag{27}
$$

$$c_\nu^{(sf)} = N^{1/4} x^{1/2} e^{i\varphi_2} \equiv N^{1/4} c_x^{(sf)},$$

$$c_\nu^{(l)} = N \left(\frac{E_C}{\omega_m}\right)^{1/2} x e^{i\varphi_2} \equiv N \left(\frac{E_C}{\omega_m}\right)^{1/2} c_x^{(l)}. \quad (28)$$

For any mechanism producing the interaction between two subsystems, the interaction Hamiltonian can be written in the form of a multiple Fourier series in φ_1, φ_2,

$$H_{int} = N \sum_{k_1 k_2} \left[g_{k_1 k_2}(N) a^{*k_1} c_\nu^{k_2} + \text{c.c.} \right] =$$

$$= N \sum_{k_1, k_2} \left[g_{k_1 k_2}^{(sf,l)}(N) n^{k_1/2} x^{\alpha k_2} \times \right.$$

$$\left. \times \exp\left(i(k_1 \varphi_1 - k_2 \varphi_2)\right) + \text{c.c.} \right], \quad (29)$$

where $\alpha_{sf} = 1/2$, $\alpha_l = 1$, and k_1 and k_2 are integers. For simplicity, we disregard the phase-independent interaction. Using Eq. (28), we obtain

$$g_{k_1 k_2}^{(sf)}(N) = g N^{-1+k_2/4},$$

$$g_{k_1 k_2}^{(l)}(N) = g N^{-1+k_2} \left(\frac{E_C}{\omega_m}\right)^{k_2/2}. \quad (30)$$

The constant g is determined by the specific mechanism producing the interaction.

We now assume that the term with the phase

$$\phi_k^r = k_{1r} \varphi_1 - k_{2r} \varphi_2,$$

which varies anomalously slowly with time, can be dropped in sum (29). This can be done under two conditions. The first condition is

$$k_{1r} \left(\frac{d\epsilon(n)}{dn}\right) = k_{2r} \left(\frac{d\varepsilon_J(x)}{dx}\right), \quad (31)$$

or equivalently,

$$x_m = \frac{k_{1r} \varepsilon_n'}{2 k_{2r} E_C}, \quad \epsilon_n' \equiv \left(\frac{d\epsilon(n)}{dn}\right). $$

The second condition is

$$\left(\frac{d^2 H_0}{dx^2}\right)_{x=x_m} (\Delta x)_{max} \gg \left(\frac{dH_{int}}{dx}\right)_{x=x_m}. \quad (32)$$

In writing this equation, we took into account that H_0 and H_{int} are functions of a single dynamic variable, e.g., x. The quantity ϵ_n' defines the level separations of the oscillation spectrum.

As is shown in what follows, condition (31) is equivalent to the condition of the minimum of the function $\varepsilon_0(n, x)$ with respect to x. When the minimum exists, it can provide the leading contribution into the thermodynamic functions.

In addition, condition (31) implies that the phase ϕ_k^r is an approximate integral of motion if the dependence on x near x_m is ignored,

$$\frac{d\phi_k^r}{dt} \approx k_{1r} \frac{\partial H_0}{\partial n} - k_{2r} \frac{\partial H_0}{\partial x} \approx 0.$$

Inequality (32) implies that the width of the near-minimum region is large at the characteristic interaction variation scale. From Eqs. (31) and (32), we can obtain that time changing the ϕ_k^r phase is proportional to $(d^2 H_0/dx^2)_m \Delta x$, where Δx is the variation of x near the x_m value. The maximum value $(\Delta x)_{max}$ specifies the width of the near-minimum region such that

$$\frac{d\phi_k^r}{dt} \sim \Delta x.$$

The estimate for $(\Delta x)_{max}$ is given in what follows. Thus, the leading term in sum (29) is given by

$$H_{int}^{(r)} = N g_k^{(sf,l)}(N; n, x) \cos \phi_k,$$
$$\phi_k = k_{1r} \varphi_1 - k_{2r} \varphi_2. \quad (33)$$

All the remaining terms in this sum are rapidly oscillating perturbations and are disregarded in this work. Here and below, the index k in g_k and ϕ_k denotes the set k_{1r}, k_{2r}.

It can be easily shown that in addition to the energy

$$H = N[\varepsilon_0(n, x) + g_k(N; n, x) \cos \phi_k],$$

the system in question has the integral of motion

$$n_0 = \frac{n}{k_{1r}} + \frac{x}{k_{2r}}, \quad \frac{dn_0}{dt} = 0.$$

Owing to this, condition (31) is equivalent to that of the minimum of $\varepsilon_0(n_0, x)$ at x_m for a given value of n_0, as mentioned above.

Using Eqs. (31)–(33), it is straightforward to write the Hamiltonian

$$H_m = H_0 + H_{int}^{(r)}$$

near the minimum to the first nonvanishing order in Δx,

$$H_m = N \left[\varepsilon_0(n_m; x_m) + \right.$$

$$\left. + \left(\frac{d^2 \varepsilon_0}{dx^2}\right)_m (\Delta x)^2 - g_{km} \cos \phi_k \right], \quad (34)$$

where

$$n_m = n_0 - \frac{k_{1r} x_m}{k_{2r}},$$

$$\left(\frac{d^2\varepsilon_0}{dx^2}\right)_m = 2E_c, \quad g_{km} = g_k^{(sf,l)}(N; n_m, x_m).$$

The terms with the derivatives of H_{int} are absent from Eq. (34) because of condition (32).

Using that

$$E_C \sim \omega_0 (a/a_0)^{2/5} N^{-3/5}$$

in the Thomas–Fermi approximation [20] (where a and a_0 are the scattering and oscillator lengths, respectively), we can represent the range $1 \ll x_m \ll N^{1/2}$ as

$$\frac{1}{N}\left(\frac{Na}{a_0}\right)^{2/5} \ll \frac{k_{1r}}{k_{2r}}\left(\frac{\epsilon'_n}{\omega_0}\right) \ll \frac{1}{N^{1/2}}\left(\frac{Na}{a_0}\right)^{2/5}. \quad (35)$$

As is known [20], the relation $(Na/a_0) \gg 1$ is valid in the Thomas–Fermi approximation. But $N^{-1/2}(Na/a_0)^{2/5} \ll 1$, and therefore, condition (35) (or equivalently, condition (31)) specifies the region of the dense oscillation spectrum, where $\epsilon'_n \ll \omega_0$. Here and in what follows, we suppose that $k_{1r} = k_{2r} = 1$ for simplicity.

From Eq. (34), the value of $(\Delta x)_{max}$ can be estimated as

$$(\Delta x)_{max} \sim \left(\frac{g_{km}}{E_C}\right)^{1/2}.$$

Hence, condition (32) becomes

$$\frac{|x_m - n_m|}{x_m n_m} \ll \left(\frac{E_C}{g_{km}(N)}\right)^{1/2}. \quad (36)$$

In what follows (see Eq. (49)) we show that the relation (36) is satisfied with macroscopic accuracy.

5. THE GROUND STATE

At a fixed value of n_0, the leading contribution to the partition function comes from the neighborhood of the minimum at $x = x_m$. The expression for $Z(n_0; x_m; T)$ is

$$Z(n_0; x_m; T) = \text{const} \int_{-\infty}^{\infty} d\Delta x \int_{-\pi}^{\pi} d\phi_k \times$$

$$\times \exp\left(-\beta H_m(n_0, \Delta x, \phi_k)\right) = \frac{\text{const}}{(\beta N E_C)^{1/2}} \times$$

$$\times \exp\left[-\beta N \varepsilon_0(N; n_m; x_m) + \ln I_0(\beta N g_{km})\right], \quad (37)$$

where $I_0(x)$ is the modified Bessel function. Equation (37) implies that the free energy of the system is given by

$$F = N\varepsilon_0(n_m; x_m) + \frac{1}{2}T \ln(\beta N E_C) - $$
$$- T \ln I_0(\beta N g_{km}), \quad (38)$$

Using Eq. (38), we obtain the equation for the $\bar{n}_m$ value realizing the minimum of the free energy,

$$\left(\frac{d\varepsilon_{0m}}{dn_m}\right)_{n_m=\bar{n}_m} = \left(\frac{dg_{km}}{dn_m}\right)_{n_m=\bar{n}_m} \frac{I_1(\beta N g_{km})}{I_0(\beta N g_{km})}, \quad (39)$$

where $I_1(x) = I'_0(x)$. In addition to $\bar{n}_m$, the thermodynamic average of $\cos\phi_k$ can be determined from Eqs. (37) or (38). This average is equal to

$$\langle\cos\phi_k\rangle_T = -\frac{\partial \ln Z}{\partial(\beta N g_{km})} =$$

$$= \frac{\partial F}{T\partial(\beta N g_{km})} = \frac{I_1(\beta N g_{km})}{I_0(\beta N g_{km})}. \quad (40)$$

The order parameters $\bar{n}_m$ and $\langle\cos\phi_k\rangle_T$ describe the new coherent state. There is a bound state of the $\bar{n}_m$ oscillation quanta and the Josephson state generated by the initial disbalance of the particle number that corresponds to the x_m value. In addition, this state has the equilibrium phase coherence factor $\langle\cos\phi_k\rangle_T$. The $\bar{n}_m \neq 0$ value provides the equilibrium distortion of the condensate shape. The above equations imply that the shape distortion is self-consistently coupled to the x_m value defining the equilibrium initial disbalance of the particle number.

At $T = 0$, the $\bar{n}_m$ value realizes the minimum of the thermodynamic energy

$$E_m = N[\varepsilon_0(N; n_m, x_m) - g_{km}(N; n_m, x_m)]. \quad (41)$$

To determine $\bar{n}_m(T = 0)$, it is suitable to use the following consideration. It is well known that the level separations ϵ'_n are slowly varying functions of n within the dense (semiclassical) spectrum. We can therefore suppose that $\epsilon'_n \approx \text{const} \equiv \omega_b \ll \omega_0$. Under this assumption, the x_m value is independent of n_m and $\bar{n}_m$ is equal to

$$\bar{n}_m^{1/2} = \frac{g^{(sf,l)}(N)x_m^{\alpha}}{2\omega_b}, \quad \alpha_{sf,l} = \frac{1}{2}, 1. \quad (42)$$

(It is worth noting that $(\partial^2 E_m/\partial n_m^2) > 0$.) Here and in what follows, the notation $g_{k=1}^{(sf,l)}(N) \equiv g^{(sf,l)}(N)$ is used. Taking Eq. (42) into account, we readily obtain

$$\bar{g}_{1m}^{(sf)} = \frac{g^{(sf)2}(N)x_m}{2\omega_b}, \quad \bar{g}_{1m}^{(l)} = \frac{g^{(l)2}(N)x_m^2}{2\omega_b},$$

$$\frac{E_m^{(sf)}}{N} = -\frac{E_J}{N} - \frac{g^{(sf)2}(N)}{8E_C}\left(1 - \frac{8E_C^2 x_m^2}{g^{(sf)2}(N)}\right),$$
$$\frac{E_m^{(l)}}{N} = -\frac{E_J}{N} - \frac{g^{(l)2}(N)x_m}{16E_C}\left(1 - \frac{16E_C^2 x_m}{g^{(l)2}(N)}\right). \quad (43)$$

The expressions for the energies imply, first, that we obtain the minimum in the region of a sufficiently dense oscillation spectrum that satisfies condition (35). The minimum corresponds to the formation of a bound state for the $\bar{n}_m, x_m$ values. Second, as can be seen from Eq. (43), the absolute minimum of E_m can be realized within the ranges

$$1 \ll x_m < \frac{g^{(sf)}(N)}{8E_C}, \quad 1 \ll x_m < \frac{g^{(l)2}(N)}{16E_C^2}. \quad (44)$$

These conditions are satisfied when the interaction matrix elements $g^{(sf,l)}(N)$ are sufficiently large. We estimate the condensate parameters that are required for the existence of the absolute minimum. In the Thomas–Fermi approximation, the inequality $g^{(sf)}(N) \gg E_C$ is valid if the total particle number is not very large, namely,

$$N \ll \frac{g}{\omega_0}\left(\frac{a_0}{a}\right)^{8/3}. \quad (45)$$

In turn, the relation $g^{(l)2}(N) \gg E_C^2$ is valid within the range

$$N^{0.1}\frac{g^2}{\Omega^{1/2}\omega_0^{3/2}}\left(\frac{a_0}{a}\right)^{0.6} \gg 1, \quad (46)$$

where we use that $E_J = \Omega N$. Condition (46) is satisfied for all admissible parameters if

$$\frac{g^2}{\Omega^{1/2}\omega_0^{3/2}} \sim 1.$$

We emphasize that the right-hand side inequalities in Eq. (44) are much stronger than the condition $x_m \ll N^{1/2}$.

The transition to the state with $\bar{n}_m \neq 0$ and $\langle\cos\phi_k\rangle_T \neq 0$ is second-order. Requiring that $\bar{n}_m \to 0$ at the transition temperature, we obtain from Eq. (39) that

$$T_c^{(sf,l)} = \frac{g^{(sf,l)2}(N)x_m^\alpha}{\omega_b}N. \quad (47)$$

The dependences of transition temperatures (47) on the total particle number are given by

$$T_c^{(sf)} \sim N^{-0.2}, \quad T_c^{(l)} \sim N^{0.8}. \quad (48)$$

The transition temperature $T_c^{(sf)}$ therefore has the macroscopic smallness in comparison with the $T_c^{(l)}$ temperature. Along with conditions (45) and (46), this fact implies that the libration Josephson state forms a bound state with the condensate oscillation rather than a self-trapping state.

6. CONCLUDING REMARKS

We have found that the interaction between the Josephson and oscillation states results in a new coherent ground state of the double-well trapped condensate. There is a bound state of the $\bar{n}_m$ oscillation quanta and the Josephson states. The latter are either two stationary states forming the macroscopic two-level subsystem or the excited Josephson states generated by a definite initial disbalance of the particle number. Both the x_m value defining the disbalance and the tunneling splitting $N\varepsilon$ of the stationary Josephson states are self-consistently coupled to the number of the oscillation quanta entering the bound state.

We emphasize that the bound states arise near the extremum points of the «unperturbed» spectrum of the system. For a two-level Josephson subsystem, there is a resonance between the «initial» tunneling splitting ε and the oscillation frequency.

For the excited Josephson states in Secs. 3 and 4, a new coherent state is formed in the neighbourhood of the minimum of the energy $\varepsilon_0(n_0; x)$ if conditions (31), (32) are satisfied.

The resonance and the minimum conditions specify the type of the interaction between the Josephson and oscillation states.

In addition, minimum condition (31) imposes essential restrictions on the spectrum of the oscillation states that can effectively interact with the Josephson degrees of freedom. This condition implies that the excited Josephson states can interact only with the oscillation states having a sufficiently large density of states (such that inequalities (35) are satisfied). For instance, in an asymmetric double-well potential, the energy $\varepsilon(n)$ of its classical oscillation states has three branches. Two branches have energies $\varepsilon_{1,2}(n) \leq V_b$, where V_b is the barrier height. There is a maximum at $\varepsilon_{1,2}(n_{max}) = V_b$. The third branch has the energy $\varepsilon_3(n) \geq V_b$ and the minimum at the energy $\varepsilon_3(n_{min}) = V_b$. Therefore, the region of the dense oscillation spectrum exists in the neighbourhood of the barrier top.

A similar situation can occur for the stationary Josephson states in Sec. 2. In the Thomas–Fermi approximation, resonance condition (4) can be satisfied within the range of the dense spectrum close to the barrier top.

We emphasize that in both cases, the bound states of the highly excited oscillation state and Josephson degrees of freedom are formed. These states can realize the absolute minimum of the thermodynamic energy, as shown in Eqs. (13) and (43).

As shown in Sec. 4, condition (32) is realized within range (36). Substituting the expressions for $\bar{n}_m$ and $\bar{g}_{1m}$ obtained above in (36), we find that this equation becomes

$$|x_m - n_m|^{(sf,l)} \ll \left(\frac{g_{1m}^{(sf,l)}(N)}{E_C} \right)^{1/2}, \qquad (49)$$

and is therefore satisfied with macroscopic accuracy.

The bound states exist at a sufficiently strong interaction determined by conditions (12) and (44). These conditions are experimentally controlled by means of either the interaction matrix elements G, g or the particle number N (see Eqs. (12), (45), and (46)).

For the stationary Josephson states in Sec. 2, the mechanism proposed above provides an essential increase of the tunneling splitting that turns out to depend on temperature. It is defined by (11). For the excited Josephson states interacting with the anharmonic oscillation, two order parameters describe the new ground state. These are the number $\bar{n}_m$ of the oscillation quanta entering the bound state and the phase coherence factor in Eq. (40).

The second-order phase transitions to states with $\bar{n}_m \neq 0$ can be observed at $T = T_c$, where the transition temperatures T_c are defined by Eqs. (14), (47), and (48).

We finally note that the appearance of bound states generates an equilibrium distortion of the condensate shape specified by the $\bar{n}_m$ values in Eqs. (11) and (42). This mechanism can provide an experimental detection of the Josephson states. The latter can be observed by changing the condensate shape.

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