ЖЭТФ, 2003, том 124, вып. 1 (7), стр. 14–27

THE SYMMETRY, INFERABLE FROM BOGOLIUBOV TRANSFORMATION, BETWEEN PROCESSES INDUCED BY A MIRROR IN 2-DIMENSIONAL AND A CHARGE IN 4-DIMENSIONAL SPACE–TIME

V. I. Ritus *

Tamm Department of Theoretical Physics, Lebedev Physical Institute, Russian Academy of Sciences
119991, Moscow, Russia

Submitted 30 December 2002

We consider the symmetry between creation of pairs of massless bosons or fermions by an accelerated mirror in $1+1$-dimensional space and emission of single photons or scalar quanta by an electric or scalar charge in $3+1$-dimensional space. The relation of Bogoliubov coefficients describing the processes generated by the mirror to Fourier components of the current or charge density implies that the spin of any disturbances bilinear in the scalar or spinor field coincides with the spin of quanta emitted by the electric or scalar charge. The mass and invariant momentum transfer of these disturbances are essential for the relation of Bogoliubov coefficients to invariant singular solutions and the Green's functions of wave equations for both $1+1$- and $3+1$-dimensional spaces and especially for the integral relations between these solutions. One of these relations leads to the coincidence of the self-action changes and vacuum–vacuum amplitudes for an accelerated mirror in 2-dimensional space-time and a charge in 4-dimensional space-time. Both invariants of the Lorentz group, spin and mass, play an essential role in the established symmetry. The symmetry embraces not only the processes of real quanta radiation, but also the processes of the mirror and charge interactions with fields carring spacelike momenta. These fields accompany their sources and determine the Bogoliubov matrix coefficients $\alpha^{B,F}_{\omega'\omega}$. It is shown that the Lorentz-invariant traces $\pm\mathrm{tr}\,\alpha^{B,F}$ describe the vector and scalar interactions of the accelerated mirror with a uniformly moving detector. This interpretation rests essentially on the relation between propagators of the waves with spacelike momenta in 2- and 4-dimensional spaces. The traces $\pm\mathrm{tr}\,\alpha^{B,F}$ coincide with the products of the mass shift $\Delta m_{1,0}$ of the accelerated electric or scalar charge and the proper time of the shift formation. The symmetry fixes the value of the bare fine structure constant $\alpha_0 = 1/4\pi$.

PACS: 11.10.Kk, 11.30.-j, 11.55.Fv, 03.65.Pm

1. INTRODUCTION

The Hawking particle production mechanism at the black hole formation is analogous to the emission from an ideal mirror accelerated in vacuum [1]. In its turn, there is a close analogy between the radiation of pairs of scalar (spinor) quanta from an accelerated mirror in $1+1$-dimensional space and the radiation of photons (scalar quanta) by an accelerated electric (scalar) charge in $3+1$-dimensional space [2, 3]. All these processes are therefore interrelated. The *in*- and *out*-sets of the wave equation solutions that are typically used for a massless scalar field in problems with moving mirrors are given by

$$\begin{aligned} \phi_{in\,\omega'} &\propto e^{-i\omega' v} - e^{-i\omega' f(u)}, \\ \phi^*_{in\,\omega'} &\propto e^{i\omega' v} - e^{i\omega' f(u)}, \end{aligned} \tag{1a}$$

$$\begin{aligned} \phi_{out\,\omega} &\propto e^{-i\omega g(v)} - e^{-i\omega u}, \\ \phi^*_{out\,\omega} &\propto e^{i\omega g(v)} - e^{i\omega u}, \end{aligned} \tag{1b}$$

with zero boundary condition

$$\phi|_{traj} = 0$$

on the mirror trajectory. Here, the variables $u = t - x$ and $v = t + x$ are used and the mirror (or charge) tra-

*E-mail: ritus@lpi.ru

jectory in the u, v plane is given by any of the two mutually inverse functions

$$v^{mir} = f(u), \quad u^{mir} = g(v).$$

We refer to [3] for the *in*- and *out*-sets of solutions of the massless Dirac equation. Dirac solutions differ from (1) by the presence of bispinor coefficients at the u- and v-plane waves. The current densities corresponding to these solutions have only tangential components at the boundary. The boundary condition for both scalar and spinor fields is therefore purely geometrical, it does not contain any dimensional parameters.

The Bogoliubov coefficients $\alpha_{\omega'\omega}$ and $\beta_{\omega'\omega}$ arise as the coefficients of the expansion of the solutions of the *out*-set in the solutions of the *in*-set; the coefficients $\alpha^*_{\omega'\omega}$ and $\mp\beta_{\omega'\omega}$ arise similarly in the inverse expansion. The upper and lower signs correspond to the scalar (Bose) and spinor (Fermi) fields. The mean number of quanta with the frequency ω and wave vector $\omega > 0$ radiated by the accelerated mirror to the right semispace is then given by the integral

$$d\bar{n}_\omega = \frac{d\omega}{2\pi} \int\limits_0^\infty \frac{d\omega'}{2\pi} |\beta_{\omega'\omega}|^2. \qquad (2)$$

At the same time, the spectra of photons and scalar quanta emitted by electric and scalar charges moving along the trajectory $x_\alpha(\tau)$ in $3+1$-dimensional space are determined by the Fourier transforms of the electric current density 4-vector $j_\alpha(k)$ and the scalar charge density $\rho(k)$,

$$s = 1, \quad j_\alpha(k) = e \int d\tau\, \dot{x}_\alpha(\tau) \exp\left(-ik^\alpha x_\alpha(\tau)\right), \qquad (3)$$

$$d\bar{n}_k^{(1)} = |j_\alpha(k_+, k_-)|^2 \frac{dk_+ dk_-}{(4\pi)^2}, \qquad (4)$$

$$s = 0, \quad \rho(k) = e \int d\tau \exp\left(-ik^\alpha x_\alpha(\tau)\right), \qquad (5)$$

$$d\bar{n}_k^{(0)} = |\rho(k_+, k_-)|^2 \frac{dk_+ dk_-}{(4\pi)^2}, \qquad (6)$$

where s and k^α are the spin and 4-momentum of the quanta,

$$k^2 = k_1^2 + k_\perp^2 - k_0^2 = 0, \quad k_\perp^2 = k_0^2 - k_1^2 = k_+ k_-,$$
$$k_\pm = k^0 \pm k^1,$$

and it is assumed in (4) and (6) that the trajectory $x^\alpha(\tau)$ has only x^0 and x^1 nontrivial components.

The symmetry between creation of Bose or Fermi pairs by the accelerated mirror in $1+1$-dimensional space and emission of single photons or scalar quanta by the electric or scalar charge in $3+1$-dimensional space consists, first of all, in the coincidence of the spectra. If we set

$$2\omega = k_+, \quad 2\omega' = k_-,$$

we have

$$|\beta^B_{\omega'\omega}|^2 = \frac{1}{e^2} |j_\alpha(k_+, k_-)|^2, \qquad |\beta^F_{\omega'\omega}|^2 = \frac{1}{e^2} |\rho(k_+, k_-)|^2. \qquad (7)$$

A more refined assertion in the Bose case is

$$\beta^{B*}_{\omega'\omega} = -\sqrt{\frac{k_+}{k_-}} \frac{j_-(k)}{e} = \sqrt{\frac{k_-}{k_+}} \frac{j_+(k)}{e} = \frac{\varepsilon_{\alpha\beta} k^\alpha j^\beta(k)}{e\sqrt{k_+ k_-}}, \qquad (8)$$

$$j_-(k) = e \int du \exp\left[\frac{i}{2}(k_+ u + k_- f(u))\right], \qquad j_+(k) = e \int dv \exp\left[\frac{i}{2}(k_- v + k_+ g(v))\right]. \qquad (9)$$

The 2-vectors $j_\alpha(k)$ and $a_\beta(k) = \varepsilon_{\alpha\beta} k^\alpha / \sqrt{k_+ k_-}$ are spacelike for timelike k^α; in a system where $k_+ = k_-$ or $\omega = \omega'$, they have only spatial components that are precisely equal to $e\beta^{B*}_{\omega'\omega}$ and 1 correspondingly.

In the Fermi case, we have

$$\beta^{F*}_{\omega'\omega} = \frac{1}{e} \rho(k). \qquad (10)$$

In Sec. 2, we underline the symmetry between analytic expressions for the Bogoliubov coefficients α and β^* and at the same time, the physical distinction between them: $\beta^{B,F*}$ is the amplitude of the source of waves that are bilinear in massless Bose or Fermi fields and carry timelike momenta, whereas $\alpha^{B,F}$ is the amplitude of the source of similar waves that carry spacelike momenta, see (14) and (15). In Secs. 3 and 4, we show that the waves with timelike momenta emitted and absorbed by the source are involved in forming the imaginary part of the source self-action. This physical picture is naturally embodied in integral relation (20) between propagators $\Delta_2(z, m)$ of virtual pairs with masses m, $\mu \leqslant m < \infty$ in 2-dimensional spacetime and the propagator $\Delta_4(z, \mu)$ of the particle in 4-dimensional spacetime. Analytic properties of the expressions obtained also allow us to define the real part of self-action. This leads to the coincidence of the self-actions and hence, of the vacuum–vacuum amplitudes of the mirror and the charge if we set $e^2 = 1$. In Sec. 5, the fields of perturbations carrying spacelike momenta are considered. These fields are defined by the matrices $\alpha^{B,F}$. Their Lorentz-invariant traces $\pm\mathrm{tr}\,\alpha^{B,F}$ are considered in Sec. 6. They describe correspondingly the vector and scalar interactions of the accelerated mirror with a uniformly moving detector in the neighborhood of the point of contact of their trajectories. In Secs. 7 and 8, the traces $\pm\mathrm{tr}\,\alpha^{B,F}$ are found for the three specific trajectories permitting analytic solutions. The general expressions for the traces are given and their ultraviolet and infrared singularities are also considered there. In these sections, we compare the found traces $\pm\mathrm{tr}\,\alpha^{B,F}$ with the mass shifts $\Delta m_{1,0}$ of the electric and scalar charges moving along the same trajectory as the mirror, but in $3+1$-dimensional space. The mass shifts $\Delta m_{1,0}$ of the charges moving along the exponential trajectory are found in Sec. 9. In Conclusions, we discuss the relation of the traces $\pm\mathrm{tr}\,\alpha^{B,F}$ to the general definition of the self-action accounting for interference effects, and draw attention to the fact that the symmetry fixes the value of the bare charge squared, $e_0^2 = 1$, which correspons to the bare fine structure constant $\alpha_0 = 1/4\pi$. The smallness and geometrical origin of this value may be interesting in quantum electrodynamics. In the Appendix, the even singular solutions of inhomogeneous wave equations with mass and momentum transfer parameters are considered. Integral relations (20) and (100) between these solutions for $1+1$- and $3+1$-dimensional spaces are very important for the symmetry considered.

2. THE PHYSICAL INTERPRETATION OF $\beta^*_{\omega'\omega}$

The absolute pair production amplitude and the single-particle scattering amplitude are related by [4]

$$\langle \mathrm{out}\,\omega''\omega|\mathrm{in}\rangle = -\sum_{\omega'}\langle \mathrm{out}\,\omega''|\omega'\mathrm{in}\rangle\,\beta^*_{\omega'\omega}. \tag{11}$$

The coefficient $\beta^*_{\omega'\omega}$ was interpreted as the amplitude of a source of a pair of massless particles potentially emitted to the right and to the left with the respective frequences ω and ω'. While the particle with the frequency ω actually escapes to the right, the particle with the frequency ω' propagates for some period of time and is then reflected by the mirror and is actually emitted to the right with an altered frequency ω'', see Fig. 1.

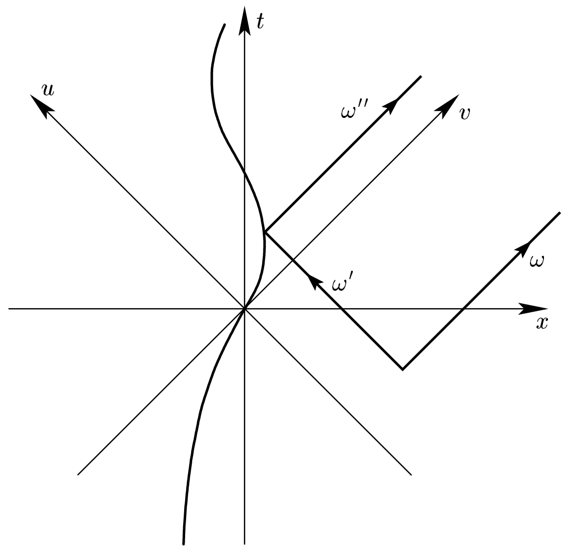

Fig. 1.

In the time interval between pair creation and reflection of the left particle, we have the virtual pair with the energy k^0, momentum k^1, and mass m,

$$k^0 = \omega + \omega', \quad k^1 = \omega - \omega', \qquad m = \sqrt{-k^2} = 2\sqrt{\omega\omega'}. \tag{12}$$

In addition to the polar timelike 2-vector k^α, very important is the axial spacelike 2-vector q^α,

$$q_\alpha = \varepsilon_{\alpha\beta}k^\beta, \quad q^0 = -k^1 = -\omega + \omega', \qquad q^1 = -k^0 = -\omega - \omega' < 0. \tag{13}$$

In terms of k^α and q^α, the symmetry between the α and β coefficients becomes clearly expressed,

$$s = 1, \quad e\beta^{B*}_{\omega'\omega} = -\frac{q_\alpha j^\alpha(k)}{\sqrt{k_+k_-}}, \qquad e\alpha^B_{\omega'\omega} = -\frac{k_\alpha j^\alpha(q)}{\sqrt{k_+k_-}}, \tag{14}$$

$$s = 0, \quad e\beta^{F*}_{\omega'\omega} = \rho(k), \quad e\alpha^F_{\omega'\omega} = \rho(q). \tag{15}$$

We note that Eqs. (3) and (5) define the current density $j^\alpha(k)$ and the charge density $\rho(k)$ as functionals of the trajectory $x^\alpha(\tau)$ and functions of any 2- or 4-vector k^α. It can be shown that in $1+1$-dimensional space, $j^\alpha(k)$ and $j^\alpha(q)$ are spacelike and timelike polar vectors respectively if k^α and q^α are timelike and spacelike vectors.

The boundary condition on the mirror leads to the appearance of vector or scalar disturbance waves bilinear in massless fields in the vacuum of the massless scalar or spinor field. There are two types of these waves:

1) waves with the amplitude $\alpha_{\omega'\omega}$ ($\alpha^*_{\omega'\omega}$) that carry a spacelike momentum directed to the left (right);

2) waves with the amplitude $\beta^*_{\omega'\omega}$ ($\beta_{\omega'\omega}$) that carry a timelike momentum with a positive (negative) frequency.

The waves with spacelike momenta appear even if the mirror is at rest or moves uniformly (the Casimir effect), while the waves with timelike momenta appear only for the accelerated mirror.

The pair of Bose (Fermi) particles has spin 1 (0) because its source is a current density vector (charge density scalar), see [5] or problem 12.15 in [6].

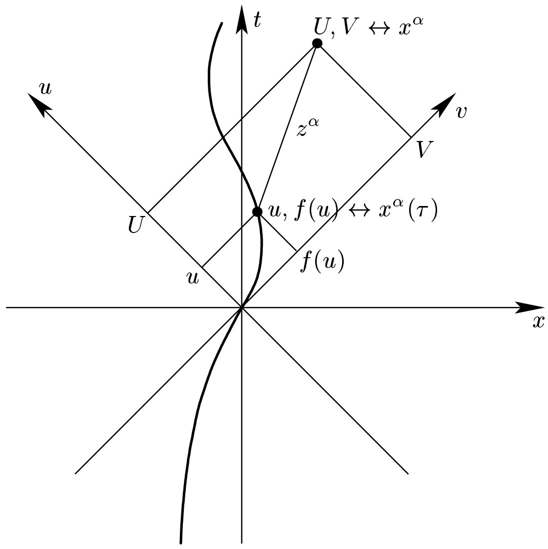

Fig. 2.

3. THE APPEARANCE OF MASS IN THE MASSLESS THEORY AND OF INVARIANT SINGULAR SOLUTIONS OF THE WAVE EQUATION WITH MASS

It follows from (8) that the bilinear in massless bose-field disturbances defined by the amplitudes $\beta^{B*}_{\omega'\omega}$ forms a positive-frequency current density vector. Its minus-component at the point U, V can be represented as

$$\iint\limits_0^\infty \frac{d\omega\, d\omega'}{(2\pi)^2}\,\frac{1}{e} j_-(k)\exp(-i\omega U - i\omega' V) = \frac{1}{8\pi^2}\times$$
$$\times\int du \int\limits_0^\infty d\rho\,\rho \int\limits_{-\infty}^{\infty} d\theta \exp\left(-i\rho(z^0\operatorname{ch}\theta - z^1\operatorname{sh}\theta)\right), \quad (16)$$

if the hyperbolic variables ρ and θ are used instead of ω and ω',

$$d\omega d\omega' = \frac{1}{2}\rho d\rho d\theta, \quad \omega = \frac{1}{2}\rho e^{\theta}, \quad \omega' = \frac{1}{2}\rho e^{-\theta},$$
$$\rho = 2\sqrt{\omega\omega'}, \quad \theta = \ln\sqrt{\frac{\omega}{\omega'}}, \quad (17)$$

and $z^\alpha = x^\alpha - x^\alpha(\tau)$, see Fig. 2. As can be seen from (12), $\rho = m$ is the mass of the pair and θ is the rapidity. The integral over rapidity in (16) is the well-known invariant positive-frequency singular function of the wave equation for 2-dimensional spacetime,

$$\int\limits_{-\infty}^{\infty} d\theta \exp\left(-im(z^0\operatorname{ch}\theta - z^1\operatorname{sh}\theta)\right) =$$
$$= -4\pi i\Delta_2^+(z,m) = 2\theta(-z^2)K_0\left(i\varepsilon(z^0)m\sqrt{-z^2}\right) +$$
$$+ 2\theta(z^2)K_0\left(m\sqrt{z^2}\right), \quad (18)$$

$$(\partial_t^2 - \partial_x^2 + m^2)\Delta_2^+(z,m) = 0. \quad (19)$$

This function describes the wave field of pairs with mass m and any possible positive-frequency momenta. It follows that the pairs are created, propagated and absorbed near the mirror within a spacelike interval of the order of m^{-1}.

Using the very important integral relation between the singular functions of wave equations for d- and $d+2$-dimensional spacetimes,

$$\Delta^f_{d+2}(z,\mu) = \frac{1}{4\pi}\int\limits_{\mu^2}^{\infty} dm^2\,\Delta^f_d(z,m), \quad (20)$$

we can represent the right-hand side of (16) as

$$-\frac{i}{4\pi}\int du \int\limits_{\mu^2\to 0}^{\infty} dm^2\,\Delta_2^+(z,m) =$$
$$= -i\int\limits_{-\infty}^{\infty} du\,\Delta_4^+(z,\mu). \quad (21)$$

The small mass μ is retained to eliminate the infrared divergence in what follows.

Similarly, the positive-frequency plus-component of the current density at the point U, V can be represented as

$$\iint\limits_0^\infty \frac{d\omega d\omega'}{(2\pi)^2}\, \frac{1}{e} j_+(k) \exp(-i\omega U - i\omega' V) = = -i\int\limits_{-\infty}^{\infty} dv\, \Delta_4^+(z,\mu). \quad (22)$$

The differentials du and dv in (21) and (22) can be replaced by $d\tau\, \dot{x}_-(\tau)$ and $d\tau\, \dot{x}_+(\tau)$.

The bilinear in massless Fermi-field disturbances defined by the amplitudes $\beta^{F*}_{\omega'\omega}$ forms a positive-frequency charge density scalar. At the point U, V, it can be represented by

$$\iint\limits_0^\infty \frac{d\omega d\omega'}{(2\pi)^2}\, \frac{1}{e} \rho(k) \exp(-i\omega U - i\omega' V) = = -i\int\limits_{-\infty}^{\infty} d\tau\, \Delta_4^+(z,\mu). \quad (23)$$

If we set the point U, V on the trajectory such that

$$U = x_-(\tau'), \quad V = x_+(\tau'), \quad z^\alpha = x^\alpha(\tau') - x^\alpha(\tau),$$

and integrate (21) over V and (22) over U, then their half sum differs from $\mathrm{tr}\,\beta^+\beta$ only by the factor i,

$$\begin{aligned} \mathrm{tr}\,\beta^{B+}\beta^B \equiv \iint\limits_0^\infty \frac{d\omega d\omega'}{(2\pi)^2} |\beta^B_{\omega'\omega}|^2 = \\ = \frac{i}{2}\iint (du\, dV + dv\, dU)\, \Delta_4^+(z,\mu) = \\ = -i\iint d\tau d\tau'\, \dot{x}_\alpha(\tau)\dot{x}^\alpha(\tau')\, \Delta_4^+(z,\mu). \quad (24) \end{aligned}$$

The real part of the function Δ^+, which is odd in z, and its imaginary part, which is even in z, are related to the causal (Feynman) function Δ^f that is even in z,

$$\begin{aligned} \Delta^+(z,\mu) = \frac{1}{2}\Delta(z,\mu) + \frac{i}{2}\Delta^1(z,\mu), \\ \mathrm{Re}\,\Delta^+ = \varepsilon(z^0)\,\mathrm{Re}\,\Delta^f, \quad \mathrm{Im}\,\Delta^+ = \mathrm{Im}\,\Delta^f, \end{aligned} \quad (25)$$

and $\mathrm{tr}\beta^{B+}\beta^B$ can therefore be written as

$$\mathrm{tr}\,(\beta^+\beta)^B = \mathrm{Im}\iint d\tau d\tau' \dot{x}_\alpha(\tau)\dot{x}^\alpha(\tau')\, \Delta_4^f(z,\mu). \quad (26)$$

$\mathrm{tr}\,\beta^{F+}\beta^F$ can be obtained from the right-hand side of (26) by the substitution

$$\dot{x}_\alpha(\tau)\dot{x}^\alpha(\tau') \to 1.$$

4. VACUUM–VACUUM AMPLITUDE $\langle \mathbf{out}|\mathbf{in}\rangle = e^{iW}$

According to DeWitt [7], Wald [8], and others (including the present author [4]),

$$2\,\mathrm{Im}\,W^{B,F} = \pm\frac{1}{2}\mathrm{tr}\,\ln(1 \pm \beta^+\beta) \quad \text{or} \quad \pm\,\mathrm{tr}\,\ln(1\pm\beta^+\beta) \quad (27)$$

in the respective cases where the particle is identical or nonidentical to the antiparticle. We confine ourselves to the last case and assume that $\mathrm{tr}\,\beta^+\beta \ll 1$. Then

$$2\,\mathrm{Im}\,W^{B,F} = = \mathrm{Im}\iint d\tau d\tau' \left\{ \begin{matrix} \dot{x}_\alpha(\tau)\dot{x}^\alpha(\tau') \\ 1 \end{matrix} \right\} \Delta_4^f(z,\mu). \quad (28)$$

We can omit the Im symbols from both sides of this equation and define the actions for Bose- and Fermi-mirrors in $1+1$-dimensional space as

$$W^{B,F} = = \frac{1}{2}\iint d\tau\, d\tau' \left\{ \begin{matrix} \dot{x}_\alpha(\tau)\dot{x}^\alpha(\tau') \\ 1 \end{matrix} \right\} \Delta_4^f(z,\mu). \quad (29)$$

We compare this with the well-known actions for electric and scalar charges in $3+1$-dimensional space,

$$W^{1,0} = = \frac{1}{2}e^2\iint d\tau d\tau' \left\{ \begin{matrix} \dot{x}_\alpha(\tau)\dot{x}^\alpha(\tau') \\ 1 \end{matrix} \right\} \Delta_4^f(z,\mu). \quad (30)$$

The symmetry would be complete if $e^2 = 1$, i.e., if the fine structure constant were $\alpha = 1/4\pi$. This «ideal» value of the fine structure constant for the charges would correspond to the ideal, geometric boundary condition on the mirror.

For the mirror trajectory with a nonzero relative velocity β_{21} of its ends (nonzero relative rapidity $\theta = \mathrm{Arth}\,\beta_{21}$), the changes of the actions due to acceleration are given by

$$\begin{aligned} \mathrm{Re}\,\Delta W^B = \frac{1}{8\pi}\left(\frac{\theta}{\mathrm{th}\,\theta} - 1\right), \\ \mathrm{Re}\,\Delta W^F = \frac{1}{8\pi}\left(1 - \frac{\theta}{\mathrm{sh}\,\theta}\right). \end{aligned} \quad (31)$$

For a uniformly accelerated mirror with the proper acceleration a, its velocity is

$$\beta(\tau) = \mathrm{th}\, a\tau,$$

where τ is the proper time. Then

$$\theta = a(\tau_2 - \tau_1)$$

and as $\tau_2 - \tau_1 \to \infty$,

$$\mathrm{Re}\,\Delta W^B = \frac{|a|}{8\pi}(\tau_2 - \tau_1). \qquad (32)$$

By definition,

$$\mathrm{Re}\,\Delta m^B = -\frac{\partial\,\mathrm{Re}\,\Delta W^B}{\partial \tau_2} = -\frac{|a|}{8\pi} \qquad (33)$$

is the self-energy shift of an accelerating Bose mirror. It differs from the mass shift of a uniformly accelerated electron only by the absence of the factor $e^2 = 4\pi\alpha$. The self-energy shift of a uniformly accelerated Fermi mirror is

$$\mathrm{Re}\,\Delta m^F = 0.$$

There are two arguments in favor of defining the action by means of the causal function $\Delta_4^f(z,\mu)$.

1. The action must represent not only the radiation of real quanta but also the self-energy and polarization effects. While the radiation effects are described by solutions of the homogeneous wave equation, the self-energy and polarization effects require solutions of the inhomogeneous wave equation, which contain information about the proper field of a source. Such solutions of the homogeneous and inhomogeneous wave equations are the functions

$$(1/2)\Delta^1 = \mathrm{Im}\,\Delta^f$$

and

$$\bar{\Delta} = \mathrm{Re}\,\Delta^f.$$

2. While the appearance of

$$(1/2)\Delta^1 \equiv \mathrm{Im}\,\Delta^f$$

in the imaginary part of the action is a consequence of mathematical transformations of the integral

$$\iint_0^\infty \frac{d\omega d\omega'}{(2\pi)^2}|\beta_{\omega'\omega}|^2$$

(transformations similar to the Plancherel theorem), the function $\bar{\Delta} \equiv \mathrm{Re}\,\Delta^f$ in the real part of the action is unique if it appears as the real part of the analytic continuation of $(i/2)\Delta^1(z,\mu)$ to negative z^2 that is even in z (as Δ^1 itself).

To conclude Secs. 3 and 4, we note that both the function $\Delta_2(z,m)$ describing the propagation of a virtual pair with the mass $m = \rho = 2\sqrt{\omega\omega'}$ in 2-dimensional space–time and the mass spectrum of these pairs arise owing to the transition from the variables ω and ω' to the hyperbolic variables ρ and θ, which reflect the Lorentzian symmetry of the problem. Further integration over the mass leads to the function $\Delta_4(z,\mu)$ that coincides with the propagator of a particle moving in 4-dimensional space–time with the mass μ equal to the least mass of virtual pairs. Thus, relation (20) appears in the framework of the present method and is immanent to the symmetry, relating the processes in two- and four-dimensional space–times.

In [9], relation (20) was obtained by the author independently of the processes considered and was required in proving that the integration variable involved in it coincides with the pair mass $m = 2\sqrt{\omega\omega'}$.

5. FORMATION OF TACHYON DISTURBANCES WITH THE INVARIANT MOMENTUM TRANSFER

The bilinear in massless Bose field perturbations that are defined by the amplitudes $\alpha^B_{\omega'\omega}$ and carry spacelike momenta to the left can be represented at the point U, V by the two current density components

$$\iint_0^\infty \frac{d\omega d\omega'}{(2\pi)^2}\frac{1}{e}j_\pm(q)\exp(i\omega U - i\omega' V) = \frac{1}{8\pi^2}\times$$
$$\times \int d\tau\,\dot{x}_\pm(\tau)\int_0^\infty d\rho\,\rho\,\times$$
$$\times\int_{-\infty}^{\infty} d\theta\exp\left[i\rho(z^0\,\mathrm{sh}\,\theta - z^1\,\mathrm{ch}\,\theta)\right] \qquad (34)$$

if we again use the change of variables in (17) and the notation

$$z^\alpha = x^\alpha - x^\alpha(\tau).$$

The integral over θ is now given by

$$\int_{-\infty}^{\infty} d\theta\exp\left[i\rho(z^0\,\mathrm{sh}\,\theta - z^1\,\mathrm{ch}\,\theta)\right] = 4\pi i\,\Delta_2^L(z,\rho) =$$
$$= 2\theta(-z^2)K_0\left(\rho\sqrt{-z^2}\right) + 2\theta(z^2)\times$$
$$\times K_0\left(i\varepsilon(z^1)\rho\sqrt{z^2}\right). \qquad (35)$$

The integrand in the left-hand side of (35) is a wave with a spacelike 2-momentum q^α,

$$q^1 = -\omega - \omega' = -\rho\,\mathrm{ch}\,\theta, \quad q^0 = -\omega + \omega' = -\rho\,\mathrm{sh}\,\theta,$$
$$\rho = \sqrt{q^2}.$$

The function $\Delta_2^L(z,\rho)$ is a superposition of plane waves with spacelike momenta directed to the left and with a fixed invariant momentum transfer $\rho = 2\sqrt{\omega\omega'}$. It satisfies the wave equation with a negative mass squared,

$$(\partial_t^2 - \partial_x^2 - \rho^2)\Delta_2^L(z,\rho) = 0. \qquad (36)$$

Using the integral relation similar to (20) (see the Appendix), we can represent the right-hand side of (34) as

$$\frac{i}{4\pi}\int d\tau\,\dot{x}_\pm(\tau)\int\limits_{\nu^2\to 0}^{\infty} d\rho^2\,\Delta_2^L(z,\rho) = \\ = -i\int d\tau\,\dot{x}_\pm(\tau)\,\Delta_4^L(z,\nu). \qquad (37)$$

The small momentum transfer ν is retained to eliminate the infrared divergence in what follows.

Similarly, the bilinear in the Fermi field disturbances that are defined by the amplitudes $\alpha^F_{\omega'\omega}$ and carry left-directed spacelike momenta forms the charge density scalar. It can be represented at the point U, V by the integral

$$\iint\limits_0^\infty \frac{d\omega d\omega'}{(2\pi)^2}\,\frac{1}{e}\rho(q)\exp(i\omega U - i\omega' V) = \\ = -i\int d\tau\,\Delta_4^L(z,\nu). \qquad (38)$$

These representations can be useful in problems close to static ones involving another characteristic length in addition to or instead of acceleration.

6. INTERPRETATION OF THE TRACES $\pm\mathrm{tr}\,\alpha^{B,F}$ OF BOGOLIUBOV COEFFICIENTS

The invariant description of the mirror trajectory in the u, v plane requires that the function

$$u^{mir} = g(v)$$

contains two positive parameters $\varkappa$ and $\varkappa'$ transforming as

$$x_+ = v, \quad x_- = u$$

and actually connects the invariant variables $\varkappa u$ and $\varkappa' v$ between themselves,

$$u^{mir} = g(v) = \frac{1}{\varkappa}G(\varkappa' v). \qquad (39)$$

Its expansion near the origin $u = v = 0$ on the trajectory is given by

$$g(v) = \frac{1}{\varkappa}\left(\varkappa' v + b\varkappa'^2 v^2 + \frac{1}{3}c\varkappa'^3 v^3 + \ldots\right), \qquad (40)$$

where $b, c, \ldots$ are some numbers. Because the mirror velocity $\beta(v)$ and the proper acceleration $a(v)$ are defined by

$$\beta(v) = \frac{1 - g'(v)}{1 + g'(v)}, \quad a(v) = -\frac{g''(v)}{2g'^{3/2}(v)}, \qquad (41)$$

the first two coefficients of expansion (40) define the mirror velocity β_0 and acceleration a_0 at zero point,

$$\beta_0 = \frac{1 - \varkappa'/\varkappa}{1 + \varkappa'/\varkappa}, \quad a_0 = -b\sqrt{\varkappa\varkappa'}. \qquad (42)$$

The absolute value of the acceleration at zero point is denoted by

$$w_0 = |b|\sqrt{\varkappa\varkappa'}.$$

We define a Lorentz-invariant trace by the formula

$$\mathrm{tr}\,\alpha = \\ = \iint\limits_0^\infty \frac{d\omega d\omega'}{(2\pi)^2}\,\alpha_{\omega'\omega}\,2\pi\,\delta\left(\sqrt{\frac{\varkappa'}{\varkappa}}\omega - \sqrt{\frac{\varkappa}{\varkappa'}}\omega'\right), \qquad (43)$$

where the Lorentz-invariant argument of the δ-function is the difference of the frequences

$$\Omega = \sqrt{\frac{\varkappa'}{\varkappa}}\omega, \quad \Omega' = \sqrt{\frac{\varkappa}{\varkappa'}}\omega' \qquad (44)$$

of the reflected and incident waves in the proper system of the mirror at the moment $u = v = 0$. In accordance with (42), the multipliers $\sqrt{\varkappa'/\varkappa}$ and $\sqrt{\varkappa/\varkappa'}$ entering (44) are the Doppler factors relating the frequences in the laboratory and proper systems. In the proper system of the mirror,

$$\Omega = \Omega' = \sqrt{\omega\omega'}.$$

In accordance with (43), $\mathrm{tr}\,\alpha$ is a Lorentz-invariant dimensionless quantity or, perhaps, has dimensionality of the action because $\hbar = 1$. We now consider its physical meaning. For this, we turn to the equality of expressions (34) and (37),

$$\iint\limits_0^\infty \frac{d\omega d\omega'}{(2\pi)^2}\,\frac{1}{e}j_\pm(q)\exp(i\omega U - i\omega' V) = \\ = -i\int d\tau\,\dot{x}_\pm(\tau)\,\Delta_4^L(z,\nu), \qquad (45)$$

where

$$z^\alpha = x^\alpha - x^\alpha(\tau), \quad x_- = U, \quad x_+ = V.$$

We put the point U, V on the tangent line to the mirror trajectory at zero point, such that

$$U = X_-(\tau') = \sqrt{\frac{\varkappa'}{\varkappa}}\,\tau', \qquad V = X_+(\tau') = \sqrt{\frac{\varkappa}{\varkappa'}}\,\tau', \tag{46}$$

where τ' is the proper time of the point on the tangent line, and integrate both sides of (45) over

$$dU = \dot X_-\,d\tau' \quad \text{or} \quad dV = \dot X_+\,d\tau'$$

for the upper or lower sign in (45) respectively. Taking Eq. (14) and current conservation into account we then obtain $\operatorname{tr}\alpha$ in the left-hand side for both the upper and lower signs in (45). In the right-hand side, we obtain the integral

$$-i\iint d\tau\,d\tau'\,\dot x_\pm(\tau)\dot X_\mp(\tau')\,\Delta_4^L(z,\nu), \qquad z^\alpha = X^\alpha(\tau') - x^\alpha(\tau), \tag{47}$$

where according to the result for the left-hand side, we can replace

$$\dot x_\pm(\tau)\dot X_\mp(\tau') = = -\dot x_\alpha(\tau)\dot X^\alpha(\tau') \mp \varepsilon_{\alpha\beta}\,\dot x^\alpha(\tau)\dot X^\beta(\tau') \tag{48}$$

with only the first term that is symmetric with respect to the permutation

$$\dot x_\alpha(\tau) \rightleftarrows \dot X_\alpha(\tau').$$

We thus obtain

$$\operatorname{tr}\alpha^B = i\iint d\tau\,d\tau'\,\dot x_\pm(\tau)\dot X_\mp(\tau')\,\Delta_4^L(z,\nu), \qquad z^\alpha = X^\alpha(\tau') - x^\alpha(\tau). \tag{49}$$

Integrating both parts of Eq. (38) along tangent line (46) similarly and taking Eqs. (15) and (43) into account, we obtain

$$\operatorname{tr}\alpha^F = -i\iint d\tau\,d\tau'\,\Delta_4^L(z,\nu), \qquad z^\alpha = X^\alpha(\tau') - x^\alpha(\tau). \tag{50}$$

For trajectories in the Minkowsky plane on the left of their tangent line at zero point, the coordinate $z^1 \geqslant 0$. In this case, $\Delta_4^L(z,\nu)$ can be replaced by the function

$$\Delta_4^{LR}(z,\nu) = \frac{1}{4\pi}\delta(z^2) - \frac{\nu}{8\pi\sqrt{z^2}}\theta(z^2) \times \left[J_1\left(\nu\sqrt{z^2}\right) - iN_1\left(\nu\sqrt{z^2}\right)\right] + i\frac{\nu}{4\pi^2\sqrt{-z^2}}\theta(-z^2)K_1\left(\nu\sqrt{-z^2}\right), \tag{51}$$

which differs from the causal function $\Delta_4^f(z,\mu)$ by complex conjugation and the replacement $\mu \to i\nu$ (or by the replacement $z^2 \to -z^2$, $\mu \to \nu$). Further details about this function are given in the Appendix.

For the above trajectories, we therefore have that

$$\pm\operatorname{tr}\alpha^{B,F} = = i\iint d\tau d\tau' \left\{\begin{array}{c} \dot x_\alpha(\tau)\dot X^\alpha(\tau') \\ 1 \end{array}\right\} \Delta_4^{LR}(z,\nu), \qquad z^\alpha = X^\alpha(\tau') - x^\alpha(\tau). \tag{52}$$

The expression obtained allows us to interpret $\pm\operatorname{tr}\alpha^{B,F}$ as a functional describing the interaction of two vector or scalar sources by the exchange of vector or scalar quanta with spacelike momenta. One of the sources moves along the mirror trajectory while the other simultaneously moves along the tangent line to the trajectory at zero point. The second source can be considered as a probe or detector of the excitation created by the accelerated mirror in the vacuum.

7. TRACES OF THE BOGOLIUBOV COEFFICIENTS FOR HYPERBOLIC AND EXPONENTIAL TRAJECTORIES

We consider $\operatorname{tr}\alpha^{B,F}$ for the hyperbolic mirror trajectory

$$u^{mir} = g(v) = \frac{\varkappa' v}{\varkappa(1-\varkappa' v)}. \tag{53}$$

Using Eqs. (14) and (4) in [3], it is not difficult to represent $\alpha^{B,F}_{\omega'\omega}$ via the Macdonald functions $K_{1,0}$,

$$\alpha^{B,F}_{\omega'\omega} = \frac{2}{\sqrt{\varkappa\varkappa'}} \times \exp\left[i\left(\frac{\omega}{\varkappa} + \frac{\omega'}{\varkappa'}\right)\right] K_{1,0}\left(2i\sqrt{\frac{\omega\omega'}{\varkappa\varkappa'}}\right). \tag{54}$$

In accordance with (43), we then have

$$\operatorname{tr}\alpha^{B,F} = \frac{1}{\pi}\int_0^\infty d\left(\frac{\omega}{\varkappa}\right)\exp\left(2i\frac{\omega}{\varkappa}\right)K_{1,0}\left(2i\frac{\omega}{\varkappa}\right) = = \frac{1}{2\pi}\int_0^\infty dz\exp(iz)\,K_{1,0}(iz). \tag{55}$$

The variable z in this integral has a simple physical meaning: it is equal to the ratio of the invariant momentum transfer to the invariant proper acceleration at zero point (but for hyperbolic motion, the acceleration is the same on the entire trajectory),

$$z = \frac{\rho}{w_0}, \quad \rho = 2\sqrt{\omega\omega'}, \quad w_0 = \sqrt{\varkappa\varkappa'}. \qquad (56)$$

The ultraviolet divergence of integral (55) is removed by subtracting the leading term of the $z \to \infty$ expansion from the integrand. The infrared divergence (in the Bose case) is removed by introducing a nonzero lower limit $\varepsilon = \nu/w_0 \ll 1$ defined by the minimal momentum transfer ν. As a result, we obtain the integral

$$\operatorname{tr}\alpha^{B,F} = \frac{1}{2\pi}\int\limits_{s\varepsilon}^{\infty} dz \left[e^{iz} K_s(iz) - \sqrt{\frac{\pi}{2iz}}\right], \qquad (57)$$
$$s = 1, 0, \quad \varepsilon \ll 1.$$

The integration contour can now be rotated to the negative imaginary semiaxis such that in the Bose case, it bypasses the singularity at zero along the arc of a circle with a small radius ε. Further calculation leads to the simple expressions

$$\operatorname{tr}\alpha^{B} = \frac{1}{2\pi}\left[-\frac{\pi}{2} - i\left(\ln\frac{2w_0}{\gamma\nu} - 1\right)\right], \qquad (58)$$
$$\nu \ll w_0, \quad \gamma = 1,781\ldots,$$

$$\operatorname{tr}\alpha^{F} = \frac{1}{2\pi} i. \qquad (59)$$

For the exponentional motion of the mirror with

$$u^{mir} = -\frac{1}{\varkappa}\ln(1 - \varkappa' v), \qquad (60)$$
$$v^{mir} = \frac{1}{\varkappa'} - \frac{1}{\varkappa'}\exp(-\varkappa u),$$

the same Eqs. (14) and (4) in [3] lead to the Bogoliubov coefficients

$$\alpha^{B}_{\omega'\omega} = \frac{1}{\varkappa}\sqrt{\frac{\omega}{\omega'}}\Gamma\left(\frac{i\omega}{\varkappa}\right)\exp\left(i\frac{\omega'}{\varkappa'} - \frac{i\omega}{\varkappa}\ln\frac{i\omega'}{\varkappa'}\right), \qquad (61)$$

$$\alpha^{F}_{\omega'\omega} = \frac{1}{\sqrt{i\varkappa\omega'}}\Gamma\left(\frac{1}{2} + \frac{i\omega}{\varkappa}\right) \times \exp\left(\frac{i\omega'}{\varkappa'} - \frac{i\omega}{\varkappa}\ln\frac{i\omega'}{\varkappa'}\right). \qquad (62)$$

The traces $\operatorname{tr}\alpha^{B,F}$ whose divergences were removed by the above prescription are given by

$$\operatorname{tr}\alpha^{\rm B} = \frac{1}{2\pi}\int\limits_{\varepsilon}^{\infty} dx \times \left[\Gamma(ix)\exp(ix - ix\ln ix) - \sqrt{\frac{2\pi}{ix}}\right], \qquad (63)$$

$$\operatorname{tr}\alpha^{\rm F} = \frac{1}{2\pi}\int\limits_{0}^{\infty} dx \times \left[\Gamma\left(\frac{1}{2} + ix\right)\frac{\exp(ix - ix\ln ix)}{\sqrt{ix}} - \sqrt{\frac{2\pi}{ix}}\right]. \qquad (64)$$

In these integrals, the variable x is equal to one fourth of z, which has the meaning of the momentum transfer in units of w_0 (as in (56)),

$$x = \frac{1}{4}z, \quad z = \frac{\rho}{w_0}, \qquad (65)$$
$$\rho = 2\sqrt{\omega\omega'}, \quad w_0 = \frac{1}{2}\sqrt{\varkappa\varkappa'}.$$

Similarly, $\varepsilon = \nu/4w_0 \ll 1$. We note that in the course of exponential motion (60), the proper acceleration increases from zero to infinity; as a function of the proper time τ, it is given by

$$a(\tau) = -\frac{w_0}{1 - w_0\tau}. \qquad (66)$$

It is now not difficult to see that the subtracted terms in integrals (63) and (64) exactly coincide with similar terms in integrals (57) if we express them through the physical variable z. In other words, up to the removal of the ultraviolet divergence from the integrals defining $\operatorname{tr}\alpha$, the asymptotic behavior of the integrands in the variable $z = \rho/w_0 \to \infty$ is described by the universal formula

$$\frac{1}{2\pi}\sqrt{\frac{\pi}{2iz}}. \qquad (67)$$

We show in the next section that this assertion is correct for any timelike trajectory in expansion (40) for which $b > 0$.

The integration contour in integrals (63) and (64) can be rotated to the negative imaginary axis bypassing the infrared singularity at zero (in the Bose case) along the arc with a radius ε. We then obtain

$$\operatorname{tr}\alpha^{B} = \frac{1}{2\pi}\left[-\frac{\pi}{2} - i\left(\ln\frac{4w_0}{\nu} - \int\limits_0^{\infty} dt\ln t B'(t)\right)\right], \quad \nu \ll w_0, \qquad (68)$$

$$\operatorname{tr}\alpha^{F} = -\frac{1}{2\pi}\int\limits_0^{\infty}\frac{dt}{\sqrt{t}}\left(\Gamma\left(\frac{1}{2} + t\right) \times \exp(t - t\ln t) - \sqrt{2\pi}\right) = \frac{1}{2\pi} i \cdot 0.8843\ldots \qquad (69)$$

In the integral in (68), the function $B'(t)$ is the derivative of the function

$$B(t) = \Gamma(1+t)\exp(t - t\ln t) - \sqrt{2\pi t}.$$

The numerical value of this integral is 2.2194 .... If we transform the imaginary part of (68) to the form of the imaginary part of (58), we obtain

$$\ln\frac{4w_0}{\nu} - 2.2194\ldots = \ln\frac{2w_0}{\gamma\nu} - 0.9491\ldots$$

Therefore, the values of $\mathrm{tr}\,\alpha^{B,F}$ for the exponential and hyperbolic motions are rather close to each other.

8. ULTRAVIOLET AND INFRARED SINGULARITIES OF $\mathrm{tr}\alpha^{B,F}$

It is not difficult to obtain the general expression for $\mathrm{tr}\,\alpha^{B,F}$ in the form of a double integral that is a functional of the mirror trajectory and is tangent to it at the point $u = v = 0$. Indeed, after substitution of the Bogoliubov coefficients

$$\begin{aligned} \alpha^B_{\omega'\omega} &= \sqrt{\frac{\omega'}{\omega}}\int\limits_{-\infty}^{\infty} dv\,\exp(i\omega' v - i\omega g(v)),\\ \alpha^F_{\omega'\omega} &= \int\limits_{-\infty}^{\infty} dv\,\sqrt{g'(v)}\exp(i\omega' v - i\omega g(v)) \end{aligned} \tag{70}$$

in (43) and a trivial integration over the frequency ω', we obtain

$$\mathrm{tr}\,\alpha^{B,F} = \frac{1}{2\pi}\int\limits_0^{\infty} d\left(\frac{\omega}{\varkappa}\right)\times \int\limits_{-\infty}^{\infty} dx\,\{1,\,\sqrt{G'(x)}\}\exp\left[-i\frac{\omega}{\varkappa}(G(x)-x)\right], \tag{71}$$

where 1 and $\sqrt{G'(x)}$ in the braces refer to the Bose and Fermi cases respectively. The Lorentz invariance of these expressions is evident. But the integral over $(\omega/\varkappa)$ diverges at the upper limit because the integrand behaves as $\sqrt{\varkappa/\omega}$ at $\omega/\varkappa\to\infty$. Indeed, the condition $|x|\ll 1$ is essential in the integral over x as $\omega/\varkappa\to\infty$. The functions $G(x)-x$ and $G'(x)$ can then be replaced by the first terms of their expansions near zero, that is, by bx^2 and 1, see (40). Consequently, at $\omega/\varkappa\to\infty$, the integral over x is given by

$$\int\limits_{-\infty}^{\infty} dx\,\exp\left(-i\frac{\omega}{\varkappa}bx^2\right) = \sqrt{\frac{\pi\varkappa}{ib\,\omega}} \tag{72}$$

in both the Bose and Fermi cases.

It is easy to show that the next term of the asymptotic expansion of the integral over x behaves as $(\varkappa/\omega)^{3/2}$. Then, after subtraction from the integral over x of the first term of its asymptotic expansion in the parameter $\omega/\varkappa\to\infty$, we make the integral over $\omega/\varkappa$ convergent at the upper limit. If we pass from the variable $\omega/\varkappa$ to the variable z,

$$\frac{\omega}{\varkappa} = \sqrt{\frac{\omega\omega'}{\varkappa\varkappa'}} = \frac{b\rho}{2w_0} = \frac{1}{2}bz, \tag{73}$$

the subtracted term in $\mathrm{tr}\,\alpha^{B,F}$ acquires the universal form

$$\frac{1}{2\pi}\int\limits_0^{\infty} dz\,\sqrt{\frac{\pi}{2iz}}. \tag{74}$$

We recall that $z = \rho/w_0$ has the meaning of the invariant momentum transfer in units of proper acceleration.

Although the expressions

$$\mathrm{tr}\,\alpha^{B,F} = \frac{1}{2\pi}\int\limits_0^{\infty} ds\left[\int\limits_{-\infty}^{\infty} dx\left\{1,\,\sqrt{G'(x)}\right\}\times \exp\left(-is(G(x)-x)\right) - \sqrt{\frac{\pi}{ibs}}\right],\quad s = \frac{\omega}{\varkappa}, \tag{75}$$

do not contain ultraviolet divergences, they can contain infrared divergences if the spectral function (the function of s in the square brackets in (75)) has the singular behavior $\propto 1/s$ as $s\to 0$. It is clear that the behavior of the spectral function near $s = \omega/\varkappa = 0$ and in the main forming region of the integral over s is determined by the behavior of the trajectory $G(x)$ far from the point of contact, where expansion (40) cannot be applied, i.e., at the distances $|x|\gtrsim 1$.

We now demonstrate the application of Eq. (75) in the example of another trajectory

$$u^{mir} = -\frac{1}{\varkappa}\ln(2-e^{\varkappa' v}),\quad G(x) = -\ln(2-e^x), \tag{76}$$

for which the spectral function can be expressed in terms of the well-known transcendental functions. This trajectory, as the hyperbolic one in (53), has two asymptotes but approaches them following an exponential, not a power-like law. Therefore, on both ends of the trajectory, the proper acceleration

$$a(v) = -\sqrt{\frac{\varkappa\varkappa'}{e^{\varkappa' v}(2-e^{\varkappa' v})}} \tag{77}$$

tends to $-\infty$ and attains the minimal value in its modulus $a_0 = -\sqrt{\varkappa\varkappa'}$ at zero point.

The integral over x in (75), in which the upper limit for trajectory (76) is equal to ln 2, is reduced to the tabular integral 2.2.5.1 in [10] after changing the variable x to $t = 1 - e^x$. As a result, we obtain

$$\operatorname{tr}\alpha^B = \frac{1}{2\pi}\int\limits_{\varepsilon}^{\infty} ds \left[\frac{\sqrt{\pi}\,\Gamma(is)}{\Gamma(\frac{1}{2}+is)} - \sqrt{\frac{\pi}{is}}\right], \tag{78}$$

$$\operatorname{tr}\alpha^F = \frac{1}{2\pi}\int\limits_{0}^{\infty} ds \left[\frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}+is)}{\Gamma(1+is)} - \sqrt{\frac{\pi}{is}}\right]. \tag{79}$$

Because the spectral function has an infrared singularity in the Bose case, the corresponding divergence of the integral over s for $\operatorname{tr}\alpha^B$ is removed by introducing a small but finite lower limit $\varepsilon = \nu/w_0$. Its physical meaning is the minimum momentum transfer in units of the acceleration at zero point.

After rotating the s-integration contour to the negative imaginary semiaxis with the singularity at zero bypassed (in the Bose case) along the arc of a circle with the radius ε, we obtain

$$\operatorname{tr}\alpha^B = \frac{1}{2\pi}\left[-\frac{\pi}{2} - i\left(\ln\frac{w_0}{\nu} - B\right)\right], \tag{80}$$

$$\operatorname{tr}\alpha^F = \frac{1}{2\pi} i \cdot F, \tag{81}$$

where positive constants B and F are defined by the integrals

$$B = \int\limits_0^{\infty} dt\, \ln t\, B'(t) = 1.887789\ldots, \qquad B(t) = \frac{\sqrt{\pi}\Gamma(1+t)}{\Gamma\left(\frac{1}{2}+t\right)} - \sqrt{\pi t}, \tag{82}$$

$$F = -\int\limits_0^{\infty} dt \left[\frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}+t\right)}{\Gamma(1+t)} - \sqrt{\frac{\pi}{t}}\right] = 1.869957\ldots \tag{83}$$

The imaginary part of (80) can be transformed to the form of the imaginary part of (57),

$$\ln\frac{w_0}{\nu} - 1.887789\ldots = \ln\frac{2w_0}{\gamma\nu} - 2.003721\ldots$$

The expressions for $\pm\operatorname{tr}\alpha^{B,F}$ obtained for the three different trajectories of the mirror are close to each other qualitatively and quantitatively, see (58), (59), (68), (69), and (80), (81). All of them have a negative imaginary part with an infrared logarithmic singularity in the Bose case. This singularity is accompanied by the appearance of the real negative part of $\operatorname{tr}\alpha^B$, namely, $\operatorname{Re}\operatorname{tr}\alpha^B = -1/4$, whereas $\operatorname{Re}\operatorname{tr}\alpha^F = 0$. Such expressions for $\pm\operatorname{tr}\alpha^{B,F}$ are typical of trajectories whose $G(x)$ function increases stronger (falls weaker) than x as x tends to the upper (lower) limit.

Because the functionals $\pm\operatorname{tr}\alpha^{B,F}$ have the meaning of the action in accordance with (52), we compare them with the changes $\Delta W_{1,0}$ of self-actions of the electric and scalar charges in hyperbolic motion [11, 12],

$$\Delta W_{1,0} = -(\tau_2 - \tau_1)\Delta m_{1,0}, \tag{84}$$

$$\Delta m_1 = \frac{e^2 w_0}{4\pi^2}\left[-\frac{\pi}{2} - i\left(\ln\frac{2w_0}{\gamma\mu} - \frac{1}{2}\right)\right], \qquad \Delta m_0 = -i\frac{e^2 w_0}{8\pi^2}. \tag{85}$$

In this motion, the proper acceleration of the charge is constant and the square of the interval between two points on the trajectory is a function of only the length of the arc connecting them,

$$\left(x_\alpha(\tau) - x_\alpha(\tau')\right)^2 = f(\tau - \tau'). \tag{86}$$

Therefore, the change of the charge self-interaction is proportional to the time interval $\tau_2 - \tau_1$ that the charge is in hyperbolic motion multiplied by the mass shift $\Delta m_{1,0}$ of the charge. The mass shift occurs because of a change of the interaction of the charge with its own field, which is essentially modified at the distances of the order of w_0^{-1} from the charge due to acceleration. In other words, the shift is formed on the arc length

$$|\tau - \tau'| \sim w_0^{-1}$$

with the center τ_c at any point of the trajectory inside the acceleration interval (τ_1, τ_2). The independence of the shift from τ_c means that it is a constant of motion. This is not so for trajectories with a variable acceleration, see Sec. 9.

Unlike $\Delta W_{1,0}$, which describes the change of interaction of the charge with itself due to acceleration, the functionals $\pm\operatorname{tr}\alpha^{B,F}$ describe the interaction of the accelerated mirror with the probe executing the uniform motion along the tangent to the trajectory of the mirror at the point where the mirror has the acceleration w_0. This interaction is transmitted by vector or scalar perturbations created by the mirror in the Bose- or Fermi-field vacuum; these perturbations carry a spacelike momentum of the order of w_0. According to (51), at the distances of the order of w_0^{-1} from the mirror, the field of these perturbations decreases exponentially in timelike directions and oscillates with a damped amplitude in spacelike directions. It can be said that such a field moves together with the mirror and is its «proper field». Hence, the probe interacts with the mirror for a time of the order of w_0^{-1}, while the charge interacts with itself all the time and feels the change of the interaction over the entire time of acceleration. It is therefore not surprising that the expressions for $\pm\mathrm{tr}\,\alpha^{B,F}$ coincide in essence with $\Delta W_{1,0}$ if we set

$$\tau_2 - \tau_1 = 2\pi/w_0, \quad e^2 = 1$$

in the latter and reverse the sign. In other words, $\pm\mathrm{tr}\,\alpha^{B,F}$ are the mass shifts of the proper field of the mirror multiplied by a characteristic proper time of their formation.

9. MASS SHIFTS OF ELECTRIC AND SCALAR CHARGES IN EXPONENTIAL MOTION

To calculate the self-actions of electric and scalar charges in exponential motion, we use Eq. (30). It is convenient to use the charge trajectory (60) in the form of a function of the proper time,

$$\begin{aligned} u^{mir}(\tau) &= -\frac{2}{\varkappa}\ln(1 - w_0\tau), \\ v^{mir}(\tau) &= \frac{1}{\varkappa'}(2w_0\tau - w_0^2\tau^2). \end{aligned} \tag{87}$$

Then

$$\begin{aligned} \dot{x}_\alpha(\tau)\dot{x}^\alpha(\tau') &= -\frac{1+z^2}{1-z^2}, \\ (x - x')^2 &= -(\tau - \tau')^2\frac{\mathrm{Arth}\,z}{z}, \\ z &= \frac{w_0(\tau - \tau')}{2 - w_0(\tau + \tau')}. \end{aligned} \tag{88}$$

We now introduce new variables $\xi = (\tau + \tau')/2$ and z instead of τ and τ'. At fixed ξ in the interval $-\infty < \xi < w_0^{-1}$, the variable z changes in the interval $-1 < z < 1$. Using the causal function Δ_4^f expressed via the Macdonald function, we obtain

$$\Delta W_1 = e^2 \int\limits_{-\infty}^{w_0^{-1}} d\xi \left(\frac{1}{w_0} - \xi\right) \times \int\limits_{-1}^{1} dz\, \dot{x}_\alpha(\tau)\dot{x}^\alpha(\tau')\, \Delta_4^f(x - x', \mu)\Big|_0^F = -\frac{e^2}{2\pi^2}\int\limits_{-\infty}^{w_0^{-1}} d\xi \int\limits_0^\infty \frac{du\,\mu}{\mathrm{sh}\,2u} \times \left\{\mathrm{ch}\,2u\frac{\mathrm{th}\,u}{u}K_1\left(i\lambda\sqrt{u\,\mathrm{th}\,u}\right) - K_1(i\lambda\,\mathrm{th}\,u)\right\}. \tag{89}$$

In the last expression, the variable $u = \mathrm{Arth}\,z$ is used instead of z and λ is a function of ξ,

$$\lambda(\xi) = 2\mu(w_0^{-1} - \xi).$$

Our problem is now to find the integral over u in the region of the variable ξ where $\lambda(\xi) \ll 1$, supposing, of course, that the infrared parameter $\mu/w_0 \ll 1$. This integral coincides, in essense, with the mass shift of the electric charge,

$$\Delta m_1 = \frac{e^2}{2\pi^2}\int\limits_0^\infty \frac{du\,\mu}{\mathrm{sh}\,2u}\left\{\mathrm{ch}\,2u\sqrt{\frac{\mathrm{th}\,u}{u}} \times K_1\left(i\lambda\sqrt{u\,\mathrm{th}\,u}\right) - K_1(i\lambda\,\mathrm{th}\,u)\right\}. \tag{90}$$

To calculate Δm_1 with $\lambda(\xi) \ll 1$, we divide the integration interval into two intervals, $0 \leqslant u \leqslant u_1$ and $u_1 \leqslant u < \infty$, by a point u_1 such that $u_1 \gg 1$, but $\lambda u_1 \ll 1$. Using the expansion of the Macdonald function at a small argument, we then obtain

$$\Delta m_1 \approx \frac{e^2 w_0}{4\pi^2(1 - w_0\xi)} \times \left\{\frac{1}{i}\int\limits_0^{u_1} du\left(\frac{\mathrm{cth}\,2u}{u} - \frac{1}{2\,\mathrm{sh}^2 u}\right) + \int\limits_{u_1}^\infty \frac{du\,\lambda}{\sqrt{u}}K_1(i\lambda u)\right\} = \frac{e^2 w_0}{4\pi^2(1 - w_0\xi)} \times \left\{-\pi - i\left(2\ln\frac{w_0}{\gamma\mu(1 - w_0\xi)} + \ln\frac{2\gamma}{\pi} + \frac{1}{2}\right)\right\}. \tag{91}$$

The mass shift Δm_0 of the scalar charge differs from (90) by the replacement $\mathrm{ch}\,2u \to -1$ in the first term in the braces and by the change of sign of the second term. Under the same condition $\lambda(\xi) \ll 1$, we then obtain

$$\Delta m_0 = -i\frac{e^2 w_0}{4\pi^2(1-w_0\xi)}\left(\ln 2 - \frac{1}{2}\right). \quad (92)$$

It follows from (91), (92), and (66) that the mass shift depends on the absolute value

$$w(\xi) = \frac{w_0}{1-w_0\xi}$$

of the proper acceleration of the mirror at the instant ξ, which may be considered as the center of the forming region of the shift. As the acceleration essentially changes on such an interval, the mass shifts in (91) and (92) do not coincide with the mass shifts of uniformly accelerated charges in (84) and (85) if we replace $w(\xi)$ with w_0. Nevertheless, rather close coincidence arises under the replacement $w(\xi) \to 0,5\,w_0$ and $w(\xi) \to 2,6\,w_0$ for Δm_1 and Δm_0 correspondingly.

10. CONCLUSION

The basis for the symmetry between processes induced by a mirror in 2-dimensional and by a charge in 4-dimensional space–time is relation (14), (15) between the Bogoliubov coefficients $\beta^{B,F}_{\omega'\omega}$ and the current density $j^\alpha(k)$ or charge density $\rho(k)$ depending on the timelike momentum k^α. The squares of these quantities represent the spectra of real pairs and particles radiated by the accelerated mirror and the charge.

In the present paper, the symmetry is extended to the self-actions of the mirror and the charge and to the corresponding vacuum–vacuum amplitudes, cf. (29) and (30). In essence, it is embodied in the discovered relation (20) between propagators of a massive pair in 2-dimensional space and of a single particle in 4-dimensional space.

Equation (29) for $W^{B,F}$ was obtained under the condition that the mean number $\mathrm{tr}\,\beta^+\beta$ of pairs created is small and the interference of two or more pairs is negligible. In the general case, $W^{B,F}$ is given by Eq. (27), which can also be written as

$$2\,\mathrm{Im}\,W^{B,F} = \pm\mathrm{tr}\,\ln(\alpha^+\alpha)^{B,F} \quad (93)$$

because

$$\alpha^+\alpha \mp \beta^+\beta = 1,$$

see [7], [4]. As follows from (27) or (93), the imaginary part of the action differs from zero and is then positive only if $\beta \neq 0$, i.e., if the radiation of real particles indeed occurs.

For $W^{B,F}$, formula (93) allows us to choose the expression

$$W^{B,F} = \pm i\,\mathrm{tr}\,\ln\,\alpha^{B,F}, \quad (94)$$

which was called natural by DeWitt [7]. But this expression is by no means unique. The expressions

$$W^{B,F} = \pm i\,\mathrm{tr}\,\ln(\alpha e^{i\gamma})^{B,F}, \quad W^{B,F} = \pm i\,\mathrm{tr}\,\ln\alpha^{B,F+}$$

have the same imaginary part. Nevertheless, Eq. (94) is interesting as the definition of both the real and imaginary parts of the self-actions $W^{B,F}$ through the Bogoliubov coefficients $\alpha^{B,F}_{\omega'\omega}$ only, which reduce to the current density $j^\alpha(q)$ or to the charge density $\rho(q)$ that depends on the spacelike momentum q^α in accordance with Eqs. (14) and (15). This implies that the field of the corresponding perturbations propagates in the vacuum together with the mirror, comoves it, and at the same time contains information about the radiation of real quanta.

Unfortunately, the author failed to find a simple integral representation for the matrix $\ln\alpha$. Nevertheless, if we again assume that the mean number of emitted particles is small, we can consider α or $i\alpha$ close to 1. Expanding $\ln i\alpha$ near $i\alpha = 1$ and confining ourselves to the first term, we then obtain

$$W^{B,F} = \pm i\,\mathrm{tr}\,\ln i\alpha^{B,F} \approx \pm i\mathrm{tr}\,(i\alpha^{B,F} - 1) = \\ = \mp\mathrm{tr}\,\alpha^{B,F} + \ldots \quad (95)$$

These qualitative arguments allow us to state that the functionals $\pm\mathrm{tr}\,\alpha^{B,F}$ are similar to the corresponding self-actions with the opposite sign and must therefore have negative imaginary parts. This is confirmed by all examples considered in Secs. 7 and 8. Nevertheless, the exact physical meaning of the quantities $\pm\mathrm{tr}\,\alpha^{B,F}$ is clearly defined by Eq. (52).

Here, we also want to focus attention on one prediction following from the symmetry between processes induced by the mirror in 2-dimensional and by the charge in 4-dimensional space–times. The symmetry predicts the value $e_0^2 = 1$ for the charge squared (in Heaviside's units), which corresponds to the fine structure constant $\alpha_0 = 1/4\pi$. Because the radiation corrections are not taken into account in both spaces and the processes in $1+1$-dimensional space are due to the purely geometrical boundary condition, it is natural to think that the above-mentioned values of the charge squared and of the fine structure constant are the unrenormalized bare values of these constants. They are therefore marked with the index 0.

It is quite interesting that the bare fine structure constant has a purely geometrical origin and that its value is small,

$$\alpha_0 = 1/4\pi \ll 1.$$

The smallness of α_0 has the essential meaning for the quantum elecrodynamics, where it a priori justifies the applicability of the perturbation theory and where the radiative corrections in accordance with the well known formula [13]

$$\alpha = \frac{\alpha_0}{1 + (\alpha_0/3\pi) N \ln(\Lambda^2/m^2)} \tag{96}$$

diminish the renormalized value of α in comparison with the unrenormalized one. Here, N is the number of charged particles with masses in the interval (m, Λ) and Λ is the upper limit of the particle energy up to which the quantum electrodynamics is correct.

APPENDIX

It is convenient to define the singular function $\Delta_d^{LR}(z,\nu)$ and the causal function $\Delta_d^f(z,\mu)$ in a d-dimensional space–time by the Fourier representation

$$\begin{aligned} \Delta_d^{LR}(z,\nu) &= \int \frac{d^d q}{(2\pi)^d} \, \frac{e^{iqz}}{q^2 - \nu^2 + i\varepsilon}, \\ \Delta_d^{f}(z,\mu) &= \int \frac{d^d q}{(2\pi)^d} \, \frac{e^{iqz}}{q^2 + \mu^2 - i\varepsilon}. \end{aligned} \tag{97}$$

These functions are the even singular solutions of the inhomogeneous wave equations

$$\begin{aligned} (-\partial^2 - \nu^2)\, \Delta^{LR}(z,\nu) &= \delta(z), \\ (-\partial^2 + \mu^2)\, \Delta^{f}(z,\mu) &= \delta(z), \end{aligned} \tag{98}$$

with opposite signs in front of the parameters ν^2 and μ^2, where ν and μ are the momentum transfer and the mass. Their proper time representations (in particular, for $d = 4$)

$$\begin{aligned} \Delta_4^{LR}(z,\nu) &= \frac{1}{(4\pi)^2} \int\limits_0^\infty \frac{ds}{s^2} \exp\left(-i\nu^2 s - \frac{iz^2}{4s}\right), \\ \Delta_4^{f}(z,\mu) &= \frac{1}{(4\pi)^2} \int\limits_0^\infty \frac{ds}{s^2} \exp\left(-i\mu^2 s + \frac{iz^2}{4s}\right) \end{aligned} \tag{99}$$

as well as the explicit expressions in terms of the Macdonald function differ by complex conjugation and by the replacement $\mu \to i\nu$ or by the replacement $z^2 \to -z^2$, $\mu \to \nu$.

The integral relation

$$\Delta_{d+2}^{LR}(z,\nu) = -\frac{1}{4\pi} \int\limits_{\nu^2}^\infty d\rho^2 \, \Delta_d^{LR}(z,\rho) \tag{100}$$

is very important for the symmetry discussed in this paper. It differs from similar relation (20) for the causal functions not only by the sign. Being written for $z^2 < 0$, it is understood for $z^2 > 0$ in the sense of analytic continuation to the lower half-plane of complex z^2. On the other hand, relation (20), being written for $z^2 > 0$, is understood for $z^2 < 0$ as the analytic continuation to the upper half-plane of complex z^2. For the Δ^+-functions, such a continuation must be carried out in the upper half-plane if $z^0 > 0$ and in the lower one if $z^0 < 0$.

The author is grateful to A. I. Nikishov and A. I. Ritus for valuable discussions and help. The work was supported by the RFBR (grants № 00-15-96566 and 02-02-16944).

REFERENCES

1. S. W. Hawking, Commun. Math. Phys. **43**, 199 (1979).
2. A. I. Nikishov and V. I. Ritus, Zh. Eksp. Teor. Fiz. **108**, 1121 (1995).
3. V. I. Ritus, Zh. Eksp. Teor. Fiz. **110**, 526 (1996).
4. V. I. Ritus, Zh. Eksp. Teor. Fiz. **114**, 46 (1998).
5. J. Schwinger, *Particles, Sources and Fields*, Addison-Wesley, Reading, Mass. (1970), Vol. 1.
6. A. P. Lightman, W. H. Press, R. H. Price, and S. A. Teukolsky, *Problem Book in Relativity and Gravitation*, Princeton University Press, Princeton, New Jersey (1975).
7. B. S. DeWitt, Phys. Rep. C **19**, 295 (1975).
8. R. M. Wald, Commun. Math. Phys. **45**, 9 (1975).
9. V. I. Ritus, Zh. Eksp. Teor. Fiz. **116**, 1523 (1999); E-print archives, hep-th/9912004.
10. A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series, Elementary Functions*, Nauka, Moscow (1981).
11. V. I. Ritus, Zh. Eksp. Teor. Fiz. **75**, 1560 (1978).
12. V. I. Ritus, Zh. Eksp. Teor. Fiz. **80**, 1288 (1981).
13. V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, *Quantum Electrodynamics*, Nauka, Moscow (1989).