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POLARITON EFFECT IN NONLINEAR PULSE PROPAGATION

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The joint influence of the polariton effect and Kerr-like nonlinearity on the propagation of optical pulses is studied. The existence of different families of envelope solitary wave solutions in the vicinity of the polariton gap is shown. The properties of solutions depend strongly on the carrier wave frequency. In particular, solitary waves inside and outside the polariton gap exhibit different velocity and amplitude dependences on their duration.

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1. INTRODUCTION

In recent years, a fast progress in the fabrication of microcavities, organic and inorganic quantum wells, etc. has resulted in a great interest in the investigation of electromagnetic properties of these new objects, including propagation of nonlinear pulses in such structures (see, e.g., [1–4]). For example, the propagation of nonlinear pulses along a quantum well imbedded in a microcavity was studied in [4] for two types of nonlinearities, a Kerr-like nonlinearity applied to envelopes of sufficiently long pulses and a self-induced transparency nonlinearity applied to short and intense pulses with the frequency close to the two-level resonance. In this study, the authors have chosen such propagation conditions that the polariton effect of the formation of the gap can be neglected in the dispersion law of an electromagnetic wave coupled to the polarization wave in the medium. The problem of the pulse propagation can then be reduced to either the nonlinear Schrödinger (NLS) equation or the sine-Gordon equation with the well-known soliton solutions. But the region of frequencies in the vicinity of the polariton gap is very important because some properties of the structures under consideration manifest themselves in this region only.

We note that related problems were already studied a long time ago in the theory a nonlinear pulse propagation through a medium in the vicinity of exciton resonances. In [5], the polariton self-induced transparency pulses were found, but it was claimed later [6, 7] that the polariton effect prevents the existence of self-induced transparency pulses. This contradiction was resolved in Refs. [8], where it was shown that polariton solitons exist due to a subtle balance of small effects, and these solutions may therefore easily be overlooked if a too rude approximation is made in the evolution equations. The authors of [9, 10] later confirmed this result in general and described some additional remarkable properties of polariton self-induced transparency pulses in the vicinity of the polariton gap beyond the perturbation theory. Similar problems have also been studied for the Kerr nonlinearity (see, e.g., Ref. [11] and references therein). But the approximations used were not justified well enough, and some properties of polariton solitons remained unclear. In [12], the governing equations for long pulses and the carrier wave frequency sufficiently far from the polariton gap were reduced to the perturbed NLS equation for the envelope and the corresponding soliton solutions were described. A closely related problem, pulse propagation in a Kerr-nonlinear medium with a singular dispersion relation was studied in [13], where bright and dark solitary wave solutions were found in the vicinity of the linear resonance.

In this paper, we thoroughly investigate polariton solitons in the case of the Kerr nonlinearity following

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the method developed by Akimoto and Ikeda [8] and show the existence of localized solutions both inside and outside the polariton gap.

2. MAIN EQUATIONS

We start with the standard equations of the classical theory of electromagnetic waves propagating through an isotropic medium (see, e.g., [14]),

$$\frac{\partial^2 \mathbf{E}}{\partial x^2} - \frac{\epsilon_0}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial^2 \mathbf{P}}{\partial t^2}, \tag{1}$$

$$\frac{\partial^2 \mathbf{P}}{\partial t^2} + \omega_T^2 \mathbf{P} + \chi \left|\mathbf{P}\right|^2 \mathbf{P} = \alpha \mathbf{E}, \tag{2}$$

where ϵ_0 is the background dielectric constant and

$$\alpha = \frac{\epsilon_0(\omega_L^2 - \omega_T^2)}{4\pi}. \tag{3}$$

These equations describe interaction of the electromagnetic field $\mathbf{E}$ and the polarization wave $\mathbf{P}$ due to the Kerr-like nonlinearity (measured by the parameter χ). We here ignore the effect of damping. The parameters ω_T^2 and ω_L^2 characterize the dispersion law $\omega = \omega(k)$ of linear waves, which is given by the equation

$$F(k,\omega) \equiv \omega_T^2 - \omega^2 - \frac{4\pi}{c^2}\frac{\alpha\omega^2}{k^2 - \epsilon_0\omega^2/c^2} = 0. \tag{4}$$

As follows from (4), the dispersion law has a gap in the frequency interval

$$\omega_T < \omega < \omega_L, \tag{5}$$

where linear waves cannot propagate. As mentioned above, the envelope function can be introduced for frequencies sufficiently far from polariton gap (5), and system (1), (2) can be reduced to the NLS equation possessing well-known soliton solutions. Here, we are interested in solutions of system (1), (2) for frequencies near and inside polariton gap (5).

We seek the solutions in the form of stationary linearly polarized waves, such that $\mathbf{E}$ and $\mathbf{P}$ can be considered as the scalar functions

$$\begin{aligned} &E(x,t) = \mathcal{E}(t - x/V)e^{i\theta}, \\ &P(x,t) = \left[u(t-x/V) - iv(t-x/V)\right]e^{i\theta}, \end{aligned} \tag{6}$$

where V is the velocity of the pulse and the phase $\theta(x,t)$ is

$$\theta(x,t) = kx - \omega t - \phi(t - x/V). \tag{7}$$

Substitution of Eqs. (6) and (7) in Eqs. (1) and (2) leads to the system of equations for the variables u, v, $\mathcal{E}$, ϕ,

$$\begin{aligned} \ddot{u} - \left(\omega^2 - \omega_T^2 + 2\omega\dot{\phi} + \dot{\phi}^2\right)u - 2(\omega + \dot{\phi})\dot{v} - \\ - \ddot{\phi}v + \chi(u^2+v^2)u = \alpha\mathcal{E}, \end{aligned} \tag{8}$$

$$\begin{aligned} \ddot{v} - \left(\omega^2 - \omega_T^2 + 2\omega\dot{\phi} + \dot{\phi}^2\right)v + \\ + 2(\omega + \dot{\phi})\dot{u} + \ddot{\phi}u + \chi(u^2+v^2)v = 0, \end{aligned} \tag{9}$$

$$\begin{aligned} \left(\frac{1}{V^2} - \frac{\epsilon_0}{c^2}\right)\ddot{\mathcal{E}} - \left[\left(k^2 - \frac{\epsilon_0}{c^2}\omega^2\right) + 2\left(\frac{k}{V} - \frac{\epsilon_0\omega}{c^2}\right)\dot{\phi} + \right. \\ \left. + \left(\frac{1}{V^2} - \frac{\epsilon_0}{c^2}\right)\dot{\phi}^2\right]\mathcal{E} = \\ = \frac{4\pi}{c^2}\left[\ddot{u} - u(\omega + \dot{\phi})^2 - \ddot{\phi}v - 2(\omega + \dot{\phi})\dot{v}\right], \end{aligned} \tag{10}$$

$$\begin{aligned} \left(\frac{1}{V^2} - \frac{\epsilon_0}{c^2}\right)\ddot{\phi}\mathcal{E} + 2\left[\frac{k}{V} - \frac{\epsilon_0\omega}{c^2} + \left(\frac{1}{V^2} - \frac{\epsilon_0}{c^2}\right)\dot{\phi}\right]\dot{\mathcal{E}} = \\ = \frac{4\pi}{c^2}\left[\ddot{v} - v(\omega + \dot{\phi})^2 + \ddot{\phi}u + 2(\omega + \dot{\phi})\dot{u}\right], \end{aligned} \tag{11}$$

where the overdot denotes the derivative with respect to $\xi = t - x/V$.

3. LINEAR APPROXIMATION

We suppose that the variables u, v, $\mathcal{E}$, and ϕ tend to zero at the tails of the pulse (i.e., at infinite $|\xi|$), and system (8)–(11) can therefore be linearized in these regions as

$$\ddot{u} - \left(\omega^2 - \omega_T^2\right)u - 2\omega\dot{v} = \alpha\mathcal{E}, \tag{12}$$

$$\ddot{v} - \left(\omega^2 - \omega_T^2\right)v + 2\omega\dot{u} = 0, \tag{13}$$

$$\begin{aligned} \left(\frac{1}{V^2} - \frac{\epsilon_0}{c^2}\right)\ddot{\mathcal{E}} - \left(k^2 - \frac{\epsilon_0}{c^2}\omega^2\right)\mathcal{E} = \\ = \frac{4\pi}{c^2}\left(\ddot{u} - \omega^2 u - 2\omega\dot{v}\right), \end{aligned} \tag{14}$$

$$2\left(\frac{k}{V} - \frac{\epsilon_0\omega}{c^2}\right)\dot{\mathcal{E}} = \frac{4\pi}{c^2}\left(\ddot{v} - \omega^2 v + 2\omega\dot{u}\right). \tag{15}$$

For the exponential dependence

$$(\mathcal{E}, u, v) = (\mathcal{E}_0, u_0, v_0)\exp(-|\xi|/\tau) \quad \text{at } |\xi| \to \infty, \tag{16}$$

where τ is the duration of the pulse, system (12)–(15) reduces to algebraic equations that define the «dispersion law» and the velocity of the pulse as functions of τ. It is convenient to introduce the variables

$$X = \frac{ck}{\omega}, \quad Y = \frac{c}{V}, \tag{17}$$

and define the characteristic parameters

$$s = \frac{1}{\omega\tau}, \quad \Lambda^2 = \frac{1}{(\omega_L^2 - \omega_T^2)\tau^2}, \tag{18}$$

such that the ratio

$$\frac{\Lambda^2}{s^2} = \frac{\omega^2}{\omega_L^2 - \omega_T^2} \equiv \Omega^2 \tag{19}$$

is independent of τ. The variable

$$\Delta^2 = \frac{\omega^2 - \omega_T^2}{\omega_L^2 - \omega_T^2} \tag{20}$$

measures the frequency in the vicinity of the polariton gap. The cases where $\Delta^2 > 1$ and $\Delta^2 < 0$ correspond to the upper and lower polariton branches respectively. Substituting Eq. (16) in Eqs. (12)–(15), we arrive at the system

$$\begin{aligned}
&\left(\Lambda^2 - \Delta^2\right) u_0 - 2s\Omega^2 v_0 = \frac{\epsilon_0}{4\pi}\mathcal{E}_0,\\
&\left(\Lambda^2 - \Delta^2\right) v_0 + 2s\Omega^2 u_0 = 0,\\
&\left(X^2 - s^2Y^2\right)\mathcal{E}_0 = \epsilon_0(1-s^2)\mathcal{E}_0 + \\
&\qquad + 4\pi[(1-s^2)u_0 + 2sv_0],\\
&2s\left(XY - \epsilon_0\right)\mathcal{E}_0 = 4\pi[(1-s^2)v_0 - 2su_0].
\end{aligned} \tag{21}$$

Eliminating u_0 and v_0 from Eqs. (21) gives the system

$$\begin{aligned}
&X^2 - s^2Y^2 = \\
&= \epsilon_0\left[1 - s^2 + \frac{(1-s^2)(\Lambda^2-\Delta^2) - 4s^2\Omega^2}{(\Lambda^2-\Delta^2)^2 + 4s^2\Omega^4}\right],\\
&XY = \epsilon_0\left[1 + \frac{\Lambda^2 - \Delta^2 + (1-s^2)\Omega^2}{(\Lambda^2-\Delta^2)^2 + 4s^2\Omega^4}\right],
\end{aligned} \tag{22}$$

solving which with Eq. (19) taken into account yields

$$\begin{aligned}
&X^2 = \left(\frac{ck}{\omega}\right)^2 = \\
&= \frac{\epsilon_0}{2}\left[(1-s^2) - \frac{(1-s^2)(\Delta^2-\Lambda^2) + 4\Lambda^2}{(\Delta^2-\Lambda^2)^2 + 4\Lambda^2\Omega^2} + \right.\\
&\qquad \left. + (1+s^2)\sqrt{\frac{(\Delta^2-\Lambda^2-1)^2 + 4\Lambda^2\Omega^2}{(\Delta^2-\Lambda^2)^2 + 4\Lambda^2\Omega^2}}\right],
\end{aligned} \tag{23}$$

$$\begin{aligned}
&Y^2 = \left(\frac{c}{V}\right)^2 = \\
&= \frac{\epsilon_0}{2s^2}\left[-(1-s^2) + \frac{(1-s^2)(\Delta^2-\Lambda^2) + 4\Lambda^2}{(\Delta^2-\Lambda^2)^2 + 4\Lambda^2\Omega^2} + \right.\\
&\qquad \left. + (1+s^2)\sqrt{\frac{(\Delta^2-\Lambda^2-1)^2 + 4\Lambda^2\Omega^2}{(\Delta^2-\Lambda^2)^2 + 4\Lambda^2\Omega^2}}\right].
\end{aligned} \tag{24}$$

In the limit of a uniform wave ($\tau \to \infty$), when $\Lambda^2 \to 0$ and $s^2 \to 0$, Eq. (23) reproduces the dispersion law of linear plane waves,

$$\left(\frac{ck}{\omega}\right)^2 = \epsilon_0\left(1 - \frac{1}{\Delta^2}\right) \geq 0, \tag{25}$$

which can be transformed into the standard form after substitution of Eq. (20). To find the velocity of the envelope of a linear wave, we must take the limit as $s^2 \to 0$ in Eq. (24), which gives

$$\left(\frac{c}{V}\right)^2 = \frac{\epsilon_0}{\Delta^6(\Delta^2-1)}\left[\Delta^2(\Delta^2-1) + \Omega^2\right]^2. \tag{26}$$

As could be expected, this velocity V coincides with the group velocity of propagating linear plane waves with dispersion law (4).

In Eqs. (23) and (24), the three parameters Δ^2, s^2, and Ω^2 depend on the frequency ω. For further investigation, it is convenient to express s^2 and Ω^2 in terms of Δ^2,

$$\Omega^2 = \Delta^2 + \kappa^2, \quad s^2 = \frac{\Lambda^2}{\Delta^2+\kappa^2}, \quad \kappa^2 = \frac{\omega_T^2}{\omega_L^2 - \omega_T^2}. \tag{27}$$

Equations (23) and (24) then become

$$\begin{aligned}
&\left(\frac{ck}{\omega}\right)^2 = \frac{1}{2}\left[1 - \frac{\Lambda^2}{\Delta^2+\kappa^2} - \right.\\
&\quad - \frac{\left(1 - \Lambda^2/(\Delta^2+\kappa^2)\right)(\Delta^2-\Lambda^2) + 4\Lambda^2}{(\Delta^2-\Lambda^2)^2 + 4\Lambda^2(\Delta^2+\kappa^2)} + \\
&\qquad + \left(1 + \frac{\Lambda^2}{\Delta^2+\kappa^2}\right)\times\\
&\qquad \left.\times\sqrt{\frac{(\Delta^2-\Lambda^2-1)^2 + 4\Lambda^2(\Delta^2+\kappa^2)}{(\Delta^2-\Lambda^2)^2 + 4\Lambda^2(\Delta^2+\kappa^2)}}\right],
\end{aligned} \tag{28}$$

$$\begin{aligned}
&\left(\frac{c}{V}\right)^2 = \frac{\Delta^2+\kappa^2}{2\Lambda^2}\left[-1 + \frac{\Lambda^2}{\Delta^2+\kappa^2} + \right.\\
&\quad + \frac{\left(1 - \Lambda^2/(\Delta^2+\kappa^2)\right)(\Delta^2-\Lambda^2) + 4\Lambda^2}{(\Delta^2-\Lambda^2)^2 + 4\Lambda^2(\Delta^2+\kappa^2)} + \\
&\qquad + \left(1 + \frac{\Lambda^2}{\Delta^2+\kappa^2}\right)\times\\
&\qquad \left.\times\sqrt{\frac{(\Delta^2-\Lambda^2-1)^2 + 4\Lambda^2(\Delta^2+\kappa^2)}{(\Delta^2-\Lambda^2)^2 + 4\Lambda^2(\Delta^2+\kappa^2)}}\right],
\end{aligned} \tag{29}$$

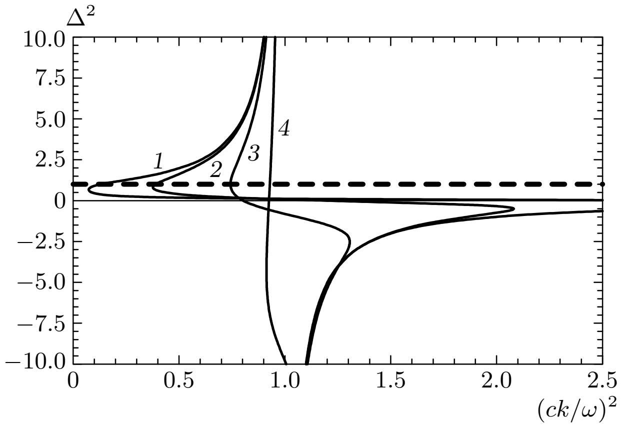

Fig. 1. Dispersion relation of the carrier wave for different values of the pulse duration τ measured by the parameter Λ (see Eq. (18)); $\Lambda^2 = 0.01$ (*1*), 0.1 (*2*), 1 (*3*), 10 (*4*)

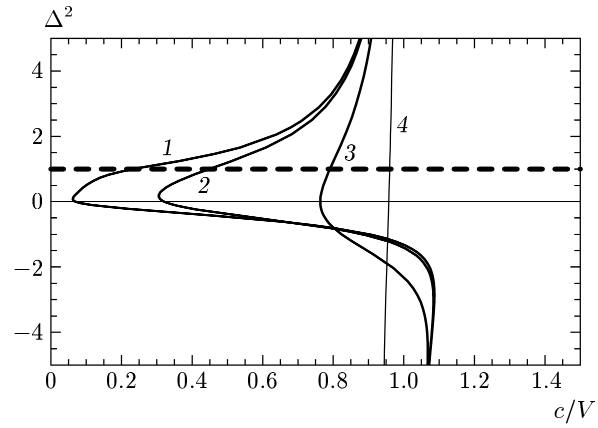

Fig. 2. Pulse velocity as a function of the carrier wave frequency for different values of the pulse duration τ measured by the parameter Λ (see Eq. (18)) $\Lambda^2 = 0.01$ (*1*), 0.1 (*2*), 1 (*3*), 10 (*4*)

where we have also put $\epsilon_0 = 1$, which is equivalent to the replacement $c \to c/\sqrt{\epsilon_0}$. We recall that in Eqs. (28) and (29), κ^2 is a constant determined by the system under consideration, the parameter Δ^2 measures the wave frequency, and the parameter Λ^2 measures the pulse duration.

Linear uniform waves cannot propagate with frequencies within polariton gap (5), or

$$0 < \Delta^2 < 1. \tag{30}$$

But at finite values of τ, two branches of the dispersion curve join into one curve. Plots of Δ^2 against $(ck/\omega)^2$ at several values of Λ^2 are shown in Fig. 1. As we can see, these curves depend essentially on the values of Λ^2, and the usual approach involving the transition to the NLS equation for the envelope function can only be applied at $\Lambda \ll 1$ and sufficiently far from the polariton gap. The velocity parameter $(c/V)^2$ of the pulse as a function of Δ^2 is shown in Fig. 2 at several values of Λ^2. It has real values even at frequencies inside polariton gap (30). It is important to note that if general nonlinear equations have a pulse solution of form (6)–(7), then its velocity must coincide with its «linear approximation» (29) calculated for the tails of the pulse.

4. SOLITON SOLUTIONS

To find soliton solutions, we return to exact equations (8)–(11) and replace differentiation with respect to $\xi = t - x/V$ by differentiation with respect to $\zeta = \xi/\tau$. Taking Eqs. (17)–(20) into account, we arrive at the system

$$\begin{aligned}\Lambda^2\ddot{u} - (\Delta^2 + 2\Lambda\sqrt{\Delta^2+\kappa^2}\dot{\phi} + \Lambda^2\dot{\phi}^2)u - \\ - 2(\Lambda\sqrt{\Delta^2+\kappa^2} + \Lambda^2\dot{\phi})\dot{v} - \\ - \Lambda^2\ddot{\phi}v + \tilde{\chi}(u^2+v^2)u = \frac{1}{4\pi}\mathcal{E},\end{aligned} \tag{31}$$

$$\begin{aligned}\Lambda^2\ddot{v} - (\Delta^2 + 2\Lambda\sqrt{\Delta^2+\kappa^2}\dot{\phi} + \Lambda^2\dot{\phi}^2)v + \\ + 2(\Lambda\sqrt{\Delta^2+\kappa^2} + \Lambda^2\dot{\phi})\dot{u} + \\ + \Lambda^2\ddot{\phi}u + \tilde{\chi}(u^2+v^2)v = 0,\end{aligned} \tag{32}$$

$$\begin{aligned}\Lambda^2\left[\left(\frac{c}{V}\right)^2 - 1\right]\ddot{\mathcal{E}} - \left\{(\Delta^2+\kappa^2)\left[\left(\frac{ck}{\omega}\right)^2 - 1\right] + \right. \\ \left. + 2\Lambda\sqrt{\Delta^2+\kappa^2}\left[\frac{c^2k}{V\omega} - 1\right]\dot{\phi} + \Lambda^2\left[\left(\frac{c}{V}\right)^2 - 1\right]\dot{\phi}^2\right\}\mathcal{E} = \\ = 4\pi\Big[\Lambda^2\ddot{u} - u(\sqrt{\Delta^2+\kappa^2} + \Lambda\dot{\phi})^2 - \\ - \Lambda^2 v\ddot{\phi} - 2(\Lambda\sqrt{\Delta^2+\kappa^2} + \Lambda^2\dot{\phi})\dot{v}\Big],\end{aligned} \tag{33}$$

$$\begin{aligned}\Lambda^2\left[\left(\frac{c}{V}\right)^2 - 1\right]\ddot{\phi}\mathcal{E} + 2\Big\{\Lambda\sqrt{\Delta^2+\kappa^2}\left(\frac{c^2k}{V\omega} - 1\right) + \\ + \Lambda^2\left[\left(\frac{c}{V}\right)^2 - 1\right]\dot{\phi}\Big\}\dot{\mathcal{E}} = \\ = 4\pi\Big[\Lambda^2\ddot{v} - v(\sqrt{\Delta^2+\kappa^2} + \Lambda\dot{\phi})^2 + \\ + \Lambda^2 u\ddot{\phi} + 2(\Lambda\sqrt{\Delta^2+\kappa^2} + \Lambda^2\dot{\phi})\dot{u}\Big],\end{aligned} \tag{34}$$

where

$$\tilde{\chi} = \frac{\chi}{\omega_L^2 - \omega_T^2}, \tag{35}$$

and the overdot now denotes the derivative with respect to $\zeta = \xi/\tau$.

We consider long pulses with $\Lambda^2 \ll \Delta^2$. For this, we introduce a small parameter ε by

$$\Lambda^2 = \varepsilon^2\Delta^2, \quad \varepsilon \ll 1, \tag{36}$$

such that Eqs. (31)–(34) can be expanded in powers of ε. Because Eqs. (28) and (29) lead to different series expansions in different intervals of Δ^2, we must consider all these cases separately.

4.1. Long pulse above the polariton gap

We begin with the case of waves with

$$\Delta^2 > 1, \tag{37}$$

for which a nonlinear pulse can be represented as the envelope of propagating linear waves with dispersion law (25). For long pulses with

$$\varepsilon^2 \ll 1, \quad \varepsilon^2 \ll \Delta^2 - 1, \tag{38}$$

the coefficients in Eqs. (33) and (34) can be represented as power series expansions in ε,

$$\left(\frac{ck}{\omega}\right)^2 - 1 = -\frac{1}{\Delta^2} + \frac{\Delta^4 - 3\kappa^2 + 4\Delta^2\kappa^2}{\Delta^4(\Delta^2-1)}\varepsilon^2 + \ldots = -\frac{1}{\Delta^2} + \alpha_2\varepsilon^2 + \ldots, \tag{39}$$

$$\Lambda^2\left[\left(\frac{c}{V}\right)^2 - 1\right] = \frac{\Delta^6 + 2\Delta^4\kappa^2 + \kappa^4}{\Delta^4(\Delta^2-1)}\varepsilon^2 + \ldots = \gamma_2\varepsilon^2 + \ldots, \tag{40}$$

$$2\Lambda\left(\frac{c^2k}{V\omega} - 1\right) = \frac{2\kappa^2}{\Delta^3}\varepsilon - \frac{4\kappa^2(\Delta^2+2\kappa^2)}{\Delta^7}\varepsilon^3 + \ldots = \beta_1\varepsilon - \beta_3\varepsilon^3 + \ldots \tag{41}$$

We suppose that the functions $\mathcal{E}$, $\dot{\phi}$, u, and v can also be represented in the form of series expansions,

$$\mathcal{E} = \varepsilon^\nu \sum_{n=0}^{\infty} \varepsilon^n \mathcal{E}_n$$

and similarly for $\dot{\phi}$, u, and v. Analysis of equations (31)–(34) and (39)–(41) shows that for self-consistency of the procedure, the series expansions of the fields must be as follows:

$$\begin{aligned} \mathcal{E} &= \varepsilon(\mathcal{E}_0 + \varepsilon^2\mathcal{E}_2 + \ldots), \\ \dot{\phi} &= \varepsilon(\theta_0 + \varepsilon^2\theta_2 + \ldots), \\ u &= \varepsilon(u_0 + \varepsilon^2 u_2 + \ldots), \\ v &= \varepsilon(\varepsilon v_1 + \varepsilon^3 v_3 + \ldots). \end{aligned} \tag{42}$$

Substitution of these expansions in Eqs. (31)–(34) then yields a sequence of equations for the coefficients $\mathcal{E}_0, \mathcal{E}_2, \theta_0, \ldots$ In the first approximation, we obtain the relations

$$u_0 = -\frac{1}{4\pi\Delta^2}\mathcal{E}_0, \quad v_1 = \frac{2\sqrt{\Delta^2+\kappa^2}}{\Delta}\dot{u}_0, \tag{43}$$

which correspond to the plane wave solution with a constant amplitude. In the next approximation, it follows from Eqs. (31) and (33) that

$$\begin{aligned} &-\ddot{\mathcal{E}}_0 + 4\frac{\sqrt{\Delta^2+\kappa^2}}{\Delta^2}\ddot{\mathcal{E}}_0 - \frac{\tilde{\chi}}{16\pi^2\Delta^6}\mathcal{E}_0^3 = \\ &= \mathcal{E}_2 + 4\pi\Delta^2 u_2 - \frac{2\sqrt{\Delta^2+\kappa}}{\Delta}\theta_0\mathcal{E}_0, \\ &\left(\gamma_2 + 1 - 4\frac{\sqrt{\Delta^2+\kappa^2}}{\Delta^2}\right)\ddot{\mathcal{E}}_0 - (\Delta^2+\kappa^2)\alpha_2\mathcal{E}_0 = \\ &= -\frac{\Delta^2+\kappa^2}{\Delta^2}\left(\mathcal{E}_2 + 4\pi\Delta^2 u_2 - \frac{2\sqrt{\Delta^2+\kappa}}{\Delta}\theta_0\mathcal{E}_0\right). \end{aligned} \tag{44}$$

Combination of these two equations yields the equation for $\mathcal{E}_0$ after simple transformations,

$$\ddot{\mathcal{E}}_0 = \mathcal{E}_0 + \frac{\tilde{\chi}}{16\pi^2\Delta^8\alpha_2}\mathcal{E}_0^3. \tag{45}$$

If $\tilde{\chi} < 0$, this equation has the soliton solution

$$\mathcal{E}_0 = \frac{a}{\operatorname{ch}\zeta}, \tag{46}$$

where

$$a = \sqrt{\frac{32\pi^2\Delta^8\alpha_2}{|\tilde{\chi}|}}, \quad \alpha_2 = \frac{\Delta^4 - 3\kappa^2 + 4\Delta^2\kappa^2}{\Delta^4(\Delta^2-1)}, \tag{47}$$

and

$$\zeta = \frac{1}{\tau}\left(t - \frac{x}{V}\right), \quad \left(\frac{c}{V}\right)^2 = \frac{(\Delta^4+\kappa^2)^2}{\Delta^6(\Delta^2-1)}. \tag{48}$$

This shows that V is equal to the group velocity given by Eq. (26) and is independent of the duration of the pulse. Because

$$\varepsilon = \frac{\Lambda}{\Delta} = \frac{1}{\sqrt{\omega^2 - \omega_T^2}\,\tau}, \quad \tilde{\chi} = \frac{\chi}{\omega_L^2 - \omega_T^2},$$

we obtain

$$\mathcal{E} = \sqrt{\frac{32\pi^2\Delta^6\alpha_2}{|\chi|}}\,\frac{1}{\tau}\,\frac{1}{\operatorname{ch}\left[\frac{1}{\tau}\left(t-\frac{x}{V}\right)\right]}. \tag{49}$$

This is the NLS type soliton solution.

We note that the above calculations were done for arbitrary values of κ. But in a majority of applications, we have

$$(\omega_L - \omega_T) \ll \omega_T,$$

that is,

$$\kappa^2 = \frac{\omega_T^2}{\omega_L^2 - \omega_T^2} \approx \frac{\omega_T}{2(\omega_L - \omega_T)} \gg 1. \tag{50}$$

Expansions (39)–(41) are then valid only under the condition that

$$\kappa^2\varepsilon^2 \approx \frac{1}{4(\omega_L - \omega_T)^2\tau^2} \ll 1, \tag{51}$$

which implies that the spectral width of the pulse of the order of $1/\tau$ is much less than the width of the gap $(\omega_L - \omega_T)$. In fact, condition (51) is already satisfied suffuciently well for

$$\frac{1}{\tau} \approx \frac{\omega_L - \omega_T}{2}.$$

4.2. Long pulse at $\Delta^2 = 1$

In this case, the pulse cannot be described by the NLS equation for the wave packet of waves with wave vectors around some nonzero value. From Eqs. (28)–(29), we have the expansions

$$\left(\frac{ck}{\omega}\right)^2 - 1 = -1 + \sqrt{1+\kappa^2}\varepsilon + \ldots = -1 + \bar{\alpha}_1\varepsilon + \ldots, \tag{52}$$

$$\Lambda^2\left[\left(\frac{c}{V}\right)^2 - 1\right] = (1+\kappa^2)^{3/2}\varepsilon - \left(2\kappa^4 + \frac{3}{2}\kappa^2 + \frac{1}{2}\right)\varepsilon^2 + \ldots = \bar{\gamma}_1\varepsilon - \bar{\gamma}_2\varepsilon^2 + \ldots, \tag{53}$$

$$2\Lambda\left(\frac{c^2k}{V\omega} - 1\right) = 2\kappa^2\varepsilon - 4(1+2\kappa^2)\kappa^2\varepsilon^3 + \ldots = \bar{\beta}_1\varepsilon - \bar{\beta}_3\varepsilon^3 + \ldots \tag{54}$$

We again seek solution of Eqs. (31)–(34) in the form of series expansions, which in this case are

$$\begin{aligned} \mathcal{E} &= \varepsilon^{1/2}(\mathcal{E}_0 + \varepsilon\mathcal{E}_1 + \ldots), \\ \dot{\phi} &= \varepsilon^{1/2}\theta_0 + \ldots, \\ u &= \varepsilon^{1/2}(u_0 + \varepsilon u_1 + \ldots), \\ v &= \varepsilon^{3/2}v_1 + \ldots. \end{aligned} \tag{55}$$

In the first approximation, we have

$$u_0 = -\frac{\mathcal{E}_0}{4\pi}, \quad v_1 = 2\sqrt{1+\kappa^2}\,\dot{u}_0. \tag{56}$$

In the next approximation, Eqs. (31) and (34) give

$$\begin{aligned} -u_1 + \tilde{\chi}u_0^3 &= \frac{\mathcal{E}_1}{4\pi}, \\ \sqrt{1+\kappa^2}\ddot{\mathcal{E}}_0 + \mathcal{E}_1 - \bar{\alpha}_1\mathcal{E}_0 &= -4\pi u_1. \end{aligned} \tag{57}$$

Hence,

$$\begin{aligned} \mathcal{E}_1 + 4\pi u_1 &= -\frac{\tilde{\chi}}{16\pi^2}\mathcal{E}_0^3, \\ \sqrt{1+\kappa^2}(\ddot{\mathcal{E}}_0 - \mathcal{E}_0) &= -(\mathcal{E}_1 + 4\pi u_1), \end{aligned} \tag{58}$$

and we arrive at the equation for $\mathcal{E}_0$,

$$\ddot{\mathcal{E}}_0 = \mathcal{E}_0 + \frac{\tilde{\chi}}{16\pi^2\sqrt{1+\kappa^2}}\mathcal{E}_0^3. \tag{59}$$

Thus, we obtain the soliton solution

$$\mathcal{E} = \frac{\varepsilon^{1/2}a}{\operatorname{ch}\left[\frac{1}{\tau}\left(t-\frac{x}{V}\right)\right]}, \quad a = \sqrt{\frac{32\pi^2(1+\kappa^2)^{1/2}}{|\tilde{\chi}|}}, \tag{60}$$

where V is given by

$$\left(\frac{c}{V}\right)^2 = 1 + \frac{(1+\kappa^2)^{3/2}}{\varepsilon}. \tag{61}$$

Taking into account that

$$\varepsilon = \Lambda = \frac{1}{\sqrt{\omega_L^2 - \omega_T^2}\,\tau},$$

we can rewrite Eqs. (60) and (61) in terms of the physical parameters

$$\begin{aligned} \mathcal{E} &= \sqrt{\frac{32\pi^2\omega_L}{|\chi|\tau}}\,\frac{1}{\operatorname{ch}\left[\frac{1}{\tau}\left(t-\frac{x}{V}\right)\right]}, \\ \left(\frac{c}{V}\right)^2 &= 1 + \frac{\omega_L^3\tau}{\omega_L^2 - \omega_T^2}. \end{aligned} \tag{62}$$

The velocity of the pulse therefore depends on τ (curves in Fig. 2 intersect the straight line $\Delta^2 = 1$ at different points depending on τ). Although the parameter κ disappeared from Eq. (62), expansions (52)–(54) are valid for $\kappa \gg 1$ provided the inequality $(\kappa\varepsilon)^2 \ll 1$ is satisfied. The ratio of the amplitude of solution (62) to that of solution (49) is of the order of magnitude

$$\frac{1}{\sqrt{\kappa\varepsilon}} \sim \sqrt{(\omega_L - \omega_T)\tau} \gg 1,$$

that is, the amplitude at the boundary of the gap is much greater than the amplitude of the soliton solution sufficiently far from the gap. This implies that the pulse must be sufficiently intense to deform the gap to such extent that the wave propagation with the frequency $\Delta^2 = 1$ becomes possible. Beyond the gap, there are linear waves that can propagate with arbitrarily small amplitudes and nonlinear effects must only compensate dispersive spreading of the wave packet built from linear waves.

4.3. Long pulse inside the polariton gap

For frequencies inside the polariton gap,

$$0 < \Delta^2 < 1, \tag{63}$$

we have the series expansions

$$\left(\frac{ck}{\omega}\right)^2 - 1 = -1 + \frac{(\Delta^4+\kappa^2)^2}{\Delta^4(\Delta^2-1)(\Delta^2+\kappa^2)}\varepsilon^2 + \ldots = -1 + \tilde{\alpha}_2\varepsilon^2 + \ldots, \tag{64}$$

$$\Lambda^2\left[\left(\frac{c}{V}\right)^2 - 1\right] = \frac{(1-\Delta^2)(\Delta^2+\kappa^2)}{\Delta^2} + \frac{\Delta^8 + 5\Delta^4\kappa^2 - 3\kappa^4 + \Delta^2\kappa^2(4\kappa^2-3)}{\Delta^4(1-\Delta^2)}\varepsilon^2 + \ldots = \tilde{\gamma}_0 + \tilde{\gamma}_2\varepsilon^2 + \ldots, \tag{65}$$

$$2\Lambda\left(\frac{c^2k}{V\omega} - 1\right) = \frac{2\kappa^2}{\Delta^3}\varepsilon - \frac{4\kappa^2(\Delta^2+2\kappa^2)}{\Delta^7}\varepsilon^3 + \ldots = \beta_1\varepsilon - \beta_3\varepsilon^3 + \ldots, \tag{66}$$

where

$$\varepsilon \ll 1 - \Delta^2, \quad \varepsilon \ll \Delta^2. \tag{67}$$

Because $\tilde{\gamma}_0 \neq 0$, the soliton solution is obtained in the first approximation, and therefore $\mathcal{E}$ and u do not have a small factor proportional to a power of ε. The equations of the first approximation are given by

$$-\Delta^2 u + \tilde{\chi}u^3 = \frac{1}{4\pi}\mathcal{E}, \tag{68}$$

$$\frac{1-\Delta^2}{\Delta^2}\ddot{\mathcal{E}} + \mathcal{E} = -4\pi u. \tag{69}$$

If we were able to express u in terms of $\mathcal{E}$ from Eq. (68) and substitute the result in Eq. (69), we would obtain the equation for $\mathcal{E}$ having solitary wave solutions. Unfortunately, that can be done only numerically except in the case where

$$|\tilde{\chi}|u^2 \ll \Delta^2. \tag{70}$$

In this limit, we have

$$u \cong -\frac{1}{4\pi\Delta^2}\mathcal{E} - \frac{\tilde{\chi}}{\Delta^2(4\pi\Delta^2)^3}\mathcal{E}^3 \tag{71}$$

and Eq. (69) takes the form

$$\ddot{\mathcal{E}} = \mathcal{E} + \frac{\tilde{\chi}}{16\pi^2\Delta^6(1-\Delta^2)}\mathcal{E}^3. \tag{72}$$

It has the soliton solution

$$\mathcal{E} = \sqrt{\frac{32\pi^2\Delta^6(1-\Delta^2)(\omega_L^2-\omega_T^2)}{|\chi|}} \times \frac{1}{\operatorname{ch}\left[\frac{1}{\tau}\left(t - \frac{x}{V}\right)\right]}, \tag{73}$$

where the pulse velocity is given by

$$\left(\frac{c}{V}\right)^2 = 1 + \frac{(1-\Delta^2)(\Delta^2+\kappa^2)(\omega^2-\omega_T^2)\tau^2}{\Delta^2}. \tag{74}$$

Using the estimate

$$u \sim \frac{a}{4\pi\Delta^2},$$

we can transform condition (70) into

$$1 - \Delta^2 \ll 1. \tag{75}$$

The frequency must therefore be sufficiently close to the upper limit of the polariton gap.

Solution (73) only applies to sufficiently long pulses

$$\varepsilon^2 \ll 1 - \Delta^2. \tag{76}$$

We note that in this case, the amplitude is independent of the pulse width τ, but its velocity strongly depends on τ. If $\kappa \gg 1$, the ratio of amplitude (73) to amplitude (49) is of the order of magnitude

$$\frac{1}{\kappa\varepsilon} \sim (\omega_L - \omega_T)\tau \gg 1,$$

and as could be expected, is much greater than the amplitude of soliton solution (62) at the boundary of the gap.

When condition (75) is not satisfied, we must solve Eqs. (68) and (69) numerically. For this, we introduce new variables

$$\mathcal{E} = \frac{E}{\sqrt{|\widetilde{\chi}|}}, \quad u = \frac{U}{\sqrt{|\widetilde{\chi}|}}, \tag{77}$$

where $\widetilde{\chi} = -|\widetilde{\chi}|$. Equations (68) and (69) then reduce to

$$\Delta^2 U + U^3 = -\frac{1}{4\pi}E, \quad \frac{1-\Delta^2}{\Delta^2}\ddot{E} + E = -4\pi U. \tag{78}$$

Numerical solution of the first equation in (78) yields the function

$$U = U(E), \tag{79}$$

substitution of which in the second equation in (78) gives the differential equation for E,

$$\frac{1-\Delta^2}{\Delta^2}\frac{d^2E}{d\zeta^2} = -E - 4\pi U(E), \tag{80}$$

or

$$\frac{d^2E}{d\zeta^2} = -\frac{\partial V(E)}{\partial E}, \tag{81}$$

where

$$V(E) = \frac{\Delta^2}{1-\Delta^2}\left(\frac{1}{2}E^2 + 4\pi\int_0^E U(E')dE'\right) \tag{82}$$

can be considered as a potential in which a particle moves in accordance with the Newton equation (81). A typical plot of potential (82) is shown in Fig. 3. The maximum value of E (the soliton amplitude a) is determined by the point $E = a$ where $V(E)$ vanishes. The solution of Eq. (80) with the initial conditions $E(0) = a$ and $E'(0) = 0$ then provides the soliton solution. We have done these calculations for several values of Δ. The results are shown in Fig. 4, where the dependence of the pulse profile E on the scaled variable $\sqrt{\Delta^2/(1-\Delta^2)}\,\zeta$ is plotted. The dependence of the amplitude on Δ is shown in Fig. 5.

5. CONCLUSION

We have studied the nonlinear optical pulse propagation in a frequency region in the vicinity of the polariton gap. The problem is described by a coupled set of the Maxwell equations for the electromagnetic field and a material equation for the macroscopic polarization allowing a Kerr-like nonlinearity. To solve this nonlinear system of equations analytically, we used

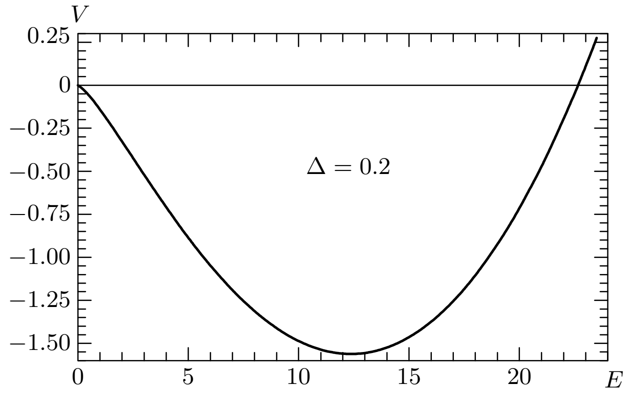

Fig. 3. The potential V as a function of the electric field amplitude for $\Delta = 0.2$

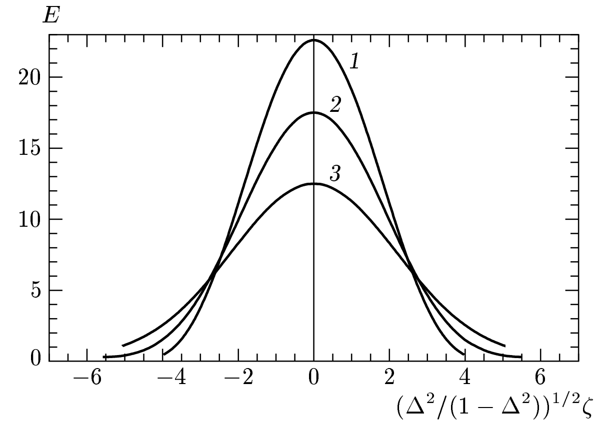

Fig. 4. Profiles of solitary wave solutions for three different values of the carrier wave frequency (see Eq. (20)) inside the polariton gap; $\Delta = 0.2$ (*1*), 0.6 (*2*), 0.8 (*3*)

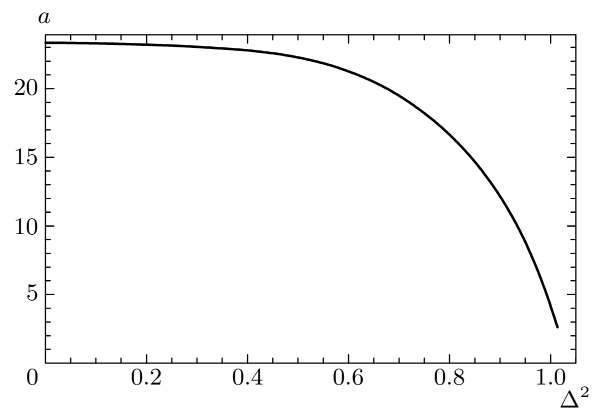

Fig. 5. Dependence of the amplitude of the solitary wave on the carrier wave frequency (see Eq. (20)) inside the polariton gap

the approach [8] based on series expansion in powers of a small parameter related to the width of the polariton gap and pulse duration. Different bright solitary wave solutions depending on the position of the carrier wave frequency with respect to the polariton gap are found and their parameters are expressed in terms of material system parameters. Outside the polariton gap, the soliton solution corresponds to the well-known soliton of the NLS equation for the envelope of a wave packet made of plane waves. But inside the polariton gap, there are no plane wave solutions and the notion of their envelope loses its physical sense. Nonetheless, solitary wave solutions can occur there with sufficiently high values of the electromagnetic field strength such that the local value of the polariton gap diminishes in the center of the polariton gap due to the Kerr nonlinearity. The difference in physical situations outside and inside the polariton gap is reflected in different dependences of the soliton amplitude on the pulse duration τ: the amplitude is independent of τ inside the gap, is proportional to $\tau^{-1/2}$ at the gap boundary, and is proportional to τ^{-1} sufficiently far from the gap.

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